ARTICLE IN PRESS YJABR:14947

Journal of Algebra ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Eigenvalue coincidences and K-orbits, I

Mark Colarusso [a,*], Sam Evens [b]

[a] Department of Mathematical Sciences, University of Wisconsin–Milwaukee, Milwaukee, WI, 53201, United States

[b] Department of Mathematics, University of Notre Dame, Notre Dame, IN, 46556, United States

ARTICLE INFO

Article history:
Received 3 February 2014
Available online xxxx
Communicated by Shrawan Kumar

MSC:
14M15
14L30
20G20

Keywords:
K-orbits on flag variety
Algebraic group actions

ABSTRACT

We study the variety $\mathfrak{g}(l)$ consisting of matrices $x \in \mathfrak{gl}(n, \mathbb{C})$ such that x and its $n-1$ by $n-1$ cutoff x_{n-1} share exactly l eigenvalues, counted with multiplicity. We determine the irreducible components of $\mathfrak{g}(l)$ by using the orbits of $GL(n-1, \mathbb{C})$ on the flag variety $\mathcal{B}$ of $\mathfrak{gl}(n, \mathbb{C})$. More precisely, let $\mathfrak{b} \in \mathcal{B}$ be a Borel subalgebra such that the orbit $GL(n-1, \mathbb{C}) \cdot \mathfrak{b}$ in $\mathcal{B}$ has codimension l. Then we show that the set $Y_{\mathfrak{b}} := \{\mathrm{Ad}(g)(x) : x \in \mathfrak{b} \cap \mathfrak{g}(l), g \in GL(n-1, \mathbb{C})\}$ is an irreducible component of $\mathfrak{g}(l)$, and every irreducible component of $\mathfrak{g}(l)$ is of the form $Y_{\mathfrak{b}}$, where $\mathfrak{b}$ lies in a $GL(n-1, \mathbb{C})$-orbit of codimension l. An important ingredient in our proof is the flatness of a variant of a morphism considered by Kostant and Wallach, and we prove this flatness assertion using an analogue of the Steinberg variety.

1. Introduction

Let x be an $n \times n$ complex matrix, and let x_{n-1} be the $n-1 \times n-1$ submatrix in the upper left corner of x. The relationship between the eigenvalues x and x_{n-1} has been an area of interest for both linear algebraists and Lie theorists for a long time. For

* Corresponding author.
E-mail addresses: colaruss@uwm.edu (M. Colarusso), sevens@nd.edu (S. Evens).

http://dx.doi.org/10.1016/j.jalgebra.2014.08.051

example, if x is Hermitian, then it is well-known that the necessarily real eigenvalues of x must interlace those of x_{n-1} (Theorem 4.3.8 of [20].) Interlacing of eigenvalues also appears in branching rules for restrictions of irreducible representations of the unitary group $U(n, \mathbb{C})$ to its subgroup $U(n-1, \mathbb{C})$ (Theorem 8.1.1 of [17]).

These two perspectives meet in the study of the Gelfand–Zeitlin integrable system on the $n \times n$ Hermitian matrices, which can be identified with the dual space of the Lie algebra of the group $U(n, \mathbb{C})$. This integrable system was constructed by Guillemin and Sternberg in [16] and is the geometric version of the classical Gelfand–Zeitlin basis for irreducible $U(n, \mathbb{C})$-representations [14]. In [16], the geometric construction of the Gelfand–Zeitlin basis makes use of the fact that both the branching rule for irreducible representations of $U(n, \mathbb{C})$ to $U(n-1, \mathbb{C})$ and the eigenvalue coincidences of x and x_{n-1} with x Hermitian satisfy the same interlacing property.

In a series of recent papers, [24,25], Kostant and Wallach have developed a complex version of the Gelfand–Zeitlin integrable system studied by Guillemin and Sternberg for the Lie algebra $\mathfrak{g} := \mathfrak{gl}(n, \mathbb{C})$ of $n \times n$ complex matrices. Their work has produced new directions in both Lie theory and linear algebra. In [24], the authors show that for an arbitrary $n \times n$ complex matrix x, the eigenvalues of x_{n-1} are independent of those of x. This observation has prompted recent work by Parlett and Strang on eigenvalue coincidences of complex matrices [31].

In this paper, we study the geometry of eigenvalue coincidences for arbitrary complex matrices using the theory of orbits of the group $K := GL(n-1, \mathbb{C}) \times GL(1, \mathbb{C})$ on the flag variety $\mathcal{B}$ of $\mathfrak{g}$. In particular, we consider the subset $\mathfrak{g}(l)$ consisting of elements $x \in \mathfrak{g}$ such that x and x_{n-1} share exactly l eigenvalues, counted with multiplicity, where $0 \leq l \leq n-1$. We study the algebraic geometry of the set $\mathfrak{g}(l)$ and determine the irreducible components of $\mathfrak{g}(l)$. This allows us to describe elements of $\mathfrak{g}(l)$ up to K-conjugacy. The proof relies on the flatness of a variant of the moment map for the Gelfand–Zeitlin system, which in turn depends on a dimension estimate proved using a variant of the Steinberg variety.

In more detail, let $G = GL(n, \mathbb{C})$ and let $\theta : G \to G$ be the involution $\theta(x) = dxd^{-1}$, where $d = \text{diag}[1, \ldots, 1, -1]$. Then $K = G^\theta$. It is well-known that K has exactly n closed orbits on the flag variety $\mathcal{B}$, and each of these closed orbits is isomorphic to the flag variety $\mathcal{B}_{n-1}$ of Borel subalgebras of $\mathfrak{gl}(n-1, \mathbb{C})$. Further, there are finitely many K-orbits on $\mathcal{B}$, and for each of these K-orbits Q, we consider its length $l(Q) = \dim(Q) - \dim(\mathcal{B}_{n-1})$. For $Q = K \cdot \mathfrak{b}_Q$, we consider the K-saturation $Y_Q := \text{Ad}(K)\mathfrak{b}_Q$ of $\mathfrak{b}_Q$, which is independent of the choice of $\mathfrak{b}_Q \in Q$.

Theorem 1.1. *The irreducible component decomposition of* $\mathfrak{g}(l)$ *is*

$$\mathfrak{g}(l) = \bigcup_{l(Q)=n-1-l} Y_Q \cap \mathfrak{g}(l). \tag{1.1}$$

A key ingredient in the proof is the observation that the variety $\mathfrak{g}(l)$ is equidimensional. To see this, we study the morphism $\Phi_n = (\chi_{n-1}, \chi_n) : \mathfrak{g} \to \mathbb{C}^{n-1} \times \mathbb{C}^n$, where for $x \in \mathfrak{g}$,

Please cite this article in press as: M. Colarusso, S. Evens, Eigenvalue coincidences and K-orbits, I, J. Algebra (2014), http://dx.doi.org/10.1016/j.jalgebra.2014.08.051

$\chi_n(x)$ is an adjoint quotient of x, and $\chi_{n-1}(x)$ is an adjoint quotient of x_{n-1} in $\mathfrak{gl}(n-1, \mathbb{C})$ (see (2.2) for a precise definition). The morphism Φ_n is a variant of the moment map for the Gelfand–Zeitlin system studied by Kostant and Wallach in [24], and we refer to it as the partial Kostant–Wallach map. We denote its nilfiber by SN_n, i.e.,

$$SN_n := \Phi_n^{-1}(0) = \{x \in \mathfrak{g} : x,\ x_{n-1} \text{ are nilpotent}\}. \tag{1.2}$$

To study SN_n, we consider the variant of the Steinberg variety given by

$$Z = \left\{ \left(x, \mathfrak{b}, \mathfrak{b}'\right) : \mathfrak{b} \in \mathcal{B},\ \mathfrak{b}' \in \mathcal{B}_{n-1} \text{ and } x \in [\mathfrak{b}, \mathfrak{b}],\ x_{n-1} \in \left[\mathfrak{b}', \mathfrak{b}'\right] \right\}$$

together with the diagram

$$\begin{array}{ccc} & Z & \\ \swarrow & & \searrow \\ \mathfrak{g} & & \mathcal{B} \times \mathcal{B}_{n-1} \end{array}$$

where the maps are projections to the appropriate factors. We identify Z with a union of conormal bundles to diagonal K-orbits in $\mathcal{B} \times \mathcal{B}_{n-1}$, and it follows that all irreducible components of Z have dimension equal to $\dim(\mathcal{B}) + \dim(\mathcal{B}_{n-1}) = n^2 - 2n + 1$. We observe that SN_n is the image of Z under projection to $\mathfrak{g}$. By elementary considerations, the dimension of each irreducible component of SN_n is at least $n^2 - 2n + 1$, and it follows that SN_n is equidimensional of dimension $n^2 - 2n + 1$. We then use homogeneity of Φ_n to prove flatness of Φ_n, from which it follows that $\mathfrak{g}(l)$ is equidimensional. We remark that flatness of Φ_n also follows from results of Ovsienko and Futorny [29,11], or of Knop [22], or of Panyushev [30]. In the case we are considering, the argument outlined above is simpler and quite different than the existing arguments in the literature. Our approach may also provide interesting applications to geometric representation theory; in particular to the construction of certain generalized Harish-Chandra modules closely related to the Gelfand–Zeitlin modules studied by Drozd, Futorny, and Ovsienko [9,13,12]. It would also be interesting to relate our methods to results of Baruch on construction of invariant distributions on $\mathfrak{g}$ [1].

Theorem 1.1 then follows from considering the interactions of K-orbits on $\mathcal{B}$ with $\mathfrak{g}$ using the Grothendieck resolution:

$$\begin{array}{ccc} & \tilde{\mathfrak{g}} = \{(x, \mathfrak{b}) : \mathfrak{b} \in \mathcal{B},\ x \in \mathfrak{b}\} & \\ \swarrow \mu & & \searrow \pi \\ \mathfrak{g} & & \mathcal{B} \end{array}$$

where μ and π are the projections onto the first and second factor respectively. We use the observation that $Y_Q = \mu(\pi^{-1}(Q))$ to study the geometry of the subsets Y_Q using properties of the geometry of K-orbits on $\mathcal{B}$ and of the morphisms π and μ. We show if $l(Q) = n - 1 - l$, then $\dim Y_Q \cap \mathfrak{g}(l) = \dim \mathfrak{g}(l)$, and $Y_Q \cap \mathfrak{g}(l)$ is closed in $\mathfrak{g}(l)$. Since $\mathfrak{g}(l)$

is equidimensional, it follows that $Y_Q \cap \mathfrak{g}(l)$ is an irreducible component of $\mathfrak{g}(l)$. We show that the varieties $Y_Q \cap \mathfrak{g}(l)$ with $l(Q) = n - 1 - l$ exhaust the irreducible components of $\mathfrak{g}(l)$ by using results from [8] and [6].

Theorem 1.1 has the following consequence. Let $\mathfrak{b}_+$ denote the Borel subalgebra consisting of upper triangular matrices. For $i = 1, \ldots, n$, let (in) be the permutation matrix corresponding to the transposition interchanging i and n, and let $\mathfrak{b}_i := \mathrm{Ad}(in)\mathfrak{b}_+$.

Corollary 1.2. *If $x \in \mathfrak{g}(l)$, then x is K-conjugate to an element in one of $l+1$ explicitly determined θ-stable parabolic subalgebras. In particular, if $x \in \mathfrak{g}(n-1)$, then x is K-conjugate to an element of $\mathfrak{b}_i$, where $i = 1, \ldots, n$.*

Using Corollary 1.2 and the geometry of the Springer resolution, we obtain an explicit description of the irreducible components of the variety SN_n introduced in Eq. (1.2). If $\mathfrak{n}_i$ is the nilradical of $\mathfrak{b}_i$, then the geometry of the Springer resolution and the equidimensionality of SN_n imply that $\mathrm{Ad}(K)\mathfrak{n}_i$ is an irreducible component of SN_n. We then use Corollary 1.2 to show that every irreducible component of SN_n is of this form (see Proposition 3.10).

This paper is part of a series of papers on K-orbits on $\mathcal{B}$ and the Gelfand–Zeitlin system on $\mathfrak{g}$. In [6], we used K-orbits to give a conceptual description of the so-called strongly regular elements in the nilfiber of the moment map of the Gelfand–Zeitlin system. These are matrices $x \in \mathfrak{g}$ such that x_i is nilpotent for all $i = 1, \ldots, n$ with the added condition that the differentials of the Gelfand–Zeitlin functions are linearly independent at x. The strongly regular elements were first studied extensively in [24], and the first author determined the strongly regular moment map fibers by explicit computation in [8]. This paper may be regarded as a step towards understanding all fibers of the Gelfand–Zeitlin moment map using the theory of K-orbits on $\mathcal{B}$. In a second paper, we will refine Corollary 1.2 to provide a standard form for all elements of $\mathfrak{g}(l)$. This uses K-orbits and a finer study of the algebraic geometry of the varieties $\mathfrak{g}(l)$. In a third paper, we will iterate the constructions in this paper to describe the subvarieties $\mathfrak{g}(l_1, \ldots, l_{n-1})$ consisting of elements $x \in \mathfrak{g}$ such that x_i and x_{i+1} share l_i eigenvalues in common counting repetition, where $0 \leq l_i \leq i$. In particular, we will give a more conceptual proof of the main result from [8] and use K-orbits to describe the geometry of arbitrary fibers of the moment map for the Gelfand–Zeitlin system.

This paper is organized as follows. In Section 2, we show that the variety SN_n is equidimensional and prove that the partial Kostant–Wallach map Φ_n is flat (see Theorem 2.3 and Proposition 2.4). We also study the geometry of the subsets Y_Q, computing their dimensions and closures (see Lemma 2.13 and Proposition 2.15). Section 3 contains the main results of the paper, where we deduce the equidimensionality of the varieties $\mathfrak{g}(l)$ and prove Theorem 1.1 (see Proposition 3.5 and Theorems 3.6 and 3.7).

The work by the second author was partially supported by NSA grant H98230-11-1-0151. We would like to thank Adam Boocher and Claudia Polini for useful discussions.

We would also like to thank the referees for useful suggestions and for alerting us to the references [22,30].

2. Preliminaries

We show flatness of the partial Kostant–Wallach morphism and recall needed results concerning K-orbits on $\mathcal{B}$.

2.1. The partial Kostant–Wallach map

For $x \in \mathfrak{g}$ and $i = 1, \ldots n$, let $x_i \in \mathfrak{gl}(i, \mathbb{C})$ denote the upper left $i \times i$ corner of the matrix x. For any $y \in \mathfrak{gl}(i, \mathbb{C})$, let $tr(y)$ denote the trace of y. For $j = 1, \ldots, i$, let $f_{i,j}(x) = tr((x_i)^j)$, which is a homogeneous function of degree j on $\mathfrak{g}$. The Gelfand–Zeitlin collection of functions is the set $J_{GZ} := \{f_{i,j}(x) : i = 1, \ldots, n, j = 1, \ldots, i\}$. The restriction of these functions to any regular adjoint orbit in $\mathfrak{g}$ produces an integrable system on the orbit [24]. Let $\chi_{i,j} : \mathfrak{gl}(i, \mathbb{C}) \to \mathbb{C}$ be the function $\chi_{i,j}(y) = tr(y^j)$, so that $f_{i,j}(x) = \chi_{i,j}(x_i)$ and $\chi_i := (\chi_{i,1}, \ldots, \chi_{i,i})$ is the adjoint quotient for $\mathfrak{gl}(i, \mathbb{C})$. The Kostant–Wallach map is the morphism given by

$$\Phi : \mathfrak{g} \to \mathbb{C}^1 \times \mathbb{C}^2 \times \cdots \times \mathbb{C}^n; \qquad \Phi(x) = \big(\chi_1(x_1), \ldots, \chi_n(x)\big). \tag{2.1}$$

We will also consider the partial Kostant–Wallach map given by the morphism

$$\Phi_n : \mathfrak{g} \to \mathbb{C}^{n-1} \times \mathbb{C}^n; \qquad \Phi_n(x) = \big(\chi_{n-1}(x_{n-1}), \chi_n(x)\big). \tag{2.2}$$

Note that

$$\Phi_n = pr \circ \Phi, \tag{2.3}$$

where $pr : \mathbb{C}^1 \times \mathbb{C}^2 \times \cdots \times \mathbb{C}^n \to \mathbb{C}^{n-1} \times \mathbb{C}^n$ is the projection on the last two factors.

Remark 2.1. By Theorem 0.1 of [24], the map Φ is surjective, and therefore Φ_n is surjective.

Notation 2.2. Throughout the paper, we use the following notation. Let $\mathbb{C}[\mathfrak{g}]$ denote the ring of regular functions on $\mathfrak{g}$. Let $I := (\{f_{ij}\}_{i=1,\ldots,n;j=1,\ldots,i})$ denote the ideal of $\mathbb{C}[\mathfrak{g}]$ generated by the Gelfand–Zeitlin functions J_{GZ}. The vanishing set $V(I)$ of I is called the variety of *strongly nilpotent matrices* and is denoted by SN:

$$SN = \{x \in \mathfrak{g} : x_i \text{ is nilpotent for } i = 1, \ldots, n\}.$$

Let $\Gamma := \mathbb{C}[\{f_{ij}\}_{i=1,\ldots,n;j=1,\ldots,i}]$ be the subring of $\mathbb{C}[\mathfrak{g}]$ generated by J_{GZ}.

Much of the paper is devoted to the study of the analogous objects obtained from the *partial Gelfand–Zeitlin functions* $J_{GZ,n} := \{f_{i,j} : i = n-1, n;\ j = 1, \ldots, i\}$. We let $I_n = (\{f_{ij}\}_{i=n-1,n;j=1,\ldots,i})$ denote the ideal of $\mathbb{C}[\mathfrak{g}]$ generated by $J_{GZ,n}$. We call the vanishing set $V(I_n)$ the variety of *partially strongly nilpotent matrices* and denote it by SN_n. Thus,

$$SN_n := \{x \in \mathfrak{g} : x,\ x_{n-1} \text{ are nilpotent}\}. \tag{2.4}$$

Finally, we let $\Gamma_n := \mathbb{C}[\{f_{ij}\}_{i=n-1,n;j=1,\ldots,i}]$ be the subring of $\mathbb{C}[\mathfrak{g}]$ generated by $J_{GZ,n}$.

Recall that if $Y \subset \mathbb{C}^m$ is a closed equidimensional subvariety of dimension $m - d$, then Y is called a complete intersection if $Y = V(f_1, \ldots, f_d)$ is the vanishing set of d functions.

Theorem 2.3. *The variety of partially strongly nilpotent matrices SN_n is a complete intersection of dimension*

$$d_n := n^2 - 2n + 1. \tag{2.5}$$

Before proving Theorem 2.3, we show how it implies the flatness of the partial Kostant–Wallach map Φ_n.

Proposition 2.4.

(1) *For all $c \in \mathbb{C}^{n-1} \times \mathbb{C}^n$, $\dim(\Phi_n^{-1}(c)) = n^2 - 2n + 1$. Thus, $\Phi_n^{-1}(c)$ is a complete intersection.*
(2) *The partial Kostant–Wallach map $\Phi_n : \mathfrak{g} \to \mathbb{C}^{2n-1}$ is a flat morphism. Thus, $\mathbb{C}[\mathfrak{g}]$ is flat over Γ_n.*

Proof. For $x \in \mathfrak{g}$, we let d_x be the maximum of the dimensions of irreducible components of $\Phi_n^{-1}(\Phi_n(x))$. For $c \in \mathbb{C}^{n-1} \times \mathbb{C}^n$, each irreducible component of $\Phi_n^{-1}(c)$ has dimension at least d_n since $\Phi_n^{-1}(c)$ is defined by $2n - 1$ equations in $\mathfrak{g}$. Hence, $d_x \geq d_n$. Since the functions $f_{i,j}$ are homogeneous, it follows that scalar multiplication by $\lambda \in \mathbb{C}^\times$ induces an isomorphism $\Phi_n^{-1}(\Phi_n(x)) \to \Phi_n^{-1}(\Phi_n(\lambda x))$. It follows that $d_x = d_{\lambda x}$. By upper semi-continuity of dimension (see for example, Proposition 4.4 of [21]), the set of $y \in \mathfrak{g}$ such that $d_y \geq d$ is closed for each integer d. It follows that $d_0 \geq d_x$. By Theorem 2.3, $d_0 = d_n$. The first assertion follows easily. The second assertion now follows by the corollary to Theorem 23.1 of [27]. □

Proof of Theorem 2.3. Let $\mathfrak{X}$ be an irreducible component of SN_n. We observed in the proof of Proposition 2.4 that $\dim \mathfrak{X} \geq d_n$. To show $\dim \mathfrak{X} \leq d_n$, we consider a generalization of the Steinberg variety (see Section 3.3 of [7]). We first recall a few facts about the cotangent bundle to the flag variety.

For the purposes of this proof, we denote the flag variety of $\mathfrak{gl}(n,\mathbb{C})$ by $\mathcal{B}_n$. We consider the form $\langle\langle\cdot,\cdot\rangle\rangle$ on $\mathfrak{g}$ given by $\langle\langle x,y\rangle\rangle = tr(xy)$ for $x,y\in\mathfrak{g}$. If $\mathfrak{b}\in\mathcal{B}_n$, the annihilator $\mathfrak{b}^\perp$ of $\mathfrak{b}$ with respect to the form $\langle\langle\cdot,\cdot\rangle\rangle$ is $\mathfrak{n}=[\mathfrak{b},\mathfrak{b}]$. We can then identify $T^*(\mathcal{B}_n)$ with the closed subset of $\mathfrak{g}\times\mathcal{B}_n$ given by:

$$T^*(\mathcal{B}_n) = \{(x,\mathfrak{b}) :\ \mathfrak{b}\in\mathcal{B}_n, x\in\mathfrak{n}\}.$$

We let $\mathfrak{g}_{n-1}=\mathfrak{gl}(n-1,\mathbb{C})$ and view $\mathfrak{g}_{n-1}$ as a subalgebra of $\mathfrak{g}$ by embedding $\mathfrak{g}_{n-1}$ in the top lefthand corner of $\mathfrak{g}$. Since $\mathfrak{g}$ is the direct sum $\mathfrak{g}=\mathfrak{g}_{n-1}\oplus\mathfrak{g}_{n-1}^\perp$, the restriction of $\langle\langle\cdot,\cdot\rangle\rangle$ to $\mathfrak{g}_{n-1}$ is non-degenerate. For a Borel subalgebra $\mathfrak{b}'\in\mathcal{B}_{n-1}$, we let $\mathfrak{n}'=[\mathfrak{b}',\mathfrak{b}']$. We consider a closed subvariety $Z\subset\mathfrak{g}\times\mathcal{B}_n\times\mathcal{B}_{n-1}$ defined as follows:

$$Z = \{(x,\mathfrak{b},\mathfrak{b}') : \mathfrak{b}\in\mathcal{B}_n,\ \mathfrak{b}'\in\mathcal{B}_{n-1} \text{ and } x\in\mathfrak{n},\ x_{n-1}\in\mathfrak{n}'\}. \tag{2.6}$$

Consider the morphism $\mu : Z\to\mathfrak{g}$, where $\mu(x,\mathfrak{b},\mathfrak{b}')=x$. Since the varieties $\mathcal{B}_n$ and $\mathcal{B}_{n-1}$ are projective, the morphism μ is proper.

We consider the closed embedding $Z\hookrightarrow T^*(\mathcal{B}_n)\times T^*(\mathcal{B}_{n-1})\cong T^*(\mathcal{B}_n\times\mathcal{B}_{n-1})$ given by $(x,\mathfrak{b},\mathfrak{b}')\to(x,-x_{n-1},\mathfrak{b},\mathfrak{b}')$. We denote the image of Z under this embedding by $\widetilde{Z}\subset T^*(\mathcal{B}_n\times\mathcal{B}_{n-1})$. Let G_{n-1} be the closed subgroup of $GL(n,\mathbb{C})$ corresponding to $\mathfrak{g}_{n-1}$. Then G_{n-1} acts diagonally on $\mathcal{B}_n\times\mathcal{B}_{n-1}$ via $k\cdot(\mathfrak{b},\mathfrak{b}')=(k\cdot\mathfrak{b},k\cdot\mathfrak{b}')$ for $k\in G_{n-1}$. We claim $\widetilde{Z}\subset T^*(\mathcal{B}_n\times\mathcal{B}_{n-1})$ is the union of conormal bundles to the G_{n-1}-diagonal orbits in $\mathcal{B}_n\times\mathcal{B}_{n-1}$. Indeed, let $(\mathfrak{b},\mathfrak{b}')\in\mathcal{B}_n\times\mathcal{B}_{n-1}$, and let Q be its G_{n-1}-orbit. Then

$$T_{(\mathfrak{b},\mathfrak{b}')}(Q) = \operatorname{span}\{(Y \bmod \mathfrak{b},\ Y \bmod \mathfrak{b}') :\ Y\in\mathfrak{g}_{n-1}\}.$$

Now let $(\lambda_1,\lambda_2)\in(\mathfrak{n},\mathfrak{n}')$ with $(\lambda_1,\lambda_2)\in(T_Q^*)(\mathcal{B}_n\times\mathcal{B}_{n-1})_{(\mathfrak{b},\mathfrak{b}')}$, the fiber of the conormal bundle to Q in $\mathcal{B}_n\times\mathcal{B}_{n-1}$ at the point $(\mathfrak{b},\mathfrak{b}')$. Then

$$\langle\langle\lambda_1,Y\rangle\rangle + \langle\langle\lambda_2,Y\rangle\rangle = 0 \quad \text{for all } Y\in\mathfrak{g}_{n-1}.$$

Thus, $\lambda_1+\lambda_2\in\mathfrak{g}_{n-1}^\perp$. But since $\lambda_2\in\mathfrak{n}'\subset\mathfrak{g}_{n-1}$, it follows that $\lambda_2=-(\lambda_1)_{n-1}$. Thus,

$$T_Q^*(\mathcal{B}_n\times\mathcal{B}_{n-1}) = \{(\mu_1,\mathfrak{b}_1,-(\mu_1)_{n-1},\mathfrak{b}_2),\ \mu_1\in\mathfrak{n}_1,\ (\mu_1)_{n-1}\in\mathfrak{n}_2,\ \text{where } (\mathfrak{b}_1,\mathfrak{b}_2)\in Q\}.$$

We recall the well-known fact that there are only finitely many G_{n-1}-diagonal orbits in $\mathcal{B}_n\times\mathcal{B}_{n-1}$, which follows from [35,4], or in a more explicit form is proved in [19]. Therefore, the irreducible component decomposition of $\widetilde{Z}$ is:

$$\widetilde{Z} = \bigcup_i \overline{T_{Q_i}^*(\mathcal{B}_n\times\mathcal{B}_{n-1})} \subset T^*(\mathcal{B}_n\times\mathcal{B}_{n-1}),$$

where i runs over the distinct G_{n-1}-diagonal orbits in $\mathcal{B}_n\times\mathcal{B}_{n-1}$. Thus, $Z\cong\widetilde{Z}$ is a closed, equidimensional subvariety of dimension

$$\dim Z = \frac{1}{2}(\dim T^*(\mathcal{B}_n\times\mathcal{B}_{n-1})) = d_n.$$

Note that $\mu : Z \to SN_n$ is surjective. Since μ is proper, for every irreducible component $\mathfrak{X} \subset SN_n$ of SN_n, we see that

$$\mathfrak{X} = \mu(Z_i) \tag{2.7}$$

for some irreducible component $Z_i \subset Z$. Since $\dim Z_i = d_n$ and $\dim \mathfrak{X} \geq d_n$, we conclude that $\dim \mathfrak{X} = d_n$. □

In Proposition 3.10, we will determine the irreducible components of SN_n explicitly.

Lemma 2.5. *Let $A = \bigoplus_{n\geq 0} A_n$ be a graded ring with $A_0 = k$ a field, and let $M = \bigoplus_{n\geq n_0} M_{n_0}$ be a graded A-module which vanishes below some index. If M is flat over A, then M is free over A.*

Proof. This result is well-known, but we include a sketch of the proof for lack of a reference. Let $I = \bigoplus_{n>0} A_n$. An easy graded version of Nakayama's lemma asserts that if M is a graded A-module as in the statement of the lemma and $I \cdot M = M$, then $M = 0$. The remainder of the proof follows by the proof of the analogous assertion for modules over a local ring (see [34], Proposition 20, page 73), with the graded version of Nakayama's Lemma playing the role of Nakayama's Lemma for local rings in the proof. □

Proposition 2.6. *(See [11].) $\mathbb{C}[\mathfrak{g}]$ is free over Γ_n.*

Proof. Apply Lemma 2.5 and Proposition 2.4 (2). □

A stronger version of this proposition is proved by much more elaborate methods in [29] and [11]. Ovsienko proves in [29] that SN is a complete intersection, and results of Futorny and Ovsienko from [11] show that Ovsienko's theorem implies that $\mathbb{C}[\mathfrak{g}]$ is free over Γ (see Notation 2.2). It then follows easily that $\mathbb{C}[\mathfrak{g}]$ is flat over Γ_n, and hence that Φ_n is flat.

Remark 2.7. We briefly discuss the connection between the result of Proposition 2.4 and the work of Knop in [22]. For the purposes of this remark only, we let $G = GL(n, \mathbb{C}) \times GL(n-1, \mathbb{C})$, and we let $H = GL(n-1, \mathbb{C})_\Delta$ be the diagonal copy of $GL(n-1, \mathbb{C})$ embedded in G. (Here we view $GL(n-1, \mathbb{C})$ as a subgroup of $GL(n, \mathbb{C})$ by embedding it in the top lefthand corner.) Then the homogeneous space G/H is spherical, i.e., a Borel subgroup of G has an open orbit on G/H. Let $\mathfrak{g} = \mathrm{Lie}(G) = \mathfrak{gl}(n, \mathbb{C}) \oplus \mathfrak{gl}(n-1, \mathbb{C})$. We identify $\mathfrak{g}$ with $\mathfrak{g}^*$ using the difference of the trace forms on $\mathfrak{gl}(n, \mathbb{C})$ and $\mathfrak{gl}(n-1, \mathbb{C})$. Let $\mathfrak{h}^\perp \subset \mathfrak{g}^* \cong \mathfrak{g}$ denote the annihilator of $\mathfrak{h} = \mathrm{Lie}(H)$. Then $\mathfrak{h}^\perp$ is identified with the subspace $\{(x, x_{n-1}) : x \in \mathfrak{gl}(n, \mathbb{C})\}$ of $\mathfrak{g}$, and hence $\mathfrak{h}^\perp \cong \mathfrak{gl}(n, \mathbb{C})$. Note also the identification $\mathfrak{g}//G \cong \mathbb{C}^{2n-1}$ given by taking unordered eigenvalues of each factor of $\mathfrak{g}$. The cotangent bundle to G/H is $T^*(G/H) = G \times_H \mathfrak{h}^\perp$. In [22], Knop studies the morphism:

$$\tilde{\Psi} : G \times_H \mathfrak{h}^\perp \to \mathfrak{g}//G,$$

which is the composition of the moment map $\mu : G \times_H \mathfrak{h}^\perp \to \mathfrak{g}$ and the adjoint quotient $\chi : \mathfrak{g} \to \mathfrak{g}//G$. Using the above identifications, it follows from definitions that the restriction of $\tilde{\Psi}$ to $\mathfrak{h}^\perp$ is the morphism Φ_n. By Remark 2.1, Φ_n is surjective, whence $\tilde{\Psi}$ is surjective. By the proof of Satz 6.6 of [22], it follows that the fibers of $\tilde{\Psi}$ are equidimensional (and $\tilde{\Psi}$ is flat). By the identity

$$\tilde{\Psi}^{-1}(c) = G \times_H \Phi_n^{-1}(c), \tag{2.8}$$

where $c \in \mathfrak{g}//G \cong \mathbb{C}^{2n-1}$, it follows that the fibers of Φ_n are equidimensional varieties of dimension $(n-1)^2$. The flatness of Φ_n now follows as in the proof of part (2) of Proposition 2.4.

Remark 2.8. We note that the morphism $\Phi_n : \mathfrak{gl}(n,\mathbb{C}) \to \mathbb{C}^{2n-1}$ is the quotient morphism by the action of $GL(n-1,\mathbb{C})$ on $\mathfrak{gl}(n,\mathbb{C})$ by conjugation where $GL(n-1,\mathbb{C})$ is embedded in $GL(n,\mathbb{C})$ as the upper left hand corner. Indeed, this follows from Korollar 7.2 of [22]. We also note the closely related Theorem 6 of [30], which implies that $\mathfrak{gl}(n,\mathbb{C})//GL(n-1,\mathbb{C}) \cong \mathbb{C}^{2n-1}$. The fact that Φ_n is a quotient morphism also follows from our results. Indeed, we can show that each fiber of Φ_n has a unique closed $GL(n-1,\mathbb{C})$-orbit, which implies that Φ_n induces a bijection $\mathfrak{gl}(n,\mathbb{C})//GL(n-1,\mathbb{C}) \to \mathbb{C}^{2n-1}$, which is an isomorphism by Zariski's Main Theorem.

Although we could have simply cited the results of Futorny and Ovsienko or Knop to prove flatness of Φ_n, we prefer our approach because of its links with geometric representation theory. In later work, we plan to use the Beilinson–Bernstein correspondence to construct generalized Harish-Chandra modules for $(\mathfrak{gl}(n,\mathbb{C}) \oplus \mathfrak{gl}(n-1,\mathbb{C}), K)$, by using the geometry of K-orbits on the product of flag varieties $\mathcal{B} \times \mathcal{B}_{n-1}$. In representation theory, this would involve using both Zuckerman functors (as in [32]) and an equivalence of categories analogous to the equivalence between category $\mathcal{O}$ and certain categories of Harish-Chandra modules ([2], 3.4). This process would produce $\mathfrak{gl}(n,\mathbb{C})$-modules with a locally finite Γ_n-action, and we hope this project will be shed new light on our larger program of the quantizing the Gelfand–Zeitlin integrable system and geometrically constructing the Gelfand–Zeitlin modules studied by Drozd, Futorny, and Ovsienko [9].

2.2. K-orbits

We recall some basic facts about K-orbits on generalized flag varieties G/P (see [26, 33,28,36,5] for more details).

Please cite this article in press as: M. Colarusso, S. Evens, Eigenvalue coincidences and K-orbits, I, J. Algebra (2014), http://dx.doi.org/10.1016/j.jalgebra.2014.08.051

By the general theory of orbits of symmetric subgroups on generalized flag varieties, K has finitely many orbits on $\mathcal{B}$. For this paper, it is useful to parametrize the orbits. To do this, we let B_+ be the upper triangular Borel subgroup of G, and identify $\mathcal{B} \cong G/B_+$ with the variety of flags in $\mathbb{C}^n$. We use the following notation for flags in $\mathbb{C}^n$. Let

$$\mathcal{F} = \big(V_0 = \{0\} \subset V_1 \subset \cdots \subset V_i \subset \cdots \subset V_n = \mathbb{C}^n\big)$$

be a flag in $\mathbb{C}^n$, with $\dim V_i = i$ and $V_i = \operatorname{span}\{v_1, \ldots, v_i\}$, with each $v_j \in \mathbb{C}^n$. We will denote the flag $\mathcal{F}$ as follows:

$$v_1 \subset v_2 \subset \cdots \subset v_i \subset v_{i+1} \subset \cdots \subset v_n.$$

We denote the standard ordered basis of $\mathbb{C}^n$ by $\{e_1, \ldots, e_n\}$, and let $E_{i,j} \in \mathfrak{g}$ be the matrix with 1 in the (i, j)-entry and 0 elsewhere.

There are n closed K-orbits on $\mathcal{B}$ (see Example 4.16 of [5]), $Q_{i,i} = K \cdot \mathfrak{b}_{i,i}$ for $i = 1, \ldots, n$, where the Borel subalgebra $\mathfrak{b}_{i,i}$ is the stabilizer of the following flag in $\mathbb{C}^n$:

$$\mathcal{F}_{i,i} = (e_1 \subset \cdots \subset e_{i-1} \subset \underbrace{e_n}_{i} \subset e_i \subset \cdots \subset e_{n-1}). \tag{2.9}$$

Note that if $i = n$, then the flag $\mathcal{F}_{i,i}$ is the standard flag $\mathcal{F}_+$:

$$\mathcal{F}_+ = (e_1 \subset \cdots \subset e_n), \tag{2.10}$$

and $\mathfrak{b}_{n,n} = \mathfrak{b}_+$ is the standard Borel subalgebra of $n \times n$ upper triangular matrices. It is not difficult to check that $K \cdot \mathfrak{b}_{i,i} = K \cdot \operatorname{Ad}(in)\mathfrak{b}_+$. If $i = 1$, then $K \cdot \mathfrak{b}_{1,1} = K \cdot \mathfrak{b}_-$, where $\mathfrak{b}_-$ is the Borel subalgebra of lower triangular matrices in $\mathfrak{g}$.

The non-closed K-orbits in $\mathcal{B}$ are the orbits $Q_{i,j} = K \cdot \mathfrak{b}_{i,j}$ for $1 \le i < j \le n$, where $\mathfrak{b}_{i,j}$ is the stabilizer of the flag in $\mathbb{C}^n$:

$$\mathcal{F}_{i,j} = (e_1 \subset \cdots \subset \underbrace{e_i + e_n}_{i} \subset e_{i+1} \subset \cdots \subset e_{j-1} \subset \underbrace{e_i}_{j} \subset e_j \subset \cdots \subset e_{n-1}). \tag{2.11}$$

There are $\binom{n}{2}$ such orbits (see Notation 4.23 and Example 4.31 of [5]).

Let w and σ be the permutation matrices corresponding respectively to the cycles $(n\, n-1 \cdots i)$ and $(i+1\, i+2 \ldots j)$, and let u_{α_i} be the Cayley transform matrix such that

$$u_{\alpha_i}(e_i) = e_i + e_{i+1}, \qquad u_{\alpha_i}(e_{i+1}) = -e_i + e_{i+1}, \quad u_{\alpha_i}(e_k) = e_k, k \ne i, i+1.$$

For $1 \le i \le j \le n$, we define:

$$v_{i,j} := \left\{ \begin{matrix} w & \text{if } i = j \\ w u_{\alpha_i} \sigma & \text{if } i \ne j \end{matrix} \right\} \tag{2.12}$$

It is easy to verify that $v_{i,j}(\mathcal{F}_+) = \mathcal{F}_{i,j}$, and thus $\mathrm{Ad}(v_{i,j})\mathfrak{b}_+ = \mathfrak{b}_{i,j}$ (see Example 4.30 of [5]).

Proposition 2.9. *The length of the K-orbit $Q_{i,j}$ is $l(Q_{i,j}) = j - i$ for any $1 \leq i \leq j \leq n$. In particular, a K-orbit $Q_{i,j}$ is closed if and only if $Q = Q_{i,i}$ for some i. The $n - l$ orbits of length l are $Q_{i,i+l}$, $i = 1, \dots, n - l$.*

Proof. The proposition follows from the calculations in Example 4.30 of [5]. □

For a parabolic subgroup P of G with Lie algebra $\mathfrak{p}$, we consider the generalized flag variety G/P, which we identify with parabolic subalgebras of type $\mathfrak{p}$ and with partial flags of type $\mathfrak{p}$. We will make use of the following notation for partial flags. Let

$$\mathcal{P} = \big(V_0 = \{0\} \subset V_1 \subset \cdots \subset V_i \subset \cdots \subset V_k = \mathbb{C}^n\big)$$

denote a k-step partial flag with $\dim V_j = i_j$ and $V_j = \mathrm{span}\{v_1, \dots, v_{i_j}\}$ for $j = 1, \dots, k$. Then we denote $\mathcal{P}$ as

$$v_1, \dots, v_{i_1} \subset v_{i_1+1}, \dots, v_{i_2} \subset \cdots \subset v_{i_{k-1}+1}, \dots, v_{i_k}.$$

In particular for $i \leq j$, we let $\mathfrak{r}_{i,j} \subset \mathfrak{g}$ denote the parabolic subalgebra which is the stabilizer of the $n - (j - i)$-step partial flag in $\mathbb{C}^n$

$$\mathcal{R}_{i,j} = (e_1 \subset e_2 \subset \cdots \subset e_{i-1} \subset e_i, \dots, e_j \subset e_{j+1} \subset \cdots \subset e_n). \tag{2.13}$$

It is easy to see that $\mathfrak{r}_{i,j}$ is the standard parabolic subalgebra generated by the Borel subalgebra $\mathfrak{b}_+$ and the negative simple root spaces $\mathfrak{g}_{-\alpha_i}, \mathfrak{g}_{-\alpha_{i+1}}, \dots, \mathfrak{g}_{-\alpha_{j-1}}$. We note that $\mathfrak{r}_{i,j}$ has Levi decomposition $\mathfrak{r}_{i,j} = \mathfrak{m} + \mathfrak{n}$, with $\mathfrak{m}$ consisting of block diagonal matrices of the form

$$\mathfrak{m} = \underbrace{\mathfrak{gl}(1,\mathbb{C}) \oplus \cdots \oplus \mathfrak{gl}(1,\mathbb{C})}_{i-1 \text{ factors}} \oplus \mathfrak{gl}(j+1-i,\mathbb{C}) \oplus \underbrace{\mathfrak{gl}(1,\mathbb{C}) \oplus \cdots \oplus \mathfrak{gl}(1,\mathbb{C})}_{n-j \text{ factors}}. \tag{2.14}$$

Let $R_{i,j}$ be the parabolic subgroup of G with Lie algebra $\mathfrak{r}_{i,j}$. Let $\mathfrak{p}_{i,j} := \mathrm{Ad}(v_{i,j})\mathfrak{r}_{i,j} \in G/R_{i,j}$, where $v_{i,j}$ is defined in (2.12). Then $\mathfrak{p}_{i,j}$ is the stabilizer of the partial flag

$$\mathcal{P}_{i,j} = (e_1 \subset e_2 \subset \cdots \subset e_{i-1} \subset e_i, \dots, e_{j-1}, e_n \subset e_j \subset \cdots \subset e_{n-1}), \tag{2.15}$$

and $\mathfrak{p}_{i,j} \in G/R_{i,j}$ is a θ-stable parabolic subalgebra of $\mathfrak{g}$. Indeed, recall that θ is given by conjugation by the diagonal matrix $d = \mathrm{diag}[1, \dots, 1, -1]$. Clearly $d(\mathcal{P}_{i,j}) = \mathcal{P}_{i,j}$, whence $\mathfrak{p}_{i,j}$ is θ-stable. Moreover, the parabolic subalgebra $\mathfrak{p}_{i,j}$ has Levi decomposition $\mathfrak{p}_{i,j} = \mathfrak{l} \oplus \mathfrak{u}$ where both $\mathfrak{l}$ and $\mathfrak{u}$ are θ-stable and $\mathfrak{l}$ is isomorphic to the Levi subalgebra in Eq. (2.14). Since $\mathfrak{p}_{i,j}$ is θ-stable, it follows from Theorem 2 of [3] that the K-orbit $Q_{\mathfrak{p}_{i,j}} = K \cdot \mathfrak{p}_{i,j}$ is closed in $G/R_{i,j}$.

Please cite this article in press as: M. Colarusso, S. Evens, Eigenvalue coincidences and K-orbits, I, J. Algebra (2014), http://dx.doi.org/10.1016/j.jalgebra.2014.08.051

For a parabolic subgroup $P \subset G$ with Lie algebra $\mathfrak{p} \subset \mathfrak{g}$, consider the partial Grothendieck resolution $\tilde{\mathfrak{g}}^{\mathfrak{p}} = \{(x, \mathfrak{r}) \in \mathfrak{g} \times G/P \mid x \in \mathfrak{r}\}$, as well as the morphisms $\mu : \tilde{\mathfrak{g}}^{\mathfrak{p}} \to \mathfrak{g}$, $\mu(x, \mathfrak{r}) = x$, and $\pi : \tilde{\mathfrak{g}}^{\mathfrak{p}} \to G/P$, $\pi(x, \mathfrak{r}) = \mathfrak{r}$. Then π is a smooth morphism of relative dimension $\dim \mathfrak{p}$ (for G/B, see Section 3.1 of [7] and Proposition III.10.4 of [18], and the general case of G/P follows by the same argument). For $\mathfrak{r} \in G/P$, let $Q_{\mathfrak{r}} = K \cdot \mathfrak{r} \subset G/P$. Then $\pi^{-1}(Q_{\mathfrak{r}})$ has dimension $\dim(Q_{\mathfrak{r}}) + \dim(\mathfrak{r})$. It is well-known that μ is proper and its restriction to $\pi^{-1}(Q_{\mathfrak{r}})$ generically has finite fibers (Proposition 3.1.34 and Example 3.1.35 of [7] for the case of G/B, and again the general case has a similar proof).

Notation 2.10. For a parabolic subalgebra $\mathfrak{r}$ with K-orbit $Q_{\mathfrak{r}} \subset G/P$, we consider the irreducible subset

$$Y_{\mathfrak{r}} := \mu\big(\pi^{-1}(Q_{\mathfrak{r}})\big) = \mathrm{Ad}(K)\mathfrak{r}. \tag{2.16}$$

To emphasize the orbit $Q_{\mathfrak{r}}$, we will also denote this set as

$$Y_{Q_{\mathfrak{r}}} := Y_{\mathfrak{r}}. \tag{2.17}$$

It follows from generic finiteness of μ that $Y_{Q_{\mathfrak{r}}}$ contains an open subset of dimension

$$\dim(Y_{Q_{\mathfrak{r}}}) := \dim \pi^{-1}(Q_{\mathfrak{r}}) = \dim \mathfrak{r} + \dim(Q_{\mathfrak{r}}) = \dim \mathfrak{r} + \dim(\mathfrak{k}/\mathfrak{k} \cap \mathfrak{r}), \tag{2.18}$$

where $\mathfrak{k} = \mathrm{Lie}(K) = \mathfrak{gl}(n-1, \mathbb{C}) \oplus \mathfrak{gl}(1, \mathbb{C})$.

Remark 2.11. Since μ is proper, when $Q_{\mathfrak{r}} = K \cdot \mathfrak{r}$ is closed in G/P, then $Y_{Q_{\mathfrak{r}}}$ is closed.

Remark 2.12. Note that

$$\mathfrak{g} = \bigcup_{Q \subset G/P} Y_Q,$$

is a partition of $\mathfrak{g}$, where the union is taken over the finitely many K-orbits in G/P.

Lemma 2.13. *Let $Q \subset G/P$ be a K-orbit. Then*

$$\overline{Y_Q} = \bigcup_{Q' \subset \overline{Q}} Y_{Q'}. \tag{2.19}$$

Proof. Since π is a smooth morphism, it is flat by Theorem III.10.2 of [18]. Thus, by Theorem VIII.4.1 of [15], $\pi^{-1}(\overline{Q}) = \overline{\pi^{-1}(Q)}$. The result follows since μ is proper. □

Please cite this article in press as: M. Colarusso, S. Evens, Eigenvalue coincidences and K-orbits, I, J. Algebra (2014), http://dx.doi.org/10.1016/j.jalgebra.2014.08.051

2.3. *Comparison of $K \cdot \mathfrak{b}_{i,j}$ and $K \cdot \mathfrak{p}_{i,j}$*

We prove a technical result that will be needed to prove our main theorem.

Remark 2.14. Note that $\mathfrak{b}_{i,j} \subset \mathfrak{p}_{i,j}$ and when $i = j$, $\mathfrak{p}_{i,i}$ is the Borel subalgebra $\mathfrak{b}_{i,i}$. To check the first assertion, note that $\mathfrak{b}_+ \subset \mathfrak{r}_{i,j}$ so that $\mathfrak{b}_{i,j} = \mathrm{Ad}(v_{i,j})\mathfrak{b}_+ \subset \mathrm{Ad}(v_{i,j})\mathfrak{r}_{i,j} = \mathfrak{p}_{i,j}$. The second assertion is verified by noting that when $i = j$, the partial flag $\mathcal{P}_{i,j}$ is the full flag $\mathcal{F}_{i,i}$.

Proposition 2.15. *Consider the K-orbits $Q_{i,j} = K \cdot \mathfrak{b}_{i,j} \subset \mathcal{B}$ and $Q_{\mathfrak{p}_{i,j}} = K \cdot \mathfrak{p}_{i,j} \subset G/P_{i,j}$, with $1 \le i \le j \le n$. Then $\dim(Y_{\mathfrak{b}_{i,j}}) = \dim(Y_{\mathfrak{p}_{i,j}})$ and $\overline{Y_{\mathfrak{b}_{i,j}}} = Y_{\mathfrak{p}_{i,j}}$.*

Proof. By definitions and Remark 2.14, $Y_{\mathfrak{b}_{i,j}}$ is a constructible subset of $Y_{\mathfrak{p}_{i,j}}$. Since $Y_{\mathfrak{p}_{i,j}}$ is closed by Remark 2.11, and irreducible by construction, it suffices to show that $\dim(Y_{\mathfrak{b}_{i,j}}) = \dim(Y_{\mathfrak{p}_{i,j}})$.

We compute the dimension of $Y_{\mathfrak{b}_{i,j}}$ using Eq. (2.18). Since $l(Q_{i,j}) = j - i$, it follows that $\dim Q_{i,j} = \dim \mathcal{B}_{n-1} + j - i$. Since $\dim(\mathcal{B}_{n-1}) = \binom{n-1}{2}$, Eq. (2.18) then implies:

$$\begin{aligned} \dim Y_{\mathfrak{b}_{i,j}} &= \dim \mathfrak{b}_{i,j} + \dim \mathcal{B}_{n-1} + l(Q_{i,j}) = \binom{n+1}{2} + \binom{n-1}{2} + l(Q_{i,j}) \\ &= n^2 - n + 1 + j - i. \end{aligned} \tag{2.20}$$

We now compute the dimension of $Y_{\mathfrak{p}_{i,j}}$. By Eq. (2.18), it follows that

$$\dim Y_{\mathfrak{p}_{i,j}} = \dim \mathfrak{p}_{i,j} + \dim \mathfrak{k} - \dim(\mathfrak{k} \cap \mathfrak{p}_{i,j}). \tag{2.21}$$

Since both $\mathfrak{l}$ and $\mathfrak{u}$ are θ-stable, it follows that $\dim \mathfrak{k} \cap \mathfrak{p}_{i,j} = \dim \mathfrak{k} \cap \mathfrak{l} + \dim \mathfrak{k} \cap \mathfrak{u}$. To compute these dimensions, it is convenient to use the following explicit matrix description of the parabolic subalgebra $\mathfrak{p}_{i,j}$, which follows from Eq. (2.15).

$$\mathfrak{p}_{i,j} = \begin{bmatrix} a_{11} & \cdots & \cdots & a_{1i-1} & a_{1i} & \cdots & a_{1j-1} & \cdots & \cdots & a_{1n-1} & a_{1n} \\ 0 & \ddots & & \vdots & \vdots & * & \vdots & * & * & \vdots & \vdots \\ \vdots & & & a_{i-1i-1} & \vdots & * & \vdots & * & * & a_{i-1n-1} & a_{i-1n} \\ & & & 0 & a_{ii} & \cdots & a_{ij-1} & * & * & a_{in-1} & a_{in} \\ & & & \vdots & \vdots & \ddots & \vdots & * & * & \vdots & \vdots \\ & & & \vdots & a_{ij-1} & \cdots & a_{j-1j-1} & \cdots & \cdots & a_{j-1n-1} & a_{j-1n} \\ & & & 0 & 0 & \cdots & 0 & a_{jj} & \cdots & a_{jn-1} & 0 \\ & & & \vdots & \vdots & & \vdots & 0 & \ddots & \vdots & \vdots \\ \vdots & & & \vdots & 0 & & 0 & 0 & 0 & a_{n-1n-1} & 0 \\ 0 & \cdots & \cdots & 0 & a_{ni} & \cdots & a_{nj-1} & a_{nj} & \cdots & a_{nn-1} & a_{nn} \end{bmatrix}. \tag{2.22}$$

Please cite this article in press as: M. Colarusso, S. Evens, Eigenvalue coincidences and K-orbits, I, J. Algebra (2014), http://dx.doi.org/10.1016/j.jalgebra.2014.08.051

Using (2.22), we observe that $\mathfrak{k} \cap \mathfrak{l} \cong \mathfrak{gl}(1, \mathbb{C})^{n-j+i} \oplus \mathfrak{gl}(j-i, \mathbb{C})$, so that $\dim \mathfrak{k} \cap \mathfrak{l} = n-j+i+(j-i)^2$. Now $\mathfrak{u} = \mathfrak{u} \cap \mathfrak{k} \oplus \mathfrak{u}^{-\theta}$, where $\mathfrak{u}^{-\theta} := \{x \in \mathfrak{u} : \theta(x) = -x\}$. Using (2.22), we see that $\mathfrak{u}^{-\theta}$ has basis $\{E_{n,j}, \dots, E_{n,n-1}, E_{1,n}, \dots, E_{i-1,n}\}$, so $\dim \mathfrak{u}^{-\theta} = n-j+i-1$. Thus, $\dim \mathfrak{u} \cap \mathfrak{k} = \dim \mathfrak{u} - (n-j+i-1)$. Putting these observations together, we obtain

$$\dim \mathfrak{k} \cap \mathfrak{p}_{i,j} = (j-i)^2 + \dim \mathfrak{u} + 1. \tag{2.23}$$

Now

$$\dim \mathfrak{p}_{i,j} = \dim \mathfrak{l} + \dim \mathfrak{u} = (j-i+1)^2 + n - j + i - 1 + \dim \mathfrak{u}$$

(see Eq. (2.14)). Thus, Eq. (2.21) implies that

$$\dim Y_{\mathfrak{p}_{i,j}} = \dim \mathfrak{k} + (j-i+1)^2 + n - j + i - 1 - (j-i)^2 - 1 = n^2 - n + 1 + j - i,$$

which agrees with (2.20), and hence completes the proof. □

Remark 2.16. It follows from Eq. (2.22) that $(\mathfrak{p}_{i,j})_{n-1} := \pi_{n-1,n}(\mathfrak{p}_{i,j})$ is a parabolic subalgebra, where $\pi_{n,n-1} : \mathfrak{g} \to \mathfrak{gl}(n-1, \mathbb{C})$ is the projection $x \mapsto x_{n-1}$. Further, with $l = j - i$, $(\mathfrak{p}_{i,j})_{n-1}$ has Levi decomposition $(\mathfrak{p}_{i,j})_{n-1} = \mathfrak{l}_{n-1} \oplus \mathfrak{u}_{n-1}$ with $\mathfrak{l}_{n-1} = \mathfrak{gl}(1, \mathbb{C})^{n-1-l} \oplus \mathfrak{gl}(l, \mathbb{C})$.

3. The varieties $\mathfrak{g}(l)$

In this section, we prove our main results.

For $x \in \mathfrak{g}$, let $\sigma(x) = \{\lambda_1, \dots, \lambda_n\}$ denote its eigenvalues, where an eigenvalue λ is listed k times if it appears with multiplicity k. Similarly, let $\sigma(x_{n-1}) = \{\mu_1, \dots, \mu_{n-1}\}$ be the eigenvalues of $x_{n-1} \in \mathfrak{gl}(n-1, \mathbb{C})$, again listed with multiplicity. For $i = n-1, n$, let $\mathfrak{h}_i \subset \mathfrak{g}_i := \mathfrak{gl}(i, \mathbb{C})$ be the standard Cartan subalgebra of diagonal matrices. We denote elements of $\mathfrak{h}_{n-1} \times \mathfrak{h}_n$ by (x, y), with $x = (x_1, \dots, x_{n-1}) \in \mathbb{C}^{n-1}$ and $y = (y_1, \dots, y_n) \in \mathbb{C}^n$ the diagonal coordinates of x and y. For $l = 0, \dots, n-1$, we define

$$(\mathfrak{h}_{n-1} \times \mathfrak{h}_n)(\geq l) = \big\{(x, y) : \exists\, 1 \leq i_1 < \cdots < i_l \leq n-1 \text{ with } x_{i_j} = y_{k_j} \\ \text{for some } 1 \leq k_1, \dots, k_l \leq n \text{ with } k_j \neq k_m\big\}.$$

Thus, $(\mathfrak{h}_{n-1} \times \mathfrak{h}_n)(\geq l)$ consists of elements of $\mathfrak{h}_{n-1} \times \mathfrak{h}_n$ with at least l coincidences in the spectrum of x and y counting repetitions. Note that $(\mathfrak{h}_{n-1} \times \mathfrak{h}_n)(\geq l)$ is a closed subvariety of $\mathfrak{h}_{n-1} \times \mathfrak{h}_n$ and is equidimensional of codimension l.

Let $W_i = W_i(\mathfrak{g}_i, \mathfrak{h}_i)$ be the Weyl group of $\mathfrak{g}_i$. Then $W_{n-1} \times W_n$ acts on $(\mathfrak{h}_{n-1} \times \mathfrak{h}_n)(\geq l)$. Consider the finite morphism $p : \mathfrak{h}_{n-1} \times \mathfrak{h}_n \to (\mathfrak{h}_{n-1} \times \mathfrak{h}_n)/(W_{n-1} \times W_n)$. Let $F_i : \mathfrak{h}_i/W_i \to \mathbb{C}^i$ be the Chevalley isomorphism, and let

$$V^{n-1,n} := \mathbb{C}^{n-1} \times \mathbb{C}^n, \quad \text{so that } F_{n-1} \times F_n : (\mathfrak{h}_{n-1} \times \mathfrak{h}_n)/(W_{n-1} \times W_n) \to V^{n-1,n}$$

Please cite this article in press as: M. Colarusso, S. Evens, Eigenvalue coincidences and K-orbits, I, J. Algebra (2014), http://dx.doi.org/10.1016/j.jalgebra.2014.08.051

is an isomorphism. The following varieties play a major role in our study of eigenvalue coincidences.

Definition 3.1. For $l = 0, \dots, n-1$, we let

$$V^{n-1,n}(\geq l) := (F_{n-1} \times F_n)\big((\mathfrak{h}_{n-1} \times \mathfrak{h}_n)(\geq l)/(W_{n-1} \times W_n)\big), \tag{3.1}$$

$$V^{n-1,n}(l) := V^{n-1,n}(\geq l) \setminus V^{n-1,n}(\geq l+1). \tag{3.2}$$

For convenience, we let $V^{n-1,n}(n) = \emptyset$.

Lemma 3.2. *The set $V^{n-1,n}(\geq l)$ is an irreducible closed subvariety of $V^{n-1,n}$ of dimension $2n-1-l$. Further, $V^{n-1,n}(l)$ is open and dense in $V^{n-1,n}(\geq l)$.*

Proof. Indeed, the set $Y := \{(x, y) \in \mathfrak{h}_{n-1} \times \mathfrak{h}_n : x_i = y_i \text{ for } i = 1, \dots, l\}$ is closed and irreducible of dimension $2n-1-l$. The first assertion follows since $(F_{n-1} \times F_n) \circ p$ is a finite morphism and $(F_{n-1} \times F_n) \circ p(Y) = V^{n-1,n}(\geq l)$. The last assertion of the lemma now follows from Eq. (3.2). □

Notation 3.3. We let

$$\mathfrak{g}(\geq l) := \Phi_n^{-1}\big(V^{n-1,n}(\geq l)\big).$$

Remark 3.4. Recall that the quotient morphism $p_i : \mathfrak{g}_i \to \mathfrak{g}_i//GL(i, \mathbb{C}) \cong \mathfrak{h}_i/W_i$ associates to $y \in \mathfrak{g}_i$ its spectrum $\sigma(y)$, and $(F_{n-1} \times F_n) \circ (p_{n-1} \times p_n) = \Phi_n$. It follows that $\mathfrak{g}(\geq l)$ consists of elements of x with at least l coincidences in the spectrum of x and x_{n-1}, counted with multiplicity.

It is routine to check that

$$\mathfrak{g}(l) := \mathfrak{g}(\geq l) \setminus \mathfrak{g}(\geq l+1) = \Phi_n^{-1}\big(V^{n-1,n}(l)\big) \tag{3.3}$$

consists of elements of $\mathfrak{g}$ with exactly l coincidences in the spectrum of x and x_{n-1}, counted with multiplicity.

Proposition 3.5.

(1) *The variety $\mathfrak{g}(\geq l)$ is equidimensional of dimension $n^2 - l$.*
(2) $\overline{\mathfrak{g}(l)} = \mathfrak{g}(\geq l) = \bigcup_{k \geq l} \mathfrak{g}(k)$.

Proof. By Proposition 2.4, the morphism Φ_n is flat. By Proposition III.9.5 and Corollary III.9.6 of [18], the variety $\mathfrak{g}(\geq l)$ is equidimensional of dimension $\dim(V^{n-1,n}(\geq l)) +$

$(n-1)^2$, which gives the first assertion by Lemma 3.2. For the second assertion, by the flatness of Φ_n, Theorem VIII.4.1 of [15], and Lemma 3.2,

$$\overline{\mathfrak{g}(l)} = \overline{\Phi_n^{-1}\big(V^{n-1,n}(l)\big)} = \Phi_n^{-1}\big(\overline{V^{n-1,n}(l)}\big) = \Phi_n^{-1}\big(V^{n-1,n}(\geq l)\big) = \mathfrak{g}(\geq l). \tag{3.4}$$

The remaining equality follows since $V^{n-1,n}(\geq l) = \bigcup_{k\geq l} V^{n-1,n}(k)$. □

We now relate the partitions $\mathfrak{g} = \bigcup \mathfrak{g}(l)$ and $\mathfrak{g} = \bigcup_{Q\subset\mathcal{B}} Y_Q$ (see Remark 2.12).

Theorem 3.6.

(1) *Consider the closed subvarieties $Y_{\mathfrak{p}_{i,j}}$ for $1 \leq i \leq j \leq n$, and let $l = j - i$. Then $Y_{\mathfrak{p}_{i,j}} \subset \mathfrak{g}(\geq n-1-l)$.*
(2) *In particular, if $Q \subset \mathcal{B}$ is a K-orbit with $l(Q) = l$, then $Y_Q \subset \mathfrak{g}(\geq n-1-l)$.*

Proof. The second statement of the theorem follows from the first statement using Propositions 2.9 and 2.15.

To prove the first statement of the theorem, let $\mathfrak{q}$ be a parabolic subalgebra of $\mathfrak{g}$ with $\mathfrak{q} \in Q_{\mathfrak{p}_{i,j}}$, and let $y \in \mathfrak{q}$. We need to show that $\Phi_n(y) \in V^{n-1,n}(\geq n-1-l)$. Since the map Φ_n is K-invariant, it is enough to show that $\Phi_n(x) \in V^{n-1,n}(\geq n-1-l)$ for $x \in \mathfrak{p}_{i,j}$.

We recall that $\Phi_n(x) = (\chi_{n-1}(x_{n-1}), \chi_n(x))$ where $\chi_i : \mathfrak{gl}(i,\mathbb{C}) \to \mathbb{C}^i$ is the adjoint quotient for $i = n-1, n$. For $x \in \mathfrak{p}_{i,j}$, let $x_{\mathfrak{l}}$ be the projection of x onto $\mathfrak{l}$ off of $\mathfrak{u}$. It is well-known that $\chi_n(x) = \chi_n(x_{\mathfrak{l}})$. Using the identification $\mathfrak{l} \cong \mathfrak{gl}(1,\mathbb{C})^{n-1-l} \oplus \mathfrak{gl}(l+1,\mathbb{C})$, we decompose $x_{\mathfrak{l}}$ as $x_{\mathfrak{l}} = x_{\mathfrak{gl}(1)^{n-1-l}} + x_{\mathfrak{gl}(l+1)}$, where $x_{\mathfrak{gl}(1)^{n-1-l}} \in \mathfrak{gl}(1,\mathbb{C})^{n-1-l}$ and $x_{\mathfrak{gl}(l+1)} \in \mathfrak{gl}(l+1,\mathbb{C})$. It follows that the coordinates of $x_{\mathfrak{gl}(1)^{n-1-l}}$ are in the spectrum of x (see (2.22)).

Recall the projection $\pi_{n,n-1} : \mathfrak{g} \to \mathfrak{g}_{n-1}$, $\pi_{n,n-1}(x) = x_{n-1}$. Recall the Levi decomposition $(\mathfrak{p}_{i,j})_{n-1} = \mathfrak{l}_{n-1} \oplus \mathfrak{u}_{n-1}$ of the parabolic subalgebra $(\mathfrak{p}_{i,j})_{n-1}$ of $\mathfrak{gl}(n-1,\mathbb{C})$ from Remark 2.16, and recall that $\mathfrak{l}_{n-1} = \mathfrak{gl}(1,\mathbb{C})^{n-1-l} \oplus \mathfrak{gl}(l,\mathbb{C})$. Thus, $\chi_{n-1}(x_{n-1}) = \chi_{n-1}((x_{n-1})_{\mathfrak{l}_{n-1}})$. We use the decomposition $(x_{n-1})_{\mathfrak{l}_{n-1}} = x_{\mathfrak{gl}(1)^{n-1-l}} + \pi_{l+1,l}(x_{\mathfrak{gl}(l+1)})$, where $\pi_{l+1,l} : \mathfrak{gl}(l+1,\mathbb{C}) \to \mathfrak{gl}(l,\mathbb{C})$ is the usual projection. It now follows easily from Remark 3.4 that $\Phi_n(x) \in V^{n-1,n}(\geq n-1-l)$, since the coordinates of $x_{\mathfrak{gl}(1)^{n-1-l}}$ are eigenvalues both for x and x_{n-1}. □

We now recall and prove our main theorem.

Theorem 3.7. *Consider the locally closed subvariety $\mathfrak{g}(n-1-l)$ for $l = 0, \dots, n-1$. Then the decomposition*

$$\mathfrak{g}(n-1-l) = \bigcup_{l(Q)=l} Y_Q \cap \mathfrak{g}(n-1-l), \tag{3.5}$$

is the irreducible component decomposition of the variety $\mathfrak{g}(n-1-l)$*, where the union is taken over all* K*-orbits* Q *of length* l *in* $\mathcal{B}$ *(cf. Theorem (1.1)).*

In fact, for $1 \leq i \leq j \leq n$ *with* $j - i = l$*, we have*

$$Y_{\mathfrak{b}_{i,j}} \cap \mathfrak{g}(n-1-l) = Y_{\mathfrak{p}_{i,j}} \cap \mathfrak{g}(n-1-l),$$

so that

$$\mathfrak{g}(n-1-l) = \bigcup_{j-i=l} Y_{\mathfrak{p}_{i,j}} \cap \mathfrak{g}(n-1-l). \tag{3.6}$$

Proof. We first claim that if $l(Q) = l$, then $Y_Q \cap \mathfrak{g}(n-1-l)$ is non-empty. By Theorem 3.6, $Y_Q \subset \mathfrak{g}(\geq n-1-l)$. Thus, if $Y_Q \cap \mathfrak{g}(n-1-l)$ were empty, then $Y_Q \subset \mathfrak{g}(\geq n-l)$. Hence, by part (1) of Proposition 3.5, $\dim(Y_Q) \leq n^2-n+l$. By Eq. (2.20), $\dim(Y_Q) = n^2-n+l+1$. This contradiction verifies the claim.

It follows from Eq. (3.3) that $\mathfrak{g}(n-1-l)$ is open in $\mathfrak{g}(\geq n-1-l)$. Thus, $Y_Q \cap \mathfrak{g}(n-1-l)$ is a non-empty Zariski open subset of Y_Q, which is irreducible since Y_Q is irreducible.

Now we claim that

$$Y_Q \cap \mathfrak{g}(n-1-l) = \overline{Y_Q} \cap \mathfrak{g}(n-1-l), \tag{3.7}$$

so that $Y_Q \cap \mathfrak{g}(n-1-l)$ is closed in $\mathfrak{g}(n-1-l)$. By Lemma 2.13, $\overline{Y_Q} = \bigcup_{Q' \subset \overline{Q}} Y_{Q'}$. Hence, if (3.7) were not an equality, there would be Q' with $l(Q') < l(Q)$ and $Y_{Q'} \cap \mathfrak{g}(n-1-l)$ nonempty. This contradicts Theorem 3.6, which asserts that $Y_{Q'} \subset \mathfrak{g}(\geq n-l)$, and hence verifies the claim. It follows that $Y_Q \cap \mathfrak{g}(n-1-l)$ is an irreducible, closed subvariety of $\mathfrak{g}(n-1-l)$ of dimension $\dim Y_Q = \dim \mathfrak{g}(n-1-l)$. Thus, $Y_Q \cap \mathfrak{g}(n-1-l)$ is an irreducible component of $\mathfrak{g}(n-1-l)$.

Since $l(Q) = l$, Proposition 2.9 implies that $Q = Q_{i,j}$ for some $i \leq j$ with $j - i = l$. Then by Proposition 2.15 and Eq. (3.7),

$$Y_{\mathfrak{b}_{i,j}} \cap \mathfrak{g}(n-1-l) = Y_{\mathfrak{p}_{i,j}} \cap \mathfrak{g}(n-1-l). \tag{3.8}$$

Let Z be an irreducible component of $\mathfrak{g}(n-1-l)$. The proof will be complete once we show that $Z = Y_{\mathfrak{p}_{i,j}} \cap \mathfrak{g}(n-1-l)$ for some i, j with $j - i = l$. To do this, consider the nonempty open set

$$U := \{x \in \mathfrak{g} : \ x_{n-1} \text{ is regular semisimple}\}.$$

Let $\widetilde{U}(n-1-l) := \mathfrak{g}(n-1-l) \cap U$.

Since $\Phi_n : \mathfrak{g} \to V^{n-1,n}$ is surjective (by Remark 2.1), it follows that $\widetilde{U}(n-1-l)$ is a nonempty Zariski open set of $\mathfrak{g}(n-1-l)$. By part (2) of Proposition 2.4 and Exercise III.9.1 of [18], $\Phi_n(U) \subset V^{n-1,n}$ is open. Thus, $V^{n-1,n}(n-1-l) \setminus \Phi_n(U)$ is a proper, closed subvariety of $V^{n-1,n}(n-1-l)$ and therefore has positive codimension

Please cite this article in press as: M. Colarusso, S. Evens, Eigenvalue coincidences and K-orbits, I, J. Algebra (2014), http://dx.doi.org/10.1016/j.jalgebra.2014.08.051

by Lemma 3.2. It follows by part (2) of Proposition 2.4 and Corollary III.9.6 of [18] that $\mathfrak{g}(n-1-l)\setminus\widetilde{U}(n-1-l) = \Phi_n^{-1}(V^{n-1,n}(n-1-l)\setminus\Phi_n(U))$ is a proper, closed subvariety of $\mathfrak{g}(n-1-l)$ of positive codimension. Since $\mathfrak{g}(n-1-l)$ is equidimensional, it follows that $Z\cap\widetilde{U}(n-1-l)$ is nonempty. Thus, it suffices to show that

$$\widetilde{U}(n-1-l) \subset \bigcup_{j-i=l} Y_{\mathfrak{p}_{i,j}} \cap \mathfrak{g}(n-1-l). \tag{3.9}$$

To prove Eq. (3.9), we consider the following subvariety of $\widetilde{U}(n-1-l)$:

$$\begin{aligned}\Xi = \big\{x \in \tilde{U}(n-1-l) :\ x_{n-1} = \operatorname{diag}[h_1,\dots,h_{n-1}],\ \text{and}\\ \sigma(x_{n-1})\cap\sigma(x) = \{h_1,\dots,h_{n-1-l}\}\big\}.\end{aligned} \tag{3.10}$$

It is easy to check that any element of $\widetilde{U}(n-1-l)$ is K-conjugate to an element in Ξ. By a linear algebra calculation from Proposition 5.9 of [8] (see Eqs. (5-11) and (5-12) in [8]), elements of Ξ are matrices of the form

$$\begin{bmatrix} h_1 & 0 & \cdots & 0 & y_1 \\ 0 & h_2 & \ddots & \vdots & \vdots \\ \vdots & & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & h_{n-1} & y_{n-1} \\ z_1 & \cdots & \cdots & z_{n-1} & w \end{bmatrix}, \tag{3.11}$$

with $h_i \neq h_j$ for $i \neq j$ and satisfying the equations:

$$\begin{aligned} z_i y_i &= 0 \quad \text{for } 1 \le i \le n-1-l \\ z_i y_i &\in \mathbb{C}^\times \quad \text{for } n-l \le i \le n-1. \end{aligned} \tag{3.12}$$

Since the varieties $Y_{\mathfrak{p}_{i,j}} \cap \mathfrak{g}(n-1-l)$ are K-stable, it suffices to prove

$$\Xi \subset \bigcup_{j-i=l} Y_{\mathfrak{p}_{i,j}} \cap \mathfrak{g}(n-1-l). \tag{3.13}$$

To prove (3.13), we need to understand the irreducible components of Ξ. For $i = 1,\dots,n-1-l$, we define an index j_i which takes on two values $j_i = U$ (U for upper) or $j_i = L$ (L for lower). Consider the subvariety $\Xi_{j_1,\dots,j_{n-1-l}} \subset \Xi$ defined by:

$$\Xi_{j_1,\dots,j_{n-1-l}} := \{x \in \Xi : z_i = 0 \text{ if } j_i = U, y_i = 0 \text{ if } j_i = L\}. \tag{3.14}$$

Then

$$\Xi = \bigcup_{j_i = U,\, L} \Xi_{j_1,\dots,j_{n-1-l}} \tag{3.15}$$

is the irreducible component decomposition of Ξ.

Please cite this article in press as: M. Colarusso, S. Evens, Eigenvalue coincidences and K-orbits, I, J. Algebra (2014), http://dx.doi.org/10.1016/j.jalgebra.2014.08.051

We now consider the irreducible variety $\Xi_{j_1,\ldots,j_{n-1-l}}$. Suppose that for the subsequence $1 \leq i_1 < \cdots < i_{k-1} \leq n-1-l$ we have $j_{i_1} = j_{i_2} = \cdots = j_{i_{k-1}} = U$ and that for the complementary subsequence $i_k < \cdots < i_{n-1-l}$ we have $j_{i_k} = j_{i_{k+1}} = \cdots = j_{i_{n-1-l}} = L$. Then a simple computation with flags shows that elements of the variety $\Xi_{j_1,\ldots,j_{n-1-l}}$ stabilize the $n-l$-step partial flag in $\mathbb{C}^n$

$$e_{i_1} \subset e_{i_2} \subset \cdots \subset e_{i_{k-1}} \subset \underbrace{e_{n-l}, \ldots, e_{n-1}, e_n}_{k} \subset e_{i_k} \subset e_{i_{k+1}} \subset \cdots \subset e_{i_{n-1-l}}. \tag{3.16}$$

(If $l=0$ the partial flag in (3.16) is a full flag with e_n in the k-th position.) It is easy to see that there is an element of K that maps the partial flag in Eq. (3.16) to the partial flag $\mathcal{P}_{k,k+l}$ in Eq. (2.15):

$$\mathcal{P}_{k,k+l} = (e_1 \subset e_2 \subset \cdots \subset e_{k-1} \subset \underbrace{e_k, \ldots, e_{k+l-1}, e_n}_{k} \subset e_{k+l} \subset \cdots \subset e_{n-1}). \tag{3.17}$$

(If $l=0$ the partial flag $\mathcal{P}_{k,k+l}$ is the full flag $\mathcal{F}_{k,k}$ (see Eq. (2.9)).) Thus, $\Xi_{j_1,\ldots,j_{n-1-l}} \subset Y_{\mathfrak{p}_{k,k+l}} \cap \mathfrak{g}(n-1-l)$. Eq. (3.15) then implies that $\Xi \subset \bigcup_{j-i=l} Y_{\mathfrak{p}_{i,j}} \cap \mathfrak{g}(n-1-l)$. $\square$

Using Theorem 3.7, we can obtain the irreducible component decomposition of the variety $\mathfrak{g}(\geq n-1-l)$ for any $l = 0, \ldots, n-1$.

Corollary 3.8. *The irreducible component decomposition of the variety* $\mathfrak{g}(\geq n-1-l)$ *is*

$$\mathfrak{g}(\geq n-1-l) = \bigcup_{j-i=l} Y_{\mathfrak{p}_{i,j}} = \bigcup_{l(Q)=l} \overline{Y_Q}. \tag{3.18}$$

Proof. Taking Zariski closures in Eq. (3.6), we obtain

$$\overline{\mathfrak{g}(n-1-l)} = \bigcup_{j-i=l} \overline{Y_{\mathfrak{p}_{i,j}} \cap \mathfrak{g}(n-1-l)} \tag{3.19}$$

is the irreducible component decomposition of the variety $\overline{\mathfrak{g}(n-1-l)}$. By Proposition 3.5, $\overline{\mathfrak{g}(n-1-l)} = \mathfrak{g}(\geq n-1-l)$, and by Theorem 3.6, $Y_{\mathfrak{p}_{i,j}} \subset \mathfrak{g}(\geq n-1-l)$. Hence $Y_{\mathfrak{p}_{i,j}} \cap \mathfrak{g}(n-1-l)$ is Zariski open in the irreducible variety $Y_{\mathfrak{p}_{i,j}}$, and is nonempty by Theorem 3.7. Therefore $\overline{Y_{\mathfrak{p}_{i,j}} \cap \mathfrak{g}(n-1-l)} = Y_{\mathfrak{p}_{i,j}}$. Eq. (3.18) now follows from Eq. (3.19) and Proposition 2.15. $\square$

Theorem 3.7 says something of particular interest to linear algebraists in the case where $l=0$. It states that the variety $\mathfrak{g}(n-1)$ consisting of elements $x \in \mathfrak{g}$ where the number of coincidences in the spectrum between x_{n-1} and x is maximal can be described in terms of closed K-orbits on $\mathcal{B}$, which are the K-orbits Q with $l(Q)=0$. It thus connects the most degenerate case of spectral coincidences to the simplest K-orbits on $\mathcal{B}$. More precisely, we have:

Corollary 3.9. *The irreducible component decomposition of the variety* $\mathfrak{g}(n-1)$ *is*

$$\mathfrak{g}(n-1) = \bigcup_{l(Q)=0} Y_Q.$$

Using Corollary 3.9 and Theorem 2.3, we obtain a precise description of the irreducible components of the variety SN_n introduced in Eq. (2.4).

Proposition 3.10. *Let* $\mathfrak{n}_{i,i} = [\mathfrak{b}_{i,i}, \mathfrak{b}_{i,i}]$ *be the nilradical of the Borel subalgebra* $\mathfrak{b}_{i,i}$ *(see Eq. 2.9). The irreducible component decomposition of* SN_n *is given by:*

$$SN_n = \bigcup_{i=1}^{n} \mathrm{Ad}(K)\mathfrak{n}_{i,i}, \tag{3.20}$$

where $\mathrm{Ad}(K)\mathfrak{n}_{i,i} \subset \mathfrak{g}$ *denotes the* K*-saturation of* $\mathfrak{n}_{i,i}$ *in* $\mathfrak{g}$*.*

Proof. We first show that $\mathrm{Ad}(K)\mathfrak{n}_{i,i}$ is an irreducible component of SN_n for $i = 1, \ldots, n$. A simple computation using the flag $\mathcal{F}_{i,i}$ in Eq. (2.9) shows that $\mathfrak{n}_{i,i} \subset SN_n$. Since SN_n is K-stable, it follows that $\mathrm{Ad}(K)\mathfrak{n}_{i,i} \subset SN_n$.

Recall the Grothendieck resolution $\widetilde{\mathfrak{g}} = \{(x, \mathfrak{b}) : x \in \mathfrak{b}\} \subset \mathfrak{g} \times \mathcal{B}$ and the morphisms $\pi : \widetilde{\mathfrak{g}} \to \mathcal{B}$, $\pi(x, \mathfrak{b}) = \mathfrak{b}$ and $\mu : \widetilde{\mathfrak{g}} \to \mathfrak{g}$, $\mu(x, \mathfrak{b}) = x$. Let $Q_{i,i} = K \cdot \mathfrak{b}_{i,i} \subset \mathcal{B}$ be the K-orbit through $\mathfrak{b}_{i,i}$. Corollary 3.1.33 of [7] gives a G-equivariant isomorphism $\widetilde{\mathfrak{g}} \cong G \times_{B_{i,i}} \mathfrak{b}_{i,i}$. Under this isomorphism $\pi^{-1}(Q_{i,i})$ is identified with the closed subvariety $K \times_{K\cap B_{i,i}} \mathfrak{b}_{i,i} \subset G \times_{B_{i,i}} \mathfrak{b}_{i,i}$. The closed subvariety $K \times_{K\cap B_{i,i}} \mathfrak{n}_{i,i} \subset K \times_{K\cap B_{i,i}} \mathfrak{b}_{i,i}$ maps surjectively under μ to $\mathrm{Ad}(K)\mathfrak{n}_{i,i}$. Since μ is proper, $\mathrm{Ad}(K)\mathfrak{n}_{i,i}$ is closed and irreducible. We also note that the restriction of μ to $K \times_{K\cap B_{i,i}} \mathfrak{n}_{i,i}$ generically has finite fibers (Proposition 3.2.14 of [7]). Thus, the same reasoning that we used in Eq. (2.20) shows that

$$\dim \mathrm{Ad}(K)\mathfrak{n}_{i,i} = \dim(K \times_{K\cap B_{i,i}} \mathfrak{n}_{i,i}) = \dim(Y_{Q_{i,i}}) - \mathrm{rk}(\mathfrak{g}) = d_n, \tag{3.21}$$

where $\mathrm{rk}(\mathfrak{g})$ denotes the rank of $\mathfrak{g}$. Thus, by Theorem 2.3, $\mathrm{Ad}(K)\mathfrak{n}_{i,i}$ is an irreducible component of SN_n.

We now show that every irreducible component of SN_n is of the form $\mathrm{Ad}(K)\mathfrak{n}_{i,i}$ for some $i = 1, \ldots, n$. It follows from definitions that $SN_n \subset \mathfrak{g}(n-1) \cap \mathcal{N}$, where $\mathcal{N} \subset \mathfrak{g}$ is the nilpotent cone in $\mathfrak{g}$. Thus, if $\mathfrak{X}$ is an irreducible component of SN_n, then $\mathfrak{X} \subset \mathrm{Ad}(K)\mathfrak{n}_{i,i}$ by Corollary 3.9. But then $\mathfrak{X} = \mathrm{Ad}(K)\mathfrak{n}_{i,i}$ by Eq. (3.21) and Theorem 2.3. □

We say that an element $x \in \mathfrak{g}$ is *n-strongly regular* if the set

$$dJ_{GZ,n}(x) := \left\{ df_{i,j}(x) : i = n-1, n; j = 1, \ldots, i \right\}$$

is linearly independent in the cotangent space $T_x^*(\mathfrak{g})$ of $\mathfrak{g}$ at x. We can use Proposition 3.10 and n-strongly regular elements to further study the ideal I_n (see Notation 2.2).

Proposition 3.11. *The ideal I_n is radical if and only if $n \leq 2$.*

Proof. The assertion is clear for $n = 1$, and we assume $n \geq 2$ in the sequel. By Theorem 18.15(a) of [10], the ideal I_n is radical if and only if the set $dJ_{GZ,n}$ is linearly independent on a dense open set of each irreducible component of $SN_n = V(I_n)$. It follows that I_n is radical if and only if each irreducible component of SN_n contains n-strongly regular elements. Let $\mathfrak{n}_+ = [\mathfrak{b}_+, \mathfrak{b}_+]$ and $\mathfrak{n}_- = [\mathfrak{b}_-, \mathfrak{b}_-]$ be the strictly upper and lower triangular matrices, respectively. By Proposition 3.10 above, SN_n has exactly n irreducible components. It follows from the discussion after Eq. (2.9) that two of them are $\mathrm{Ad}(K)\mathfrak{n}_+$ and $\mathrm{Ad}(K)\mathfrak{n}_-$. We claim that $\mathrm{Ad}(K)\mathfrak{n}_+$ and $\mathrm{Ad}(K)\mathfrak{n}_-$ are the only irreducible components of SN_n which contain n-strongly regular elements. To see this, we view $\mathfrak{g}_{n-1}$ as the top lefthand corner of $\mathfrak{g}$. It follows from a well-known result of Kostant (Theorem 9 of [23]) that $x_i \in \mathfrak{g}_i$ is regular if and only if the set $\{df_{i,j}(x) : j = 1, \dots, i\}$ is linearly independent. If $x_i \in \mathfrak{g}_i$ is regular, and we identify $T_x^*(\mathfrak{g}) = \mathfrak{g}^*$ with $\mathfrak{g}$ using the trace form $\langle\langle x, y\rangle\rangle = tr(xy)$, then

$$\mathrm{span}\{df_{i,j}(x) : j = 1, \dots, i\} = \mathfrak{z}_{\mathfrak{g}_i}(x_i),$$

where $\mathfrak{z}_{\mathfrak{g}_i}(x_i)$ denotes the centralizer of x_i in $\mathfrak{g}_i$. Thus, it follows that $x \in \mathfrak{g}$ is n-strongly regular if and only if x satisfies the following two conditions:

$$\begin{aligned} &(1) \quad x \in \mathfrak{g} \text{ and } x_{n-1} \in \mathfrak{g}_{n-1} \text{ are regular;} \quad \text{and} \\ &(2) \quad \mathfrak{z}_{\mathfrak{g}_{n-1}}(x_{n-1}) \cap \mathfrak{z}_{\mathfrak{g}}(x) = 0. \end{aligned} \tag{3.22}$$

Propositions 3.10 and 3.11 of [6] imply that the only components of SN_n which contain elements satisfying the conditions in (3.22) are $\mathrm{Ad}(K)\mathfrak{n}_+$ and $\mathrm{Ad}(K)\mathfrak{n}_-$. The assertion follows. □

See Remark 1.1 of [29] for a related observation, which follows also from the analysis proving our claim.

References

[1] Ehud Moshe Baruch, A proof of Kirillov's conjecture, Ann. of Math. (2) 158 (1) (2003) 207–252.
[2] W. Borho, J.-L. Brylinski, Differential operators on homogeneous spaces. III. Characteristic varieties of Harish-Chandra modules and of primitive ideals, Invent. Math. 80 (1) (1985) 1–68.
[3] Michel Brion, Aloysius G. Helminck, On orbit closures of symmetric subgroups in flag varieties, Canad. J. Math. 52 (2) (2000) 265–292.
[4] M. Brion, Classification des espaces homogènes sphériques, Compos. Math. 63 (2) (1987) 189–208.
[5] Mark Colarusso, Sam Evens, The Gelfand–Zeitlin integrable system and K-orbits on the flag variety, in: Symmetry: Representation Theory and Its Applications, in: Progr. Math., Birkhäuser, Boston, 2014, 34 pages.
[6] Mark Colarusso, Sam Evens, K-orbits on the flag variety and strongly regular nilpotent matrices, Selecta Math. (N.S.) 18 (1) (2012) 159–177.
[7] Neil Chriss, Victor Ginzburg, Representation Theory and Complex Geometry, Birkhäuser Boston Inc., Boston, MA, 1997.
[8] Mark Colarusso, The orbit structure of the Gelfand–Zeitlin group on $n \times n$ matrices, Pacific J. Math. 250 (1) (2011) 109–138.

Please cite this article in press as: M. Colarusso, S. Evens, Eigenvalue coincidences and K-orbits, I, J. Algebra (2014), http://dx.doi.org/10.1016/j.jalgebra.2014.08.051

[9] Yu.A. Drozd, V.M. Futorny, S.A. Ovsienko, Harish-Chandra subalgebras and Gel'fand–Zetlin modules, in: Finite-Dimensional Algebras and Related Topics, Ottawa, ON, 1992, in: NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 424, Kluwer Acad. Publ., Dordrecht, 1994, pp. 79–93.
[10] David Eisenbud, Commutative Algebra, Grad. Texts in Math., vol. 150, Springer-Verlag, New York, 1995, with a view toward algebraic geometry.
[11] Vyacheslav Futorny, Serge Ovsienko, Kostant's theorem for special filtered algebras, Bull. Lond. Math. Soc. 37 (2) (2005) 187–199.
[12] Vyacheslav Futorny, Serge Ovsienko, Fibers of characters in Gelfand–Tsetlin categories, Trans. Amer. Math. Soc. 366 (8) (2014) 4173–4208.
[13] Vyacheslav Futorny, Harish-Chandra categories and Kostant's theorem, Resenhas 6 (2–3) (2004) 177–186.
[14] I.M. Gel'fand, M.L. Cetlin, Finite-dimensional representations of groups of orthogonal matrices, Dokl. Akad. Nauk SSSR (N.S.) 71 (1950) 1017–1020.
[15] Alexander Grothendieck, Revêtements Étales et groupe fondamental (SGA 1), in: Séminaire de Géométrie Algébrique du Bois Marie, 1960–1961, Augmenté de deux exposés de Michèle Raynaud, in: Doc. Math. (Paris) [Mathematical Documents (Paris)], vol. 3, Société Mathématique de France, Paris, 2003, with two exposés by Michèle Raynaud.
[16] V. Guillemin, S. Sternberg, The Gel'fand–Cetlin system and quantization of the complex flag manifolds, J. Funct. Anal. 52 (1) (1983) 106–128.
[17] Roe Goodman, Nolan R. Wallach, Representations and Invariants of the Classical Groups, Encyclopedia Math. Appl., vol. 68, Cambridge University Press, Cambridge, 1998.
[18] Robin Hartshorne, Algebraic Geometry, Grad. Texts in Math., vol. 52, Springer-Verlag, New York, 1977.
[19] Takashi Hashimoto, B_{n-1}-orbits on the flag variety GL_n/B_n, Geom. Dedicata 105 (2004) 13–27.
[20] Roger A. Horn, Charles R. Johnson, Matrix Analysis, Cambridge University Press, Cambridge, 1985.
[21] James E. Humphreys, Linear Algebraic Groups, Grad. Texts in Math., vol. 21, Springer-Verlag, New York, 1975.
[22] Friedrich Knop, Weylgruppe und Momentabbildung, Invent. Math. 99 (1) (1990) 1–23.
[23] Bertram Kostant, Lie group representations on polynomial rings, Amer. J. Math. 85 (1963) 327–404.
[24] Bertram Kostant, Nolan Wallach, Gelfand–Zeitlin theory from the perspective of classical mechanics. I, in: Studies in Lie Theory, in: Progr. Math., vol. 243, Birkhäuser Boston, Boston, MA, 2006, pp. 319–364.
[25] Bertram Kostant, Nolan Wallach, Gelfand–Zeitlin theory from the perspective of classical mechanics. II, in: The Unity of Mathematics, in: Progr. Math., vol. 244, Birkhäuser Boston, Boston, MA, 2006, pp. 387–420.
[26] Toshihiko Matsuki, The orbits of affine symmetric spaces under the action of minimal parabolic subgroups, J. Math. Soc. Japan 31 (2) (1979) 331–357.
[27] Hideyuki Matsumura, Commutative Ring Theory, Cambridge Stud. Adv. Math., vol. 8, Cambridge University Press, Cambridge, 1986, translated from the Japanese by M. Reid.
[28] Toshihiko Matsuki, Toshio Ōshima, Embeddings of discrete series into principal series, in: The Orbit Method in Representation Theory, Copenhagen, 1988, in: Progr. Math., vol. 82, Birkhäuser Boston, Boston, MA, 1990, pp. 147–175.
[29] Serge Ovsienko, Strongly nilpotent matrices and Gelfand–Zetlin modules, in: Special Issue on Linear Algebra Methods in Representation Theory, Linear Algebra Appl. 365 (2003) 349–367.
[30] D.I. Panyushev, Complexity and rank of homogeneous spaces, Geom. Dedicata 34 (3) (1990) 249–269.
[31] Beresford Parlett, Gilbert Strang, Matrices with prescribed Ritz values, Linear Algebra Appl. 428 (7) (2008) 1725–1739.
[32] Ivan Penkov, Vera Serganova, Gregg Zuckerman, On the existence of $(\mathfrak{g}, \mathfrak{l})$-modules of finite type, Duke Math. J. 125 (2) (2004) 329–349.
[33] R.W. Richardson, T.A. Springer, The Bruhat order on symmetric varieties, Geom. Dedicata 35 (1–3) (1990) 389–436.
[34] Jean-Pierre Serre, Local Algebra, Springer Monogr. Math., Springer-Verlag, Berlin, 2000, translated from the French by CheeWhye Chin and revised by the author.
[35] É.A. Vinberg, B.N. Kimel'fel'd, Homogeneous domains on flag manifolds and spherical subsets of semisimple Lie groups, Funktsional. Anal. i Prilozhen. 12 (3) (1978) 12–19, 96.
[36] Atsuko Yamamoto, Orbits in the flag variety and images of the moment map for classical groups. I, Represent. Theory 1 (1997) 329–404 (electronic).

Please cite this article in press as: M. Colarusso, S. Evens, Eigenvalue coincidences and K-orbits, I, J. Algebra (2014), http://dx.doi.org/10.1016/j.jalgebra.2014.08.051