Equivalent Defintions of e^x

It is well-known that for all complex numbers *z*,

$$\lim_{N\to\infty}\left(1+\frac{z}{N}\right)^N = \lim_{N\to\infty} 1+\sum_{k=1}^{N}\frac{1}{k!}z^k.$$

What follows below is a direct proof of this fact in the case where *z* is a positive real number. The proof is elementary in that it does not depend on limits superior and inferior, but instead on the Pinching Theorem.

Lemma 198

Suppose that k and N are positive integers and $k \le N$.Then

$$\left(1-\frac{1}{N}\right)\left(1-\frac{2}{N}\right)\cdots\left(1-\frac{k}{N}\right) \ge 1-\frac{k(k+1)}{2N}$$

Proof: The lemma is clearly true if

$$1-\frac{k(k+1)}{2N}\le 0$$

since the righthand side expression is never negative. Therefore, suppose that

$$1-\frac{k(k+1)}{2N} > 0.$$

In this case the proof is by induction on *k* along with the observation that for $0 < u, v < 1$, $(1-u)(1-v) > 1-(u+v)$. **QED**

It follows directly from the Binomial Theorem that for $N \ge 2$ and *x* > 0

$$\left(1+\frac{x}{N}\right)^N = 1+x+\sum_{k=2}^{N}\frac{1}{k!}\left(1-\frac{1}{N}\right)\cdots\left(1-\frac{k-1}{N}\right)x^k$$

so on the one hand we have

$$\left(1+\frac{x}{N}\right)^N \le 1+x+\sum_{k=2}^{N}\frac{1}{k!}x^k$$

while by applying the Lemma we have

$$\left(1+\frac{x}{N}\right)^N \ge 1+x+\sum_{k=2}^{N}\frac{1}{k!}\left(1-\frac{k(k-1)}{2N}\right)x^k$$

so that

$$\left|\left(1+\frac{x}{N}\right)^N - \left(\sum_{k=0}^{N}\frac{1}{k!}x^k\right)\right| \le \frac{1}{2N}\sum_{k=0}^{N-2}\frac{1}{k!}x^k$$

Since it is readily established that

$$\lim_{N\to\infty}\sum_{k=0}^{N}\frac{1}{k!}x^k$$

exists for positive x by using comparison to a geometric series, we see that

$$\left(1+\frac{x}{N}\right)^N$$

has the same limit as $N \to \infty$.