Superiority of Root Test over Ratio Test

Suppose that $\{a(j), j = 0, \ldots\}$ is a sequence of strictly positive terms with *a*(0) = 1.

Put $A(n) = a_0 + \ldots + a_n$. The weak form of the ratio test says that if

$$\lim_{j \to \infty} \frac{a(j)}{a(j-1)} = R < 1$$

then

$$\lim_{n \to \infty} A(n) \in (1, \infty),$$

while if

$$\lim_{j \to \infty} \frac{a(j)}{a(j-1)} = R > 1$$

then

$$\lim_{n \to \infty} A(n) = \infty.$$

The weak form of the root test says that if

$$\lim_{j \to \infty} \sqrt[j]{a(j)} = S < 1$$

then

$$\lim_{n \to \infty} A(n) \in (1, \infty),$$

while if

$$\lim_{j \to \infty} \sqrt[j]{a(j)} = S > 1$$

then

$$\lim_{n \to \infty} A(n) = \infty.$$

Using elementary properties of logarithms these tests can be reformulated to say:

Ratio Test: If

$$\lim_{j\to\infty} \log\left(\frac{a(j)}{a(j-1)}\right) = r < 0$$

then

$$\lim_{n\to\infty} A(n) \in (1,\infty),$$

while if

$$\lim_{j\to\infty} \log\left(\frac{a(j)}{a(j-1)}\right) = r > 0$$

then

$$\lim_{n\to\infty} A(n) = \infty.$$

Root Test: If

$$\lim_{j\to\infty} \frac{1}{j}\log(a(j)) = s < 0$$

then

$$\lim_{n\to\infty} A(n) \in (1,\infty),$$

while if

$$\lim_{j\to\infty} \frac{1}{j}\log(a(j)) = s > 0$$

then

$$\lim_{n\to\infty} A(n) = \infty.$$

Since

$$a(j) = \prod_{k=1}^{j} \frac{a(k)}{a(k-1)}$$

we have

$$\frac{1}{j}\log(a(j)) = \frac{1}{j}\sum_{k=1}^{j}\log\left(\frac{a(k)}{a(k-1)}\right)$$

so if

$$\lim_{j\to\infty}\log\left(\frac{a(j)}{a(j-1)}\right) = r$$

then

$$\lim_{n\to\infty}\frac{1}{j}\log(a(j)) = r.$$

So the weak form of the root test will decide the convergence/divergence of $A(n)$ whenever the weak form of the ratio test does, and potentially will decide the convergence/divergence of $A(n)$ when the weak form of the ratio test fails to give any information. The demonstration above makes it clear why: The ratio test only uses one ratio, while the root test uses all the ratios. One might say, the root test is better because it remembers it roots.