MARK COLARUSSO'S RESEARCH STATEMENT

1. Overview

I work in the fields of Lie theory, algebraic geometry, Poisson geometry and geometric representation theory. The bulk of my work focuses on Gelfand-Zeitlin theory, specifically the geometry of complex Gelfand-Zeitlin integrable systems and their applications to representation theory, algebraic groups, and Poisson geometry.

Gelfand-Zeitlin (GZ) theory began in the 1950's when Gelfand and Zeitlin found a canonical way of obtaining a basis for irreducible, finite dimensional representations of the complex general linear group and orthogonal groups [GC50a], [GC50b]. Since then Gelfand-Zeitlin theory has developed in many directions including geometry [GS83],[KM05], [Har06], combinatorics [DLM04], representation theory [Mol00], and applied matrix theory [PS08].

The geometric version of GZ theory began with the work of Guillemin and Sternberg in the early 1980's [GS83]. They produced an integrable system on conjugacy clases of Hermitian matrices that is related to the basis constructed by Gelfand and Zeitlin via geometric quantization. In 2006, Kostant and Wallach [KW06a],[KW06b] started a new chapter in GZ theory by developing a complexified version of the GZ integrable system on the Lie algebra of $n \times n$ complex matrices. This is the foundation for my work in [Col11, Col09, CE10, CE12, CE15, CE14, CL16, CEa].

In this research statement, we describe our work in studying the geometry of the GZ integrable systems on the complex general linear Lie algebra $\mathfrak{gl}(n, \mathbb{C})$ and on the complex orthogonal Lie algebra $\mathfrak{so}(n, \mathbb{C})$. The latter system was introduced by the author in [Col09]. With the goal of quantizing these systems, we describe the Lagrangian foliation of adjoint orbits for the general linear GZ system, and show that it is algebraically integrable (see Sections 3.1, 3.3). The quantum version of complex GZ systems is a category of infinite dimensional representations called Gelfand-Zeitlin modules (see Section 2). One of our main goals is to produce GZ modules geometrically using a construction similar to the Beilinson-Bernstein classification of classical Harish-Chandra modules [BB81]. With this in mind, we describe how the geometry of the moment map fibres for both orthogonal and general linear GZ systems can be understood using the theory of orbits of a symmetric subgroup on the flag variety and spherical pairs (see Section 3.4). This allows us to understand the geometry of orthogonal GZ systems and paves a way for a geometric construction of a category of generalized Harish-Chandra modules closed related to GZ modules using a generalization of the Beilinson-Bernstein approach (see Section 4). In the final sections, we describe infinite dimensional and nonlinear analogues of GZ systems and their applications to the study of direct limit groups and Poisson Lie groups respectively (see Sections 5.1 and 5.2 respectively).

2. Motivation: Gelfand-Zeitlin Modules

The problem of classifying all irreducible representations of a finite dimensional, complex Lie algebra $\mathfrak{g}$ is a wild problem. Even classifying irreducible representations of $\mathfrak{g}$ where the centre

$Z(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ acts by a character is intractable. Irrreducible Gelfand-Zeitlin modules are more a refined subcategory of such modules for $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ or $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$ for which an approach at classification appears to be possible. In more detail, for $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$, $\mathfrak{so}(n, \mathbb{C})$, we have a chain of subalgebras

$$\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots \subset \mathfrak{g}_i \subset \cdots \subset \mathfrak{g}_n = \mathfrak{g}, \tag{2.1}$$

where $\mathfrak{g}_i = \mathfrak{gl}(i, \mathbb{C})$, $\mathfrak{so}(i, \mathbb{C})$, respectively embedded in the $i \times i$ upper left hand corner of $\mathfrak{g}$. This chain of inclusions allows us to form an associative, abelian subalgebra of the enveloping algebra $U(\mathfrak{g})$ generated by the centres $Z(\mathfrak{g}_i)$ of $U(\mathfrak{g}_i)$ for $i = 1, \ldots, n$:

$$\Gamma := \langle Z(\mathfrak{g}_1) \ldots Z(\mathfrak{g}) \rangle \cong Z(\mathfrak{g}_1) \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} Z(\mathfrak{g}) \tag{2.2}$$

The associative algebra Γ is called the *Gelfand-Zeitlin* algebra.

Definition 2.1. A Gelfand-Zeitlin module M (GZ module) is a module for $U(\mathfrak{g})$ which is the sum of generalized Γ-eigenspaces.

Gelfand-Zeitlin modules were first introduced by Drozd, Futorny, and Ovsienko [DFO94] and have since been studied extensively [FOS11, FO07, Ovs03, Ovs02, MO98, Kho05, Ram12]. GZ modules are of interest in infinite dimensional representation theory for several reasons. For $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ the category of GZ modules can be viewed as a natural generalization of the category of weight modules studied by Fernando and Mathieu [Mat00],[Fer90]. In both orthogonal and general linear cases, GZ modules are also so natural generalizations of so-called generalized Harish-Chandra modules studied by Zuckerman, Penkov, and Serganova [PZ04b, PZ12, PZ04a, PZ07, PSZ04, PS12]. Many interesting results concerning generalized Harish-Chandra modules and GZ modules exists, but they are largely on a case-by-case basis and involve some difficult computations. One of the our main projects funded in our recent NSA grant proposal is to develop a geometric construction of GZ modules that encompasses the different cases studied by Futorny and his collaborators in one setting. This construction will be possible because of our understanding of complex GZ integrable systems and related actions of algebraic groups.

3. Geometry of GZ integrable system

3.1. **Integrable Systems and Quantization.** An integrable system on a symplectic manifold M of dimension $2n$ is a collection of n independent functions $\{F_1, \ldots, F_n\}$ such that $\{F_i, F_j\} = 0$ for $1 \leq i, j \leq n$. Here $\{\cdot, \cdot\}$ denotes the Poisson bracket on the algebra of smooth (or analytic) functions ton M. A particularly important role is played by the *moment* map:

$$\Phi : M \to F^n \text{ given by } \Phi(m) = (F_1(m), \ldots, F_n(m)). \tag{3.1}$$

It is an easy calculation in symplectic geometry to see that regular level sets of Φ are Lagrangian submanifolds of M (i.e. maximal isotropic submanifolds).

Of course, $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C}), \mathfrak{so}(n, \mathbb{C})$ is not a symplectic complex manifold, but a Poisson manifold when equipped with the Lie-Poisson structure, so the ring of analytic functions on $\mathfrak{g}$ comes equipped with a Poisson bracket $\{\cdot, \cdot\}$. This Poisson structure is fairly straightforward to describe. Let $<< \cdot, \cdot >>$ be the non-degnerate bilinear form on $\mathfrak{g}$ given by $<< x, y >>= \mathrm{tr}(xy)$, where $\mathrm{tr}(\cdot)$ denotes the trace form. The Poisson bracket of the linear functions $f_x(z) = \mathrm{tr}(xz)$ and $f_y(z) = \mathrm{tr}(yz)$ is just their Lie bracket, i.e.

$$\{f_x, f_y\} = f_{[x,y]}. \tag{3.2}$$

The Lie-Poisson structure is then given by extending (3.2) to all polynomials functions via the Leibniz rule and then to all smooth (or analytic) functions via the Stone-Weierstrass theorem. For example, if $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ and x_{ij} is the function on $\mathfrak{g}$ such that $x_{ij}(A) = A_{ij}$ for any $n \times n$ matrix A, then for any functions f and g on $\mathfrak{g}$, we have

$$\{f, g\} = \sum \frac{\partial f}{\partial x_{ij}} \frac{\partial g}{\partial x_{kl}} (\delta_{ik} x_{lj} - \delta_{lj} x_{ik}). \tag{3.3}$$

The Lie-Poisson structure is an example of a linear Poisson structure, since the Poisson bracket of two linear functions on $\mathfrak{g}$ is again linear (see (3.2)). We will discuss GZ theory for nonlinear Poisson structures in Section 5.2. It is well known that any Poisson manifold is foliated by symplectic submanifolds. In the case of $\mathfrak{g}$ with the Lie-Poisson structure these are just the $G = GL(n, \mathbb{C})$, $SO(n, \mathbb{C})$-adjoint orbits with the classical Kostant-Kirillov symplectic structure.

The idea of quantization comes from the passage from classical to quantum mechanics. In the Hamiltonian formalism of classical mechanics, one speaks of a phase space for the system where the trajectory and momentum of the system evolve with time. For motion with n degrees of freedom without constraints, the phase space is the cotangent bundle of $\mathbb{R}^n$, $T^*\mathbb{R}^n$ with position coordinates $q_1, \ldots, q_n$ and momentum coordinates $p_1, \ldots, p_n$. Recall that $T^*\mathbb{R}^n$ is naturally a symplectic manifold and the coordinates have the canonical Poisson commutation relations:

$$\{q_i, q_j\} = \{p_i, p_j\} = 0 \text{ and } \{p_j, q_i\} = \delta_{ij}. \tag{3.4}$$

In the formalism of quantum mechanics, matters has a wave nature rather than a particulate nature and phase space is replaced by a Hilbert space $L^2(q_1, \ldots, q_n)$ of probability distributions that in some sense describe how the matter is distributed throughout space. The momenta $p_1, \ldots, p_n$ are no longer functions on the space but differential operators $\frac{\partial}{\partial q_i}$ which act as self-adjoint operators on the Hilbert space. Notice that the Hilbert space has "half" the coordinates of the original classical phase space. This roughly corresponds to "cutting out" the Lagrangian submanifolds corresponding to level sets of the momentum functions $p_1, \ldots, p_n$ which form an integrable system on $T^*\mathbb{R}^n$ (The term "moment map" in the definition of the map in (3.1) comes from the fact that many classical integrable systems were given by momenta along various axes of rotation.)

Roughly speaking, there are two forms of quantization in mathematics: geometric quantization and deformation quantization. Geometric quantization attaches to a symplectic manifold M a Hilbert space in a vast generalization of the physical construction described above. A key ingredient in this construction is a foliation of M into Lagrangian submanifolds as might be given by the presence of an integrable system. Further, if M has an action of a Lie group G, then G acts on the resulting Hilbert space producing a unitary representation of G. On the other hand, deformation quantization replaces the Poisson algebra of functions on a symplectic or Poisson manifold M with a non-commutative, associative algebra that deforms to the Poisson algebra in the classical limit (see for example [Cat05].) In the example of $T^*\mathbb{R}^n$, the quantization of the Poisson algebra of polynomial functions on $T^*\mathbb{R}^n$ with commutation relations given in (3.4) is the Weyl algebra of differential operators: $\{q_i, \frac{\partial}{\partial q_j}\}$ on $L^2(q_1, \ldots, q_n)$ which have the commutation relations:

$$[q_i, q_j] = 0 = [\frac{\partial}{\partial q_i}, \frac{\partial}{\partial q_j}] \text{ and } [\frac{\partial}{\partial q_j}, q_i] = \delta_{ij}.$$

As another example of deformation quantization, consider the Lie-Poisson structure. If we think of the symmetric algebra $S(\mathfrak{g})$ of $\mathfrak{g}$ as polynomial functions on $\mathfrak{g}$ using the trace form, then we can realize the Lie-Poisson structure described in (3.3) as a kind of deformation of the associative algebra structure on the enveloping algebra $U(\mathfrak{g})$ (see [CG97], for example.) For instance, the associative commutator of two elements $x,\, y \in \mathfrak{g} \subset U(\mathfrak{g})$ is just their Lie bracket cf (3.2). This construction can also be thought of as replacing functions on the Poisson manifold by differential operators, since we can think of think of $U(\mathfrak{g})$ as left-invariant differential operators on G. In this philosophy of quantization, irreducible modules for $U(\mathfrak{g})$ correspond to Lagrangian subvarieties of adjoint orbits. Thus, it is natural to expect that Lagrangian foliation given by the complex GZ systems correspond to irreducible GZ modules. We will discuss in Section 4, how we will use the theory of differential operators on the flag variety to geometrically construct a category of $U(\mathfrak{g})$-modules closely related to GZ modules in our effort to quantize the GZ systems.

3.2. **Overview of GZ systems and early results.** The complex GZ systems on $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{C})$, $\mathfrak{so}(n,\mathbb{C})$ are constructed in an analogous way to the GZ algebra in (2.2). We start with the chain of subalgebras in (2.1). Let $G_i \cong GL(i,\mathbb{C})$, $SO(i,\mathbb{C})$ respectively be the closed subgroup corresponding to the subalgebra $\mathfrak{g}_i$. Let $\mathbb{C}[\mathfrak{g}_i]^{G_i}$ be the adjoint invariant polynomials on the subalgebra $\mathfrak{g}_i$, and let $\mathbb{C}[\mathfrak{g}]$ denote the ring of polynomial functions on $\mathfrak{g}$. Consider the subalgebra of $\mathbb{C}[\mathfrak{g}]$ generated by the rings $\mathbb{C}[\mathfrak{g}_i]^{G_i}$ for all $i = 1, \ldots, n$, i.e.

$$J(\mathfrak{g}) := \langle \mathbb{C}[\mathfrak{g}_1]^{G_1} \ldots \mathbb{C}[\mathfrak{g}]^{G} \rangle \cong \mathbb{C}[\mathfrak{g}_1]^{G_1} \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathbb{C}[\mathfrak{g}]^{G}. \tag{3.5}$$

The algebra $J(\mathfrak{g})$ is the classical analogue of the Gelfand-Zeitlin algebra in (2.2). Choosing an algebraically independent set of generators

$$J_{GZ} := \{f_{i,j};\, i = 1, \ldots, n;\, j = 1, \ldots, r_i = \operatorname{rank}(\mathfrak{g}_i)\}, \tag{3.6}$$

of the rings $\mathbb{C}[\mathfrak{g}_i]^{G_i}$ gives rise to the Gelfand-Zeitlin integrable system. For example, if $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{C})$, we can take where $f_{i,j}(x) =$ coefficient of t^{j-1} in the characteristic polynomial of x_i, where x_i is the $i \times i$ submatrix in the upper left hand corner of x. The restriction of the functions J_{GZ} to any regular G-adjoint orbit in $\mathfrak{g}$ forms an integrable system in the symplectic sense ([KW06a],[Col09],[CEb]).

The moment map for the GZ system, Φ, is often referred to in the literature as the *Kostant-Wallach* map (KW map):

$$\Phi : \mathfrak{g} \to \mathbb{C}^{r_1} \times \cdots \times \mathbb{C}^{r_i} \times \cdots \times \mathbb{C}^{r_n}, \text{ given by } \Phi(x) = (f_{i,j}(x))_{i=1,\ldots,n}^{j=1,\ldots,r_i}. \tag{3.7}$$

Remark 3.1. In the case of $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{C})$, the fibres of Φ are the closed subvarities of $\mathfrak{g}$:

$$F = \{x \in \mathfrak{g} : x_i \text{ has eigenvalues } \lambda_{i1}, \ldots, \lambda_{ii}, i = 1, \ldots, n\}, \tag{3.8}$$

where the λ_{ij} are fixed complex numbers and the eigenvalues x_i are listed with multiplicity. If $\mathfrak{g} = \mathfrak{so}(n,\mathbb{C})$ the description of the KW map fibres is slightly more complicated. If i is even, then x_i is required to have a prescribed set of eigenvalues and a prescribed Pfaffian.

If $f \in J_{GZ}$, the corresponding Hamiltonian vector field ξ_f is complete, and the abelian Lie algebra of GZ vector fields:

$$\mathfrak{a} = \{\xi_f : f \in J_{GZ}\} \tag{3.9}$$

integrates to a holomorphic action of $\mathbb{C}^d$ given by the joint flows of the vector fields ξ_f, $f \in J_{GZ}$, where d is the half the dimension of a regular G adjoint orbit on $\mathfrak{g}$. The generic orbits of

$\mathbb{C}^d$ form the irreducible components of the regular levels sets of the KW map in (3.7) and therefore are Lagrangian submanifolds of regular adjoint orbits. We call an element $x \in \mathfrak{g}$ *strongly regular* if the joint flows of the GZ vector fields through that element are Lagrangian and denote the strongly regular locus by $\mathfrak{g}_{sreg}$. We denote the strongly regular locus of each fibre by $F_{sreg} = F \cap \mathfrak{g}_{sreg}$. Given the importance of Lagrangian subvarieties in the study of quantization, we turn our attention to the study of $\mathfrak{g}_{sreg}$ and F_{sreg}.

The action of the flows of the GZ vector fields is too difficult to describe directly. We resolve this issue by locally algebraically integrating GZ vector fields on F_{sreg} to an action of a commutative algebra group.

(1) [Col11] (For $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$) The flows of the GZ vector fields on F_{sreg} coincide with an algebraic action of an abelian, algebraic group Z_F acting freely on F_{sreg}. This action is much easier to understand than the action of the GZ vector fields and allows us to develop angle coordinates for the GZ integrable system and determine the number of Lagrangian components in each fibre. (The answer is slightly technical but the number of components in F_{sreg} is related to the number of eigenvalue coincidences between the submatrices x_i and x_{i+1}.)

(2) [Col09] (For $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$). We construct the GZ integrable system J_{GZ} an integrate it to an action of a holomorphic group. We locally algebraically integrate the system on a special class of KW fibres F extending results of [KW06a] to the orthogonal case.

3.3. **Algebraic integrability of GZ vector fields and Poisson resolutions.** There are two drawbacks to this approach. In [Col11], the algebraic action of the group Z_F on F_{sreg} does not in general extend to a larger variety. The issue is that the definition of this action requires being able to write down the eigenvalues of each submatrix x_i in a prescribed order as algebraic functions of the submatrix. On the fibre F, the eigenvalues of x_i are fixed (see Equation 3.8), but in general there is no way to do this. Thus, (1) only gives a local picture of the Lagrangian foliation given by the GZ system and ideally we would like a global picture of this foliation. In the language of integrable systems, we would like both angle and *action* coordinates for the GZ system. (Action coordinates are dual to the angle coordinates given by the flows of the integrable system and can be very useful in constructing a quantization.) In [CE10], we algebraically integrate the GZ vector fields over the strongly regular set, generalizing work of Kostant-Wallach in [KW06b].

This is accomplished in three steps in [CE10]:

(1) We stratify the strongly regular locus $\mathfrak{g}_{sreg}$ by smooth, irreducible subvarieties $X_{\mathcal{D}}$ that are invariant under the action of the GZ flows. The varieties $X_{\mathcal{D}}$ are defined using decomposition classes, which were introduced by Borho and Kraft [BK79] and play a major role in the study of algebraic groups.
(2) For each stratum $X_{\mathcal{D}}$, we construct a smooth, irreducible étale covering $\hat{\mathfrak{g}}_{\mathcal{D}} \to X_{\mathcal{D}}$ using ideas of Broer [Bro98a],[Bro98b]. The covering has the advantage that the ordered eigenvalues of the matrices x_i are algebraic functions of the matrix entries.
(3) We use Poisson geometry, namely the theory of Poisson reduction, to endow the covering $\hat{\mathfrak{g}}_{\mathcal{D}}$ with the structure of a Poisson variety and lift the Lie algebra of GZ vector fields in (3.9) to $\hat{\mathfrak{g}}_{\mathcal{D}}$ where they integrate to a *global* action action of an algebraic group $Z_{\mathcal{D}}$ on the covering

$\hat{\mathfrak{g}}_{\mathcal{D}}$. The action of $Z_{\mathcal{D}}$ on $\hat{\mathfrak{g}}_{\mathcal{D}}$ pushes down to the local action of Z_F on F_{sreg} constructed in [Col11]. The global action of $Z_{\mathcal{D}}$ on $\hat{\mathfrak{g}}_{\mathcal{D}}$ is much easier to understand than the corresponding local action. In fact, the Poisson structure we construct on $\hat{\mathfrak{g}}_{\mathcal{D}}$ facilitates the separation of GZ flows into semisimple and nilpotent parts (cf [KKS78], [EL07]).

Remark 3.2.

The $Z_{\mathcal{D}}$-action allows us to trivialize the Lagrangian foliation of the GZ system on $\hat{\mathfrak{g}}_{\mathcal{D}}$. Using this trivialization, we should be able to produce action coordinates for the GZ integrable system.

3.4. **G_{n-1}-orbits on the flag variety and eigenvalue coincidences.** Our results in [Col11],[CE10] concern the GZ system on $\mathfrak{gl}(n,\mathbb{C})$. The problem is that the computations needed to define the action of Z_F on an arbitrary fibre F_{sreg} in [Col11] become intractable in the orthogonal case. To understand the geometry of the GZ systems for orthogonal Lie algebras, a more conceptual approach is needed that covers both orthogonal and general linear GZ systems. This approach makes use of ideas from algebraic group theory and geometric invariant theory, especially the theory of spherical varieties and the theory of orbits of a symmetric subgroup on the flag variety. It also has the advantage that it introduces a natural setting for the geometric construction of a category of $U(\mathfrak{g})$-modules closely related to GZ modules (see Section 4 below).

We begin by studying the *partial KW map*:

$$\Phi_n : \mathfrak{g} \to \mathbb{C}^{r_{n-1}} \times \mathbb{C}^{r_n};\ \Phi_n(x) = (f_{n-1,1}(x), \dots, f_{n,r_{n-1}}(x), f_{n,1}(x), \dots, f_{n,r_n}(x)) \tag{3.10}$$

defined by using algebraically independent generators of the rings $\mathbb{C}[\mathfrak{g}_{n-1}]^{G_{n-1}}$ and $\mathbb{C}[\mathfrak{g}]^G$ (cf (3.7)). We can then use information concerning partial KW maps Φ_i, $i = 2, \dots, n$ to obtain information about the full KW map Φ and the geometry of the GZ system. Using this approach we can understand $\mathfrak{g}_{sreg}$ for both general linear and orthogonal systems and geometrically construct both Lagrangian and singular components of KW map fibres in both cases [CE12],[CEa],[CEb].

Results of Knop [Kno90] imply that the morphism Φ_n is a geometric invariant theory (GIT) quotient for the action of G_{n-1} on $\mathfrak{g}$ by conjugation. This unlocks the full toolbox of geometric invariant theory to study the geometry of the map Φ_n and its fibres, which is not available for the KW map Φ. Namely, we can use the Luna slice theorem [Lun73] to reduce the study of an arbitrary fibre of Φ_n to the nilfibre:

$$\Phi_n^{-1}(0) := \{x \in \mathfrak{g} : x, x_{n-1} \text{ are nilpotent}\}. \tag{3.11}$$

The study of this partial nilfibre involves a surprising connection between the theory of G_{n-1}-orbits on the flag variety $\mathcal{B}$ of $\mathfrak{g}$ and GZ geometry. Note that G_{n-1} is essentially a symmetric subgroup of G and therefore acts on $\mathcal{B}$ with finitely many orbits (see for example [Spr85],[RS90]). The geometry of these orbits plays a major role in the geometric construction of classical Harish-Chandra modules via the Beilinson-Bernstein classification [Vog83], [HMSW87], [Col85].

The main result which allows us to understand $\Phi_n^{-1}(0)$ involves the geometry of eigenvalues coincidence varieties and appears in [CE15] for $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{C})$ and [CEa] for $\mathfrak{g} = \mathfrak{so}(n,\mathbb{C})$.

Theorem 3.3. *Let*

$$\mathfrak{g}(\geq l) := \{x \in \mathfrak{g} : x \text{ and } x_{n-1} \text{ share at least } l \text{ eigenvalues counting repetitions}\}.$$

Recalling that the flag variety $\mathcal{B}$ of $\mathfrak{g}$ is the variety of all Borel subalgebras of $\mathfrak{g}$, and the action of G_{n-1} on $\mathcal{B}$ is simply conjugation of Borel subalgebras by G_{n-1}. Then there is a 1-1

correspondence
(3.12)
Conjugacy classes of Borel subalgebras of codimension l in $\mathcal{B} \longleftrightarrow$ irreducible components of $\mathfrak{g}(\geq l)$.

Remark 3.4. *Theorem 3.3 can be used to obtain a standard form for matrices x that have a fixed number of eigenvalue coincidences with x_{n-1}. This is a basic problem in numerical linear algebra* [PS08].

The correspondence in (3.12) gives a 1-1 correspondence

$$\text{Closed } G_{n-1}\text{-conjugacy classes in } \mathcal{B} \longleftrightarrow \text{ irreducible components of } \Phi_n^{-1}(0). \tag{3.13}$$

(see [CE15], Prop 3.10 and [CEa], Theorem 4.23.) Using the results in (3.12) and (3.13), we can:

(A) Use the Luna slice theorem to compute the structure of an arbitrary fibre F_n of Φ_n, i.e. describe all irreducible components of F_n in terms of G_{n-1}-orbits on $\mathcal{B}$ and identify points in F_n where the GZ vector flows are Lagrangian ([CEb]). From which we can:

(1) Describe the structure of F_{sreg} in the orthogonal case for any fibre F extending [Col11] to the orthogonal case ([CEb]):
(2) Geometrically construct certain generic fibres F of Φ and understand the degenerate (i.e. non-Lagrangian) GZ flows on such fibres F. We can realize certain degenerate GZ flows as orbits of a local algebraic group action as in [Col11] for both general linear and orthogonal cases [CEb].

(B) We can describe an open, dense subset of the full nilfibre $\Phi^{-1}(0)$:

$$\Phi^{-1}(0) = \{x \in \mathfrak{g} : x_i \text{ is nilpotent for all } i\}. \tag{3.14}$$

The variety $\Phi^{-1}(0)$ is referred to in the literature as the variety of strongly nilpotent matrices [Ovs03], [PS08] and is an important object of study in applied linear algebra [PS08], [SP09]. In more detail,

(1) For $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$, in [CE12], we describe all irreducible components of $\Phi^{-1}(0)_{sreg}$ as nilpotent elements of special Borel subalgebras. These Borel subalgebras are constructed using G_{i-1}-orbits on the flag variety $\mathcal{B}_i$ of $\mathfrak{g}_i$ which correspond to holomorphic or anti-holomorphic discrete series representations of the real Lie group $U(i-1,1)$ via the Beilinson-Bernstein correspondence for $i = 2, \ldots, n$.
(2) Using ideas of [CE12] and [CEa], we construct all components of $\Phi^{-1}(0)$ in both orthogonal and general linear cases containing elements $x \in \mathfrak{g}$ such that the submatrix x_i is regular for all i. These components are nilpotent elements of Borel subalgebras constructed using closed G_{i-1}-conjugacy classes in the flag variety of $\mathcal{B}_i$ of $\mathfrak{g}_i$ for $i = 1, \ldots, n$, [CEb] (cf (3.13)).

Problem 3.5. (1) Using the new results of [CEa] and [CEb] extend our work in [CE10] to the orthogonal case and develop action and angle coordinates for both orthogonal and general linear GZ systems.
(2) Describe the geometry of all fibres F (including points) in $F \setminus F_{sreg}$ and degenerate GZ flows on the fibres in (A2) above.

Of particular importance in (2) of Problem 3.5 would be to develop a complete description of the nilfibre $\Phi^{-1}(0)$. This will be important for the study of GZ modules for the following reason: The associated variety of any GZ module M is a subvariety of $\Phi^{-1}(0)$. We plan to approach this problem using the theory of deformations of schemes. We have observed in several examples that if we start with a suitably generic fibre F as in (A2) above that we can deform its irreducible components algebraically to obtain all of the irreducible components of $\Phi^{-1}(0)$. We have also observed that we can compute the scheme-theoretic multiplicities of these components using the deformation. This tells us exactly how the strongly regular elements are distributed in $\Phi^{-1}(0)$ and gives us information about degenerate GZ flows, which should also play a role in the geometric study of GZ modules as they do in the compact case in [GS83]. We should also be able to use these deformations to show that $\Phi^{-1}(0)$ is an equidimensional variety which would imply the KW map is flat. This would provide a more geometric and conceptual proof of the main result of [Ovs03] and extend Ovsienko's result to the orthogonal case.

The appearance of G_{n-1}-orbits on $\mathcal{B}$ in the description of the partial KW map and in the construction of components of $\Phi^{-1}(0)$ in [CE12] and [CEa] indicates much deeper geometry is lurking behind the surface of the GZ integrable systems. This geometry involves the theory of spherical pairs. A *spherical pair* (G, H) is an algebraic group G and a subgroup $H \subset G$ with the property that H acts on the flag variety of $\mathfrak{g} = \mathrm{Lie}(G)$ with finitely many orbits. Thus, spherical pairs are natural generalizations of symmetric pairs. It turns out that the G_{n-1}-action on $\mathfrak{g}$ by conjugation can be identified with the coisotropy representation of the spherical pair $(G \times G_{n-1}, (G_{n-1})_{\Delta})$, where $(G_{n-1})_{\Delta}$ is the diagonal copy of G_{n-1} in the product. This allows us to use the theory of spherical varieties to study Φ_n further [CEa]. Using this theory and ideas of Panyushev [Pan90], we prove a variant of Kostant's theorem concerning the linear independence of differentials to characterize regular elements [Kos63]. We can then simplify the criterion for the GZ flows to be Lagrangian [CEa]. This simplification makes the strongly regular locus much easier to understand and allows us to identify a previously unknown class of strongly regular elements in the orthogonal case that were inaccessible using the computations of [Col11] and [Col09]. Furthermore, the appearance of the spherical pair $(G \times G_{n-1}, (G_{n-1})_{\Delta})$ in the study of partial KW fibres provides a pathway for the geometric construction of GZ modules.

4. Geometric Construction of GZ modules and partial GZ modules

In this section, we will make use of the following notation:

Notation 4.1. A $(\mathfrak{g}, A)$-module is a module for the enveloping algebra of a Lie algebra $\mathfrak{g}$ which is the sum of finite dimensional A-representations, where A is either an algebraic group or an associative subalgebra of $U(\mathfrak{g})$. We can think of $(\mathfrak{g}, A)$-modules as generalized Harish-Chandra (HC) modules for the pair $(\mathfrak{g}, A)$. For example, a GZ module is a $(\mathfrak{g}, \Gamma)$-module, where Γ is the GZ algebra in (2.2).

Ideally, one would like a geometric construction of GZ modules analogous to the Beilinson-Bernstein classification of $(\mathfrak{g}, K)$-modules where K is a symmetric subgroup of the adjoint group of G. The Beilinson-Bernstein classification realizes $(\mathfrak{g}, K)$-modules as global sections of sheaves of differential operators on the flag variety $\mathcal{B}$ of $\mathfrak{g}$ which are supported on the closure of orbits of K-orbits on $\mathcal{B}$ of $\mathfrak{g}$ [Mil93], [Sch05]. Crucial in this classification is the fact that K acts on $\mathcal{B}$ with finitely many orbits. One of the major problems with attempting an analogous construction in the GZ situation is that the GZ algebra Γ in doesn't integrate to an algebraic group that acts on $\mathcal{B}$ with finitely many orbits. The resolution to this problem is suggested by the geometric methods for constructing KW map fibres outlined in the previous section, where we use the G_{n-1}-action on $\mathcal{B}$ and the geometry of the partial KW fibres to construct the fibres of the KW map Φ. This suggests that on the quantum side of the picture, we first look at *partial GZ modules*. These are the category of $(\mathfrak{g}, \Gamma_n)$-modules where Γ_n is the partial GZ algebra:

$$\Gamma_n \cong Z(\mathfrak{g}_{n-1}) \otimes_{\mathbb{C}} Z(\mathfrak{g}) \tag{4.1}$$

Even though quantization is not a precise procedure, there should be a similar approach in representation theory to construct GZ modules from partial GZ modules. This approach seems especially promising because the category of partial GZ modules is intimately related to the category of $(\mathfrak{g} \oplus \mathfrak{g}_{n-1}, (G_{n-1})_\Delta)$-modules, i.e. Harish-Chandra modules for the spherical pair that appears in the study of the partial KW map.

Theorem 4.2. *The category $(\mathfrak{g}\oplus\mathfrak{g}_{n-1}, (G_{n-1})_\Delta)$-modules is equivalent to a subcategory of partial GZ modules.*

This theorem is proven by generalizing a result of Borho-Brylinski and Jantzen concerning the equivalence of categories between Harish-Chandra modules for complex groups and category $\mathcal{O}$ to this setting [Jan83, BB85]. Our work on the geometry of the partial KW map in Section 3.4 will be very important in understanding the category of $(\mathfrak{g}\oplus\mathfrak{g}_{n-1}, (G_{n-1})_\Delta)$-modules in much the same way that Kostant-Rallis theory and the geometry of the K-action on $\mathfrak{p}$ was important for understanding classical $(\mathfrak{g}, K)$-modules [KR71],[Vog91] through theory of associated varieties. In fact, the associated variety of an $(\mathfrak{g} \oplus \mathfrak{g}_{n-1}, (G_{n-1})_\Delta)$-module is a subvariety of the partial KW nilfibre $\Phi_n^{-1}(0)$ in (3.11).

For any spherical pair (G, H) it is in theory possible to construct HC modules for the pair $(U(\mathfrak{g}), H)$ by using a Beilinson-Bernstein type construction. However, to develop a concrete understanding of these modules, it is necessary to understand the geometry of the orbits of H on $\mathcal{B}$. Specifically, one needs to understand the closure relations between the orbits and their individual geometry. Even in the case where H is a symmetric subgroup of G, this is a very hard problem (see for example [RS90], [RS93],[CE14]), and very little is understood in the general spherical case ([Kno95],[Bri01],[GP]). However, in our manuscript [CEc], we obtain a complete description of the H-orbits on $\mathcal{B}$ for our special spherical pair $(G \times G_{n-1}, (G_{n-1})_\Delta)$. In [CEc], we show that each orbit has the natural structure of a fibre bundle and can obtain the closure relations between the orbits. The former point implies the standard modules in the category of $(\mathfrak{g} \oplus \mathfrak{g}_{n-1}, (G_{n-1})_\Delta)$-modules can be obtained from parabolic induction. In the future, we plan on using this work to:

Problem 4.3. (1) Geometrically construct the category of HC modules for the spherical pair $(G \times G_{n-1}, (G_{n-1})_\Delta)$ using a generalization of the Beilinson-Bernstein approach.
(2) Use (1) and Theorem 4.2 to construct partial GZ modules geometrically. Is it possible to construct all partial GZ modules this way? If not, can we describe the subcategory of partial GZ modules that we obtain in this way?
(3) Construct full GZ modules from partial GZ modules using an inductive procedure that lines up with our construction of full KW map fibres from partial KW fibres in Section 3.4.

Remark 4.4. In fact, in certain cases some of the partial GZ modules we obtain from Theorem 4.2 are full GZ modules. So we already have a way of geometrically constructing some examples of GZ modules. It would be interesting to understand how these special examples fit into the broader category of GZ modules and how they relate to the geometry of the GZ integrable system.

5. Other projects: Infinite Dimensional and Nonlinear GZ theory

In these two brief sections, we discuss infinite dimensional and nonlinear analogues of GZ systems. To study the infinite dimensional analogue, we had to develop the necessary Poisson geometry in infinite dimensions which is of interest in its own right. The nonlinear GZ systems require the use of Poisson Lie groups.

5.1. **Lie-Poisson theory for direct limit Lie algebras.** In recent work with Michael Lau [CL16], we have developed the fundamentals of a Lie-Poisson theory for direct limits of complex algebraic groups and Lie algebras. Direct limit groups and algebras play a very important role in representation theory and integrable systems. Categorification of direct limit Lie algebras has recently been used to study finite-dimensional representations of Lie superalgebras [Bru03]. Applications of direct limit groups notably include early work on infinite-dimensional integrable systems, including the KP hierarchy (see [KR87]).

Developing Poisson geometry for direct limit algebras and groups requires the use of ind-varieties and pro-varieties, basic objects in infinite dimensional algebraic geometry [Kum02], [MN02]. In infinite dimensions, there is no guarantee that a Poisson variety (or manifold) is foliated by symplectic subvarieties [OR03]. Even in the comparatively well-behaved setting of Banach Lie groups, it is not known whether the coadjoint orbits of G on a predual $\mathfrak{g}_*$ of its Lie algebra possess a symplectic structure. In [CL16], we show that in the setting of direct limit groups and algebras, Lie-Poisson theory works as in the finite dimensional setting. Our main theorem is:

Theorem 5.1. [CL16]
Let $G = \varinjlim G_n$ be a direct limit of complex algebraic groups G_n, and let $\mathfrak{g}^* = \varprojlim \mathfrak{g}_n^*$ be the dual space of its Lie algebra. Let $\lambda \in \mathfrak{g}^*$, and let $G \cdot \lambda$ be the coadjoint orbit of λ. Then

1. *The space $\mathfrak{g}^*$ naturally has the structure of a Poisson provariety.*
2. *$G \cdot \lambda$ has the structure of a weak symplectic ind-subvariety of $\mathfrak{g}^*$.*
3. *The symplectic structure on $G \cdot \lambda$ is compatible with the Poisson structure on $\mathfrak{g}^*$.*

Thus, the coadjoint orbits of G on $\mathfrak{g}^*$ form a weak symplectic foliation of the Poisson provariety $\mathfrak{g}^*$.

We apply our results to the specific setting of $G = GL(\infty) = \varinjlim GL(n, \mathbb{C})$ and $\mathfrak{g}^* = M(\infty) = \varprojlim \mathfrak{gl}(n, \mathbb{C})$, the space of infinite complex matrices with arbitrary entries. We construct a Gelfand-Zeitlin integrable system on $M(\infty)$, which generalizes the one constructed by Kostant and Wallach on $\mathfrak{gl}(n, \mathbb{C})$ [KW06a]. The system integrates to an action of a direct limit group $A(\infty)$ on $M(\infty)$. For $X \in M(\infty)$ denotes its $A(\infty)$-orbit by $A(\infty) \cdot X$. We prove:

Theorem 5.2. *For generic $X \in M(\infty)$, $A(\infty) \cdot X \subset GL(\infty) \cdot X$ is an irreducible, Lagrangian ind-subvariety of $GL(\infty) \cdot X$ with respect to the weak symplectic form on $GL(\infty) \cdot X$ given in Part (1) of Theorem 5.1.*

This result is much trickier than the corresponding result in finite dimensions. In the infinite dimensional setting, it is not automatic that the generic orbits of an integrable system are Lagrangian. To prove this theorem, we had to develop a Lagrangian calculus of the coadjoint orbits of $G = \varinjlim G_n$ on $\mathfrak{g}^* = \varprojlim \mathfrak{g}_n^*$.

We hope the results of this paper will provide the foundation for developing geometric constructions of representations of direct limit groups.

Problem 5.3. (1) Use our work in [CL16] to come up with a symplectic interpretation of the Bott-Borel-Weil theorem in [DPW02] using geometric quantization.
(2) There are natural direct limit analogues of GZ modules. It would be interesting to use the geometry of the infinite dimensional GZ system described in [CL16] to study these modules geometrically as in the finite dimensional case.

5.2. **Nonlinear GZ theory.** A Poisson Lie group (G, π_G) is a Lie group with a compatible Poisson structure $\pi_G \in \wedge^2 TG$. Poisson Lie groups were invented by Drinfeld [Dri87] as a classical analogue to quantum groups, and there are relations between the geometry of Poisson Lie groups and the representation theory of quantized universal enveloping algebras $U_q(\mathfrak{g})$ (see for example [DCKP93, HL93], [Jos95]).

For a Poisson Lie group (G, π_G), the Poisson structure $\pi_G \in \wedge^2 TG$ is nonlinear and is much more difficult to understand than linear Poisson structures such as the Lie-Poisson structure on $\mathfrak{g}^*$ where $\mathfrak{g} = \mathrm{Lie}(G)$. However, it appears that GZ systems can be used to construct explicit equivalences between π_G and the Lie-Poisson structure on $\mathfrak{g}^*$ for certain Poisson Lie groups. This approach begins with the work of Ginzburg and Weinstein [GW92]. Any compact Lie group K has a natural Poisson Lie group structure, giving rise to a dual Poisson Lie group K^* [LW90]. In [GW92], the authors produce Poisson diffeomorphisms $K^* \to \mathfrak{k}^*$, where $\mathfrak{k} = \mathrm{Lie}(K)$ and $\mathfrak{k}^*$ is endowed with the linear Lie-Poisson structure. However, these diffeomorphisms are not constructed explicitly. In the mid 1990's, Flaschka and Raitu constructed a GZ integrable system on the Poisson Lie group $U(n, \mathbb{C})^*$ [FR96], which is a nonlinear version of the GZ system on $\mathfrak{u}(n, \mathbb{C})^*$ in [GS83]. In [AM07], Alekseev and Meinrenken construct a canonical Ginzburg-Weinstein diffeomorphism $U(n, \mathbb{C})^* \to \mathfrak{u}(n, \mathbb{C})^*$ which intertwines the action of the two GZ systems. In joint work with Sam Evens, we are working to extend the results of Ginzburg-Weistnein and Alekseev-Meinrenken to the setting of complex Poisson Lie groups.

We consider $G = GL(n, \mathbb{C})$ as a Poisson Lie group with the standard Drinfeld Poisson structure $\pi_G \in \wedge^2 TG$ and its dual Poisson Lie group $G^* = GL(n, \mathbb{C})^*$ constructed using the standard Manin triple for $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ (see [KS98],[LW90]).

Theorem 5.4. (1) *There is a natural GZ integrable system on $GL(n, \mathbb{C})^*$ which can be viewed as a nonlinear version of the system constructed by Kostant and Wallach.*
(2) *There is a natural local diffeomorphism $GL(n, \mathbb{C})^* \to \mathfrak{gl}(n, \mathbb{C})$ which intertwines the action of the two GZ systems.*

Problem 5.5. (1) Use the equivalence between the nonlinear GZ system on $GL(n, \mathbb{C})^*$ and the linear GZ system on $\mathfrak{gl}(n, \mathbb{C})$ to construct a complexified Ginzburg-Weinstein diffeomorphism from a quotient of $GL(n, \mathbb{C})^*$ to the linear Poisson space $\mathfrak{gl}(n, \mathbb{C})^* \cong \mathfrak{gl}(n, \mathbb{C})$.

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