Kepler's Laws from the Universal Law of Gravitation
An example of the power of vector analysis
adapted from *Calculus with Analytic Geometry*
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In what follows **boldface** indicates vector quantities.

Let $\mathbf{r}$ be the vector from the mass M to the mass m, and assume $\mathbf{r} \neq \mathbf{0}$.

The universal law of gravitation states that the force $\mathbf{F}$ acting on m due to M obeys

$$\mathbf{F} = -\frac{GmM}{r^2}\mathbf{u} \tag{1}$$

where $\mathbf{u}$ is the unit vector in the direction of $\mathbf{r}$. r will denote the magnitude of $\mathbf{r}$, $\mathbf{v}$ and $\mathbf{a}$ will denote the first and second time derivatives of $\mathbf{r}$ and v and a will denote their magnitudes respectively.

Since

$$\mathbf{F} = m\mathbf{a}$$

substituting into (1) we have

$$\mathbf{a} = -\frac{GM}{r^2}\mathbf{u}, \tag{2}$$

an expression independent of m!

Hence

$$\mathbf{a} \times \mathbf{r} = \mathbf{0}. \tag{3}$$

Therefore

$$(\mathbf{r} \times \mathbf{v})' = (\mathbf{v} \times \mathbf{v}) + (\mathbf{r} \times \mathbf{a}) = \mathbf{0} \tag{4}$$

so that we know there is some $\mathbf{c}$ so that

$$\mathbf{r} \times \mathbf{v} = \mathbf{c} \tag{5}$$

Therefore, either $\mathbf{c} = \mathbf{0}$ or $\mathbf{r}$ is always perpendicular to $\mathbf{c}$. In the latter case, the motion of m is in the plane containing M and perpendicular to $\mathbf{c}$. In the former case, the motion is linear, which is clearly false for planets, or the planet is stationary, also false.

Now we have

$$\mathbf{v} = (r\mathbf{u})' = r\mathbf{u}' + r'\mathbf{u} \tag{6}$$

and

$$\mathbf{u} \cdot \mathbf{u}' = 0.$$

Substituting (6) into (5) we get

$$\mathbf{c} = r\mathbf{u} \times (r\mathbf{u}' + r'\mathbf{u}) = r^2(\mathbf{u} \times \mathbf{u}') = r^2||\mathbf{u}'||. \tag{7}$$

Therefore, by combining (3), (4) and (7) we get

$$\begin{aligned}
\mathbf{a} \times \mathbf{c} &= \left(\frac{-GM}{r^2}\mathbf{u}\right) \times \left[r^2(\mathbf{u} \times \mathbf{u}')\right] \\
&= -GM\left[\mathbf{u} \times (\mathbf{u} \times \mathbf{u}')\right] \\
&= -GM\left[(\mathbf{u} \cdot \mathbf{u}')\mathbf{u} - (\mathbf{u} \cdot \mathbf{u})\mathbf{u}'\right] \\
&= GM\mathbf{u}'
\end{aligned}$$

On the other hand, since $\mathbf{c}$ is a constant vector we have

$$\mathbf{a} \times \mathbf{c} = (\mathbf{v} \times \mathbf{c})'$$

so

$$(\mathbf{v} \times \mathbf{c})' = GM\mathbf{u}',$$

or, by integrating from time equal to 0,

$$\mathbf{v} \times \mathbf{c} = GM\mathbf{u} + \mathbf{b}$$

for some constant vector $\mathbf{b}$. Since $\mathbf{b}$ is the difference between two vectors which are both perpendicular to $\mathbf{c}$ then $\mathbf{b}$ is perpendicular to $\mathbf{c}$ as well. Hence $\mathbf{r}$ and $\mathbf{b}$ are co-planar for all time.

Observe now that $\mathbf{c} \cdot \mathbf{c}$ is a positive constant, and

$$\begin{aligned} \mathbf{c} \cdot \mathbf{c} &= (\mathbf{r} \times \mathbf{v}) \cdot \mathbf{c} \\ &= \mathbf{r} \cdot (\mathbf{v} \times \mathbf{c}) \\ &= \mathbf{r} \cdot \left(\frac{GM}{r} \mathbf{r} \right) + \mathbf{r} \cdot \mathbf{b} \\ &= GMr + \mathbf{r} \cdot \mathbf{b} \end{aligned}$$

so

$$\mathbf{c} \cdot \mathbf{c} = GMr + \mathbf{r} \cdot \mathbf{b}. \tag{8}$$

Since $\mathbf{c}$ is constant, if $\mathbf{b} = \mathbf{0}$, then r is constant, and the motion lies on a circle.

If $\mathbf{b} \neq \mathbf{0}$ then (8) tells us that

$$r = \frac{c^2}{GM + b\cos(\theta)}, \tag{9}$$

where c is the length of $\mathbf{c}$, b is the length of $\mathbf{b}$ and θ is the angle between $\mathbf{r}$ and $\mathbf{b}$. This now makes sense because neither $\mathbf{r}$ nor $\mathbf{b}$ can be $\mathbf{0}$.

It is well-known that (9) is the equation of either an ellipse, parabola or hyperbola in polar coordinates, with one focus at the polar origin and the (major) axis of symmetry along the polar axis, which in this case is $\mathbf{b}$. This establishes the first of Kepler's Laws. In particular, the orbits of the planets are elliptical because they have been observed not to be linear, circular, parabolic or hyperbolic.

For the second law, we know that the total area swept out as θ changes from a to b is

$$\frac{1}{2} \int_a^b r^2 \, d\theta.$$

Fix a and let b vary. Denote this function of b by A. Then from the fundamental theorem of calculus we have $A'(\theta) = r^2/2$. By the chain rule, since θ is a function of t (time), we have

$$\frac{dA(\theta(t))}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}. \tag{10}$$

By projecting $\mathbf{u}$ onto any pair of fixed orthonormal vectors in the plane perpendicular to $\mathbf{c}$, say $\mathbf{x}$ and $\mathbf{y}$ and differentiating with respect to t we see that

$$||\mathbf{u}'|| = \left| \frac{d\theta}{dt} \right|.$$

By observation of actual orbits, $d\theta/dt$ does not change sign, and we may assume a convention by which it is postive. Therefore it follows from (7) and (10) that

$$\frac{dA(\theta(t))}{dt} = \frac{c}{2},$$

which is Kepler's Second Law. Kepler's Third Law now follows from Kepler's Second Law and purely geometric properties of ellipses.