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Lie–Poisson theory for direct limit Lie algebras [☆]

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ABSTRACT

In the first half of this paper, we develop the fundamentals of Lie–Poisson theory for direct limits $G = \varinjlim G_n$ of complex algebraic groups and their Lie algebras $\mathfrak{g} = \varinjlim \mathfrak{g}_n$. We describe the Poisson pro- and ind-variety structures on $\mathfrak{g}^* = \varprojlim \mathfrak{g}_n^*$ and the coadjoint orbits of G, respectively. While the existence of symplectic foliations remains an open question for most infinite-dimensional Poisson manifolds, we show that for direct limit algebras, the coadjoint orbits give a weak symplectic foliation of the Poisson provariety $\mathfrak{g}^*$.

The second half of the paper applies our general results to the concrete setting of $G = GL(\infty)$ and $\mathfrak{g}^* = M(\infty)$, the space of infinite-by-infinite complex matrices with arbitrary entries. We use the Poisson structure of $\mathfrak{g}^*$ to construct an integrable system on $M(\infty)$ that generalizes the Gelfand–Zeitlin system on $\mathfrak{gl}(n, \mathbb{C})$ to the infinite-dimensional setting. We further show that this integrable system integrates to a global action of a direct limit group on $M(\infty)$, whose generic orbits are Lagrangian ind-subvarieties of the coadjoint orbits of $GL(\infty)$ on $M(\infty)$.

1. Introduction

The interaction between Lie theory and Poisson geometry plays an important role in much of modern mathematics and mathematical physics; it is of central importance in geometric representation theory, integrable systems, and classical mechanics. Given a finite-dimensional real or complex Lie group G, there is a canonical Lie–Poisson structure on the dual space $\mathfrak{g}^*$ of its Lie algebra $\mathfrak{g}$. The starting point of Lie–Poisson theory is the observation that the symplectic leaves of $\mathfrak{g}^*$ are the coadjoint orbits of the identity component G^0 of G equipped with the Kostant–Kirillov symplectic form. This symplectic structure is the cornerstone of the orbit method in representation theory and plays an important role in deformation quantization.

The main goal of this paper is to extend this theory to direct limit Lie algebras. Given a direct limit group G with Lie algebra $\mathfrak{g}$, we define an analogous Lie–Poisson structure on the dual space $\mathfrak{g}^*$ and construct a

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symplectic foliation using the coadjoint action of G on $\mathfrak{g}^*$. There is an extensive literature concerning direct limit groups and their Lie algebras, root systems, and representations, though there has been little study of Poisson geometry in this context. See for example, [1,3,8,7,9,12,21,23,25] and the references therein. Categorification of direct limit Lie algebras has recently been used to study finite-dimensional representations of Lie superalgebras [2]. Applications of direct limit groups notably include early work on infinite-dimensional integrable systems, including the KP hierarchy. See [17], for instance. In particular, when $\mathfrak{g} = \mathfrak{gl}(\infty) := \varinjlim \mathfrak{gl}(n, \mathbb{C})$, we use our new Lie–Poisson structure to define an infinite-dimensional analogue of the Gelfand–Zeitlin integrable system on $\mathfrak{gl}(\infty)^*$.

In more detail, let $\{(G_n, \iota_{nm})\}_{n\in\mathbb{N}}$ be a directed system of complex affine algebraic groups G_n for which the transition maps $\iota_{nm} : G_n \hookrightarrow G_m$ are homomorphic embeddings of algebraic groups. The direct limit group $G := \varinjlim G_n$ has the structure of an ind-variety and its Lie algebra $\mathfrak{g} = \varinjlim \mathfrak{g}_n$ is a direct limit Lie algebra. The algebraic dual $\mathfrak{g}^* = \varprojlim \mathfrak{g}_n^*$ is a provariety, an inverse limit in the category of varieties. We show that $\mathfrak{g}^*$ has a natural Poisson structure inherited from the Lie–Poisson structure of each $\mathfrak{g}_n^*$ and compute its characteristic distribution. This construction requires understanding subtle aspects about the geometry of provarieties including their structure sheaves, tangent spaces, and morphisms (Propositions 2.3, 2.7, and Theorem 2.10).

In infinite dimensions, there is no guarantee that the characteristic distribution of a Poisson manifold is integrable nor that its leaves possess a symplectic structure. Even in the comparatively well-behaved setting of Banach Lie groups G, it is not known whether the coadjoint orbits of G on a predual $\mathfrak{g}_*$ of its Lie algebra $\mathfrak{g}$ are weakly symplectic [24]. One of the main results of this paper is to show that the coadjoint orbits of a direct limit group G on the dual of its Lie algebra $\mathfrak{g}^*$ form a symplectic foliation of $\mathfrak{g}^*$ which is tangent to the characteristic distribution of $\mathfrak{g}^*$.

Theorem 1.1. *(See Proposition 4.7 and Theorem 4.12.) Let $G = \varinjlim G_n$ be a direct limit group, and let $\mathfrak{g}^* = \varprojlim \mathfrak{g}_n^*$ be the dual space of its Lie algebra. Let $\lambda \in \mathfrak{g}^*$, and let $G \cdot \lambda$ be the coadjoint orbit of λ. Then*

(1) *$G \cdot \lambda$ has the structure of a weak symplectic ind-subvariety of $\mathfrak{g}^*$.*
(2) *$G\cdot\lambda$ is tangent to the characteristic distribution of $\mathfrak{g}^*$, and the symplectic structure on $G\cdot\lambda$ is compatible with the Poisson structure on $\mathfrak{g}^*$.*

To prove Part (1), we observe that the coadjoint orbit $G \cdot \lambda$ inherits an ind-variety structure from G via:

$$G \cdot \lambda = \varinjlim G_n \cdot \lambda.$$

Since $\lambda \in \mathfrak{g}^* = \varprojlim \mathfrak{g}_n^*$, we can represent λ as an infinite sequence, $\lambda = (\lambda_1, \dots, \lambda_n, \dots)$ with $\lambda_n \in \mathfrak{g}_n^*$. Each variety $G_n \cdot \lambda$ inherits a 2-form from the Kostant–Kirillov form on the G_n-coadjoint orbit of λ_n, and we show that these 2-forms glue to give a non-degenerate, closed two form on $G \cdot \lambda$ (Proposition 4.7).

Part (2) of the theorem is much more difficult to prove than the analogous result in finite dimensions and requires relating the ind-variety structure of $G \cdot \lambda$ to the provariety structure of $\mathfrak{g}^*$. We use our basic results about morphisms of provarieties and ind-varieties (Propositions 2.19 and 3.1) to show that $G\cdot\lambda$ is an ind-subvariety of $\mathfrak{g}^*$ whose tangent space agrees with the characteristic distribution of $\mathfrak{g}^*$. The compatibility of the Poisson structure on $\mathfrak{g}^*$ with the symplectic structure of $G \cdot \lambda$ requires an explicit understanding of the anchor map of $\mathfrak{g}^*$ and its kernel (Propositions 2.37 and 2.42).

In the second half of the paper, we apply our results to the case where G is the group $GL(\infty) := \varinjlim GL(n, \mathbb{C})$ with the Lie algebra $\mathfrak{g} = \mathfrak{gl}(\infty) = \varinjlim \mathfrak{gl}(n, \mathbb{C})$ of infinite-by-infinite complex matrices with only finitely many nonzero entries. The dual space $\mathfrak{g}^*$ is the Poisson provariety $M(\infty)$ of all infinite-by-infinite complex matrices. We construct an infinite-dimensional analogue of the Gelfand–Zeitlin integrable system on $M(\infty)$ which generalizes the one constructed by Kostant and Wallach on $\mathfrak{gl}(n, \mathbb{C})$ in [19,20].

In more detail, we identify $M(\infty)$ with the set of infinite sequences:

$$M(\infty) := \{X = (X(1), X(2), X(3), \dots) : X(n) \in \mathfrak{gl}(n, \mathbb{C}) \text{ and } X(n+1)_n = X(n)\},$$

where $X(n+1)_n$ denotes the $n \times n$ upper left corner of $X(n+1) \in \mathfrak{gl}(n+1, \mathbb{C})$. For any $n \in \mathbb{N}$ and $j = 1, \dots, n$, let f_{nj} be the function on $M(\infty)$ given by $f_{nj}(X) = tr(X(n)^j)$, where $tr(\cdot)$ denotes the trace function. The algebra generated by the collection of functions

$$J_\infty := \{f_{nj}(X) : n \in \mathbb{N},\, j = 1, \dots, n\}$$

is then a maximal Poisson-commutative subalgebra of the space of global regular functions on $M(\infty)$ (Proposition 5.5 and Remark 5.6). Moreover, the corresponding Lie algebra of Hamiltonian vector fields $\mathfrak{a}(\infty)$ is infinite dimensional and integrates to a global action of a direct limit group $A(\infty)$ which preserves the coadjoint orbits of $GL(\infty)$ on $M(\infty)$, but does not act algebraically on $M(\infty)$. Following [19], we say that an element $X \in M(\infty)$ is *strongly regular* if the differentials of the functions in J_∞ are independent at X. It follows easily from work of the first author that the set of strongly regular elements of $M(\infty)$ is non-empty (Example 5.15). Despite the $A(\infty)$-action on $M(\infty)$ not being algebraic, we show that any strongly regular $A(\infty)$-orbit on $M(\infty)$ is an algebraic ind-subvariety of the corresponding $GL(\infty)$-coadjoint orbit which is Lagrangian with respect to the weak symplectic form constructed in Part (1) of Theorem 1.1. The following theorem generalizes one of the main results of Kostant and Wallach (cf. [19, Theorem 3.36]) to the direct limit setting.

Theorem 1.2. *(See Theorem 5.18.) Let $X \in M(\infty)$ be strongly regular. Then $A(\infty) \cdot X \subset GL(\infty) \cdot X$ is an irreducible, Lagrangian ind-subvariety of $GL(\infty) \cdot X$.*

As was the case with Theorem 1.1, the infinite-dimensional setting contains difficulties that are not present in finite dimensions. The foremost being that it is not automatic that the generic leaves of an integrable system are Lagrangian. To circumvent this difficulty, we have to develop a Lagrangian calculus for the weakly symplectic ind-varieties $G \cdot \lambda$ (Proposition 4.19).

In the philosophy of quantization, Lagrangian submanifolds of $\mathfrak{g}^*$ correspond to irreducible representations of G. For the group of $n \times n$ unitary matrices, Guillemin and Sternberg have used the Gelfand–Zeitlin system to obtain a new quantization consistent with the Bott–Borel–Weil construction [13]. It would be interesting to apply the geometric methods and results concerning the infinite dimensional Gelfand–Zeitlin system developed in this paper to study the representations of direct limit groups geometrically. Dimitrov, Penkov, and Wolf have given the beautiful and nontrivial analogue of the Bott–Borel–Weil theorem for direct limit groups [9]. In the future, we plan to reinterpret the results of [9] using the Lie–Poisson theory developed in the first half of this paper. The quantum analogue of the finite dimensional Gelfand–Zeitlin system on $\mathfrak{gl}(n, \mathbb{C})$ are the Gelfand–Zeitlin modules introduced by Drozd, Futorny, and Ovsienko [10]. These modules have natural direct limit analogues, and we plan to use Theorem 1.2 and the geometry of the Gelfand–Zeitlin system on $M(\infty)$ to study them geometrically.

The paper is organized as follows. In Section 2, we study general provarieties $X = \varprojlim X_n$, where X_n is a finite-dimensional variety defined over an arbitrary algebraically closed field F of characteristic zero. We define a structure sheaf $\mathcal{O}_X$ which makes the pair $(X, \mathcal{O}_X)$ into a locally ringed space and describe the tangent space of X (Propositions 2.3 and 2.7, Theorem 2.10). In Section 2.3, we study morphisms of provarieties and prove Proposition 2.19. In Section 2.4, we specialize to the case where each X_n is a Poisson variety and show that $\mathcal{O}_X$ is a sheaf of Poisson algebras (Proposition–Definition 2.34). The provariety structure on $\mathfrak{g}^*$ is described in Example 2.27, and its Lie–Poisson structure is obtained in Example 2.38. In Section 3, we review basic facts about ind-varieties and describe the ind-variety structure of the coadjoint orbits $G \cdot \lambda$ (Proposition 3.9 and Corollary 3.10). In Section 4, we develop the weak symplectic form on

$G \cdot \lambda$ and prove Theorem 1.1. In Section 5, we construct the Gelfand–Zeitlin integrable system on $M(\infty)$ and prove Theorem 1.2.

Notation. Throughout this paper, $\mathbb{N}$ and $\mathbb{C}$ will denote the positive integers and complex numbers, respectively.

2. Provarieties

2.1. The structure sheaf of a provariety

Let $\{(X_n, p_{nm})\}_{n\in\mathbb{N}}$ be an inverse system of irreducible varieties over an algebraically closed field F of characteristic zero with surjective transition morphisms: $p_{nm} : X_n \to X_m$ for $n \geq m$. We call the inverse limit $X = \varprojlim X_n$ of such a system (X_n, p_{nm}) a *provariety*. Another introduction to provarieties may be found in [22]. They do not assume that their inverse system of varieties is countable. We will only consider countable inverse systems of varieties, and the exposition here is self-contained.

As a topological space, X has the inverse limit topology. A basis for this topology is the collection of sets

$$\mathcal{B} = \{p_n^{-1}(U_n) : U_n \subset X_n \text{ is open}\}.$$

We construct a structure sheaf $\mathcal{O}_X$ on X which makes $(X, \mathcal{O}_X)$ into a locally ringed space. We begin by defining a $\mathcal{B}$-presheaf $\widetilde{\mathcal{O}_X}$ of F-algebras on X, i.e. a presheaf whose sections $\widetilde{\mathcal{O}_X}(U)$ are defined only for $U \in \mathcal{B}$. Suppose $U \in \mathcal{B}$ with $U = p_n^{-1}(U_n)$ for some open subset $U_n \subseteq X_n$. The inverse system $\{(X_k, p_{\ell k})\}_{\ell\geq k\geq n}$ gives rise to a directed system of F-algebras $\{\mathcal{O}_{X_k}(p_{kn}^{-1}(U_n)), p_{\ell k}^*\}_{\ell\geq k\geq n}$. Since the transition maps $p_{\ell k}$ are surjective for all pairs $\ell \geq k$, it follows that the canonical projections $p_k : X \to X_k$ are surjective for all k. Thus, $\mathcal{O}_{X_k}(p_{kn}^{-1}(U_n)) \cong p_k^*\mathcal{O}_{X_k}(p_{kn}^{-1}(U_n))$ and we can define:

$$\widetilde{\mathcal{O}_X}(U) := \varinjlim_{k\geq n} p_k^*\mathcal{O}_{X_k}(p_{kn}^{-1}(U_n)). \tag{2.1}$$

We claim that (2.1) makes $\widetilde{\mathcal{O}_X}$ into a $\mathcal{B}$-presheaf. Indeed, suppose we have $V \subseteq U$ with $V, U \in \mathcal{B}$. Let $V = p_\ell^{-1}(U_\ell)$ for $U_\ell \subseteq X_\ell$ open. We define the restriction maps

$$\rho_{UV} : \widetilde{\mathcal{O}_X}(U) \to \widetilde{\mathcal{O}_X}(V)$$

as follows. Suppose $f \in \widetilde{\mathcal{O}_X}(U)$. Then $f = p_k^* f_k$ for some $f_k \in \mathcal{O}_{X_k}(p_{kn}^{-1}(U_n))$ and $k \geq n$. Let $m \geq \ell, k$. Then $f = p_m^* p_{mk}^* f_k$ with $p_{mk}^* f_k \in \mathcal{O}_{X_m}(p_{mn}^{-1}(U_n))$. Since p_m is surjective, $p_{m\ell}^{-1}(U_\ell) \subseteq p_{mn}^{-1}(U_n)$. We can therefore define

$$\rho_{UV}(f) := p_m^*(p_{mk}^* f_k|_{p_{m\ell}^{-1}(U_\ell)}) \in \widetilde{\mathcal{O}_X}(V),$$

where $(p_{mk}^* f_k)|_{p_{m\ell}^{-1}(U_\ell)}$ denotes the restriction of $p_{mk}^* f_k \in \mathcal{O}_{X_m}(p_{mn}^{-1}(U_n))$ to $p_{m\ell}^{-1}(U_\ell)$. One can verify that ρ_{UV} is well defined and that for $W \subseteq V \subseteq U$ with $W \in \mathcal{B}$, we have $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$. Note also that $\rho_{UU} = Id_{\widetilde{\mathcal{O}_X}(U)}$. Thus, $\widetilde{\mathcal{O}_X}$ is a $\mathcal{B}$-presheaf of F-algebras. Since inverse limits exist in the category of F-algebras, we can form a presheaf on all of X by setting

$$\mathcal{O}_X(U) := \varprojlim_{V\subseteq U,\, V\in\mathcal{B}} \widetilde{\mathcal{O}_X}(V) \tag{2.2}$$

for each open set $U \subseteq X$. It follows from the universal property of the inverse limit that $\mathcal{O}_X$ is a presheaf on X, and $\mathcal{O}_X(U) = \widetilde{\mathcal{O}_X}(U)$ for $U \in \mathcal{B}$. Moreover, $\mathcal{O}_X$ is in fact a sheaf on X.

Proposition 2.3. *The presheaf $\mathcal{O}_X$ on X is a sheaf of F-algebras on X.*

Proof. It follows from [11, Proposition I-12(i)] that it suffices to check the sheaf axioms on sections $\mathcal{O}_X(U)$ with $U \in \mathcal{B}$. Accordingly, let $U \in \mathcal{B}$ with $U = p_n^{-1}(U_n)$ with $U_n \subseteq X_n$ open, and let $\bigcup_{i\in I} p_i^{-1}(U_i) = U$ be an open cover of U by basic open sets of X. Suppose that for each $i \in I$, we are given $f_i \in \mathcal{O}_X(p_i^{-1}(U_i))$ such that

$$f_i|_{p_i^{-1}(U_i)\cap p_j^{-1}(U_j)} = f_j|_{p_i^{-1}(U_i)\cap p_j^{-1}(U_j)} \tag{2.4}$$

for every i, $j \in I$. Let $F(X_n)$ be the function field of X_n. Consider the field:

$$F(X) := \varinjlim p_n^* F(X_n). \tag{2.5}$$

Equation (2.4) implies that the functions f_i with $i \in I$ define the same element $g \in F(X)$. Without loss of generality, we may assume that $g = p_n^* g_n$ for $g_n \in F(X_n)$. We claim that $g_n \in \mathcal{O}_{X_n}(U_n)$. By construction, $g|_{p_i^{-1}(U_i)} = f_i$ for all i. Now let $x \in p_n^{-1}(U_n)$, then $x \in p_i^{-1}(U_i)$ for some $i \in I$. We have

$$f_i(x) = g(x) = g_n(x_n),$$

where $x_n = p_n(x)$. Since $p_n : \varprojlim X_k \to X_n$ is surjective, $g_n \in F(X_n)$ is defined at all points of $U_n \subset X_n$. Thus, $g_n \in \mathcal{O}_{X_n}(U_n)$, so that $g = p_n^* g_n \in \mathcal{O}_X(U)$.

Since the varieties X_n are irreducible for all n, the restriction maps $\rho_{p_n^{-1}(U_n),p_i^{-1}(U_i)}$ are injective. Indeed, suppose that $f \in \mathcal{O}_X(p_n^{-1}(U_n))$ with $f|_{p_i^{-1}(U_i)} = 0$ for some $i \in I$. Then there exist $k \geq n$, i and a regular function $f_k \in \mathcal{O}_{X_k}(p_{kn}^{-1}(U_n))$ such that $f = p_k^* f_k$ and $f_k|_{p_{ki}^{-1}(U_i)} = 0$. But then since $p_{ki}^{-1}(U_i) \subseteq p_{kn}^{-1}(U_n)$ is open and $p_{kn}^{-1}(U_n)$ is irreducible, it follows that $f_k = 0$ and hence $f = 0$. $\square$

Proposition 2.3 implies that $(X, \mathcal{O}_X)$ is a ringed space. Since stalks $\mathcal{O}_{X,x}$ can be computed using basic open sets, Equation (2.1) implies that

$$\mathcal{O}_{X,x} = \varinjlim p_n^* \mathcal{O}_{X_n,x_n} \cong \varinjlim \mathcal{O}_{X_n,x_n}, \tag{2.6}$$

where $x_n = p_n(x)$. Equation (2.6) implies that $(X, \mathcal{O}_X)$ is a locally ringed space.

Proposition 2.7. *Let $X = \varprojlim X_n$ be a provariety, and let $x \in X$ with $x_n = p_n(x)$. Let $\mathfrak{m}_{x_n}$ be the unique maximal ideal of the local ring $\mathcal{O}_{X_n,x_n}$. Then the stalk $\mathcal{O}_{X,x}$ of the sheaf $\mathcal{O}_X$ at $x \in X$ is a local ring with maximal ideal $\mathfrak{m} = \varinjlim p_n^* \mathfrak{m}_{x_n}$.*

The proposition follows immediately from the following general fact.

Lemma 2.8. *Suppose $\{(A_n, \mathfrak{m}_n, \phi_{nm})\}_{n\in\mathbb{N}}$ is a directed system of local rings with $\mathfrak{m}_n \subset A_n$ the unique maximal ideal and local homomorphisms $\phi_{nm} : A_n \to A_m$ for $n \leq m$. Then the direct limit $A = \varinjlim A_n$ is a local ring with unique maximal ideal $\mathfrak{m} = \varinjlim \mathfrak{m}_n$.*

Proof. Let $\mathfrak{m} = \varinjlim \mathfrak{m}_n$ and $a \in A \setminus \mathfrak{m}$. Abusing notation, we also denote by A_n and $\mathfrak{m}_n$, the images of A_n and $\mathfrak{m}_n$ in $\varinjlim A_n$ respectively. It follows that $a \in A_i \setminus \mathfrak{m}_i$ for some i, whence $a \in A$ is a unit. $\square$

2.2. Tangent spaces to provarieties

Let $(X, \mathcal{O}_X)$ be a provariety. Since $(X, \mathcal{O}_X)$ is a locally ringed space, we can define the Zariski tangent space $T_x(X)$ of a point $x \in X$ as:

$$T_x(X) := (\mathfrak{m}_x/\mathfrak{m}_x^2)^*, \tag{2.9}$$

where $(\mathfrak{m}_x/\mathfrak{m}_x^2)^*$ is the dual of the infinite dimensional F-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$. It is easy to see that $T_x(X)$ can be identified with the space of all F-linear point derivations of the F-algebra $\mathcal{O}_{X,x}$ at the point $x \in X$. The F-vector space $T_x(X)$ is also an inverse limit.

Theorem 2.10. *Let $\{(X_n, p_{nm})\}$ be an inverse system of varieties, and let $X = \varprojlim X_n$ be the corresponding provariety. There is a canonical isomorphism of F-vector spaces:*

$$T_x(X) \cong \varprojlim T_{x_n}(X_n), \tag{2.11}$$

where $x_n = p_n(x)$ for each $x \in X$. That is, the following diagram commutes:

$$\begin{array}{ccc} \varprojlim T_{x_n}(X_n) & \cong & T_x(X) \\ \downarrow \pi_k & & \downarrow (dp_k)_x \\ T_{x_k}(X_k) & = & T_{x_k}(X_k), \end{array} \tag{2.12}$$

where $\pi_k : \varprojlim T_{x_n}(X_n) \to T_{x_k}(X_k)$ is the canonical projection.

Proof. Let $x \in X$, and let $x_n = p_n(x)$ for $n \in \mathbb{N}$. The inverse system $\{(X_n, p_{nm})\}$ gives rise to an inverse system $\{T_{x_n}(X_n), (dp_{nm})_{x_n}\}$. We can then form the inverse limit $\varprojlim T_{x_n}(X_n)$.

By Proposition 2.7, $\mathfrak{m}_x = \varinjlim p_n^*\mathfrak{m}_{x_n}$, where $\mathfrak{m}_{x_n} \subset \mathcal{O}_{X_n,x_n}$ is the unique maximal ideal of $\mathcal{O}_{X_n,x_n}$. It follows that $\mathfrak{m}_x^2 = \varinjlim p_n^*(\mathfrak{m}_{x_n}^2)$. Indeed, suppose that $f \in \mathfrak{m}_x^2$. Then f is a finite sum $f = \sum_{n,m}(p_n^* f_n)(p_m^* g_m)$, with $f_n \in \mathfrak{m}_{x_n}$ and $g_m \in \mathfrak{m}_{x_m}$. If we let γ be the maximum over all indices n and m appearing in this sum, then $f = \sum_{\text{finite}} p_\gamma^*(f_\gamma g_\gamma)$, where $f_\gamma = p_{\gamma n}^* f_n$ and $g_\gamma = p_{\gamma m}^* g_m$. But then $f \in \varinjlim p_n^*(\mathfrak{m}_{x_n}^2)$. It is easy to see that this argument is independent of the choice of indices used to represent f. Thus, $\mathfrak{m}_x^2 \subseteq \varinjlim p_n^*(\mathfrak{m}_{x_n}^2)$ and the other inclusion is clear. Therefore,

$$\begin{aligned} \varprojlim T_{x_n}(X_n) &= \varprojlim (\mathfrak{m}_{x_n}/\mathfrak{m}_{x_n}^2)^* \\ &\cong (\varinjlim (\mathfrak{m}_{x_n}/\mathfrak{m}_{x_n}^2))^* \\ &\cong (\varinjlim \mathfrak{m}_{x_n}/\varinjlim \mathfrak{m}_{x_n}^2)^* \\ &\cong (\mathfrak{m}_x/\mathfrak{m}_x^2)^* \\ &= T_x(X). \end{aligned} \tag{2.13}$$

The commutativity of Diagram (2.12) now follows from a simple computation. □

Remark 2.14. If the transition maps p_{nm} are assumed to be surjective submersions for all n, m, then $T_x(X) = \varprojlim T_{x_n}(X_n)$ has the structure of a provariety as in Section 2.1.

Definition 2.15. We call a derivation of the sheaf of F-algebras $\mathcal{O}_X$, $\xi : \mathcal{O}_X \to \mathcal{O}_X$ a (global) vector field on X. It follows from definitions that for each $x \in X$, ξ induces a point derivation of the stalk $\xi_x : \mathcal{O}_{X,x} \to F$, so that for all $x \in X$, $\xi_x \in T_x(X)$.

Definition 2.16. For $x \in X = \varprojlim X_n$, we define the cotangent space at x to be

$$T_x^*(X) := \varinjlim T_{x_n}^*(X_n),$$

where $T_{x_n}^*(X_n)$ is the contangent space at $x_n = p_n(x)$ of X_n. Observe that $(T_x^*(X))^* = T_x(X)$ by Theorem 2.10.

2.3. Morphisms of provarieties

In this section, we show that the provariety constructed in Section 2.1 is an inverse limit in the category of locally ringed spaces. We first observe that the canonical projection maps: $p_k : X = \varprojlim X_n \to X_k$ are morphisms of locally ringed spaces with differentials $dp_k = \pi_k : T(X) = \varprojlim T(X_n) \to T(X_k)$. (See (2.12).) This follows immediately from Equation (2.1), Proposition 2.7, and Theorem 2.10. The following basic lemma, which appears without proof in [11], will be used to establish the main result of this section.

Lemma 2.17. *Let X be a topological space, and let $\mathcal{B}$ be a basis for the topology on X. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of F-algebras on X. Suppose that for any $U \in \mathcal{B}$, we have a homomorphism of F-algebras:*

$$\Phi_U : \mathcal{F}(U) \to \mathcal{G}(U),$$

such that if $W \subseteq U$ with $W \in \mathcal{B}$, then the following diagram commutes:

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\Phi_U"] \arrow[d, "\rho^{\mathcal{F}}_{UW}"] & \mathcal{G}(U) \arrow[d, "\rho^{\mathcal{G}}_{UW}"] \\ \mathcal{F}(W) \arrow[r, "\Phi_W"] & \mathcal{G}(W). \end{tikzcd} \tag{2.18}$$

(i.e. Φ is a morphism of the $\mathcal{B}$-presheaves associated to the sheaves $\mathcal{F}$ and $\mathcal{G}$.) Then Φ lifts to a morphism of sheaves $\tilde{\Phi} : \mathcal{F} \to \mathcal{G}$ such that $\widetilde{\Phi}_U = \Phi_U$ for $U \in \mathcal{B}$.

Proof. Let $V \subseteq X$ be open. Then since $\mathcal{F}$ and $\mathcal{G}$ are sheaves of F-algebras,

$$\mathcal{F}(V) \cong \varprojlim_{U \subseteq V, U \in \mathcal{B}} \mathcal{F}(U) \text{ and } \mathcal{G}(V) \cong \varprojlim_{U \subseteq V, U \in \mathcal{B}} \mathcal{G}(U).$$

Since the diagram in (2.18) is commutative, the universal property of inverse limits gives a morphism:

$$\widetilde{\Phi}_U := \varprojlim_{U \subseteq V} \Phi_U : \mathcal{F}(V) \to \mathcal{G}(V).$$

It is easy to see that $\tilde{\Phi}$ is a morphism of sheaves with the desired property. □

We now state and prove the main result of this section.

Proposition 2.19. *Let $\{(X_n, p_{nk})\}$ be an inverse system of varieties with surjective transition morphisms, and let $\mathcal{O}_{X_n}$ be the structure sheaf of X_n. Let $(X = \varprojlim X_n, \mathcal{O}_X)$ be the corresponding provariety. Let $(Y, \mathcal{O}_Y)$ be a locally ringed space. Suppose we are given morphisms of locally ringed spaces $\{f_n\}_{n\in\mathbb{N}}$ with $f_n : (Y, \mathcal{O}_Y) \to (X_n, \mathcal{O}_{X_n})$ such that for any $m \geq n$ the following diagram commutes:*

$$\begin{tikzcd} (Y, \mathcal{O}_Y) \arrow[d, "f_n"] \arrow[dr, "f_m"] & \\ (X_n, \mathcal{O}_{X_n}) & (X_m, \mathcal{O}_{X_m}). \arrow[l, "p_{mn}"] \end{tikzcd} \tag{2.20}$$

Then the map $f := \varprojlim f_n$ is a morphism of locally ringed spaces

$$f : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X).$$

Moreover, for any $y \in Y$, the differential $(df)_y : T_y(Y) \to T_{f(y)}(X)$ is given by:

$$(df)_y = \varprojlim (df_n)_y. \tag{2.21}$$

Proof. Since the diagram in (2.20) is commutative, the universal property of inverse limits gives us a map of sets $f := \varprojlim f_n : Y \to \varprojlim X_n = X$. Since X has the inverse limit topology, it follows that f is continuous.

We claim that f induces a morphism of sheaves of F-algebras on X, $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_Y$. To show this, we use Lemma 2.17. Let $U \subseteq X$ be a basic open set. Then $U = p_n^{-1}(U_n)$ for some open set $U_n \subseteq X_n$. Since the comorphism $p_n^* : \mathcal{O}_{X_n} \to (p_n)_*\mathcal{O}_X$ is injective for all n, the commutativity of Diagram (2.20) implies that the following diagram is also commutative:

$$\begin{tikzcd} & \mathcal{O}_Y(f_n^{-1}(U_n)) \\ p_n^*\mathcal{O}_{X_n}(p_n^{-1}(U_n)) \arrow[ur, "f_n^\sharp \circ (p_n^*)^{-1}"] \arrow[r, "p_m^* \circ p_{mn}^* \circ (p_n^*)^{-1}"] & p_m^*\mathcal{O}_{X_m}(p_{mn}^{-1}(U_n)). \arrow[u, "f_m^\sharp \circ (p_m^*)^{-1}"'] \end{tikzcd} \tag{2.22}$$

For ease of notation, let $\tilde{f_m} := f_m^\sharp \circ (p_m^*)^{-1}$ for each $m \in \mathbb{N}$. It follows from Diagram (2.22) and the universal property of direct limits that

$$\varinjlim_{m \geq n} \tilde{f_m} : \mathcal{O}_X(U) = \varinjlim_{m \geq n} p_m^*\mathcal{O}_{X_m}(p_{mn}^{-1}(U_n)) \to \mathcal{O}_Y(f_n^{-1}(U_n)) = f_*\mathcal{O}_Y(p_n^{-1}(U_n)) \tag{2.23}$$

is a homomorphism of F-algebras. It is easy to see that this homomorphism is compatible with restriction maps of the $\mathcal{B}$-presheaf $\widetilde{\mathcal{O}_X}$. Thus, by Lemma 2.17 we obtain a morphism of sheaves of F-algebras:

$$f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_Y.$$

Since the maps f_n are morphisms of locally ringed spaces, it follows from Proposition 2.7 that $(f, f^\sharp)$ is a morphism of locally ringed spaces.

We now compute the differential of f. The diagram in (2.20) gives rise to a commutative diagram

$$\begin{tikzcd} T_y(Y) \arrow[d, "(df_n)_y"] \arrow[dr, "(df_m)_y"] & \\ T_{f_n(y)}(X_n) & T_{f_m(y)}(X_m) \arrow[l, "(dp_{mn})_{f_m(y)}"'] \end{tikzcd} \tag{2.24}$$

for any $y \in Y$. It follows from Theorem 2.10 that

$$(df)_y = \varprojlim (df_n)_y : T_y(Y) \to \varprojlim T_{f_n(y)}(X_n) = T_{f(y)}(X). \qquad \square$$

Corollary 2.25. *Let $\{(X_n, p_{nk})_{n\in\mathbb{N}}\}$ and $\{(Y_n, q_{nk})\}_{n\in\mathbb{N}}$ be inverse systems of varieties with surjective transition morphisms, and let $(X = \varprojlim X_n, \mathcal{O}_X)$ and $(Y = \varprojlim Y_n, \mathcal{O}_Y)$ be the corresponding provarieties. Suppose that for each $n \in \mathbb{N}$, we have morphisms $f_n : X_n \to Y_n$ such that for any $m \geq n$ the following diagram commutes:*

$$\begin{tikzcd} X_m \arrow[r, "f_m"] \arrow[d, "p_{mn}"'] & Y_m \arrow[d, "q_{mn}"] \\ X_n \arrow[r, "f_n"] & Y_n. \end{tikzcd} \tag{2.26}$$

Then the map $f = \varprojlim f_n : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of locally ringed spaces with differential:

$$df = \varprojlim df_n : \varprojlim T(X_n) \to \varprojlim T(Y_n).$$

Proof. The hypotheses of the corollary imply that the maps

$$\tilde{f}_n : \varprojlim X_n \xrightarrow{p_n} X_n \xrightarrow{f_n} Y_n$$

are morphisms of locally ringed spaces satisfying the conditions of Proposition 2.19. It then follows that $\varprojlim \tilde{f}_n = \varprojlim f_n$ is a morphism of locally ringed spaces with differential $\varprojlim d\tilde{f}_n = \varprojlim df_n$. $\square$

Example 2.27. For $n \in \mathbb{N}$, let $\mathfrak{g}_n$ be a finite dimensional Lie algebra over $\mathbb{C}$. Suppose we have a chain

$$\mathfrak{g}_1 \xrightarrow{j_{12}} \mathfrak{g}_2 \xrightarrow{j_{23}} \mathfrak{g}_3 \to \cdots \to \mathfrak{g}_n \xrightarrow{j_{n,n+1}} \cdots, \tag{2.28}$$

where $j_{n,n+1} : \mathfrak{g}_n \to \mathfrak{g}_{n+1}$ is an injective homomorphism of Lie algebras. The direct limit $\mathfrak{g} := \varinjlim \mathfrak{g}_n$ is naturally a Lie algebra, and the full vector space dual $\mathfrak{g}^* = \varprojlim \mathfrak{g}_n^*$ is a provariety. For $\lambda \in \mathfrak{g}^*$, the tangent space at λ is naturally the provariety:

$$T_\lambda(\mathfrak{g}^*) = \varprojlim T_{\lambda_n}(\mathfrak{g}_n^*) = \mathfrak{g}^*$$

by Theorem 2.10 and Remark 2.14. Similarly, we can identify the cotangent space at $\lambda \in \mathfrak{g}^*$ with the Lie algebra $\mathfrak{g}$ as a vector space:

$$T_\lambda^*(\mathfrak{g}^*) = \varinjlim T_{\lambda_n}^*(\mathfrak{g}_n^*) = \varinjlim (\mathfrak{g}_n^*)^* \cong \mathfrak{g}. \tag{2.29}$$

Suppose that for each $n \in \mathbb{N}$, the Lie algebra $\mathfrak{g}_n$ is reductive with non-degenerate, associative form $\ll \cdot,\cdot \gg$. Then we can use the form $\ll \cdot,\cdot \gg$ to identify $\mathfrak{g}_n$ with $\mathfrak{g}_n^*$, giving the vector space

$$\tilde{\mathfrak{g}} := \varprojlim \mathfrak{g}_n \tag{2.30}$$

the structure of a provariety. By Corollary 2.25, $\tilde{\mathfrak{g}} \cong \mathfrak{g}^*$ as provarieties.

In particular, consider the case where $\mathfrak{g}_n = \mathfrak{gl}(n, \mathbb{C})$ is the Lie algebra of $n \times n$ complex matrices. For $X \in \mathfrak{g}_n$, let $j_{n,n+1}(X)$ be the $(n+1) \times (n+1)$ matrix with $(j_{n,n+1}(X))_{kj} = X_{kj}$ for $k, j \in \{1, \ldots, n\}$ and $(j_{n,n+1}(X))_{kj} = 0$ otherwise. Then $\mathfrak{g} = \mathfrak{gl}(\infty)$ is the Lie algebra of infinite-by-infinite complex matrices with only finitely many non-zero entries. Moreover, the Lie algebra $\mathfrak{g}_n$ is reductive with non-degenerate, associative form $\ll X, Y \gg = tr(XY)$, where $tr(\cdot)$ denotes the trace function. Using the trace form, the map $j_{n,n+1}^* : \mathfrak{g}_{n+1}^* \to \mathfrak{g}_n^*$ is identified with the map $p_{n+1,n} : \mathfrak{g}_{n+1} \to \mathfrak{g}_n$, where $p_{n+1,n}(X) = X_n$, and X_n is the $n \times n$ submatrix in the upper left-hand corner of $X \in \mathfrak{g}_{n+1}$. We denote the dual space of $\mathfrak{g}$, $\tilde{\mathfrak{g}}$ defined in Equation (2.30) as $M(\infty)$. Thus,

$$M(\infty) := \{(X(1), X(2), \ldots, X(n), X(n+1), \ldots,) : X(n) \in \mathfrak{g}_n \text{ and } X(n+1)_n = X(n)\}. \tag{2.31}$$

The provariety $M(\infty)$ is naturally isomorphic to the vector space of infinite-by-infinite complex matrices with arbitrary entries.

A similar construction works for any classical direct limit Lie algebra. For example, if $\mathfrak{g}_n = \mathfrak{so}(n, \mathbb{C})$ is the Lie algebra of $n \times n$ complex skew-symmetric matrices, then $\mathfrak{so}(\infty) := \varinjlim \mathfrak{g}_n$ is the Lie algebra of infinite-by-infinite skew-symmetric matrices with only finitely many nonzero entries. The dual space $\tilde{\mathfrak{g}} \cong \mathfrak{so}(\infty)^*$ is the provariety of infinite-by-infinite complex skew-symmetric matrices.

We will see in the next section that the Lie–Poisson structure of $\mathfrak{g}_n^*$ has a natural generalization to the provariety $\mathfrak{g}^* = \varprojlim \mathfrak{g}_n^*$.

2.4. Poisson provarieties

We briefly recall some basic definitions from Poisson geometry. A variety X is a *Poisson variety* if the structure sheaf $\mathcal{O}_X$ is a sheaf of Poisson algebras. That is to say that for each open subset $U \subseteq X$, $\mathcal{O}_X(U)$ is a Poisson algebra and the restriction maps $\rho_{UV} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ are homomorphisms of Poisson algebras. This is equivalent to specifying a regular bivector field $\pi \in \wedge^2 TX$, whose Schouten–Nijenhuis bracket $[\pi, \pi] = 0$. We have the relation

$$\{f, g\}(x) = \pi_x(df_x, dg_x), \tag{2.32}$$

for $x \in X$ and $f, g \in \mathcal{O}_X(X)$, where $\{\cdot,\cdot\}$ denotes the Poisson bracket on $\mathcal{O}_X(X)$. For a regular function $f \in \mathcal{O}_X(X)$, we define the Hamiltonian vector field ξ_f by

$$\xi_f(g) = \{f, g\},$$

for $g \in \mathcal{O}_X(X)$. The Poisson bivector π defines a bundle map $\widetilde{\pi} : T^*(X) \to T(X)$, given by

$$\widetilde{\pi}(\lambda)(\mu) = \pi(\lambda, \mu),$$

for $\lambda, \mu \in T^*(X)$. It follows from (2.32) that $\widetilde{\pi}(df) = \xi_f$. We refer to $\widetilde{\pi}$ as the *anchor map*.

Given two Poisson varieties (X_1, π_1), (X_2, π_2), a morphism $\phi : X_1 \to X_2$ is said to be *Poisson* if the comorphism $\phi^\sharp : \mathcal{O}_{X_2} \to \phi_* \mathcal{O}_{X_1}$ is a morphism of sheaves of Poisson algebras. In particular, we say that $(X_1, \pi_1) \subset (X_2, \pi_2)$ is a Poisson subvariety if the inclusion map $i : X_1 \to X_2$ is Poisson.

Let (X_n, p_{nm}) be an inverse system of varieties with surjective transition morphisms. Suppose that each of the varieties X_n is Poisson and the morphisms $p_{nm} : X_n \to X_m$ are Poisson. We claim that the structure sheaf $\mathcal{O}_X$ constructed in Section 2.1 is a sheaf of Poisson algebras. We begin with the following lemma whose proof is elementary.

Lemma 2.33.

(1) *Let (A_n, ϕ_{nm}) be a directed system of Poisson algebras. That is, A_n is a Poisson algebra for each n and $\phi_{nm} : A_n \to A_m$ is a homomorphism of Poisson algebras for each $n \leq m$. Then the direct limit $A = \varinjlim A_n$ has a natural Poisson algebra structure and is a direct limit in the category of Poisson algebras.*
(2) *Let (B_n, ψ_{nm}) be an inverse system of Poisson algebras. Then $\varprojlim B_n$ has the structure of a Poisson algebra and is the inverse limit in the category of Poisson algebras.*

Proposition–Definition 2.34. *Let (X_n, p_{nm}) be an inverse system of Poisson varieties X_n, with surjective Poisson morphisms p_{nm}, and let $(X = \varprojlim X_n, \mathcal{O}_X)$ be the corresponding provariety. Then the structure sheaf $\mathcal{O}_X$ constructed in Section 2.1 is a sheaf of Poisson algebras. We call $X = \varprojlim X_n$ a* Poisson provariety.

Proof. It follows from (2.1) and Part (1) of Lemma 2.33 that the $\mathcal{B}$-presheaf $\widetilde{\mathcal{O}_X}$ is a $\mathcal{B}$-presheaf of Poisson algebras. Part (2) of Lemma 2.33 and Equation (2.2) then imply that the sheaf $\mathcal{O}_X$ is a sheaf of Poisson algebras. $\square$

The following lemma will play an important role in the constructions that follow.

Lemma 2.35. *Let $\{V_n, \phi_{nm}\}$ be a directed system of vector spaces. Then for any $k \in \mathbb{N}$*

$$\left(\bigwedge^k \varinjlim_n V_n\right)^* \cong \varprojlim_n \left[\left(\bigwedge^k V_n\right)^*\right].$$

Proof. By universal properties of direct limits, there exist $\phi_\ell : V_\ell \to \varinjlim_n V_n$ compatible with transition functions $\phi_{\ell m} : V_\ell \to V_m$ for all $\ell \le m$. These define maps $\wedge^k \phi_\ell : \wedge^k V_\ell \to \wedge^k \varinjlim_n V_n$ compatible with the transition maps $\wedge^k \phi_{\ell m} : \wedge^k V_\ell \to \wedge^k V_m$. This induces a map $\varinjlim_n \wedge^k \phi_n : \varinjlim_n \wedge^k V_n \to \wedge^k \varinjlim_n V_n$. Dualizing, we obtain the desired map

$$\psi : \left(\bigwedge^k \varinjlim_n V_n\right)^* \to \left(\varinjlim_n \bigwedge^k V_n\right)^* = \varprojlim_n \left[\left(\bigwedge^k V_n\right)^*\right].$$

It is straightforward to verify that ψ is a vector space isomorphism. Concretely, $\psi(f) = (f_1, f_2, \ldots)$, where

$$f_n = f \circ \wedge^k \phi_n,$$

for each n. $\square$

Let $X = \varprojlim X_n$ be a Poisson provariety. As in the finite dimensional case, the Hamiltonian vector field ξ_f of f is defined by $\xi_f(g) = \{f, g\}$ for any f, $g \in \mathcal{O}_X(X)$. The cotangent space $T_x^*(X) = \varinjlim T_{x_n}^*(X_n)$ is spanned by the differentials df_x of global functions $f \in \mathcal{O}_X(X) \cong \varinjlim \mathcal{O}_{X_n}(X_n)$. Thus, for each $x \in X$, the Poisson bracket $\{\cdot,\cdot\}$ defines an element $\pi_{X,x} \in \left(\wedge^2 T_x^* X\right)^*$ given by

$$\pi_{X,x}(df_x, dg_x) := \{f, g\}(x), \tag{2.36}$$

cf. (2.32). By Lemma 2.35, we can view $\pi_{X,x}$ as an element of $\varprojlim \wedge^2 T_{x_n} X_n$ at each $x \in X$. We define the Poisson bivector of X, π_X to be the element of $\varprojlim \wedge^2 T X_n$ whose value at each $x \in X$ is given by (2.36). The bivector π_X is an inverse limit of the Poisson bivector on each X_n.

Proposition 2.37. *Let $X = \varprojlim X_n$ be a Poisson provariety, and let $\pi_n \in \wedge^2 T X_n$ be the bivector fields defining the Poisson structure on X_n. Then $\pi_X = \varprojlim \pi_n \in \varprojlim \wedge^2 T X_n$. For each $x \in X$, the anchor map $\widetilde{\pi}_{X,x} : T_x^* X \to T_x X$ is*

$$\widetilde{\pi}_{X,x}(\lambda_n) = \left(dp_{n1}\widetilde{\pi}_{n,x_n}(\lambda_n), dp_{n2}\widetilde{\pi}_{n,x_n}(\lambda_n), \ldots, \widetilde{\pi}_{n,x_n}(\lambda_n), \widetilde{\pi}_{n+1,x_{n+1}}(dp_{n+1,n}^*\lambda_n), \ldots\right),$$

for $\lambda_n \in \varinjlim T_{x_n}^* X_n$, a representative of $\lambda_n \in T_{x_n}^*(X_n)$ in the direct limit.

Proof. This is an elementary computation using the definition of the Poisson bracket $\{\cdot,\cdot\}$ on X. $\square$

Example 2.38. For $n \in \mathbb{N}$, let $\mathfrak{g}_n$ be a finite dimensional, complex Lie algebra. Then $\mathfrak{g}_n^*$ is a Poisson variety with the Lie–Poisson structure. The Poisson bracket of linear functions x_n, $y_n \in \mathfrak{g}_n$ is given by their Lie bracket, i.e.

$$\{x_n, y_n\}(\mu_n) = \mu_n([x_n, y_n]), \tag{2.39}$$

for $\mu_n \in \mathfrak{g}_n^*$ (see for example, Section 1.3, [5]). We denote the corresponding bivector by $\pi_n \in \wedge^2 T\mathfrak{g}_n^*$. We let ad^* denote the coadjoint action of $\mathfrak{g}_n$ on $\mathfrak{g}_n^*$. Equation (2.39) implies the anchor map $\widetilde{\pi}_n$ for the Lie–Poisson structure on $\mathfrak{g}_n^*$ is given by

$$\widetilde{\pi}_{n,\mu_n}(x_n) = -\,\mathrm{ad}^*(x_n)\mu_n. \tag{2.40}$$

Now suppose we have a chain of Lie algebras as in Equation (2.28) of Example 2.27:

$$\mathfrak{g}_1 \xrightarrow{j_{12}} \mathfrak{g}_2 \xrightarrow{j_{23}} \mathfrak{g}_3 \to \cdots \to \mathfrak{g}_n \xrightarrow{j_{n,n+1}} \cdots,$$

and let $\mathfrak{g} := \varinjlim \mathfrak{g}_n$ be the corresponding direct limit Lie algebra. Since the homomorphisms $j_{n,n+1} : \mathfrak{g}_n \to \mathfrak{g}_{n+1}$ are inclusions, their pullbacks $p_{n+1,n} : \mathfrak{g}_{n+1}^* \to \mathfrak{g}_n^*$ are Poisson submersions with respect to the Lie–Poisson structures on $\mathfrak{g}_{n+1}^*$ and $\mathfrak{g}_n^*$. Thus, $\mathfrak{g}^* = \varprojlim \mathfrak{g}_n^*$ is a Poisson provariety with bivector $\pi_{\mathfrak{g}^*} = \varprojlim \pi_n$.

For $\mu \in \mathfrak{g}^*$, we identify the cotangent space $T_\mu^*(\mathfrak{g}^*)$ with $\mathfrak{g}$ as in (2.29). Then Proposition 2.37 and Equation (2.40) imply that the anchor map is

$$\widetilde{\pi}_{\mathfrak{g}^*,\mu}(x_n) = \left(-\operatorname{ad}^*(x_n)\mu_n|_{\mathfrak{g}_1}, \ldots, -\operatorname{ad}^*(x_n)\mu_n|_{\mathfrak{g}_{n-1}}, -\operatorname{ad}^*(x_n)\mu_n, \ldots, -\operatorname{ad}^*(x_n)\mu_k, \ldots\right), \tag{2.41}$$

for $x_n \in \mathfrak{g}_n \subset \mathfrak{g}$, $\mu \in \mathfrak{g}^*$, and where $-\operatorname{ad}^*(x_n)\mu_n|_{\mathfrak{g}_\ell}$ denotes the restriction of the linear functional $-\operatorname{ad}^*(x_n)\mu_n \in \mathfrak{g}_n^*$ to $\mathfrak{g}_\ell$ for $\ell < n$.

By Equation (2.41), the kernel of the anchor map consists precisely of the covectors $x_n \in T_{\mu_n}^*\mathfrak{g}_n^* \subseteq T_\mu^*\mathfrak{g}^* = \mathfrak{g}$ whose coadjoint action $\operatorname{ad}^*(x_n)$ annihilates μ_k for $k \geq n$. For $k \geq n$, let $\mathfrak{g}_n^{\mu_k} := \{x_n \in \mathfrak{g}_n : \operatorname{ad}^*(x_n)\mu_k = 0\}$ denote the annihilator of μ_k in $\mathfrak{g}_n$.

Proposition 2.42. *Let $\mu \in \mathfrak{g}^*$ and let Ker $\widetilde{\pi_{\mathfrak{g}^*}}_\mu$ be the kernel of the anchor map $\widetilde{\pi_{\mathfrak{g}^*}}$ at μ. Then*

$$\mathit{Ker}\ \widetilde{\pi_{\mathfrak{g}^*}}_\mu = \varinjlim_n \bigcap_{k \geq n} \mathfrak{g}_n^{\mu_k}. \tag{2.43}$$

In the case where $\mathfrak{g}_n$ is reductive with adjoint group G_n, we can use the non-degenerate G_n-equivariant form $\ll \cdot,\cdot \gg$ on $\mathfrak{g}_n$ to transfer the Lie–Poisson structure of $\mathfrak{g}_n^*$ to $\mathfrak{g}_n$. The coadjoint action of G_n on $\mathfrak{g}_n^*$ is then identified with the adjoint action of G_n on $\mathfrak{g}_n$. The induced maps $p_{n+1,n} : \mathfrak{g}_{n+1} \to \mathfrak{g}_n$ are Poisson submersions and the provariety $\tilde{\mathfrak{g}} = \varprojlim \mathfrak{g}_n$ defined in Equation (2.30) is a Poisson provariety. For example, the provariety $M(\infty)$ defined in Equation (2.31) is a Poisson provariety.

Let X be a Poisson provariety with bivector $\pi_X \in \varprojlim \wedge^2 TX_n$. For each $x \in X$, consider the subspace

$$\mathfrak{X}(X)_x = \{(\xi_f)_x : f \in \mathcal{O}_X(X)\} = \operatorname{Im}\{\tilde{\pi}_{X,x}(T_x^*(X))\} \subseteq T_x(X). \tag{2.44}$$

As in the finite dimensional case, we refer to the union $\mathfrak{X}(X) = \bigcup_{x\in X} \mathfrak{X}(X)_x$ as the *characteristic distribution* of X.

If $(X, \mathcal{O}_X)$ is a finite dimensional, non-singular, Poisson variety over $\mathbb{C}$ then $\mathfrak{X}(X)$ is an integrable distribution. Its leaves are immersed Poisson analytic submanifolds of $(S, \{\cdot,\cdot\}_S)$ where the Poisson bracket on S is induced by a symplectic form ω_S on S (see Chapter 2, [26] for example). The Poisson submanifolds $(S, \{\cdot,\cdot\}_S)$ are referred to as *symplectic leaves* of X. For example, let $\mathfrak{g}_n^*$ be the dual space of a finite dimensional Lie algebra over $\mathbb{C}$ with the Lie–Poisson structure π_n as in Example 2.38. Then the symplectic leaves of $(\mathfrak{g}_n^*, \pi_n)$ are the coadjoint orbits of G_n on $\mathfrak{g}_n^*$ equipped with Kostant–Kirillov symplectic structure, where G_n is any connected Lie group with Lie algebra $\mathfrak{g}_n$ (see Proposition 3.1, [26]).

In infinite dimensions, it is not known whether the characteristic distribution is integrable even for the case of Banach–Poisson manifolds [24]. In Section 4, we show that for the dual $\mathfrak{g}^*$ of a direct limit Lie algebra $\mathfrak{g}$, the characteristic distribution is integrable, and that the symplectic foliation of $\mathfrak{g}^*$ is given by the coadjoint orbits of an Ind-group G on $\mathfrak{g}^*$ with Lie algebra $\mathfrak{g}$. For this, we need to study ind-varieties and direct limit groups in more detail.

3. Ind-groups

3.1. Basic definitions

In this section, we recall some basic facts about ind-varieties. For further reading, see [18].

For each $n \in \mathbb{N}$, let X_n be a finite dimensional variety defined over the field F. Suppose for any $m \in \mathbb{N}$ with $n \leq m$, we have a locally closed embedding $i_{nm} : X_n \to X_m$. We call the direct limit $X := \varinjlim X_n$ of the varieties $\{X_n\}_{n\in\mathbb{N}}$, an *ind-variety*.[1]

As a topological space X is endowed with the final topology (i.e. the finest topology for which the inclusion maps $\iota_n : X_n \hookrightarrow X$ are continuous), so that $U \subset X$ is open if and only if $U \cap X_n$ is open for all $n \in \mathbb{N}$. It is easy to see that $Z \subset X$ is closed if and only if $Z \cap X_n$ is closed for all $n \in \mathbb{N}$. An ind-variety X is said to be irreducible if its underlying topological space is irreducible. One notes that if $X = \varinjlim X_n$ with X_n irreducible for all n, then X is irreducible.

For any open set $U \subseteq X$, the structure sheaf is given by $\mathcal{O}_X(U) = \varprojlim \mathcal{O}_{X_n}(U_n)$, where $U_n = U \cap X_n$. (When there is no ambiguity, we identify X_n with its image $\iota_n(X_n) \subseteq X$.) A map $f : X \to Y$ is a *morphism of ind-varieties* if there is a strictly increasing function $m : \mathbb{N} \to \mathbb{N}$, such that the restriction f_n of f to $X_n \subseteq X$ is a morphism of varieties $f_n : X_n \to Y_{m(n)}$. The map f induces a morphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$. Two ind-variety structures on the same set X are said to be equivalent if the identity map $I_X : X \to X$ is an isomorphism of ind-varieties. We will not distinguish between equivalent ind-variety structures.

The product $X \times Y$ of two ind-varieties X and Y is naturally an ind-variety, by viewing $X \times Y = \varinjlim (X \times Y)_n$, where $(X \times Y)_n = X_n \times Y_n$, and the transition maps $\iota_{nm} : (X \times Y)_n \to (X \times Y)_m$ are given by $\iota_{nm} = \iota^X_{nm} \times \iota^Y_{nm}$, where $\iota^X_{nm} : X_n \to X_m$ and $\iota^Y_{nm} : Y_n \to Y_m$ are the corresponding transition maps for X and Y.

Given an element x of an ind-variety $X = \varinjlim X_n$, there exists $k \in \mathbb{N}$ so that $x \in X_\ell$ for all $\ell \geq k$. We define the *tangent space* $T_x(X)$ to X at x to be $T_x(X) = \varinjlim_{\ell \geq k} T_x(X_\ell)$. For a morphism of ind-varieties $f : X \to Y$, the differential $(df)_x$ at $x \in X$ is given by

$$(df)_x = \varinjlim_{\ell \geq k} (df_\ell)_x : \varinjlim_{\ell \geq k} T_x(X_\ell) \to \varinjlim_{\ell \geq k} T_{f(x)}(Y_{m(\ell)}),$$

where $f_\ell : X_\ell \to Y_{m(\ell)}$ is the morphism obtained by restricting f to X_ℓ.

The next proposition asserts that an ind-variety is a direct limit in the category of ringed spaces.

Proposition 3.1. *Let $(X = \varinjlim X_n, \mathcal{O}_X)$ be an ind-variety and let $(Y, \mathcal{O}_Y)$ be a locally ringed space. For each $i \in \mathbb{N}$, suppose we have morphisms of locally ringed spaces*

$$f_n : (X_n, \mathcal{O}_{X_n}) \to (Y, \mathcal{O}_Y)$$

such that the following diagram commutes.

$$\begin{tikzcd}
(Y, \mathcal{O}_Y) & \\
(X_n, \mathcal{O}_{X_n}) \arrow[u, "f_n"] \arrow[r, "i_{nm}"] & (X_m, \mathcal{O}_{X_m}). \arrow[ul, "f_m"']
\end{tikzcd} \tag{3.2}$$

Then $f := \varinjlim f_n : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces with differential: $df = \varinjlim df_n : T(X) \to T(Y)$.

[1] The traditional definition of an ind-variety stipulates that the embeddings $i_{nm} : X_n \to X_m$ are closed (see for example [18,9]). We require a slightly more general notion for the objects we consider.

Proof. By the universal property of the direct limit, there is a map of sets $f := \varinjlim f_n : \varinjlim X_n \to Y$. We note that f is continuous, since $X = \varinjlim X_n = \bigcup_{n\in\mathbb{N}} X_n$ has the final topology and each of the maps $f_n : X_n \to Y$ are continuous.

We claim that f induces a morphism of sheaves of F-algebras on Y, $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$. The commutative diagram in (3.2) gives rises to a commutative diagram of morphisms of sheaves of F-algebras on Y:

$$\begin{tikzcd} \mathcal{O}_Y \arrow[d, "f_n^\sharp"'] \arrow[drr, "f_m^\sharp"] & & \\ f_{n,*}\mathcal{O}_{X_n} & & f_{m,*}\mathcal{O}_{X_m}, \arrow[ll, "i_{nm}^\sharp"'] \end{tikzcd} \tag{3.3}$$

and $\{f_{m,*}\mathcal{O}_{X_m}, i_{nm}^\sharp\}$ is an inverse system of sheaves of F-algebras on Y. By Exercise II 1.12, [14], $\varprojlim f_{n,*}\mathcal{O}_{X_n}$ is a sheaf of F-algebras on Y, which satisfies the universal property of inverse limits in the category of sheaves of F-algebras on Y. Thus, we get a morphism of sheaves of F-algebras on Y:

$$\varprojlim f_n^\sharp : \mathcal{O}_Y \to \varprojlim f_{n,*}\mathcal{O}_{X_n}.$$

It follows from definitions that $\varprojlim f_{n,*}\mathcal{O}_{X_n} = f_*\mathcal{O}_X$. If we let $f^\sharp := \varprojlim f_n^\sharp$, then $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces.

We now compute the differential df. Let $x \in X_n \subset X$. The commutative diagram in (3.2) yields a commutative diagram:

$$\begin{tikzcd} T(Y)_{f(x)} & & \\ T_x(X_n) \arrow[u, "(df_n)_x"] \arrow[rr, "(di_{nm})_x"] & & T_x(X_m). \arrow[ull, "(df_m)_x"'] \end{tikzcd} \tag{3.4}$$

By the universal property of direct limits, we obtain a map:

$$\varinjlim_{m\geq n} (df_m)_x : \varinjlim_{m\geq n} T_x(X_m) \to T_{f(x)}(Y).$$

Since $T_x(X) = \varinjlim_{m\geq n} T_x(X_m)$, we have $(df)_x = \varinjlim_{m\geq n} (df_m)_x$. $\square$

3.2. *Affine direct limit groups*

Let $\{G_n, i_{nm}\}_{m\geq n\in\mathbb{N}}$ be a directed system of affine algebraic groups, and let $i_{nm} : G_n \to G_m$ be a homomorphic embedding of algebraic groups. Then the image of G_n is closed in G_m (see for example, Section 7.2, [15]). The (affine) direct limit group $G = \varinjlim G_n$ is then naturally an ind-variety.

For $G = \varinjlim G_n$ a direct limit group, we consider the tangent space at the identity, $T_e(G)$. We have $T_e(G) = \varinjlim T_e(G_n) \cong \varinjlim \mathfrak{g}_n$, where $\mathfrak{g}_n = \mathrm{Lie}(G_n) \cong T_e(G_n)$, and we think of $\mathrm{Lie}(G_n)$ as the Lie algebra of right invariant vector fields on G_n. The ind-variety $\mathfrak{g} := \varinjlim \mathfrak{g}_n = \mathrm{Lie}(G)$ is a direct limit Lie algebra (see Example 2.27).

Example 3.5. For each $n \in \mathbb{N}$, let $G_n := GL(n, \mathbb{C})$ be the group of $n \times n$ invertible matrices over the complex numbers. We can embed G_n in G_{n+1} via the map

$$i_{nn+1} : g \hookrightarrow \begin{bmatrix} g & 0 \\ 0 & 1 \end{bmatrix} \in G_{n+1}.$$

This map is clearly a closed embedding, so we can form the direct limit group:

$$GL(\infty) := \varinjlim G_n. \tag{3.6}$$

Of course, $\mathrm{Lie}(GL(\infty)) = \mathfrak{gl}(\infty) = \varinjlim \mathfrak{gl}(n, \mathbb{C})$ is the direct limit Lie algebra discussed in Example 2.27.

An *algebraic action* of a direct limit group G on an ind-variety V is a morphism of ind-varieties $f : G \times V \to V$ such that each restriction $f_n = f|_{G_n \times V_n}$ defines an algebraic action of G_n on V_n, and the following diagram commutes:

$$\begin{tikzcd} G_n \times V_n \arrow[r, "f_n"] \arrow[d, "\iota^G_{nm} \times \iota^V_{nm}"'] & V_n \arrow[d, "\iota^V_{nm}"] \\ G_m \times V_m \arrow[r, "f_m"'] & V_m \end{tikzcd},$$

i.e. $f = \varinjlim f_n$. If the algebraic action of G on V is transitive, we say that V is a *homogeneous space* for G. If each V_n is a vector space over the base field F, then V is an *algebraic representation*. Any algebraic representation $\rho : G \times V \to V$ induces a representation $d\rho : \mathfrak{g} \times V \to V$ of $\mathfrak{g}$ by differentiation.

Example 3.7. The adjoint representation $\mathrm{Ad} : G \times \mathfrak{g} \to \mathfrak{g}$ defines an algebraic representation of G on $\mathfrak{g}$, and its differential $\mathrm{ad} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is the adjoint representation of $\mathfrak{g}$:

$$\mathrm{ad}(X)(Y) = [X, Y]_k,$$

where $X \in \mathfrak{g}_n$, $Y \in \mathfrak{g}_m$, $k = \max\{n, m\}$, and $[X, Y]_k$ is the bracket of X and Y thought of as elements of $\mathfrak{g}_k$.

The directed system $\iota_{nm} : \mathfrak{g}_n \to \mathfrak{g}_m$ induces an inverse system $\iota^*_{nm} : \mathfrak{g}^*_m \to \mathfrak{g}^*_n$ for $n \leq m$. The transition maps $\iota^*_{nm} : \mathfrak{g}^*_m \to \mathfrak{g}^*_n$ are G_n-equivariant with respect to the coadjoint action of $G_n \subset G_m$ on $\mathfrak{g}^*_m$ and $\mathfrak{g}^*_n$. Thus, we obtain an action of G on the dual space of its Lie algebra $\mathfrak{g}^* = \varprojlim \mathfrak{g}^*_n$, which we refer to as the coadjoint action of G on $\mathfrak{g}^*$. Concretely, let $\lambda = (\lambda_1, \dots, \lambda_n, \lambda_{n+1}, \dots,) \in \mathfrak{g}^*$. For $g \in G$, there exists $n > 0$ so that $g \in G_n$, and then

$$\mathrm{Ad}^*(g) \cdot \lambda = ((\mathrm{Ad}^*(g) \cdot \lambda_n)|_{\mathfrak{g}_1}, \dots, (\mathrm{Ad}^*(g) \cdot \lambda_n)|_{\mathfrak{g}_{n-1}}, \mathrm{Ad}^*(g) \cdot \lambda_n, \dots, \mathrm{Ad}^*(g) \cdot \lambda_k, \dots), \tag{3.8}$$

where $(\mathrm{Ad}^*(g) \cdot \lambda_n)|_{\mathfrak{g}_j}$ denotes the restriction of $\mathrm{Ad}^*(g) \cdot \lambda_n \in \mathfrak{g}^*_n$ to $\mathfrak{g}_j$ for $j < n$.

As has already been discussed in Section 2.4, $\mathfrak{g}^*$ is a Poisson provariety. In the next section, we will see that the coadjoint orbits of G on $\mathfrak{g}^*$ form a weak symplectic foliation of the Poisson provariety $\mathfrak{g}^*$. To do this, we first need to endow the coadjoint orbits described in (3.8) with the structure of a G-homogeneous ind-variety in a natural way. The key ingredient is the following proposition.

Proposition 3.9. *Let H be a closed subgroup of a direct limit group G. Then H is a direct limit group, and the quotient space G/H is an ind-variety and thus a homogeneous space for G. For any $g \in G$, the tangent space $T_{gH}(G/H)$ can be identified with the ind-variety $\mathfrak{g}/\mathrm{Ad}(g)\mathfrak{h}$.*

Conversely, if G acts transitively on a nonempty set X and the isotropy group G^x of any $x \in X$ is closed, then X can naturally be given the structure of an ind-variety by identifying X with the G-homogeneous ind-variety G/G^x. The resulting ind-variety structure on X is independent of the choice of point $x \in X$.

Proof. Write $G = \varinjlim G_n$ and $H_n = H \cap G_n$ for each $n \in \mathbb{N}$. Then $H = \varinjlim H_n$ is naturally a direct limit subgroup of G, and we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & H_n \arrow[r] \arrow[d, "\iota_{nm}"'] & G_n \arrow[r] \arrow[d, "\iota_{nm}"'] & G_n/H_n \arrow[r] \arrow[d, "\iota_{nm}"'] & 0 \\
0 \arrow[r] & H_m \arrow[r] & G_m \arrow[r] & G_m/H_m \arrow[r] & 0
\end{tikzcd}$$

where ι_{nm} denotes the transition map $G_n \to G_m$ as well as its restriction to H_n and the induced map on the quotient $i_{nm} : G_n/H_n \to G_m/H_m$. The transition maps $i_{nm} : G_n/H_n \to G_m/H_m$ are locally closed embeddings. By exactness of the direct limit functor, the sequence

$$0 \to \varinjlim H_n \to \varinjlim G_n \to \varinjlim G_n/H_n \to 0$$

is exact, so $\varinjlim G_n/H_n \cong \varinjlim G_n/\varinjlim H_n = G/H$ is naturally an ind-variety. It follows from definitions that the action of G on G/H is algebraic, so that G/H is a G-homogeneous space.

Let $gH \in G/H$ and consider the tangent space $T_{gH}(G/H)$. By our discussion above, gH can be identified with a unique element $g_nH_n \in \varinjlim_k G_k/H_k$. It follows that

$$T_{gH}(G/H) = \varinjlim_{k\geq n} T_{g_nH_k}(G_k/H_k) = \varinjlim_{k\geq n} \mathfrak{g}_k/\mathrm{Ad}(g_n)\mathfrak{h}_k \cong \varinjlim_{k\geq n} \mathfrak{g}_k/\varinjlim_{k\geq n} \mathrm{Ad}(g_n)\mathfrak{h}_k = \mathfrak{g}/\mathrm{Ad}(g)\mathfrak{h},$$

where we have used right invariant vector fields to identify the tangent space $T_{xH_k}(G_k/H_k)$ with $\mathfrak{g}_k/\mathrm{Ad}(x)\mathfrak{h}_k$ for any $x \in G_k$.

Conversely, suppose that G acts on a nonempty set X. Let $x \in X$. Then $X = G \cdot x = \bigcup_{n=1}^{\infty} G_n \cdot x$. Since G^x is closed, $G_n^x = G_n \cap G^x$ is closed for each n. Thus, $G_n \cdot x$ can be given the structure of a variety such that $G_n \cdot x \cong G_n/G_n^x$ as algebraic varieties. Thus,

$$X = \varinjlim G_n \cdot x \cong \varinjlim G_n/G_n^x \cong \varinjlim G_n/\varinjlim G_n^x = G/G^x$$

has the structure of G-homogeneous ind-variety. It is easy to see that the choice of any other point $y \in X$ produces an equivalent ind-variety structure on X. □

For a point $\lambda \in \mathfrak{g}^*$, we denote its coadjoint orbit by $G\cdot\lambda$. Using Proposition 3.9, we can endow $G\cdot\lambda$ with the structure of a G-homogeneous ind-variety.

Corollary 3.10. *Let $\lambda = (\lambda_1, \ldots, \lambda_n, \ldots, \lambda_k, \ldots) \in \mathfrak{g}^*$, with $\lambda_k \in \mathfrak{g}_k^*$, and let $G\cdot\lambda \subset \mathfrak{g}^*$ denote the coadjoint orbit through λ. Then the isotropy group of λ, G^λ is given by*

$$G^\lambda = \varinjlim G_n^\lambda \text{ where } G_n^\lambda = G^\lambda \cap G_n = \bigcap_{k\geq n} G_n^{\lambda_k}, \tag{3.11}$$

where $G_n^{\lambda_k}$ is the isotropy group of $\lambda_k \in \mathfrak{g}_k^*$ under the coadjoint action of $G_n \subset G_k$.

Thus, G_n^λ is closed, so that

$$G\cdot\lambda = \varinjlim G_n\cdot\lambda \cong \varinjlim G_n/G_n^\lambda \cong G/G^\lambda \tag{3.12}$$

has the structure of a G-homogeneous ind-variety. For any $\mu \in G\cdot\lambda$, we have

$$T_\mu(G\cdot\lambda) = \mathfrak{g}/\mathfrak{g}^\mu = \varinjlim \mathfrak{g}_n/\mathfrak{g}_n^\mu = T_\mu(G\cdot\mu), \tag{3.13}$$

where $\mathfrak{g}^\mu = Lie(G^\mu)$ with $G^\mu \subset G$ the isotropy group of μ.

Proof. We need only verify that

$$G_n^\lambda = \bigcap_{k \geq n} G_n^{\lambda_k}, \tag{3.14}$$

since the other statements of the corollary then follow immediately from Proposition 3.9. But (3.14) follows from the definition of the coadjoint action in Equation (3.8). □

4. Symplectic foliation of $\mathfrak{g}^*$

4.1. Kostant–Kirillov form

Throughout this section, let $G = \varinjlim G_n$ be an (affine) direct limit group, with G_n a connected, complex, affine algebraic group. Let $\lambda = (\lambda_1, \dots, \lambda_n, \dots, \lambda_k, \dots)$ be an element of the dual $\mathfrak{g}^* = \varprojlim \mathfrak{g}_n^*$ of the Lie algebra $\mathfrak{g} = \varinjlim \mathfrak{g}_n$ of G. Since G_n is connected for each n, the coadjoint orbit $G \cdot \lambda = \varinjlim G_n \cdot \lambda$ is irreducible. In this section, we develop an analogue of the Kostant–Kirillov form on $G \cdot \lambda$.

We now construct a 2-form on $G \cdot \lambda$. That is to say, that for each $\mu \in G \cdot \lambda$, we construct an element $(\omega_\infty)_\mu \in (\wedge^2 T_\mu(G \cdot \lambda))^*$, which is closed with respect to a natural exterior derivative on $(\wedge^2 T(G \cdot \lambda))^*$. By Equation (3.13), it suffices to define ω_∞ at $\mu = \lambda$.

For each $n \in \mathbb{N}$, we have a natural projection $p_n : G_n \cdot \lambda \cong G_n/G_n^\lambda \to G_n/G_n^{\lambda_n} \cong G_n \cdot \lambda_n$, where $G_n \cdot \lambda_n \subset \mathfrak{g}_n^*$ is the G_n-coadjoint orbit of $\lambda_n \in \mathfrak{g}_n^*$. Consider the diagram

$$\begin{array}{ccc} G_n \cdot \lambda & \xhookrightarrow{\iota_{n,n+1}} & G_{n+1} \cdot \lambda \\ \downarrow p_n & & \downarrow p_{n+1} \\ G_n \cdot \lambda_n & & G_{n+1} \cdot \lambda_{n+1}. \end{array} \tag{4.1}$$

The map $p_n : G_n \cdot \lambda \to G_n \cdot \lambda_n$ is easily seen to be a surjective submersion with differential at $\lambda \in \mathfrak{g}^*$

$$(dp_n)_\lambda : \mathfrak{g}_n/\mathfrak{g}_n^\lambda \to \mathfrak{g}_n/\mathfrak{g}_n^{\lambda_n} \text{ given by } (dp_n)_\lambda(X + \mathfrak{g}_n^\lambda) = X + \mathfrak{g}_n^{\lambda_n},$$

for $X \in \mathfrak{g}_n$. For $n \in \mathbb{N}$, let ω_n be the Kostant–Kirillov form on the coadjoint orbit $G_n \cdot \lambda_n$. We claim that

$$d\iota_{n,n+1}^*(dp_{n+1}^*\omega_{n+1})_\lambda = (dp_n^*\omega_n)_\lambda. \tag{4.2}$$

Indeed, let $X + \mathfrak{g}_n^\lambda,\ Y + \mathfrak{g}_n^\lambda \in \mathfrak{g}_n/\mathfrak{g}_n^\lambda$. It is straightforward to verify that

$$d\iota_{n,n+1}^*(dp_{n+1}^*\omega_{n+1})_\lambda(X + \mathfrak{g}_n^\lambda, Y + \mathfrak{g}_n^\lambda) = \lambda_{n+1}([X, Y]).$$

Similarly, $dp_n^*\omega_n(X + \mathfrak{g}_n^\lambda, Y + \mathfrak{g}_n^\lambda) = \lambda_n([X, Y])$. Since $\lambda_{n+1}|_{\mathfrak{g}_n} = \lambda_n$, these expressions agree. Thus, by Lemma 2.35, we can define an element of the inverse limit $\varprojlim \wedge^2 T^*(G_n \cdot \lambda) \cong (\wedge^2 \varinjlim T(G_n \cdot \lambda))^* = (\wedge^2 T(G \cdot \lambda))^*$ by

$$\omega_\infty := \varprojlim dp_n^*\omega_n = (dp_1^*\omega_1, dp_2^*\omega_2, dp_3^*\omega_3, \dots). \tag{4.3}$$

By Lemma 2.35, the alternating k-forms on $T(G \cdot \lambda)$ can be identified with elements of the space $\varprojlim_n \bigwedge^k T^*(G_n \cdot \lambda)$. We consider the following bicomplex, where $d_{k,n}$ are the exterior derivatives and the $\wedge^k d\iota_{n,n+1}^*$ are obtained from pullbacks of the transition maps $\iota_{n,n+1} :\ G_n \to G_{n+1}$ in the directed system defining G:

$$
\begin{tikzcd}
\vdots \arrow[d, "d_{k-1,1}"'] & \vdots \arrow[d, "d_{k-1,2}"'] & \vdots \arrow[d, "d_{k-1,3}"'] & \\
\wedge^k T^*(G_1\cdot\lambda) \arrow[d, "d_{k,1}"'] & \wedge^k T^*(G_2\cdot\lambda) \arrow[l, "\wedge^k d\iota_{12}^*"'] \arrow[d, "d_{k,2}"'] & \wedge^k T^*(G_3\cdot\lambda) \arrow[l, "\wedge^k d\iota_{23}^*"'] \arrow[d, "d_{k,3}"'] & \cdots \arrow[l, "\wedge^k d\iota_{34}^*"'] \\
\wedge^{k+1} T^*(G_1\cdot\lambda) \arrow[d, "d_{k+1,1}"'] & \wedge^{k+1} T^*(G_2\cdot\lambda) \arrow[l, "\wedge^{k+1} d\iota_{12}^*"'] \arrow[d, "d_{k+1,2}"'] & \wedge^{k+1} T^*(G_3\cdot\lambda) \arrow[l, "\wedge^{k+1} d\iota_{23}^*"'] \arrow[d, "d_{k+1,3}"'] & \cdots \arrow[l, "\wedge^{k+1} d\iota_{34}^*"'] \\
\wedge^{k+2} T^*(G_1\cdot\lambda) \arrow[d, "d_{k+2,1}"'] & \wedge^{k+2} T^*(G_2\cdot\lambda) \arrow[l, "\wedge^{k+2} d\iota_{12}^*"'] \arrow[d, "d_{k+2,2}"'] & \wedge^{k+2} T^*(G_3\cdot\lambda) \arrow[l, "\wedge^{k+2} d\iota_{23}^*"'] \arrow[d, "d_{k+2,3}"'] & \cdots \arrow[l, "\wedge^{k+2} d\iota_{34}^*"'] \\
\vdots & \vdots & \vdots &
\end{tikzcd}
\tag{4.4}
$$

It is straightforward to verify that all the squares in the bicomplex (4.4) commute. Thus, there is a map

$$d_{k,\infty}:\ \varprojlim_n \bigwedge^k T^*(G_n\cdot\lambda) \to \varprojlim_n \bigwedge^{k+1} T^*(G_n\cdot\lambda),$$

for each $k \geq 0$, given by

$$d_{k,\infty}(\alpha_1,\alpha_2,\alpha_3,\ldots) = (d_{k,1}(\alpha_1), d_{k,2}(\alpha_2), d_{k,3}(\alpha_3),\ldots), \tag{4.5}$$

where $(\alpha_1,\alpha_2,\alpha_3,\ldots) \in \varprojlim_n \bigwedge^k T^*(G_n\cdot\lambda)$. In particular, $d_{0,\infty} : \mathcal{O}(G\cdot\lambda) \to T^*(G\cdot\lambda)$ coincides with the usual notion of the differential for functions in $\mathcal{O}(G\cdot\lambda) = \varprojlim \mathcal{O}(G_n\cdot\lambda)$.

The 2-form $\omega_\infty \in (\wedge^2 T(G\cdot\lambda))^*$ induces a map $\widetilde{\omega_\infty} : T(G\cdot\lambda) \to T^*(G\cdot\lambda)$ given by:

$$(\widetilde{\omega_\infty})_\mu(Y)(Z) = \omega_{\infty,\mu}(Y,Z) \text{ for } \mu \in G\cdot\lambda,\ Y,\ Z \in T_\lambda(G\cdot\mu) = T_\mu(G\cdot\mu). \tag{4.6}$$

Following [24], we call ω_∞ a *weak symplectic* form on $G\cdot\lambda$ if the following two conditions are satisfied:

(1) The form ω_∞ is closed with respect to the differential $d_{2,\infty}$ defined in Equation (4.5).

(2) For each $\mu \in G\cdot\lambda$, the map $(\widetilde{\omega_\infty})_\mu$ defined in (4.6) is an injective, regular linear map from the linear ind-variety $T_\mu(G\cdot\mu) = \varinjlim T_\mu(G_n\cdot\mu_n)$ to the linear provariety $T^*_\mu(G\cdot\mu) = \varprojlim T^*_\mu(G_n\cdot\mu_n)$ (see Remark 2.14).

Proposition 4.7. *For $\lambda \in \mathfrak{g}^*$, the coadjoint orbit $(G\cdot\lambda,\omega_\infty)$ is a weak symplectic ind-variety. If we identify $T_\lambda(G\cdot\lambda) \cong \mathfrak{g}/\mathfrak{g}^\lambda$, then ω_∞ is given by the formula*

$$(\omega_\infty)_\lambda(X+\mathfrak{g}^\lambda, Y+\mathfrak{g}^\lambda) = \lambda([X,Y]), \tag{4.8}$$

for X, $Y \in \mathfrak{g}$.

Proof. Equation (4.8) follows directly from the definition of ω_∞. We now show that ω_∞ satisfies (2) in the definition of a weak symplectic form. Without loss of generality, we may assume $\mu = \lambda$. Consider the map $(\widetilde{\omega_\infty})_\lambda$ defined in (4.6). We first show that $(\widetilde{\omega_\infty})_\lambda$ is injective. Suppose that $X \in \mathfrak{g}_n$ is such that $(\omega_\infty)_\lambda(X+\mathfrak{g}_n^\lambda, Y+\mathfrak{g}_k^\lambda) = 0$ for all $Y \in \mathfrak{g}_k$ and $k \geq 1$. For $k \geq n$, Equation (4.8) implies that $\lambda_k([X,Y]) = 0$ for all $Y \in \mathfrak{g}_k$. Thus, $X \in \bigcap_{k\geq n} \mathfrak{g}_n^{\lambda_k} = \mathfrak{g}_n^\lambda$.

We now show that $(\widetilde{\omega_\infty})_\lambda$ is a morphism. By Propositions 2.19 and 3.1, $(\widetilde{\omega_\infty})_\lambda$ is a morphism if for every $m, k \in \mathbb{N}$, the following composition of maps

$$\mathfrak{g}_m/\mathfrak{g}_m^\lambda \hookrightarrow \mathfrak{g}/\mathfrak{g}^\lambda \xrightarrow{(\widetilde{\omega_\infty})_\lambda} \varprojlim_n T_\lambda^*(\mathfrak{g}_n/\mathfrak{g}_n^\lambda) \xrightarrow{p_k} T_\lambda^*(\mathfrak{g}_k/\mathfrak{g}_k^\lambda)$$

is a morphism of finite dimensional affine varieties. This is an elementary computation using (4.8).

We now show that the 2-form ω_∞ is closed with respect to the differential $d_{2,\infty}$ defined in (4.5). Since the Kostant–Kirillov form ω_n on $G_n\cdot\lambda_n$ is closed, we have $d_n\omega_n = 0$, where $d_n : \bigwedge^2 T^*(G_n\cdot\lambda_n) \to \bigwedge^3 T^*(G_n\cdot\lambda_n)$ is the exterior derivative on $G_n \cdot \lambda_n = G_n/G_n^{\lambda_n}$. Thus,

$$\begin{aligned} d_{2,\infty} &= d_{2,\infty}(dp_1^*\omega_1, dp_2^*\omega_2, dp_3^*\omega_3, \dots) \\ &= (d_{2,1}(dp_1^*\omega_1),\ d_{2,2}(dp_2^*\omega_2),\ d_{2,3}(dp_3^*\omega_3),\ \dots) \\ &= (dp_1^*(d_1\omega_1),\ dp_2^*(d_2\omega_2),\ dp_3^*(d_3\omega_3), \dots) \\ &= (dp_1^*(0), dp_2^*(0), dp_3^*(0), \dots) \\ &= 0, \end{aligned}$$

so the 2-form ω_∞ is closed. $\square$

Remark 4.9. Suppose that $G = \varinjlim G_n$, where G_n is a reductive algebraic group. Then $\mathrm{Lie}(G_n) = \mathfrak{g}_n$ is reductive with a non-degenerate, $\mathrm{Ad}(G_n)$-invariant form $\ll \cdot,\cdot \gg$, which allows us to identify $\mathfrak{g}_n$ with $\mathfrak{g}_n^*$. The induced isomorphism $\mathfrak{g}^* \cong \widetilde{\mathfrak{g}} = \varprojlim \mathfrak{g}_n$ is equivariant with respect to the coadjoint action of G on $\mathfrak{g}^*$ and the adjoint action of G on $\widetilde{\mathfrak{g}}$:

$$\mathrm{Ad}(g)(x_1, x_2, x_3, \dots) = \left(\mathrm{Ad}(g)x_n|_{\mathfrak{g}_1}, \dots, \mathrm{Ad}(g)x_n|_{\mathfrak{g}_{n-1}}, \mathrm{Ad}(g)x_n, \mathrm{Ad}(g)x_{n+1}, \dots\right),$$

where $g \in G_n$. In particular, we can transfer the symplectic form on coadjoint orbits in $\mathfrak{g}^*$ to adjoint orbits in $\widetilde{\mathfrak{g}}$.

In the next theorem, we will consider the inclusion of the coadjoint orbits into the provariety $\mathfrak{g}^*$ and the compatibility of the symplectic structure on coadjoint orbits with the Poisson structure on $\mathfrak{g}^*$. Consider the natural inclusion

$$i = \varinjlim i_n :\ \varinjlim G_n \cdot \lambda \hookrightarrow \mathfrak{g}^*, \tag{4.10}$$

where $i_n : G_n/G_n^\lambda \hookrightarrow \mathfrak{g}^*$ is given by

$$i_n(g_nG_n^\lambda) = (\mathrm{Ad}^*(g_n)\lambda_n|_{\mathfrak{g}_1}, \dots, (\mathrm{Ad}^*(g_n)\lambda_n)|_{\mathfrak{g}_{n-1}}, \mathrm{Ad}^*(g_n)\lambda_n, \mathrm{Ad}^*(g_n)\lambda_{n+1}, \dots). \tag{4.11}$$

Via the map i, the coadjoint orbits $G \cdot \lambda = \varinjlim G_n \cdot \lambda$ are irreducible, immersed ind-subvarieties that are tangent to the characteristic distribution $\mathfrak{X}(\mathfrak{g}^*)$ defined in (2.44). More precisely, we have the following theorem.

Theorem 4.12. (1) *The natural inclusion $i :\ G \cdot \lambda \hookrightarrow \mathfrak{g}^*$ is an injective immersion of the irreducible ind-variety $G \cdot \lambda$ into the provariety $\mathfrak{g}^*$.*

(2) *The coadjoint orbits are tangent to the characteristic distribution of the Poisson provariety $(\mathfrak{g}^*, \pi_{\mathfrak{g}^*})$:*

$$(di)_\lambda(T_\lambda(G \cdot \lambda)) = \mathfrak{X}(\mathfrak{g}^*)_\lambda, \tag{4.13}$$

(3) *The symplectic form ω_∞ on $G \cdot \lambda$ is consistent with the Poisson structure of $\mathfrak{g}^*$, i.e.*

$$\omega_{\infty,\lambda}(Y,Z) = \pi_{\mathfrak{g}^*,\lambda}([\widetilde{\pi_{\mathfrak{g}^*,\lambda}}]^{-1} \circ di_\lambda(Y), [\widetilde{\pi_{\mathfrak{g}^*,\lambda}}]^{-1} \circ di_\lambda(Z)) \tag{4.14}$$

where $[\widetilde{\pi_{\mathfrak{g}^*,\lambda}}]$ is the bijective morphism $[\widetilde{\pi_{\mathfrak{g}^*,\lambda}}] : T^*_\lambda(\mathfrak{g}^*)/Ker\, \widetilde{\pi_{\mathfrak{g}^*,\lambda}} \to \mathfrak{X}(\mathfrak{g}^*)_\lambda$ induced by the anchor map $\widetilde{\pi_{\mathfrak{g}^*,\lambda}}$ (see Equation (2.41)).

Proof. Written explicitly, the inclusion i_n in (4.11) is simply the map $\varprojlim_j i_{nj}$, where $i_{nj} : G_n/G_n^\lambda \to \mathfrak{g}_j^*$ is given by

$$i_{nj} : g_n G_n^\lambda \mapsto \mathrm{Ad}^*(g_n)\lambda_j, \tag{4.15}$$

for all $j \geq n$, and

$$i_{nj} : g_n G_n^\lambda \mapsto (\mathrm{Ad}^*(g_n)\lambda_n)|_{\mathfrak{g}_j}, \tag{4.16}$$

for $j < n$. By Propositions 3.1 and 2.19, the map $i : G \cdot \lambda \hookrightarrow \mathfrak{g}^*$ is a morphism if the maps i_{jn} are morphisms for all $j, n \in \mathbb{N}$. This follows from the universal property of the geometric quotient G_n/G_n^λ.

By Propositions 3.1 and 2.19, it follows that the differential

$$di : T(G\cdot\lambda) = \varinjlim_n T(G_n \cdot \lambda) \to T(\mathfrak{g}^*) = \varprojlim_j T(\mathfrak{g}_j^*) \text{ is precisely } di = \varinjlim_n \varprojlim_j di_{nj}.$$

Using Equations (4.15) and (4.16), we see that $(di_n)_\lambda : \mathfrak{g}_n/\mathfrak{g}_n^\lambda \to T_\lambda(\mathfrak{g}^*)$ is given by

$$(di_n)_\lambda(X_n + \mathfrak{g}_n^\lambda) = (\mathrm{ad}^*(X_n)\lambda_1|_{\mathfrak{g}_1}, \dots, (\mathrm{ad}^*(X_n)\lambda_n)|_{\mathfrak{g}_{n-1}}, \mathrm{ad}^*(X_n)\lambda_n, \dots, \mathrm{ad}^*(X_n)\lambda_k, \dots). \tag{4.17}$$

From Equation (4.17), it follows that $(di)_\lambda = \varinjlim (di_n)_\lambda$ is injective. Thus, $G\cdot\lambda$ is an immersed ind-subvariety of $\mathfrak{g}^*$.

Part (2) follows directly from Equations (2.41) and (4.17). Indeed,

$$\begin{aligned}
\mathfrak{X}(\mathfrak{g}^*)_\lambda &= \mathrm{Im}\ \widetilde{\pi}_{\mathfrak{g}^*,\lambda} \\
&= \varinjlim_n ((\mathrm{ad}^*(\mathfrak{g}_n)\lambda_n|_{\mathfrak{g}_1}, \dots, (\mathrm{ad}^*(\mathfrak{g}_n)\lambda_n)|_{\mathfrak{g}_{n-1}}, \mathrm{ad}^*(\mathfrak{g}_n)\lambda_n, \dots, \mathrm{ad}^*(\mathfrak{g}_n)\lambda_k, \dots) \\
&= di_\lambda(T_\lambda(G\cdot\lambda)).
\end{aligned}$$

Finally, we show that (4.14) holds. Without loss of generality, we may assume that $Y = Y_m + \mathfrak{g}_m^\lambda$ and $Z = Z_n + \mathfrak{g}_n^\lambda$ with $Y_m \in \mathfrak{g}_m$, $Z_n \in \mathfrak{g}_n$, and $n \geq m$. By (4.8), the left-hand side of (4.14) is $\lambda_n([Y_m, Z_n]_n)$, where $[Y_m, Z_n]_n$ denotes the Lie bracket of Y_m, Z_n as elements of $\mathfrak{g}_n$.

To compute the right-hand side of (4.14), note that Ker $\widetilde{\pi_{\mathfrak{g}^*,\lambda}} = \mathfrak{g}^\lambda$, by (2.43) and (3.11). Then (2.41) and (4.17) imply that $[\widetilde{\pi_{\mathfrak{g}^*,\lambda}}]^{-1} \circ di_\lambda$ is the identity map on $\mathfrak{g}/\mathfrak{g}^\lambda$. Therefore,

$$\pi_{\mathfrak{g}^*,\mu}([\widetilde{\pi_{\mathfrak{g}^*,\lambda}}]^{-1} \circ di_\lambda(Y), [\widetilde{\pi_{\mathfrak{g}^*,\lambda}}]^{-1} \circ di_\lambda(Z)) = \pi_{\mathfrak{g}^*,\lambda}(Y_m + \mathfrak{g}_m^\lambda, Z_n + \mathfrak{g}_n^\lambda).$$

Proposition 2.37 implies that $\pi_{\mathfrak{g}^*,\lambda} = \varprojlim \pi_{\mathfrak{g}_n^*,\lambda_n}$, where $\pi_{\mathfrak{g}_n^*,\lambda_n}$ is the bivector for the Lie–Poisson structure on $\mathfrak{g}_n^*$ evaluated at λ_n. Thus, $\pi_{\mathfrak{g}^*,\lambda}(Y_m + \mathfrak{g}_m^\lambda, Z_n + \mathfrak{g}_n^\lambda) = \lambda_n([Y_m, Z_n]_n)$, and Equation (4.14) holds. □

Equation (4.14) lets us define Hamiltonian vector fields for functions on $G\cdot\lambda$ obtained as pullbacks of functions on $\mathfrak{g}^*$, giving a Poisson algebra structure on the set of such functions. The following proposition is a restatement of Proposition 7.2 in [24]. The proof given there carries over to our case.

Proposition 4.18. *Let $\lambda \in \mathfrak{g}^*$, let $(G \cdot \lambda, \omega_\infty)$ be the coadjoint orbit through λ, and let $i : (G \cdot \lambda, \omega_\infty) \hookrightarrow \mathfrak{g}^*$ be the inclusion morphism given in Equation (4.10).*

(1) *Let $U \subset \mathfrak{g}^*$ be open and suppose $\mu \in i^{-1}(U) \subset G \cdot \lambda$. Let $f \in \mathcal{O}_{\mathfrak{g}^*}(U)$, so that $f \circ i \in \mathcal{O}_{G\cdot\lambda}(i^{-1}(U))$. The differential $d(f \circ i)(\mu) \in T^*_\mu(G \cdot \lambda)$ is given by:*

$$d(f \circ i)(\mu) = \omega_{\infty,\mu}(di_\mu^{-1}(\xi_f)_\mu, \cdot).$$

(2) *Let $U \subset \mathfrak{g}^*$ be open. Then $i^*\mathcal{O}_{\mathfrak{g}^*}(U) \subset \mathcal{O}_{G\cdot\lambda}(i^{-1}(U))$ has the structure of a Poisson algebra with Poisson bracket given by:*

$$\{f \circ i, g \circ i\}_\infty(\mu) := \omega_{\infty,\mu}((di_\mu)^{-1}(\xi_f)_\mu, (di_\mu)^{-1}(\xi_g)_\mu).$$

The pullback $i^* : (\mathcal{O}_{\mathfrak{g}^*}(U), \{\cdot,\cdot\}) \to (i^*\mathcal{O}_{\mathfrak{g}^*}(U), \{\cdot,\cdot\}_\infty)$ is a homomorphism of Poisson algebras.

We end this section with a discussion of the Lagrangian calculus of a coadjoint orbit $G \cdot \lambda$ that will be useful in Section 5.2.

Proposition 4.19. *Let $\mathcal{L} \subseteq G \cdot \lambda$ be an ind-subvariety, so that $\mathcal{L} = \bigcup_{n=1}^\infty \mathcal{L}_n$ with $\mathcal{L}_n := \mathcal{L} \cap (G_n \cdot \lambda)$ a locally closed subvariety of $G_n \cdot \lambda$. Let $p_n : G_n \cdot \lambda \to G_n \cdot \lambda_n$ be the projection. Define $\tilde{\mathcal{L}_n} := p_n(\mathcal{L}_n)$ and suppose that $\tilde{\mathcal{L}_n}$ satisfies the following conditions:*

(1) *$\tilde{\mathcal{L}_n} \subseteq G_n \cdot \lambda_n$ is a subvariety.*
(2) *$dp_n T(\mathcal{L}_n) = T(\tilde{\mathcal{L}_n})$.*
(3) *$dp_n^{-1}(T(\tilde{\mathcal{L}_n})) \subseteq T(\mathcal{L}_n)$.*
(4) *$\tilde{\mathcal{L}_n} \subseteq (G_n \cdot \lambda_n, \omega_n)$ is Lagrangian.*

Then $\mathcal{L} \subseteq (G \cdot \lambda, \omega_\infty)$ is a Lagrangian ind-subvariety.

Proof. Fix $\mu \in \mathcal{L}$ and let $\ell \geq 1$ be such that $\mu \in G_\ell \cdot \lambda$, but $\mu \notin G_k \cdot \lambda$ for any $k < \ell$. Note that for any $n \geq \ell$, we have $G_n \cdot \lambda = G_n \cdot \mu$, so that $T_\mu(\mathcal{L}) = \varinjlim_{n \geq \ell} T_\mu(\mathcal{L}_n) \subset \varinjlim_{n \geq \ell} T_\mu(G_n \cdot \mu)$. We show that $T_\mu(\mathcal{L}) = T_\mu(\mathcal{L})^\perp$, where $T_\mu(\mathcal{L})^\perp$ denotes the annihilator of $T_\mu(\mathcal{L})$ in $T_\mu(G \cdot \mu)$ with respect to the weak symplectic form ω_∞.

We first show that $\mathcal{L}$ is coisotropic, i.e., that $T_\mu(\mathcal{L})^\perp \subseteq T_\mu(\mathcal{L})$. Let $\xi \in T_\mu(\mathcal{L})^\perp$, with $\xi = \xi_n + \mathfrak{g}_n^\mu$ for some $\xi_n \in \mathfrak{g}_n$ and $n \geq 1$. We consider $(\omega_\infty)_\mu(\xi_n + \mathfrak{g}_n^\mu, L_k + \mathfrak{g}_k^\mu)$, where $k \geq \ell$. Suppose $n \leq \ell$. Then by definition of ω_∞, we have

$$(\omega_\infty)_\mu(\xi_n + \mathfrak{g}_n^\mu, T_\mu(\mathcal{L})) = (\omega_\infty)_\mu(di_{n\ell}(\xi) + \mathfrak{g}_\ell^\mu, T_\mu(\mathcal{L})),$$

where $di_{n\ell}$ denotes the differential of the inclusion $i_{n\ell} : G_n \cdot \mu \to G_\ell \cdot \mu$. We can thus assume, without loss of generality, that $n \geq \ell$.

Note that $(\omega_\infty)_\mu(\xi_n + \mathfrak{g}_n^\mu, T_\mu(\mathcal{L})) = 0$, so $(\omega_\infty)_\mu(\xi_n + \mathfrak{g}_n^\mu, T_\mu(\mathcal{L}_k)) = 0$ for all $k \geq \ell$. In particular,

$$\begin{aligned}(\omega_\infty)_\mu(\xi_n + \mathfrak{g}_n^\mu, L_n + \mathfrak{g}_n^\mu) &= (\omega_n)_{\mu_n}(\xi_n + \mathfrak{g}_n^{\mu_n}, dp_n(L_n + \mathfrak{g}_n^\mu)) \\ &= 0,\end{aligned}$$

for all $L_n + \mathfrak{g}_n^\mu \in T_\mu(\mathcal{L}_n)$. By (2), $dp_n T_\mu(\mathcal{L}_n) = T_{\mu_n}(\tilde{\mathcal{L}_n})$, so $\xi_n + \mathfrak{g}_n^{\mu_n} \in T_{\mu_n}(\tilde{\mathcal{L}_n})^\perp$. By (4), $\tilde{\mathcal{L}_n}$ is Lagrangian in $G_n \cdot \mu_n = G_n \cdot \lambda_n$, whence $T_{\mu_n}(\tilde{\mathcal{L}_n})^\perp = T_{\mu_n}(\tilde{\mathcal{L}_n})$. Thus,

$$\xi = \xi_n + \mathfrak{g}_n^\mu \in dp_n^{-1}(T_{\mu_n}(\tilde{\mathcal{L}_n})) \subseteq T_\mu(\mathcal{L}_n) \subseteq T_\mu(\mathcal{L})$$

by (3).

We now show that $\mathcal{L}$ is isotropic. Suppose that $\xi = \xi_n + \mathfrak{g}_n^\mu \in T_\mu(\mathcal{L}_n)$ with $n \geq \ell$. Consider $(\omega_\infty)_\mu(\xi, L_k + \mathfrak{g}_k^\mu)$ for $L_k + \mathfrak{g}_k^\mu \in T_\mu(\mathcal{L}_k)$, with $k \geq \ell$. As before, if $k \leq n$, we can identify $L_k + \mathfrak{g}_k^\mu$ with its pushforward in $T_\mu(\mathcal{L}_n)$. This lets us reduce to the case where $k \geq n$. Identifying ξ with its image $d\iota_{nk}(\xi) = \xi_n + \mathfrak{g}_k^\mu \in T_\mu(\mathcal{L}_k)$, we have

$$\begin{aligned}(\omega_\infty)_\mu(\xi, L_k + \mathfrak{g}_k^\mu) &= (\omega_k)_{\mu_k}(dp_k(\xi_n + \mathfrak{g}_k^\mu), dp_k(L_k + \mathfrak{g}_k^\mu)) \\ &= (\omega_k)_{\mu_k}(\xi_n + \mathfrak{g}_k^{\mu_k}, L_k + \mathfrak{g}_k^{\mu_k}).\end{aligned}$$

But $dp_k T_\mu(\mathcal{L}_k) = T_{\mu_k}(\tilde{\mathcal{L}}_k)$ by (2), and $\tilde{\mathcal{L}}_k$ is Lagrangian in $G_k \cdot \lambda_k = G_k \cdot \mu_k$. Thus, $T_{\mu_k}(\tilde{\mathcal{L}}_k) \subseteq T_{\mu_k}(\tilde{\mathcal{L}}_k)^\perp$, so $(\omega_\infty)_\mu(\xi, L_k + \mathfrak{g}_k^\mu) = 0$. Since $k \geq \ell$ is arbitrary, we have $T_\mu(\mathcal{L}) \subseteq T_\mu(\mathcal{L})^\perp$, and $\mathcal{L}$ is Lagrangian. □

5. Gelfand–Zeitlin integrable system on $M(\infty)$

5.1. The group $A(\infty)$

In this section, we study the analogue of the Gelfand–Zeitlin[2] collection of functions for the Poisson provariety $M(\infty)$ defined in Example 2.27. We show that the corresponding Lie algebra of Hamiltonian vector fields integrates to the action of a direct limit group $A(\infty)$ on $M(\infty)$ whose generic orbits form Lagrangian ind-subvarieties of the corresponding adjoint orbit. We begin by recalling some facts about the Gelfand–Zeitlin integrable system on $\mathfrak{g}_n = \mathfrak{gl}(n, \mathbb{C})$ constructed by Kostant and Wallach in [19].

We denote by $\mathbb{C}[\mathfrak{g}_n]$ the polynomial functions on $\mathfrak{g}_n$. For $i = 1, \ldots, n$ and $j = 1, \ldots, i$, we let $f_{ij} \in \mathbb{C}[\mathfrak{g}_n]$ be the polynomial $f_{ij}(X) = tr(X_i^j)$, where X_i is the $i \times i$ submatrix in the upper left-hand corner of $X \in \mathfrak{g}_n$, and $tr(\cdot)$ denotes the trace function on $\mathfrak{g}_n$. If $\mathbb{C}[\mathfrak{g}_n]^{G_n}$ denotes the $\mathrm{Ad}(G_n)$-invariant polynomials on $\mathfrak{g}_n$, then $\mathbb{C}[\mathfrak{g}_n]^{G_n}$ is the polynomial ring $\mathbb{C}[f_{n1}, \ldots, f_{nn}]$. Consider the Hamiltonian vector field $\xi_{f_{ij}}$ on $\mathfrak{g}_n$. For $X \in \mathfrak{g}_n$, $(df_{ij})_X \in T_X^*(\mathfrak{g}_n) = \mathfrak{g}_n^*$. We can use the trace form $\ll X, Z \gg = tr(XZ)$ on $\mathfrak{g}_n$ to identify the differential $(df_{ij})_X$ with an element $\nabla f_{ij}(X) \in \mathfrak{g}_n$. The element $\nabla f_{ij}(X)$ is determined by its pairing against $Z \in \mathfrak{g}_n$ by the formula

$$\ll \nabla f_{ij}(X), Z \gg = \frac{d}{dt}|_{t=0} f_{ij}(X + tZ) = (df_{ij})_X(Z).$$

We compute

$$\nabla f_{ij}(X) = jX_i^{j-1} \in \mathfrak{g}_i \hookrightarrow \mathfrak{g}_n, \tag{5.1}$$

where $\mathfrak{g}_i$ is embedded in the top left-hand corner of $\mathfrak{g}_n$ (see Example 2.27). It follows that

$$(\xi_{f_{ij}})_X = -[jX_i^{j-1}, X] \tag{5.2}$$

(cf. Equation (2.40)). Note that if $i = n$, then $\xi_{f_{nj}} = 0$ for all $j = 1, \ldots, n$, since $f_{nj} \in \mathbb{C}[\mathfrak{g}_n]^{G_n}$ is a Casimir function for the Lie–Poisson structure on $\mathfrak{g}_n$. The Gelfand–Zeitlin collection of functions on $\mathfrak{g}_n$ is $J_{GZ} := \{f_{ij} : 1 \leq j \leq i \leq n\}$. The functions J_{GZ} are Poisson commutative and their restriction to a regular adjoint orbit of G_n on $\mathfrak{g}_n$ forms an integrable system [19].

We let

$$\mathfrak{a}(n) := \mathrm{span}\{\xi_{f_{ij}} : 1 \leq j \leq i \leq n-1\}$$

[2] Alternate spellings of Zeitlin include Cetlin, Tsetlin, Tzetlin, and Zetlin. In this paper, we follow the convention from the earlier work of the first author.

be the corresponding Lie algebra of Gelfand–Zeitlin vector fields on $\mathfrak{g}_n$. Then $\mathfrak{a}(n)$ is an abelian Lie algebra of dimension $\binom{n}{2}$. Moreover, the Lie algebra $\mathfrak{a}(n)$ integrates to an analytic action of $A(n) := \mathbb{C}^{\binom{n}{2}}$ on $\mathfrak{g}_n$ (see [19, Section 3]). This action can be described as follows. We take $\underline{t} = (\underline{t}_1, \ldots, \underline{t}_{n-1}) \in \mathbb{C}^1 \times \cdots \times \mathbb{C}^{n-1} = \mathbb{C}^{\binom{n}{2}}$ as coordinates on $A(n)$, where $\underline{t}_i = (t_{i1}, \ldots, t_{ii}) \in \mathbb{C}^i$ for $1 \leq i \leq n-1$. In these coordinates, the action of $A(n)$ on $\mathfrak{g}_n$ is given by

$$a \cdot X = \mathrm{Ad}(\exp(t_{1,1})) \cdot \ldots \cdot \mathrm{Ad}(\exp(jt_{i,j}X_i^{j-1})) \cdot \ldots \cdot \mathrm{Ad}(\exp((n-1)t_{n-1,n-1}X_{n-1}^{n-2})) \cdot X, \tag{5.3}$$

for all $1 \leq j \leq i \leq n-1$ and $X \in \mathfrak{g}_n$. Since the Gelfand–Zeitlin functions Poisson commute, $A(n) \cdot X \subset G_n \cdot X$ is an isotropic submanifold. For each $X \in \mathfrak{g}_n$, it follows from Equation (5.2) that

$$T_X(A(n) \cdot X) = \mathfrak{a}(n)_X = \mathrm{span}\{[X_i^{j-1}, X] : 1 \leq j \leq i \leq n-1\}. \tag{5.4}$$

We now define an infinite dimensional Gelfand–Zeitlin system J_∞ for the provariety $M(\infty)$ by pulling back the Gelfand–Zeitlin functions f_{ij} to $M(\infty)$. We recall from Example 2.27 that $M(\infty)$ can be identified with the space of sequences:

$$M(\infty) = \{X = (X(1), X(2), \ldots, X(n), X(n+1), \ldots,) : X(n) \in \mathfrak{g}_n \text{ and } X(n+1)_n = X(n)\}.$$

We have a natural morphism of locally ringed spaces $p_n : M(\infty) \to \mathfrak{g}_n$, given by $p_n(X) = X(n)$. Also, the morphism p_n is Poisson with respect to the Poisson provariety structure on $M(\infty)$ given in Example 2.38 and the Lie–Poisson structure on $\mathfrak{g}_n$. Let

$$J_\infty := \{p_n^* f_{nj} : n \in \mathbb{N}, j = 1, \ldots, n\}.$$

Proposition 5.5. *The set J_∞ of Gelfand–Zeitlin functions on $M(\infty)$ is Poisson commutative.*

Proof. For the purposes of this proof, we will denote the Poisson bracket on $M(\infty)$ by $\{\cdot,\cdot\}_\infty$ and the Poisson bracket on $\mathfrak{g}_n$ by $\{\cdot,\cdot\}_n$. Consider $p_n^* f_{nj} \in J_\infty$ for $n \in \mathbb{N}$. Note that for any $m \leq n$, and any $1 \leq k \leq m$, we have $\{p_n^* f_{nj}, p_m^* f_{mk}\}_\infty = 0$. Indeed, $p_m^* f_{mk} = p_n^* p_{nm}^* f_{mk}$ so that $\{p_n^* f_{nj}, p_m^* f_{mk}\}_\infty = \{p_n^* f_{nj}, p_n^* p_{nm}^* f_{mk}\}_\infty = p_n^*\{f_{nj}, p_{nm}^* f_{mk}\}_n$. But $\{f_{nj}, p_{nm}^* f_{mk}\}_n = 0$, since elements of $\mathbb{C}[\mathfrak{g}_n]^{G_n}$ are Casimir functions for the Lie–Poisson structure on $\mathfrak{g}_n$. A completely analogous argument shows that if $m > n$, $\{p_m^* f_{mk}, p_n^* f_{nj}\}_\infty = 0$. $\square$

Remark 5.6. In fact, it can be shown that the functions J_∞ generate a maximal Poisson commutative subalgebra of $\mathbb{C}[M(\infty)] := \varinjlim \mathbb{C}[\mathfrak{g}_n]$.

We consider the abelian Lie algebra of Hamiltonian vector fields on $M(\infty)$,

$$\mathfrak{a}(\infty) := \mathrm{span}\{\xi_f : f \in J_\infty\}. \tag{5.7}$$

Let $f = p_n^* f_{n,j}$, we compute ξ_f. It follows from definitions that $(\xi_f)_X = \widetilde{\pi}_{\infty,X}(dp_n^* f_{n,j})$, where $\widetilde{\pi}_{\infty,X}$ is the anchor map for the Poisson structure π_∞ on $M(\infty)$ evaluated at $X = (X_1, \ldots, X_n, \ldots, X_k, \ldots,) \in M(\infty)$. Equation (2.41) implies that

$$(\xi_f)_X = (0, \ldots, 0, \underbrace{-[jX_n^{j-1}, X_{n+1}]}_{n+1}, \ldots, -[jX_n^{j-1}, X_k], \ldots). \tag{5.8}$$

We now construct an action of a direct limit group $A(\infty) := \mathbb{C}^\infty$ on $M(\infty)$ whose generic orbits on $M(\infty)$ are tangent to the Lie algebra of Hamiltonian vector fields $\mathfrak{a}(\infty)$.

For each $n \in \mathbb{N}$, we have a natural homomorphism

$$\phi_{n,n+1} : A(n) \hookrightarrow A(n+1) \text{ given by } \phi_{n,n+1}(\underline{t}_1, \dots, \underline{t}_{n-1}) = (\underline{t}_1, \dots, \underline{t}_{n-1}, (0, \dots, 0)).$$

The maps $\phi_{n,n+1}$ are clearly closed embeddings, and thus the direct limit

$$A(\infty) := \varinjlim A(n) = \bigcup_{n \geq 1} A(n)$$

naturally has the structure of a direct limit group. For each $n \geq 1$, it follows from Equation (5.3) that $A(n)$ acts on $M(\infty)$ via:

$$\begin{aligned} a \cdot X = (X_1, t_{11} \cdot X_2, \dots, (t_{11}, \dots, t_{n-2,n-2}) \cdot X_{n-1}, (t_{11}, \dots, t_{n-1,n-1}) \cdot X_n, \dots, \\ (t_{11}, \dots, t_{n-1,n-1}) \cdot X_k, \dots). \end{aligned} \tag{5.9}$$

Observe that the diagram

$$\begin{tikzcd} A(n) \times M(\infty) \arrow[r] \arrow[d, "\phi_{n,n+1} \times Id"'] & M(\infty) \arrow[d, "Id"] \\ A(n+1) \times M(\infty) \arrow[r] & M(\infty). \end{tikzcd}$$

is commutative, where the horizontal maps are given by (5.9). We therefore obtain an action of $A(\infty)$ on $M(\infty)$. Note that $A(\infty) \cdot X \subseteq i(GL(\infty) \cdot X) \subseteq M(\infty)$. However, this is not an algebraic action of $A(\infty) = \mathbb{C}^\infty$ on $M(\infty)$.

5.2. *Strongly regular orbits*

We now show that the generic $A(\infty)$-orbits on $M(\infty)$ form Lagrangian subvarieties of the corresponding $GL(\infty)$-adjoint orbit with respect to the symplectic form ω_∞ constructed in Section 4. We first recall the conditions for an $A(n)$-orbit on $\mathfrak{gl}(n, \mathbb{C})$ to be generic. An element $X \in \mathfrak{g}_n$ is said to be *strongly regular* if the differentials $\{(df_{ij})_X : 1 \leq j \leq i \leq n\}$ are linearly independent (see [19, Theorem 2.7]). We denote the set of strongly regular elements of $\mathfrak{g}_n$ by $(\mathfrak{g}_n)_{sreg}$.

Strongly regular elements may be characterized as follows:

Proposition 5.10. *(See [19, Proposition 2.7 and Theorem 2.14].) The following statements are equivalent.*

(1) *$X \in \mathfrak{g}_n$ is strongly regular.*
(2) *The tangent vectors $\{(\xi_{f_{ij}})_X ;\ i = 1, \dots, n-1,\ j = 1, \dots, i\}$ are linearly independent.*
(3) *The elements $X_i \in \mathfrak{g}_i$ are regular for all $i = 1, \dots, n$ and $\mathfrak{z}_{\mathfrak{g}_i}(X_i) \cap \mathfrak{z}_{\mathfrak{g}_{i+1}}(X_{i+1}) = 0$ for $i = 1, \dots, n-1$, where $\mathfrak{z}_{\mathfrak{g}_i}(X_i)$ denotes the centralizer of X_i in $\mathfrak{g}_i$.*
(4) *The $A(n)$-orbit of X, $A(n) \cdot X$ is a Lagrangian subvariety of the adjoint orbit $G_n \cdot X$. In particular, $\dim A(n) \cdot X = \binom{n}{2}$.*

Remark 5.11. For $i = 1, \dots, n$, let $Z_{G_i}(X_i)$ denote the centralizer in G_i of X_i, so that $\mathrm{Lie}(Z_{G_i}(X_i)) = \mathfrak{z}_{\mathfrak{g}_i}(X_i)$. For any $i = 1, \dots, n-1$, it is easy to see that $\mathfrak{z}_{\mathfrak{g}_i}(X_i) \cap \mathfrak{z}_{\mathfrak{g}_{i+1}}(X_{i+1}) = 0$ if and only if $Z_{G_i}(X_i) \cap Z_{G_{i+1}}(X_{i+1}) = Id_{i+1}$, where Id_{i+1} denotes the $(i+1) \times (i+1)$ identity matrix (see [4, Lemma 5.12]).

The notion of strong regularity generalizes naturally to the action of $A(\infty)$ on $M(\infty)$.

Definition 5.12. We say that $X \in M(\infty)$ is *strongly regular* if the differentials $\{(df)_X : f \in J_\infty\}$ are linearly independent at X. We denote the set of strongly regular elements in $M(\infty)$ by $M(\infty)_{sreg}$.

It is easy to see that $X = (X_1, \dots, X_n, \dots) \in M(\infty)_{sreg}$ if and only if $X_n \in (\mathfrak{g}_n)_{sreg}$ for all n. So that we have

$$M(\infty)_{sreg} = \varprojlim (\mathfrak{g}_n)_{sreg}. \tag{5.13}$$

Remark 5.14. One can show that $M(\infty)_{sreg}$ is a dense subset of $M(\infty)$ with empty interior.

Using results of the first author, we can easily create examples of strongly regular elements.

Example 5.15. Let $M(\infty)_\theta \subseteq M(\infty)$ be the set

$$M(\infty)_\theta = \{X \in M(\infty) : \sigma(X_i) \cap \sigma(X_{i+1}) = \emptyset \text{ for each } i \in \mathbb{N}\}, \tag{5.16}$$

where $\sigma(X_i)$ denotes the spectrum of X_i. It follows from [6, Theorem 5.5] that $M(\infty)_\theta \subseteq M(\infty)_{sreg}$.

We also have the following characterization of strongly regular elements of $M(\infty)$.

Proposition 5.17. *Let $X = (X_1, \dots, X_n, \dots, X_k, \dots) \in M(\infty)$. Then the following conditions are equivalent:*

(1) *X is strongly regular.*
(2) *For all $i \in \mathbb{N}$, X_i is regular and $Z_{G_i}(X_i) \cap Z_{G_{i+1}}(X_{i+1}) = Id_{i+1}$.*
(3) *The tangent vectors $\mathfrak{a}(\infty)_X := \{(\xi_f)_X : f \in J_\infty\}$ are linearly independent.*

Proof. The equivalence of statements (1) and (2) follows from Equation (5.13) and Part (3) of Proposition 5.10 along with Remark 5.11. We now see that (1) is equivalent to (3). If $X \in M(\infty)_{sreg}$, then for any $n \in \mathbb{N}$, we have $\bigcap_{k \geq n} \mathfrak{z}_{\mathfrak{g}_k}(X_k) = 0$ by Part (3) of Proposition 5.10. Thus, by Proposition 2.42, we have that $(\widetilde{\pi_\infty})_X$ is injective, which implies (3). That (3) implies (1) is trivial. □

Proposition 5.17 and the existence of strongly regular elements immediately imply that the Lie algebra $\mathfrak{a}(\infty)$ is infinite dimensional.

The main result of this section is the following theorem.

Theorem 5.18. *Let $X \in M(\infty)_{sreg}$. Let $i : GL(\infty) \cdot X \hookrightarrow M(\infty)$ be the inclusion morphism in (4.10). Then*

(1) *The set $i^{-1}(A(\infty) \cdot X)$ naturally has the structure of an irreducible ind-subvariety of $GL(\infty) \cdot X$. Thus, $A(\infty) \cdot X \subset M(\infty)$ is an immersed irreducible ind-subvariety.*
(2) *The ind-subvariety $i^{-1}(A(\infty) \cdot X) \subseteq GL(\infty) \cdot X$ is Lagrangian with respect to the weak symplectic form ω_∞ on $GL(\infty) \cdot X$.*
(3) *For any $Y \in A(\infty) \cdot X$, we have*

$$T_Y(A(\infty) \cdot Y) = \mathfrak{a}(\infty)_Y, \tag{5.19}$$

so that the strongly regular $A(\infty)$-orbits in $M(\infty)$ are tangent to the Lie algebra $\mathfrak{a}(\infty)$ of Hamiltonian vector fields defined in Equation (5.7).

Proof. Let $X \in M(\infty)_{sreg}$. It follows from Part (2) of Proposition 5.17 that for each $n \in \mathbb{N}$, we have $G_n^X = \bigcap_{k \geq n} Z_{G_n}(X_k) = Id_n$, where Id_n denotes the $n \times n$ identity matrix. Thus,

$$GL(\infty) \cdot X = \varinjlim G_n/G_n^X = \varinjlim G_n = GL(\infty).$$

We claim

$$i^{-1}(A(\infty) \cdot X) \cap G_n = i_n^{-1}(A(\infty) \cdot X) = Z_{G_1}(X_1) \cdots Z_{G_n}(X_n), \tag{5.20}$$

where $i_n : G_n \to M(\infty)$ is the morphism in Equation (4.11), and

$$Z_{G_1}(X_1) \cdots Z_{G_n}(X_n) = \{g \in G_n : \ g = z_1 \cdots z_n \text{ with } z_i \in Z_{G_i}(X_i)\}.$$

For ease of notation, we denote $Z_{G_1}(X_1) \cdots Z_{G_n}(X_n)$ by $\mathcal{Z}_n$. Indeed, suppose that $g_n \in i_n^{-1}(A(\infty) \cdot X)$. Then

$$((\mathrm{Ad}(g_n)X_n)_1, \ldots, (\mathrm{Ad}(g_n)X_n)_{n-1}, \mathrm{Ad}(g_n)X_n, \ldots, \mathrm{Ad}(g_n)X_k, \ldots,) = a_m \cdot X \text{ for some } a_m \in A(m),$$

with $m \geq 1$. Let $a_m = (\underline{t}_1, \ldots, \underline{t}_{m-1}) \in \mathbb{C}^{\binom{m}{2}}$. By Equation (5.3), $A(m)$ acts on $\mathfrak{g}_m$ via

$$a_m \cdot X_m = \mathrm{Ad}(h_{m-1})X_m, \text{ where } h_{m-1} = z_1 \cdots z_{m-1} \in G_{m-1}, \text{ with}$$
$$z_i = \exp(t_{i1} Id_i) \cdots \exp(it_{ii} X_i^{i-1}) \in Z_{G_i}(X_i) \subset G_i.$$

First, suppose that $m > n$, and consider $(\mathrm{Ad}(h_{m-1})X_m)_{n+1}$. Since $z_i \in Z_{G_i}(X_i)$ for $i = 1, \ldots, m-1$, it follows that

$$(\mathrm{Ad}(h_{m-1})X_m)_{n+1} = \mathrm{Ad}(z_1 \ldots z_n)X_{n+1} = \mathrm{Ad}(g_n)X_{n+1}.$$

Since $X \in M(\infty)_{sreg}$, we have $g_n = z_1 \ldots z_n$. The case where $m < n$ is analogous. Thus, $i_n^{-1}(A(\infty) \cdot X) \subseteq \mathcal{Z}_n$.

We now show that $\mathcal{Z}_n \subset i_n^{-1}(A(\infty) \cdot X)$. Since $X \in M(\infty)_{sreg}$, X_i is regular for all i by Proposition 5.17. Whence, $Z_{G_i}(X_i)$ is an abelian, connected algebraic group (Proposition 14, [16]). Therefore, the exponential map, $\exp : \mathfrak{z}_{\mathfrak{g}_i}(X_i) \to Z_{G_i}(X_i)$ is a surjective homomorphism of algebraic groups. It is well-known that $\mathfrak{z}_{\mathfrak{g}_i}(X_i) = \mathrm{span}\{Id_i, \ldots, X_i^{i-1}\}$ for $X_i \in \mathfrak{g}_i$ regular. It follows that any $z \in Z_{G_i}(X_i)$ can be written as $z = \exp(t_{i1} Id_i) \ldots \exp(t_{ii} X_i^{i-1})$, for some $t_{ij} \in \mathbb{C}$. The inclusion $\mathcal{Z}_n \subseteq i_n^{-1}(A(\infty) \cdot X)$ now follows from Equations (5.3) and (5.9).

Now we claim that $\mathcal{Z}_n$ is a smooth subvariety of G_n of dimension $\binom{n+1}{2}$. It follows from our discussion above that $\mathrm{Ad}(Z_{G_1}(X_1) \cdots Z_{G_{n-1}}(X_{n-1}))X_n \subseteq G_n \cdot X_n$ coincides with the $A(n)$-orbit of X_n. Moreover, Theorem 3.12, [19] implies that $A(n) \cdot X_n$ is an irreducible, non-singular variety of dimension $\binom{n}{2}$. If $p_n : G_n \to G_n \cdot X_n$ denotes the orbit map, then Proposition III 10.4, [14] implies that p_n is a smooth morphism of relative dimension $\dim Z_{G_n}(X_n) = n$. Since the diagram

$$\begin{tikzcd} \mathcal{Z}_n \arrow[r] \arrow[d, "p_n"'] & G_n \arrow[d, "p_n"] \\ A(n) \cdot X_n \arrow[r] & G_n \cdot X_n \end{tikzcd}$$

is Cartesian, it follows from Proposition III 10.1 (b), [14] that $\mathcal{Z}_n$ is a non-singular variety of dimension $\binom{n+1}{2}$. Thus, $i^{-1}(A(\infty) \cdot X) = \bigcup_{n=1}^{\infty} \mathcal{Z}_n$ is an irreducible ind-subvariety of $GL(\infty) \cong GL(\infty) \cdot X$, and $A(\infty) \cdot X = i(\bigcup_{n=1}^{\infty} \mathcal{Z}_n)$ is an irreducible, immersed ind-subvariety of $M(\infty)$.

We now compute the tangent space $T_z(\mathcal{Z}_n)$ for $z \in \mathcal{Z}_n$ and show that $i^{-1}(A(\infty) \cdot X) \subset GL(\infty) \cdot X$ is Lagrangian. Represent $z = z_1 \dots z_n$ with $z_i \in Z_{G_i}(X_i)$ for $i = 1, \dots, n$. Let $Y \in M(\infty)$ be given by $Y = \mathrm{Ad}(z)X$, so that $Y_n = \mathrm{Ad}(z_1 \dots z_{n-1})X_n$. Then $Y \in A(\infty) \cdot X$ and $Y_n \in A(n) \cdot X_n$. It follows from Equation (5.4) that

$$(dp_n^{-1})_z(T_{Y_n}(A(n) \cdot Y_n)) = \mathrm{span}\{Y_i^j : 1 \le j \le i \le n\} = \mathrm{span}\{(df_{ij})_{Y_n} : 1 \le j \le i \le n\}.$$

It follows from the definition of strong regularity that $\dim \mathrm{span}\{Y_i^j : 1 \le j \le i \le n\} = \binom{n+1}{2}$. Since $\dim \mathcal{Z}_n = \binom{n+1}{2}$ and $\mathcal{Z}_n$ is non-singular, we have

$$T_z(\mathcal{Z}_n) = \mathrm{span}\{Y_i^j : 1 \le j \le i \le n\}. \tag{5.21}$$

Part (2) of the theorem now follows immediately from Proposition 4.19 and Part (4) of Proposition 5.10. Part (3) of the theorem follows from Equation (5.21) along with Equations (5.8) and (4.17). □

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