DESCRIPTION OF PROPOSED RESEARCH

MARK COLARUSSO

1. Overview

This proposal discusses three research projects in areas of geometric and combinatorial Lie theory. The first project two projects, which are joint with Sam Evens, concern the application of the theory of Gelfand-Zeitlin (GZ) integrable systems to problems in representation theory and Poisson Lie theory. The theory of integrable systems has long had important applications in Lie theory ([Kos79, Eti07, ES01, GS83, Har06, Kos09, KW06a, KW06b, Col11, CE10, CE12]). The various GZ integrable systems shed light on the geometric construction of representations using the theory of quantization [GS83, KW06a, KW06b], the geometry of Poisson Lie groups [AM07, FR96, GY09], properties of quantum algebras [MT00, DEFK10, AR12], and the geometry of matrices with complicated spectral properties [PS08, Ovs03, SP09]. All of these areas are investigated in this proposal. The third project develops a new method of computing generalized Littlewood-Richardson coefficients for arbitrary representations of classical groups using the theory of dual pairs and Harish-Chandra modules. This project is joint work with Jeb Willenbring.

The first GZ system was developed by Guillemin and Sternberg on the dual to the Lie algebra of the unitary group $\mathfrak{u}(n,\mathbb{C})^*$ [GS83]. In [GS83], the authors give a geometric construction of the classical Gelfand-Zeitlin basis for irreducible representations of $U(n,\mathbb{C})$ [GC50b, GC50a]. This is achieved by constructing a geometric quantization of an integral coadjoint orbit in $\mathfrak{u}(n,\mathbb{C})^*$ using the Lagrangian foliation of the orbit induced by the GZ integrable system. In 2006, Kostant and Wallach developed a complexified version of this integrable system for $\mathfrak{gl}(n,\mathbb{C})$ [KW06a, KW06b]. In [Col09], we developed an analogous system on the complex orthogonal Lie algebra $\mathfrak{so}(n,\mathbb{C})$. We briefly review the construction of the complex GZ systems.

Let $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{C})$, $\mathfrak{so}(n,\mathbb{C})$. We view $\mathfrak{g} \cong \mathfrak{g}^*$ as a Poisson manifold with the Lie-Poisson structure. We embed $\mathfrak{g}_i := \mathfrak{gl}(i,\mathbb{C}), \mathfrak{so}(i,\mathbb{C})$ into $\mathfrak{g}$ in the top lefthand corner for $i = 1,\ldots,n-1$. (Here we think of $\mathfrak{so}(n,\mathbb{C})$ as $n\times n$ skew-symmetric matrices.) We let $G_i \cong GL(i,\mathbb{C})$, $SO(i,\mathbb{C})$ be the corresponding algebraic subgroup of $G = GL(n,\mathbb{C})$, $SO(n,\mathbb{C})$ respectively. Let $\mathbb{C}[\mathfrak{g}]$ denote the algebra of polynomials on $\mathfrak{g}$. The subalgebra $J(\mathfrak{g})$ of $\mathbb{C}[\mathfrak{g}]$ generated by the G_i-adjoint invariant polynomials $\mathbb{C}[\mathfrak{g}_i]^{G_i}$ for $i = 1,\ldots,n$ is maximal Poisson commutative and is a polynomial algebra in $\frac{\dim G - \mathrm{rk}(\mathfrak{g})}{2} + \mathrm{rk}(\mathfrak{g})$ generators:

$$J(\mathfrak{g}) \cong \mathbb{C}[\mathfrak{g}_1]^{G_1} \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathbb{C}[\mathfrak{g}]^{G}. \tag{1.1}$$

The GZ integrable system on $\mathfrak{g}$ is then given by choosing a specific set of algebraically independent generators for $J(\mathfrak{g})$ (see Section 2.1).

Throughout the proposal, we will make heavy use of the following notation. Let $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$ and let $\mathfrak{t} \subset U(\mathfrak{g})$ be an associative subalgebra. We say a module M for $U(\mathfrak{g})$ is a $(U(\mathfrak{g}), \mathfrak{t})$-module if the subalgebra $\mathfrak{t}$ acts *locally finitely* on M. One of the main goals of this proposal is to construct certain $(U(\mathfrak{g}), \mathfrak{t})$-modules by quantizing the GZ integrable system. The maximal Poisson commutative algebra $J(\mathfrak{g}) \subset \mathbb{C}[\mathfrak{g}]$ in Equation 1.1 quantizes to a maximal abelian subalgebra $\Gamma \subset U(\mathfrak{g})$. The algebra Γ is the so-called Gelfand-Zeitlin subalgebra of $U(\mathfrak{g})$:

$$\Gamma \cong Z(\mathfrak{g}_1) \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} Z(\mathfrak{g}), \tag{1.2}$$

where $Z(\mathfrak{g}_i) \subset U(\mathfrak{g}_i)$ denotes the centre of $U(\mathfrak{g}_i)$ for $i = 1, \ldots, n$. It appears that the quantum version of the complex GZ system is the category of $(U(\mathfrak{g}), \Gamma)$-modules studied by Drozd, Futorny, Ovsienko, and others [DFO94, FOS11, FO07, Ovs03, Ovs02, MO98, Kho05, Ram12]. These are often called Gelfand-Zeitlin modules (GZ modules) in the literature. It is a theorem of Futorny and Ovsienko, Corollary 5.3, [FO14] that any Γ-type occurs in a GZ module with finite multiplicity, i.e. GZ modules are necessarily Γ-admissible. Thus, $(U(\mathfrak{g}), \Gamma)$-modules are a generalization of the generalized Harish-Chandra modules, which have been studied extensively by Zuckerman, Penkov, and Serganova [PZ04b, PZ12, PZ04a, PZ07, PSZ04, PS12].

An often studied example of generalized Harish-Chandra modules are admissible $(U(\mathfrak{g}), \mathfrak{h})$-modules where $\mathfrak{h}$ is a Cartan subalgebra $\mathfrak{g}$, i.e. generalized weight modules. Irreducible, admissible $(U(\mathfrak{g}), \mathfrak{h})$-modules are simply $\mathfrak{h}$-weight modules with finite dimensional $\mathfrak{h}$-weight spaces. These have been classified for any complex reductive Lie algebra by Mathieu building on work of Fernando [Mat00],[Fer90]. For $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$, it is easy to see that any simple, admissible $(U(\mathfrak{g}), \mathfrak{h})$-module is a $(U(\mathfrak{g}), \Gamma)$-module. This follows from that fact that the standard diagonal Cartan subalgebra $\mathfrak{h} \subset \Gamma$. Moreover, any irreducible GZ module for $U(\mathfrak{gl}(n, \mathbb{C}))$ is easily seen to be an $\mathfrak{h}$-weight module; however its weight spaces are in general infinite dimensional. Thus, the category of GZ modules for $\mathfrak{gl}(n, \mathbb{C})$ can be viewed as a natural generalization of the category of admissible $(U(\mathfrak{g}), \mathfrak{h})$-modules. The study of GZ modules for $\mathfrak{gl}(n, \mathbb{C})$ will shed light on $\mathfrak{h}$-weight modules with infinite dimensional weight spaces about which little is known. In the orthogonal setting, the GZ algebra Γ does not contain a Cartan subalgebra, and the category of GZ modules appears to be an entirely new category of $U(\mathfrak{so}(n, \mathbb{C}))$-modules with a locally a finite action of the centre of $U(\mathfrak{so}(n, \mathbb{C}))$.

Futorny and his collaborators have studied GZ modules from the perspective of combinatorics and non-commutative algebra [DFO94, FOS11, FGRa, FGRb]. Futorny, Grantcharov, and Ramirez have obtained a classification of irreducible "generic" GZ modules for $\mathfrak{gl}(n, \mathbb{C})$ from a combinatorial point of view in terms of so-called "generic " GZ tableaux [FGRa]. These new results are very interesting, but involve some difficult explicit combinatorial computations. Beyond the generic cases, very few general results exist and a complete classification is still a long way off. In the orthogonal setting even less is known with only the most generic examples computed using the orthogonal analogue of GZ tableaux in [Maz01]. We hope our approach to studying GZ modules will unify the different combinatorial and algebraic methods that have been employed so far under one framework and eventually provide a complete classification of GZ modules for both orthogonal and general linear Lie algebras.

In more detail, we plan to use the theory of $\mathcal{D}$-modules and deformation quantization along with our ongoing study of the geometry of the GZ system [Col11, Col09, CE10, CE12, CEa] to construct GZ modules geometrically. In the philosophy of quantization, irreducible representations with fixed central character correspond to Lagranian submanifolds of adjoint orbits. In Section 2.1, we completely describe the Lagrangian foliation of $\mathfrak{gl}(n,\mathbb{C})$ given by the GZ integrable system. We also mention what is understood about the Lagrangian foliation in the orthogonal case (see Remark 2.5).

Ideally, one would like a geometric construction of $(U(\mathfrak{g}),\Gamma)$-modules analogous to the Beilinson-Bernstein classification of $(U(\mathfrak{g}),K)$-modules with fixed (anti-dominant) central character, where K is a symmetric subgroup of G [BB81, Mil93, Sch05, Sch91]. One of the major problems with this type of construction in the Gelfand-Zeitlin situation is that the algebra Γ does not integrate to an algebraic group that acts on the flag variety of $\mathfrak{g}$ with finitely many orbits. However, we find an unexpected connection between the geometry of the GZ system and of orbits of a symmetric subgroup K of $GL(n,\mathbb{C}), SO(n,\mathbb{C})$ on the flag variety $\mathcal{B}$ of $\mathfrak{gl}(n,\mathbb{C})$ and $\mathfrak{so}(n,\mathbb{C})$ respectively, [CE12, CEa, CEb]. This connection was first realized for $\mathfrak{gl}(n,\mathbb{C})$, [CE12]. In Section 2.2, we show how the theory of $K_i := GL(i-1,\mathbb{C})\times GL(1,\mathbb{C})$-orbits on the flag variety of $\mathfrak{gl}(i,\mathbb{C})$ can be used to inductively construct the generic components of the nilfibre of the moment map of the system on $\mathfrak{gl}(n,\mathbb{C})$. This allows us to approach the problem of understanding the associated variety of a GZ module in a more conceptual fashion and suggests a geometric approach to the construction of GZ modules.

Motivated by our inductive construction of components of the nilfibre in Section 2.2, we begin our study of GZ modules by studying a closely related category of "partial" GZ modules. These are $U(\mathfrak{g})$-modules on which the *partial* GZ-algebra, Γ_n:

$$\Gamma_n \cong Z(\mathfrak{g}_{n-1}) \otimes_{\mathbb{C}} Z(\mathfrak{g}). \tag{1.3}$$

acts locally finitely. The category of these modules is related to the category of $(\mathfrak{g}\oplus\mathfrak{g}_{n-1},(G_{n-1})_\Delta)$-modules, where $(G_{n-1})_\Delta\subset G\times G_{n-1}$ is the diagonal copy of $G_{n-1} = GL(n-1,\mathbb{C})$, $SO(n-1,\mathbb{C})$ in the product. The pair $(\mathfrak{g}\oplus\mathfrak{g}_{n-1},(G_{n-1})_\Delta)$ is spherical, i.e. $(G_{n-1})_\Delta$ acts on the product of flag varieties $G/B\times G_{n-1}/B_{n-1}$ with finitely many orbits. Thus, we can construct $(\mathfrak{g}\oplus\mathfrak{g}_{n-1},(G_{n-1})_\Delta)$-modules using an analogue of the Beilinson-Bernstein classification. Generalizing ideas of Borho and Brylinski [BB85] we propose to geometrically construct $(U(\mathfrak{g}),\Gamma_n)$-modules and even some $(U(\mathfrak{g}),\Gamma)$-modules from $(\mathfrak{g}\oplus\mathfrak{g}_{n-1},(G_{n-1})_\Delta)$-modules. We discuss this project in Section 3.

The next project in the proposal describes how the geometry of GZ integrable systems can be used to study Poisson Lie groups and quantum algebras. We construct a nonlinear version of the GZ system constructed by Kostant and Wallach. We propose to use this

GZ system to extend Ginzburg-Weinstein theory to the setting of complex Poisson Lie groups and to generalize the Joseph-Letzter theorem for $U_q(\mathfrak{g})$ to the Gelfand-Zeitlin setting by showing that $U_q(\mathfrak{g})$ is free over the quantum analogue of the algebra Γ in (1.1). The methods used here should also allow us to obtain a new proof of the Joseph-Letzter theorem using the theory of Poisson Lie groups and deformation quantization. We hope in the future to extend these ideas to other quantum algebras such as the Yangian algebra of $\mathfrak{gl}(n,\mathbb{C})$, $Y(\mathfrak{gl}(n,\mathbb{C}))$ [Mol07]. This project is discussed in Section 4.

The third project in this proposal represents a new direction of research for us. In joint work with Jeb Willenbring, we develop a new method to compute generalized Littlewood-Richardson (L-R) coefficients of classical groups. L-R theory has played an enormous role in several different areas of mathematics in recent years. This includes Knutson and Tao's solution to the Horn problem using puzzles and honeycombs [KT99, KT01, KTW04], schubert calculus (see for example [Kre07]), and group theory.

Most expositions of L-R theory for $GL(n,\mathbb{C})$ reduce to the case where all the representations involved have polynomial matrix coefficients. Of course, many finite dimensional representations do not have this property (e.g. the adjoint representation). The usual way to make this reduction is to tensor with a sufficiently high power of the determinant. However, this is not the only way to organize the combinatorics.

We present a generalization of Littlewood-Richardson theory describing the multiplicities of irreducible $GL(n,\mathbb{C})$-representations in a tensor product of an arbitrary number of *rational* representations of $GL(n,\mathbb{C})$. Using the dual pair $(GL(n,\mathbb{C}),\mathfrak{gl}(p,\mathbb{C})\oplus\mathfrak{gl}(q,\mathbb{C}))$ we show that these multiplicities are given by branching multiplicities between certain irreducible Harish-Chandra modules. These branching multiplicities are easily computed using an abstraction of what is known as a *contingency table* in statistics. As we will see, this new approach introduces some new convex geometry in to the computation of generalized L-R coefficients. It also has the potential to easily generalize to other classical groups. This project is discussed in Section 5.

2. The geometry of the GZ system on $\mathfrak{gl}(n,\mathbb{C})$

2.1. **Background.** We now describe the GZ system on $\mathfrak{g}=\mathfrak{gl}(n,\mathbb{C})$ in more detail. We also describe some of our early results concerning the geometry of the system [Col11]. For $i=1,\ldots,n$ and $j=1,\ldots,i$, let $f_{i,j}\in\mathbb{C}[\mathfrak{g}]$ be the function $f_{i,j}(x)=tr(x_i^j)$, where x_i is the $i\times i$ submatrix in the upper left corner of x and $tr(\cdot)$ denotes the trace function. We define the GZ collection of functions:

$$J_{GZ}:=\{f_{ij}(x): i=1,\ldots,n,\ j=1,\ldots i\}. \tag{2.1}$$

The functions J_{GZ} are algebraically independent and generate the maximal Poisson commutative algebra $J(\mathfrak{g})$ in (1.1). Moreover, the restriction of these functions to each regular adjoint orbit in $\mathfrak{g}$ forms an integrable system on the orbit [KW06a]. (A different choice of generators for $J(\mathfrak{g})$ results in an equivalent integrable system on $\mathfrak{g}$.)

To understand the Lagrangian foliation of $\mathfrak{g}$ given by the GZ system, we consider the moment map for the GZ system, which we refer to as the Kostant-Wallach (KW) map:

$$\Phi : \mathfrak{g} \to \mathbb{C}^{\binom{n+1}{2}}, \Phi(x) = (f_{11}(x_1), \dots, f_{ij}(x_i), \dots, f_{nn}(x)), \tag{2.2}$$

Remark 2.1. For $x \in \mathfrak{g}$, let $\sigma(x_i) = \{\lambda_1, \dots, \lambda_i\}$ denote the spectrum of x_i where the eigenvalues λ_j are listed with repetition. For $c \in \mathbb{C}^{\binom{n+1}{2}}$, we observe that $x, y \in \Phi^{-1}(c)$ if and only if $\sigma(x_i) = \sigma(y_i)$ for all $i = 1, \dots, n$.

Regular level sets of the Kostant-Wallach map in (2.2) are Lagrangian submanifolds of regular G-adjoint orbits on $\mathfrak{g}$. We therefore consider the open subvariety of elements of $\mathfrak{g}$ where the differentials of the GZ functions in (2.1) are linearly independent.

Definition 2.2. *An element $x \in \mathfrak{g}$ is said to be* strongly regular *if the set*

$$dJ_{GZ}(x) := \{df_{ij}(x) : i = 1, \dots, n,\ j = 1, \dots, i\}.$$

is linearly independent. We let $\mathfrak{g}_{sreg} \subset \mathfrak{g}$ denote the set of strongly regular elements.

Kostant and Wallach show that the map Φ is surjective and every fibre of Φ contains strongly regular elements. For $c \in \mathbb{C}^{\binom{n+1}{2}}$, let $\Phi^{-1}(c)_{sreg} := \mathfrak{g}_{sreg} \cap \Phi^{-1}(c)$. The irreducible components of the variety $\Phi^{-1}(c)_{sreg}$ are smooth Lagrangian subvarieties of regular adjoint orbits. These components can be described as generic orbits of an action of analytic group $A \cong \mathbb{C}^{\binom{n}{2}}$ on $\mathfrak{g}$. The group A is the simply connected complex Lie group corresponding to the abelian Lie algebra of Hamiltonian vector fields of the GZ functions, i.e.

$$\mathfrak{a} := \{\xi_f : f \in J_{GZ}\} \tag{2.3}$$

In [Col11], we describe the A-orbit structure of $\Phi^{-1}(c)_{sreg}$. This is achieved by integrating the Lie algebra $\mathfrak{a}$ of GZ vector fields in (2.3) locally on $\Phi^{-1}(c)_{sreg}$ to the action of a connected, commutative algebraic group.

Theorem 2.3. *[Theorem 5.11,* [Col11]*]*

(1) *Let $x \in \mathfrak{g}_{sreg}$ and suppose that x_i and x_{i+1} have j_i eigenvalues in common not counting repetitions, for $i = 1, \dots, n-1$. Then $\Phi^{-1}(\Phi(x))_{sreg}$ contains exactly $2^{\sum_{i=1}^{n-1} j_i}$ A-orbits.*
(2) *For $i = 1, \dots, n-1$, let Z_i denote the centralizer in G_i of the Jordan form of x_i. There is a free algebraic action of the group $Z := Z_1 \times \cdots \times Z_{n-1}$ on $\Phi^{-1}(\Phi(x))_{sreg}$ whose orbits coincide with the orbits of A in (1).*

A special case of Theorem 2.3 will play an important role in the next few sections. Let $\Phi^{-1}(0) := \Phi^{-1}(0, \dots, 0)$ denote the nilfibre of the KW map and denote its strongly regular points by $\Phi^{-1}(0)_{sreg}$.

Corollary 2.4. *The strongly regular nilfibre $\Phi^{-1}(0)_{sreg}$ contains exactly 2^{n-1} A-orbits.*

Remark 2.5. The definition of the orthogonal GZ system is analogous, see [Col09] for details. The moment map or orthogonal Kostant-Wallach map is also defined similarly: $\Phi(x) = (\chi_1(x_1), \ldots, \chi_i(x_i), \ldots, \chi_n(x))$, where χ_i is the adjoint quotient for $\mathfrak{so}(i, \mathbb{C})$. For $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$ the analogue of the Lie algebra $\mathfrak{a}$ in (2.3) also integrates to an action of a holomorphic group on $\mathfrak{g}$ whose dimension is half the dimension of a regular G-adjoint orbit on $\mathfrak{g}$. However, the analogue of Theorem 2.3 is known only for certain fibres of Φ which contain certain regular semisimple elements of $\mathfrak{so}(n, \mathbb{C})$ (see Theorem 3.2, [Col09].)

To construct $(U(\mathfrak{so}(n, \mathbb{C})), \Gamma)$-modules geometrically, we would like to extend Theorem 2.3 to the orthogonal case.

Problem 2.6. Describe the strongly regular fibres of the ananlogue of the KW map for $\mathfrak{so}(n, \mathbb{C})$.

This problem is much more difficult than the case of $\mathfrak{gl}(n, \mathbb{C})$. The reason being that the methods in [Col11] do not easily extend to the case of $\mathfrak{so}(n, \mathbb{C})$ except for some special generic fibres. It is not even known if the KW map for $\mathfrak{so}(n, \mathbb{C})$ is surjective. Moreover, it follows from our recent work (Corollary 4.20 , [CEb]) that not every fibre of Φ contains strongly regular elements. We begin to address some of these questions in Section 3 by studying a "partial" version of the KW map using the geometry of spherical pairs (see Equation 3.1).

We now describe how the theory of orbits a symmetric subgroup of $GL(n, \mathbb{C})$ on its flag variety can be used to understand the structure of $\Phi^{-1}(0)_{sreg}$ for $\mathfrak{gl}(n, \mathbb{C})$.

2.2. K **-orbits on the flag variety and KW fibres.** It follows from Remark 2.1 that the nilfibre $\Phi^{-1}(0)$ of the Kostant-Wallach map is given by:

$$SN := \Phi^{-1}(0) = \{x \in \mathfrak{g} : x_i \in \mathfrak{g}_i \text{ is nilpotent for } i = 1, \ldots, n\}. \tag{2.4}$$

The variety SN is often referred to in the literature as *strongly nilpotent matrices*. These matrices have been an object of interest for both linear algebraists and Lie theorists for many years [Ovs03, PS08]. In [CE12], we realize the 2^{n-1} A-orbits in $\Phi^{-1}(0)_{sreg}$ given in Corollary 2.4 as the varieties of regular nilpotent elements of 2^{n-1} Borel subalgebras of $\mathfrak{g}$ constructed using $K_i := GL(i-1, \mathbb{C}) \times GL(1, \mathbb{C})$-orbits on the flag variety $\mathcal{B}_i$ of $\mathfrak{g}_i$ for $i = 1, \ldots, n$. For $i = 1, \ldots, n$, let Q_i be the K_i-orbit on $\mathcal{B}_i$ through the $i \times i$ upper or lower triangular matrices. Using these orbits, we inductively construct 2^{n-1} Borel subalgebras $\mathfrak{b}_{Q_1,\ldots,Q_n}$ which have the property that the projection of $\mathfrak{b}_{Q_1,\ldots,Q_n}$ onto its $i \times i$ upper left corner is a Borel subalgebra of $\mathfrak{g}_i$ which is in the K_i-orbit Q_i. We denote by $\mathfrak{n}^{reg}_{Q_1,\ldots,Q_n}$ the regular nilpotent elements of $\mathfrak{b}_{Q_1,\ldots,Q_n}$. The main result of [CE12] is the following theorem.

Theorem 2.7. *[*[CE12]*, Theorem 4.5] The irreducible component decomposition of the nilfibre* $\Phi^{-1}(0)_{sreg}$ *is precisely*

$$\Phi^{-1}(0)_{sreg} = \coprod \mathfrak{n}^{reg}_{Q_1,\ldots,Q_n}. \tag{2.5}$$

Example 2.8. For $\mathfrak{g} = \mathfrak{gl}(3, \mathbb{C})$, Theorem 2.7 implies that the four irreducible components of $\Phi^{-1}(0)_{sreg}$ are:

$$\mathfrak{n}^{reg}_{Q_-,Q_-} = \begin{bmatrix} 0 & 0 & 0 \\ a_1 & 0 & 0 \\ a_3 & a_2 & 0 \end{bmatrix} \qquad \mathfrak{n}^{reg}_{Q_+,Q_+} = \begin{bmatrix} 0 & a_1 & a_3 \\ 0 & 0 & a_2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathfrak{n}^{reg}_{Q_+,Q_-} = \begin{bmatrix} 0 & a_1 & 0 \\ 0 & 0 & 0 \\ a_2 & a_3 & 0 \end{bmatrix} \qquad \mathfrak{n}^{reg}_{Q_-,Q_+} = \begin{bmatrix} 0 & 0 & a_1 \\ a_2 & 0 & a_3 \\ 0 & 0 & 0 \end{bmatrix},$$

where a_1, $a_2 \in \mathbb{C}^\times$ and $a_3 \in \mathbb{C}$.

Understanding $\Phi^{-1}(0)_{sreg}$ and more generally, $\Phi^{-1}(0) = SN$ geometrically is very important in the study of GZ modules for the following reason.

Proposition 2.9. *The associated variety of a GZ module is a subvariety of SN.*

(A completely analogous statement is true for $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$ with SN defined as in Equation (2.4). See Remark 2.5.)

Building on our work in [Col11, CE12], we propose the following problems for future research.

Problem 2.10. For $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ and $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$:

(1) Describe all irreducible components of SN.
(2) Describe all strongly regular fibres of the KW map using K_i-orbits on $\mathcal{B}_i$.
(3) Describe the topology of the strongly regular set $\mathfrak{g}_{sreg}$.

We describe now an approach to problem (1). As a first step in understanding (3), we prove

Theorem 2.11 (Theorem 5.3, [CE12])**.** *Every Borel subalgbera $\mathfrak{b} \subset \mathfrak{gl}(n, \mathbb{C})$ contains strongly regular elements.*

We will return to problems (2) and (3) in more detail in Section 3.

2.3. **Geometric Description of strongly nilpotent matrices.** For $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$, Ovsienko [Ovs03] shows that SN is a complete intersection of dimension $\binom{n}{2}$. It follows that the KW map Φ is flat [Mat86]. Let I_{GZ} be the ideal generated by the GZ functions J_{GZ} in (2.1). Then we note that $SN = V(I_{GZ})$. One of the major difficulties in studying the variety SN is that the ideal $I_{GZ} \subset \mathbb{C}[\mathfrak{gl}(n, \mathbb{C})]$ is not radical if $n \geq 3$. Indeed, Theorem 18.15 (a), [Eis95] implies that I_{GZ} is radical if and only if $\Phi^{-1}(0)_{sreg}$ is dense in $\Phi^{-1}(0)$. For example, in the case of $n = 3$, SN contains an irreducible component of the form

$$\left\{ \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & y \\ r & s & 0 \end{bmatrix} : \; xy + rs = 0 \right\}. \tag{2.6}$$

But it follows from Theorem 2.7 that $\overline{\Phi^{-1}(0)_{sreg}} = \coprod_{i=1}^{4} \mathfrak{n}_i$, where $\mathfrak{n}_i$ is one of the four nilradicals $\mathfrak{n}_{Q_\pm, Q_\pm}$ in Example 2.8.

However, the ideal I_{GZ} is stable under Poisson bracket and the integrability of characteristics theorem (Theorem 1.5.17, [CG97]) implies that SN is a coisotropic subvariety of the Poisson variety $\mathfrak{g}$. We claim that SN is a Lagrangian subvariety of $\mathfrak{g}$. Recall that the components of $\Phi^{-1}(0)_{sreg}$ are Lagrangian. In the case where $n = 3$, one can show directly that the component in (2.6) is Lagrangian. We believe this should be true in general.

Problem 2.12. (1) Show that SN is a Lagrangian subvariety of the Poisson variety $\mathfrak{g}$ for both $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ and $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$.
(2) Construct irreducible $(U(\mathfrak{g}), \Gamma)$-modules whose associated varieties are the Lagrangian components of SN.

The solution to the first problem should provide a more conceptual proof of Ovsienko's result and the flatness of Φ. Since SN is coisotropic to show SN is Lagrangian, it suffices to show that every irreducible component of SN contains a regular element. We will discuss an approach to the second problem in the next section.

3. Partial GZ modules

In this section, we work in the general setting of $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ or $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$. There are several things that make the category of $(U(\mathfrak{g}), \Gamma)$-modules very difficult to understand:

(1) Given an arbitrary GZ module M, Γ need not act semisimply on M.
(2) There are no obvious "standard" modules in the category of $(U(\mathfrak{g}), \Gamma)$-modules. For example, given a multiplicative character χ of Γ, the "Verma" module
$$V_\chi := U(\mathfrak{g}) \otimes_\Gamma \mathbb{C}_\chi,$$
may have several irreducible quotients.
(3) The category of GZ modules does not contain enough projectives. Therefore homological techniques like the Zuckerman functor are not readily applicable to construct $(U(\mathfrak{g}), \Gamma)$-modules. (See [Zuc12] for an exposition of the Zuckerman functor and its application to the study of generalized Harish-Chandra modules.)
(4) As we mentioned in Section 1, the algebra Γ does not integrate to a group that acts on the flag variety of $\mathfrak{g}$ with finitely many orbits. So a direct Beilinson-Bernstein approach is also not possible.

Roughly speaking, a GZ module M is generic if Γ acts semisimply on M. As mentioned in Section 1, Futorny and his collaborators have classified nearly all irreducible generic GZ modules using combinatorial and algebraic methods [FGRa]. Modules on which Γ does not act semisimply are called *singular*. Very little is known about singular GZ modules outside of some special cases [FGRb]. The combinatorial methods used in the generic case become much more complicated and cumbersome in the singular case. To our knowledge nothing is known about singular modules in the orthogonal case. Considering the difficulties in understanding the category of $(U(\mathfrak{g}),\Gamma)$-modules directly, we propose to first study $(U(\mathfrak{g}),\Gamma_n)$-modules or "partial GZ modules" (see (1.3)). This approach is suggested by the geometric methods outlined in Sections 2.1 and 2.2, where components of strongly fibres are constructed by studying the relation between adjacent submatrices x_i and x_{i+1} and K_i-orbits on $\mathcal{B}_i$ for each i. Even though quantization is not a precise procedure, there should be a similar approach in representation theory to construct $(U(\mathfrak{g}),\Gamma)$-modules inductively from $(U(\mathfrak{g}),\Gamma_n)$-modules. This approach appears promising, because we can develop a more comprehensive approach to studying $(U(\mathfrak{g}),\Gamma_n)$-modules for both the $\mathfrak{gl}(n,\mathbb{C})$ and $\mathfrak{so}(n,\mathbb{C})$ using geometric methods.

Remark 3.1. The discussion in the Section 2.2 suggests a connection between Harish-Chandra modules for the pair $(U(\mathfrak{gl}(n,\mathbb{C})),K)$, $K = GL(n-1,\mathbb{C})\times GL(1,\mathbb{C})$ and $(U(\mathfrak{gl}(n,\mathbb{C})),\Gamma)$-modules. In fact, for $\mathfrak{g}=\mathfrak{gl}(n,\mathbb{C})$, $\mathfrak{so}(n,\mathbb{C})$ and $K = GL(n-1,\mathbb{C})\times GL(1,\mathbb{C})$, $K = SO(n-1,\mathbb{C})$ respectively, any admissible $(\mathfrak{g},K)$-module with fixed central character is easily seen to be a $(U(\mathfrak{g}),\Gamma)$-module on which Γ acts semisimply. However, such $(\mathfrak{g},K)$-modules represent only a very small class of "generic" GZ modules. Even for the case $\mathfrak{g}=\mathfrak{gl}(3,\mathbb{C})$, there are several examples of generic GZ modules in which Γ acts semisimply, but which are not locally finite for $GL(2,\mathbb{C})$.

In the partial GZ situation, any $(U(\mathfrak{g}),\Gamma_n)$-module with a semisimple, admissible Γ_n-action is an admissible $(\mathfrak{g},K)$-module, but this represents a very special class of partial GZ modules. Many $(U(\mathfrak{g}),\Gamma)$-modules are not even admissible $(U(\mathfrak{g}),\Gamma_n)$-modules. A comprehensive understanding of the category of GZ modules requires a broader view than the classical theory of $(\mathfrak{g},K)$-modules.

3.1. **Multiplicity free spherical pairs and partial GZ modules.** We can still approach $(U(\mathfrak{g}),\Gamma_n)$-modules in a geometric fashion not by using classical Harish-Chandra modules for the pair $(\mathfrak{g},K)$, but by studying a category of modules for a related spherical pair $(\mathfrak{g}\oplus\mathfrak{g}_{n-1},(G_{n-1})_\Delta)$, where $(G_{n-1})_\Delta$ denotes the diagonal copy of $G_{n-1}=GL(n-1,\mathbb{C})$ or $SO(n-1,\mathbb{C})$ in the product $G\times G_{n-1}$. This pair is spherical precisely because the branching rule from G to G_{n-1} is multiplicity free. Thus, we call $(\mathfrak{g}\oplus\mathfrak{g}_{n-1},(G_{n-1})_\Delta)$ a *multiplicity free spherical pair*. We study the geometry and invariant theory of these pairs extensively in our preprint [CEb].

To connect $(\mathfrak{g}\oplus\mathfrak{g}_{n-1},(G_{n-1})_\Delta)$-modules to partial GZ modules, we plan to generalize a result of Borho-Brylinski, [BB85] and Jantzen, [Jan83] giving an equivalence of categories between Harish-Chandra bimodules with fixed central character and a category of highest weight modules. Let $B_{n-1}\subset G_{n-1}$ be a Borel subgroup. Let $\mathrm{Mod}(\mathfrak{g},B_{n-1})^\chi$ be the category of finitely generated $\mathfrak{g}$ modules with a locally finite action of B_{n-1} and fixed central character χ.

Claim 3.2. *[cf Theorem 3.4,* [BB85]*; Theorem 6.27,* [Jan83]*] There is an equivalence of categories*

$$Mod(\mathfrak{g} \oplus \mathfrak{g}_{n-1}, (G_{n-1})_\Delta)^\chi \leftrightarrow Mod(\mathfrak{g}, B_{n-1})^\chi.$$

The proof of claim 3.2 should work the same way as the analogous result in [BB85] and [Jan83]. Jantzen proves the corresponding result for Harish-Chandra bimodules using algebraic techniques. Borho and Bryliniski reinterpret his result in terms of Beilinson-Bernstein localization. Both of their techniques should generalize fairly easily to our case. The connection with partial GZ modules is:

Proposition 3.3. *Let M be a $(U(\mathfrak{g}), B_{n-1})$-module with fixed central character. Then M has a locally finite action of Γ_n. However, M is not necessarily Γ_n-admissible nor does Γ_n necessarily act semisimply on M. (Thus, an arbitrary $(U(\mathfrak{g}), B_{n-1})$-modules is not necessarily a classical $(U(\mathfrak{g}), G_{n-1})$-module. See Remark 3.1.)*

We summarize our strategy to construct $(U(\mathfrak{g}), \Gamma_n)$-modules and $(U(\mathfrak{g}), \Gamma)$-modules.
Program:

(1) Geometrically construct $(\mathfrak{g} \oplus \mathfrak{g}_{n-1}, (G_{n-1})_\Delta)$-modules using $(G_{n-1})_\Delta$-equivariant $\mathcal{D}$-modules on the product of flag varieties $G/B \times G_{n-1}/B_{n-1}$.
(2) Use the categorical equivalence in Claim 3.2 to understand the category of $(U(\mathfrak{g}), B_{n-1})$-modules.
(3) Use Proposition 3.3 to construct $(U(\mathfrak{g}), \Gamma_n)$-modules.
(4) Construct $(U(\mathfrak{g}), \Gamma)$-modules inductively from $(U(\mathfrak{g}), \Gamma_n)$-modules.

Remark 3.4. In the representation theory of real semisimple groups, irreducible representations are unique irreducible quotients of standard modules which are induced from parabolic subalgebras. In the Beilinson-Bernstein correspondence, this is seen in the Cayley transforms done from a closed K-orbit. We expect something similar works for $(\mathfrak{g} \oplus \mathfrak{g}_{n-1}, (G_{n-1})_\Delta)$-modules and from this can hope to show that $(\mathfrak{g} \oplus \mathfrak{g}_{n-1}, (G_{n-1})_\Delta)$-modules come from parabolic induction. Thus, using Theorem 3.2 and Proposition 3.3 we may be able to realize $(U(\mathfrak{g}), \Gamma_n)$-modules via parabolic induction and get around the apparent lack of standard modules for the GZ category described in beginning of the previous section.

The construction in (1) involves understanding local systems on $(G_{n-1})_\Delta$-orbits in $G/B \times G_{n-1}/B_{n-1}$. In the case of a symmetric subgroup K acting on G/B, there is extensive literature concerning the geometry and combinatorics of these orbits (see for example, [RS93, RS90, Vog83, MŌ90, CEc]). However, no general theory seems to exist in the spherical case. For $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ and $B_{n-1} \subset G_{n-1}$, where B_{n-1} is the standard Borel subgroup of upper triangular matrices, the B_{n-1}-orbits on G/B are described in [Has04]. However, this description is very combinatorial and difficult to use from a geometric perspective. Our preliminary work indicates that we can describe the geometry of the $(G_{n-1})_\Delta$-orbits on $G/B \times G_{n-1}/B_{n-1}$ directly by developing an analogue of the monoid action considered in [RS90].

Problem 3.5. Develop a parametrization of $(G_{n-1})_\Delta$-orbits on $G/B \times G_{n-1}/B_{n-1}$ for our multiplicity free spherical pairs.

The second and third steps of the program should not be very difficult. The real "uncharted waters" of the program lie in the fourth step. However, in some special cases the partial GZ-modules we create in this program turn out to be $(U(\mathfrak{g}), \Gamma)$-modules.

Suppose that $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ and $\mathfrak{b} = \mathfrak{b}_{Q_1,\dots,Q_n}$ is one of the Borel subalgebras appearing in Theorem 2.7. Then for all $i = 1, \dots, n$, the $i \times i$ corner of $\mathfrak{b}$, $\mathfrak{b}_i = \mathfrak{b}_{Q_1,\dots,Q_i}$, is a Borel subalgebra of $\mathfrak{gl}(i, \mathbb{C})$. We can use a slightly more general version of Proposition 3.3 to show that any $(U(\mathfrak{g}), B_{Q_1,\dots,Q_{n-1}})$-module has a locally finite action of the full GZ algebra Γ. The analogue of the Borel subalgebras $\mathfrak{b}_{Q_1,\dots,Q_n}$ exist in the orthogonal case, even though the analogue of Theorem 2.7 does not hold in that situation. Thus, Claim 3.2 and Proposition 3.3 give us a way of constructing some $(U(\mathfrak{g}), \Gamma)$-modules geometrically. On the other hand if $\mathfrak{b}_{n-1}$ is an arbitrary Borel of $\mathfrak{gl}(n-1, \mathbb{C})$ then a $(U(\mathfrak{g}), B_{n-1})$-module need not be a $(U(\mathfrak{g}), \Gamma)$-module. Also, it is easy to construct examples of $(U(\mathfrak{g}), \Gamma)$-modules which are not $(U(\mathfrak{g}), B_{Q_1,\dots,Q_{n-1}})$-modules. The production of other types of $(U(\mathfrak{g}), \Gamma)$-modules will still require an inductive construction starting with $(U(\mathfrak{g}), \Gamma_n)$-modules as in part (4) of the program.

The starting point for the entire program requires a deeper understanding of the geometry of the spherical pair $(\mathfrak{g} \oplus \mathfrak{g}_{n-1}, (G_{n-1})_\Delta)$ [CEb].

3.2. **Kostant-Rallis theory for mulitiplicity free spherical pairs.** When (G, K) is a symmetric pair, the theory of the K-action on $\mathfrak{p}$ developed by Kostant and Rallis [KR71] plays an important role in the geometric study of classical $(\mathfrak{g}, K)$-modules (see for example, [Vog91]). We develop an analogous theory for the coisotropy representation of spherical pairs $(\mathfrak{g} \oplus \mathfrak{g}_{n-1}, (G_{n-1})_\Delta)$. Recall that the coisotropy representation is the action of $(G_{n-1})_\Delta$ on the cotangent space of the identity coset $T^*_{e(G_{n-1})_\Delta}((G \times G_{n-1})/(G_{n-1})_\Delta)$. It is easy to see that this can be identified with the action of G_{n-1} on $\mathfrak{g}$ via conjugation. The study of this action is intimately tied up with GZ theory. In fact, our work in this section can be viewed as generalizing our results in Section 2.2 to describe all fibres of the KW map using $GL(i, \mathbb{C})$-orbits on $\mathcal{B}_i$ and extending that theory to the orthogonal case.

For $i = n-1, n$, let $\mathbb{C}[\mathfrak{g}_i]^{G_i} = \mathbb{C}[f_{i1}, \dots, f_{ir_i}]$, where $r_i = \text{rank}(\mathfrak{g}_i)$. Define the *partial Kostant-Wallach map*

(3.1)
$$\Phi_n : \mathfrak{g} \to \mathbb{C}^{r_{n-1}} \times \mathbb{C}^{r_n};\ \Phi_n(x) = (f_{n-1,1}(x_{n-1}), \dots, f_{n-1,r_{n-1}}(x_{n-1}), f_{n1}(x), \dots, f_{nr_n}(x)).$$

Proposition 3.6. *[Proposition 2.2,* [CEb]*]*

1. *The partial KW map Φ_n is a flat morphism.*
2. *The morphism Φ_n is a GIT quotient for the G_{n-1}-action on $\mathfrak{g}$ by conjugation. In particular, Φ_n is surjective.*

This proposition follows from work of Knop [Kno90], or our own work [CEa] for $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ (see Proposition 2.4 and Remark 2.8, [CEa]).

Definition 3.7. *We will call an element of* $\mathfrak{g}$ *n-strongly regular if* $\dim \operatorname{Ad}(G_{n-1}) \cdot x = \dim G_{n-1}$ *is maximal (cf Definition 2.2).*

We denote the set of n-strongly regular elements of $\mathfrak{g}$ by $\mathfrak{g}_{nsreg}$. We prove the following result which is an analogue of Theorem 9, [Kos63] or Theorem 13, [KR71].

Theorem 3.8. *[Theorem 4.6,* [CEb]*]*
$x \in \mathfrak{g}_{nsreg}$ *if and only if the differentials* $\{df_{i,j}(x) : i = n-1, n,\ j = 1, \ldots, r_i\}$ *are linearly independent.*

Theorem 3.8 implies the following surprising result in linear algebra. Let $x \in \mathfrak{g}$, and let $\mathfrak{z}_{\mathfrak{g}_i}(x_i)$ denote the centralizer of x_i in $\mathfrak{g}_i$ for $i = n-1, n$.

Corollary 3.9. *[Corollary 4.7,* [CEb]*] Let* $x \in \mathfrak{g}$ *with* $\mathfrak{z}_{\mathfrak{g}_{n-1}}(x_{n-1}) \cap \mathfrak{z}_{\mathfrak{g}}(x) = 0$. *Then* $x_{n-1} \in \mathfrak{g}_{n-1}$ *and* $x \in \mathfrak{g}$ *are both regular.*

Corollary 3.9 can be used to simplify Kostant and Wallach's original definition of strongly regular elements (see Definition 2.2). It should help us to better understand the topology of the strongly regular set.

3.2.1. *Eigenvalue Coincidences and coisotropy represention.* To study the GIT quotient $\Phi_n : \mathfrak{g} \to \mathfrak{g}//G_{n-1}$ further, we study the relation between the eigenvalues of x and x_{n-1} in $\mathfrak{g}$. This is a fundamental problem in linear algebra [PS08], [SP09], and we have many new results in this direction [CEa, CEb]. For $i = n-1, n$, let $\sigma(x_i)$ be the spectrum of x_i. For $l = 0, \ldots, r_{n-1}$, we define the eigenvalue coincidence variety:

$$\mathfrak{g}(\geq l) := \{x \in \mathfrak{g} : |\sigma(x) \cap \sigma(x_{n-1})| \geq l\}, \tag{3.2}$$

where the intersection in spectra is counted with repetitions. When $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$ coincidences come in pairs $\lambda_i = \pm\mu_j$. Each pair is counted only once, and we only allow zero to have even multiplicity as an eigenvalue of $\mathfrak{so}(2l+1, \mathbb{C})$. Not that any fibre of Φ_n is contained in a variety $\mathfrak{g}(\geq l)$ for some $l = 0, \ldots, r_{n-1}$.

There is a surprising connection between the structure of the varieties $\mathfrak{g}(\geq l)$ and the geometry of G_{n-1}-orbits on the flag variety of $\mathfrak{g}$ (cf Section 2.2.). For $Q = G_{n-1} \cdot \mathfrak{b}$ a G_{n-1}-orbit on G/B, let $Y_Q := \operatorname{Ad}(G_{n-1}) \cdot \mathfrak{b} \subset \mathfrak{g}$ be the G_{n-1}-saturation of $\mathfrak{b}$ in $\mathfrak{g}$.

Theorem 3.10. *[Theorem 1.1,* [CEa, CEb]*] The irreducible component decomposition of the variety* $\mathfrak{g}(\geq l)$*:*

$$\mathfrak{g}(\geq l) = \bigcup_{codim(Q)=l} \overline{Y_Q}. \tag{3.3}$$

In particular,

$$\mathfrak{g}(\geq r_{n-1}) = \bigcup_{Q\ closed} Y_Q \tag{3.4}$$

Equation 3.4 is of particular interest in linear algebra, because it connects the most degenerate case of spectral coincidences to the simplest G_{n-1}-orbits on $\mathcal{B}$. Using Equation 3.4, we can describe the nilfibre of Φ_n (cf Equation 2.4):

$$SN_n := \Phi_n^{-1}(0) = \{x \in \mathfrak{g} : x, x_{n-1} \text{ are nilpotent}\}. \tag{3.5}$$

Proposition 3.11. *[Proposition 3.10,* [CEa]*, Theorem 4.18,* [CEb]*] Let $Q = G_{n-1} \cdot \mathfrak{b}$ be a closed G_{n-1}-orbit and let $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$.*

$$SN_n = \bigcup_{Q \text{ closed}} \operatorname{Ad}(G_{n-1}) \cdot \mathfrak{n}. \tag{3.6}$$

Fact: If M is a $(U(\mathfrak{g}), \Gamma_n)$-module, then $\operatorname{Ass}(M) \subset SN_n$, where $\operatorname{Ass}(M)$ denotes the associated variety of M.

Problem 3.12. Geometrically construct irreducible partial GZ modules whose associated varieties are irreducible components of SN_n.

Using the four-step program that we described above and an analogue of [BB85], Theorem 4.8 (c) we should be able to construct such modules.

In [CEb], using Theorem 3.10:

(1) We describe all closed G_{n-1}-orbits on $\mathfrak{g}$. (Section 4.3, [CEb])
(2) We show that the Zariski dense set

$$\mathfrak{g}(0) := \{x \in \mathfrak{g} : \sigma(x) \cap \sigma(x_{n-1}) = \emptyset\}.$$

consists of n-strongly regular elements whose G_{n-1}-orbits are closed.
(3) We show that a fibre of Φ_n is a single G_{n-1}-orbit if and only if it contains an element of $\mathfrak{g}(0)$. (Theorem 4.13, Corollary 4.14, [CEb].) (i.e. Elements of $\mathfrak{g}(0)$ play the role of the elements in $\mathfrak{a}^{reg}$ in the Kostant-Rallis situation.)
(4) We show that $\Phi_n^{-1}(0)$ contains no n-strongly regular elements in the orthogonal case (cf Remark 2.5). (Corollary 4.20, [CEb]).

Remark 3.13. We can describe the closed G_{n-1}-orbits fairly easily without computing a Cartan subspace for the coisotropy representation as in Theorem 6, [Pan90].

Remark 3.14. Even though there are some striking similarities between the G_{n-1}-action on $\mathfrak{g}$ and the K-action on $\mathfrak{p}$, the G_{n-1}-action is not polar in the sense of Kac and Dadok, [DK85]. There are also some striking differences. Namely G_{n-1} does not act on SN_n with finitely many orbits, i.e. the G_{n-1}-action on $\mathfrak{g}$ is not visible.

Problem 3.15. (1) Describe G_{n-1}-orbit structure of all fibres of Φ_n using GIT theory.
(2) Use geometry of the coisotropy representation to construct $(\mathfrak{g} \oplus \mathfrak{g}_{n-1}, (G_{n-1})_{\Delta})$-modules.

A solution to (1) appears to be especially close at hand.

3.3. **Jordan Decomposition for the coisotropy rep.** We briefly recall how the usual Jordan decomposition allows us to understand the fibres of the adjoint quotient $\chi : \mathfrak{g} \to \mathfrak{g}//G$ for a complex, reductive Lie algebra $\mathfrak{g}$. For $x \in \mathfrak{g}$, $x = s + n$ with $\mathrm{Ad}(G) \cdot s$ closed and $\chi(s+n) = \chi(s)$. Let $L = Z_G(s)$ and $\mathfrak{l} = \mathfrak{z}_{\mathfrak{g}}(s)$. Then the fibre $\chi^{-1}(\chi(x))$ has the structure of a fibre bundle

$$\chi^{-1}(\chi(x)) := G \times_L s + \mathcal{N}_{\mathfrak{l}},$$

where $\mathcal{N}_{\mathfrak{l}}$ is the nilpotent cone in $\mathfrak{l}$. Thus, to understand the adjoint quotient it is enough to understand the closed G-orbits and the orbits on the nilpotent cone. It appears that a similar kind of Jordan decomposition holds for the coisotropy representation of the multiplicity free spherical pairs $(\mathfrak{g} \oplus \mathfrak{g}_{n-1}, (G_{n-1})_\Delta)$. For now we consider the case $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{C})$. Given $x \in \mathfrak{g}$, we can write $x = x_c + x_n$ where $G_{n-1} \cdot x_c$ is closed and $\Phi_n(x) = \Phi_n(x_c)$. Now $Z_{G_{n-1}}(x_c)$ is reductive and furthermore $Z_{G_{n-1}}(x_c) = GL(\nu_1,\mathbb{C}) \times \cdots \times GL(\nu_r,\mathbb{C})$ for some $r \leq n-1$, even though x_c is not semisimple in the usual sense. Further, $x_n \in \Phi^{-1}_{\nu_1+1}(0) \times \cdots \times \Phi^{-1}_{\nu_r+1}(0)$, where Φ_{ν_i+1} is the partial KW map on $\mathfrak{gl}(\nu_i+1,\mathbb{C})$ (see Equation 3.1). We have a surjection:

$$G_{n-1} \times_{Z_{G_{n-1}}(x_c)} x_c + \Phi^{-1}_{\nu_1+1}(0) \times \cdots \times \Phi^{-1}_{\nu_r+1}(0) \to \Phi_n^{-1}(\Phi_n(x)).$$

Although this map is not in general an isomorphism, its fibres are finite and easily understood. Thus, we can reduce the study of G_{n-1}-orbits on $\mathfrak{g}$ to understanding the closed G_{n-1}-orbits and the G_{n-1}-orbits in $\Phi_n^{-1}(0)$ both of which we understand. An initial examination of examples in $\mathfrak{g} = \mathfrak{so}(n,\mathbb{C})$ case indicates that analogous result also holds in that setting.

The results in Section 3.2 and the solution to (1) of Problem 3.15 outlined here are all "partial KW" versions of solutions to the problems in Section 2 concerning the full Kostant-Wallach map Φ (see Problems 2.6, 2.10, and 2.12). Using these results and inductive methods like those developed to prove Theorem 2.7, we should be able to solve these problems and use the results as a geometric model from which to inductively construct $(U(\mathfrak{g}),\Gamma)$-modules from $(U(\mathfrak{g}),\Gamma_n)$-modules.

4. Nonlinear GZ theory: Applications to Poisson Lie groups and quantum algebras

In joint work with Sam Evens, we have constructed a nonlinear version of the Gelfand-Zeitlin integrable system constructed by Kostant and Wallach. The nonlinear system requires the use of Poisson Lie groups. A Poisson Lie group (G,π_G) is a Lie group with a compatible Poisson structure $\pi_G \in \wedge^2 TG$. Poisson Lie groups were invented by Drinfeld [Dri87] as a classical analogue to quantum groups, and there are relations between the geometry of Poisson Lie groups and the representation theory of quantized universal enveloping algebras $U_q(\mathfrak{g})$ (see for example [DCKP93, HL93], [Jos95]).

We consider $G = GL(n,\mathbb{C})$ as a Poisson Lie group with the standard Drinfeld Poisson structure $\pi_G \in \wedge^2 TG$ and its dual Poisson Lie group $G^* = GL(n,\mathbb{C})^*$ constructed using the standard Manin triple for $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{C})$ (see [KS98],[LW90]). The construction of the

GZ system on G^* requires very different ideas than the construction of the system on $\mathfrak{g}^*$. We make use of the Poisson-Lie theorem [KS98] and the work of Evens and Lu [EL07] to produce a maximal Poisson commuting algebra of functions $J(G^*)$ on G^* that is the nonlinear analogue of the algebra of functions $J(\mathfrak{g})$ in (1.1). Choosing a generating set for $J(G^*)$:

$$J_{nl} = \{\phi_{i,j} : i = 1, \dots, n,\ j = 1, \dots, i\} \tag{4.1}$$

yields an integrable system on the Poisson manifold G^*.

Theorem 4.1. *The functions J_{nl} form an integrable system on the Poisson manifold G^*. Moreover, the Hamiltonian vector fields ξ_f, $f \in J_{nl}$ are complete and simultaneously integrate to an action of $A_{nl} := \mathbb{C}^{\binom{n}{2}}$ on G^*. (cf the discussion at the beginning of Section 2.)*

We call an element $g \in G^*$ strongly regular if its A_{nl}-orbit is of maximal dimension $\binom{n}{2}$. Strongly regular A_{nl}-orbits form Lagrangian submanifolds of generic symplectic leaves of G^*.

To understand the geometry of the GZ system on G^*, we relate it to the GZ system on $\mathfrak{g}^* \cong \mathfrak{g}$. Let $\mathcal{U} \subset G$ be the open Bruhat cell in G. It is easy to show that the group A obtained by integrating the Lie algebra of GZ vector fields in (2.3) preserves $\mathcal{U}$, where $\mathcal{U}$ is viewed as a subset of $\mathfrak{g}$. In [EL07], Evens and Lu construct a Poisson structure on G such that the morphism $m : G^* \to \mathcal{U}$ defined using the standard Manin triple for G^* is a 2^n Poisson covering. Let Ψ denote the moment map for the nonlinear GZ system.

Proposition 4.2. (1) *We have a Cartesian diagram:*

$$\begin{array}{ccc} G^* & \xrightarrow{\quad m \quad} & \mathcal{U} \\ \downarrow{\scriptstyle\Psi} & & \downarrow{\scriptstyle\Phi|_{\mathcal{U}}} \\ (\mathbb{C}^\times)^n \times \mathbb{C}^{\binom{n}{2}} & \xrightarrow{\quad \alpha \quad} & (\mathbb{C}^\times)^n \times \mathbb{C}^{\binom{n}{2}}, \end{array} \tag{4.2}$$

where α is a 2^n-covering, and $\Phi|_{\mathcal{U}}$ is the restriction of the Kostant-Wallach map in (2.2) to the big cell $\mathcal{U}$.

(2) *The Poisson covering m intertwines the action of the nonlinear GZ group A_{nl} on G^* with the action of the linear GZ group A on $\mathfrak{g}$.*

Proposition 4.2 opens the door for the study of varying problems in Poisson Lie theory and quantum algebra using the nonlinear GZ system on G^*.

Problem 4.3. (1) Use the GZ system on $GL(n, \mathbb{C})^*$ to extend Ginzburg-Weinstein theory to the setting of complex Poisson Lie groups.

(2) Prove an analogue of the Joseph-Letzter theorem for the quantum version of the GZ subalgebra of $U_q(\mathfrak{g})$.

We outline briefly how our work above lends itself to the study of these problems.

4.1. **Ginzburg-Weinstein theory.** For a Poisson Lie group (G, π_G), the Poisson structure $\pi_G \in \wedge^2 TG$ is nonlinear and is much more difficult to understand than linear Poisson structures such as the Lie-Poisson structure on $\mathfrak{g}^*$. However, it appears that GZ systems can be used to construct explicit equivalences between π_G and the Lie-Poisson structure on $\mathfrak{g}^*$ for certain Poisson Lie groups. This approach begins with the work of Ginzburg and Weinstein [GW92]. Any compact Lie group K has a natural Poisson Lie group structure giving rise to a dual Poisson Lie group K^* [LW90]. In [GW92], the authors produce Poisson diffeomorphisms $K^* \to \mathfrak{k}^*$, where $\mathfrak{k} = \text{Lie}(K)$ and $\mathfrak{k}^*$ is endowed with the linear Lie-Poisson structure. However, these diffeomorphisms are not constructed explicitly. In the mid 1990's, Flaschka and Raitu constructed a GZ integrable system on the Poisson Lie group $U(n, \mathbb{C})^*$ [FR96], which is a nonlinear version of the GZ system on $\mathfrak{u}(n, \mathbb{C})^*$ in [GS83]. In [AM07], Alekseev and Meinrenken construct a canonical Ginzburg-Weinstein diffeomorphism $U(n, \mathbb{C})^* \to \mathfrak{u}(n, \mathbb{C})^*$ which intertwines the action of the two GZ systems. The GZ system that we have constructed on $GL(n, \mathbb{C})^*$ can be viewed as a complexified version of the one constructed on $U(n, \mathbb{C})^*$ by Flaschka and Raitu. A key step in the construction of Alekseev and Meinrenken is the equivalence between the linear and nonlinear compact GZ systems. Part (2) of Proposition 4.2 proves the analogue of this result in the complex case.

Problem 4.4. Use the equivalence between the nonlinear GZ system on G^* and the linear GZ system on $\mathfrak{g}^*$ to construct a complexified Ginzburg-Weinstein diffeomorphism from a quotient of G^* to the linear Poisson space $\mathfrak{g}^*$.

4.2. **Joseph-Letzter theorem in the GZ setting.** Let $\mathfrak{g}$ be a reductive complex Lie algebra. It is a celebrated theorem of Kostant that $U(\mathfrak{g})$ is free over its centre $Z(\mathfrak{g})$ [Kos63]. This can be proven geometrically in the following way. The adjoint quotient $\chi : \mathfrak{g} \to \mathfrak{g}//G$ is a flat morphism. Thus, $\mathbb{C}[\mathfrak{g}]$ is a flat $\mathbb{C}[\mathfrak{g}]^G$-module. Since $\mathbb{C}[\mathfrak{g}]^G$ is a polynomial ring and therefore graded, it follows that $\mathbb{C}[\mathfrak{g}]$ is free as a $\mathbb{C}[\mathfrak{g}]^G$-module (see for example Lemma 2.5, [CEa]). Using the the fact that $\text{Gr}(U(\mathfrak{g})) \cong \mathbb{C}[\mathfrak{g}^*] \cong \mathbb{C}[\mathfrak{g}]$, it is then easy to prove that $U(\mathfrak{g})$ is free over $Z(\mathfrak{g})$ using some basic results about filtered algebras (see for example Corollary 6.7.13, [CG97]). In recent work Futorny and Ovsienko [FO05] have shown that $U(\mathfrak{gl}(n, \mathbb{C}))$ is free over the GZ algebra Γ building on the work of Ovsienko in [Ovs03]. We seek to generalize their result to the quantum setting using Poisson Lie theory and deformation quantization.

Let $U_q(\mathfrak{g})$ be the quantized universal enveloping algebra of $\mathfrak{g}$. In [JL94], Joseph and Letzter prove an analogue of Kostant's theorem for $\mathcal{F}(U_q(\mathfrak{g}))$, the subalgebra of $U_q(\mathfrak{g})$ which is locally finite under the adjoint action of $U_q(\mathfrak{g})$. We seek to prove that $\mathcal{F}(U_q(\mathfrak{g}))$ is free over the quantized GZ algebra

$$\Gamma_q := \langle Z_q(\mathfrak{g}_1) \dots Z_q(\mathfrak{g}_i) \dots Z_q(\mathfrak{g}) \rangle, \tag{4.3}$$

where $Z_q(\mathfrak{g}_i)$ is the centre of $U_q(\mathfrak{g}_i) \subset U_q(\mathfrak{g})$ and $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$, using deformation quantization and our results concerning the GZ system on G^*. Using similar techniques we can also reprove the Joseph-Letzter theorem for any reductive algebraic group G with simply connected derived subgroup.

By the results of Futorny and Ovisenko [FO05], we know that $\mathbb{C}[\mathfrak{g}]$ is free as a $J(\mathfrak{g})$-module. It then follows from part (1) of Proposition 4.2 that

$$\mathbb{C}[G^*] \text{ is free as a } J(G^*)\text{-module.} \tag{4.4}$$

The Poisson dual group G^* is the semiclassical limit of the quantum group $\mathbb{C}_q[G^*]$. The latter is known to be isomorphic to $U_q(\mathfrak{g})$ as a Hopf algebra. Thus, using (4.4), we hope to show that:

Problem 4.5. $\mathcal{F}(U_q(\mathfrak{g}))$ is free over Γ_q.

The passage from (4.4) to a proof of Problem 4.5 will be much more difficult than in the classical case. The issue is that $U_q(\mathfrak{g})$ is not just a $\mathbb{C}[q]$-module, but a $\mathbb{C}[q, q^{-1}]$-module. So a simple filteration argument involving the classical limit may not be possible. Rather a specialization argument (i.e. setting $q = 1$ might be necessary.) This is technically much more cumbersome, but should give us the desired result.

Now let G be any reductive algebraic group with simply connected derived subgroup. Using the Poisson structure on G developed by Evens-Lu [EL07], results of Richardson [Ric79] on the structure of $\mathbb{C}[G]^G$, and an analogue of Proposition 4.2, we can show that $\mathbb{C}[G^*]$ is free over the ring $\mathcal{C}$ generated by its Casimir functions. With this result we should be able to prove the Joseph-Letzter theorem using deformation quantization.

5. Tensor product multiplicities for rational representations of classical groups via contingency tables

We now discuss our joint work with Jeb Willenbring in developing a new approach to the computation of generalized Littlewood-Richardson (L-R) coefficients for $GL(n, \mathbb{C})$ using the theory of dual pairs and branching rules for Harish-Chandra modules. We expect a preprint to appear on the arxiv very soon.

In more detail, let $\lambda^+ = (\alpha_1 \geq \cdots \geq \alpha_l)$ and $\lambda^- = (\beta_1 \geq \cdots \geq \beta_k)$ be partitions with $\ell(\lambda_+) + \ell(\lambda_-) \leq n$, where $\ell(\lambda_+) = l$ denotes the *length or depth* of the partition. Then the n-tuple:

$$(\lambda^+, \lambda^-) := (\alpha_1 \geq \cdots \geq \alpha_l > 0 \geq \cdots \geq 0 > -\beta_k \geq \cdots \geq -\beta_1),$$

is a dominant weight for $GL(n, \mathbb{C})$. Let $F_n^{(\lambda^+,\lambda^-)} :=$ irrep of highest weight (λ_+, λ_-). For partitions $\mu_+^{(i)}$ and $\mu_-^{(i)}$, with $\ell(\mu_+^{(i)}) = p_i$, $\ell(\mu_-^{(i)}) = q_i$, and $p_i + q_i \leq n$, we consider a vector of partitions:

$$(\underline{\mu_+}, \underline{\mu_-}) := ((\mu_+^{(1)}, \mu_-^{(1)}), \ldots, (\mu_+^{(r)}, \mu_-^{(r)})).$$

Define:

$$F_n^{(\underline{\mu_+},\underline{\mu_-})} := F_n^{(\mu_+^{(1)},\mu_-^{(1)})} \otimes \cdots \otimes F_n^{(\mu_+^{(r)},\mu_-^{(r)})}. \tag{5.1}$$

We compute the generalized L-R coefficient for the rational reps:

$$c_{(\underline{\mu_+},\underline{\mu_-})}^{(\lambda_+,\lambda_-)} := [F_n^{(\lambda_+,\lambda_-)} : F_n^{(\underline{\mu_+},\underline{\mu_-})}] = \mathrm{Hom}_{GL(n,\mathbb{C})}(F_n^{(\lambda_+,\lambda_-)}, F_n^{(\underline{\mu_+},\underline{\mu_-})}) \tag{5.2}$$

using the theory of dual pairs, see-saw reciprocity, and Harish-Chandra modules.

Remark 5.1. The computation of the branching rule $[F_n^{(\lambda_+,\lambda_-)} : F_n^{(\mu_+,\mu_-)} \otimes F_n^{(\gamma_+,\gamma_-)}]$ first appears in King's paper [Kin71] as Equation (4.6) together with (4.15) (see also [Kin90] for a broader exposition.) King works from a largely combinatorial point of view and does not use the theory of dual pairs of Howe. Howe, Tan, and Willenbring compute branching rules for all classical symmetric pairs $K \subset G$ in [HTW05] using dual pairs, but this does not cover the case of arbitrary tensor products in (5.2).

In our work a key role is played by *contingency tables of partitions.*

Definition 5.2. *Let*

$$\Lambda := \begin{bmatrix} 0 & \lambda_{12} & \dots & \dots & \lambda_{1r} \\ \lambda_{21} & 0 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \lambda_{r-1r} \\ \lambda_{r1} & \dots & \dots & \lambda_{rr-1} & 0 \end{bmatrix},$$

with $\lambda_{i,j}$ a partition and 0 the empty partition.

Then Λ is said to be a contingency table of partitions *with contingencies*

$\mu_+^{(1)}, \dots, \mu_+^{(r)};\ \mu_-^{(1)}, \dots, \mu_-^{(r)}$ *if and only if the generalized L-R coefficients:*

$$c^{(\mu_+^{(i)},0)}_{(Row_{(i)}(\Lambda),0)} \neq 0, \ \text{and}\ c^{(\mu_-^{(j)},0)}_{(Col_{(j)}(\Lambda),0)} \neq 0 \text{ for all } i,\, j = 1, \dots, r. \tag{5.3}$$

Note that the representations involved in computing the coefficients in (5.3) are all polynomial representations. We compute (5.2) by counting the number of contingency tables of partitions and computing the product of generalized L-R coefficients in (5.3). Our main results is:

Theorem 5.3. *Let $c^0_{(\underline{\mu_+},\underline{\mu_-})} := [F_n^{(0,0)} : F_n^{(\underline{\mu_+},\underline{\mu_-})}] = (F_n^{(\underline{\mu_+},\underline{\mu_-})})^G$ and suppose $n \geq \sum_{i=1}^r \ell(\mu_+^{(i)}) + \ell(\mu_-^{(i)}) = \sum_{i=1}^r (p_i + q_i)$. Then*

$$c^0_{(\underline{\mu_+},\underline{\mu_-})} = \sum \left(\prod_{i,\,j=1}^{r} c^{\mu_+^{(i)}}_{Row_i(\Lambda)} c^{\mu_-^{(j)}}_{Col_{(j)}(\Lambda)} \right), \tag{5.4}$$

where the sum is taken over all contingency tables Λ with contingencies $\mu_+^{(1)}, \dots, \mu_+^{(r)};\ \mu_-^{(1)}, \dots, \mu_-^{(r)}$.

Remark 5.4.
(1) Note that $c^0_{(\underline{\mu_+},\underline{\mu_-})}$ is enough to compute arbitrary $c^{(\lambda_+,\lambda_-)}_{(\underline{\mu_+},\underline{\mu_-})}$, since $c^{(\lambda_+,\lambda_-)}_{(\underline{\mu_+},\underline{\mu_-})} = [F_n^{(0,0)} : (F_n^{(\lambda_+,\lambda_-)})^* \otimes F_n^{(\underline{\mu_+},\underline{\mu_-})}]$.
(2) An analogue of this result can be used to compute arbitrary tensor product multiplicities for other classical groups. For example, to compute generalized L-R

coefficients for $O(n,\mathbb{C})$, we have an analogous result using symmetric contingency tables.

5.1. **Applications and special cases:** We briefly present a few special cases of Theorem 5.3 that explain the use of the word contingency table. Consider the case where $\mu_+^{(i)} = c_i \in \mathbb{N}$ and $\mu_-^{(i)} = d_i \in \mathbb{N}$. In this setting, the generalized Pieri rule tells us that:

$$c^{c_i}_{\mathrm{Row}_i(\Lambda)} = 1 \text{ if and only if } \lambda_{ij} \in \mathbb{C} \text{ and } \sum_{j=1}^{r} \lambda_{ij} = c_i.$$

Similarly,

$$c^{d_j}_{\mathrm{Col}_j(\Lambda)} = 1 \text{ if and only if } \lambda_{ij} \in \mathbb{C} \text{ and } \sum_{i=1}^{r} \lambda_{ij} = d_j.$$

So in this case the matrix Λ is an actual contingency matrix with zeroes down the diagonal and contingencies $c_1, \ldots, c_r;\ d_1, \ldots, d_r$.

Corollary 5.5. *Let $(\underline{\mu_+}, \underline{\mu_-}) = ((c_1,d_1), \ldots, (c_r,d_r))$. Then $c^0_{(\underline{\mu_+},\underline{\mu_-})} = \#$ of $r \times r$ contingency tables with zeroes on the main diagonal and contingencies $c_1, \ldots, c_r;\ d_1, \ldots, d_r$.*

Using Corollary 5.5, we can use contingency tables to compute spaces of $GL(n,\mathbb{C})$ intertwining operators. (Recall that for any $k \in \mathbb{N}$, $F_n^k \cong S^k(\mathbb{C}^n)$.)

Corollary 5.6.

$$\dim Hom_{GL(n)}(S^{d_1}(\mathbb{C}^n) \otimes \cdots \otimes S^{d_r}(\mathbb{C}^n), S^{c_1}(\mathbb{C}^n) \otimes \cdots \otimes S^{c_r}(\mathbb{C}^n)) =$$

the number of $r \times r$ contingency tables with contingencies $c_1, \ldots, c_r;\ d_1, \ldots, d_r$.

The number of such contingency tables forms a convex set. Thus, Corollaries 5.5 and 5.6 introduce new convex geometry into the computation of L-R coefficients.

5.2. **Summary of methods and the stable range.** We outline briefly our approach to proving Theorem 5.3 using Howe duality and the role played by assumption that $n \geq \sum_{i=1}^{r}(p_i + q_i)$.

Let $V = M_{n,p+q}(\mathbb{C})$ be the $n \times (p+q)$ complex matrices, and $P(V)$ be the algebra of polynomial functions on V. Then $GL(n,\mathbb{C})$ acts naturally on $P(V)$ on the left by $g \cdot f(x \oplus y) = f(g^T x \oplus g^{-1} y)$. The commutant of this action in the algebra of polynomial differential operators on V can be identified as an associative algebra with a quotient of the universal enveloping algebra of $U(\mathfrak{gl}(p+q,\mathbb{C}))$. Thus, we have a multiplicity free decomposition of V as $GL(n,\mathbb{C}) \times \mathfrak{gl}(p+q,\mathbb{C})$-module:

$$\mathcal{P}(V) \cong \bigoplus_{(\lambda_+,\lambda_-)} F_n^{(\lambda^+,\lambda^-)} \otimes \widetilde{F}_{p,q}^{(\lambda_+,\lambda_-)}, \tag{5.5}$$

where the sum is over all dominant weights (λ_+, λ_-) with $\ell(\lambda_+) \leq p$, $\ell(\lambda_-) \leq q$, and $\ell(\lambda_+) + \ell(\lambda_-) \leq n$, [EHW83],[KV78],[How89]. The $\mathfrak{gl}(p+q,\mathbb{C})$-module $\widetilde{F}_{p,q}^{(\lambda_+,\lambda_-)}$ is a

Harish-Chandra module for the pair $(\mathfrak{gl}(p+q,\mathbb{C}), GL(p,\mathbb{C}) \times GL(q,\mathbb{C}))$ with lowest $K = GL(p,\mathbb{C}) \times GL(q,\mathbb{C})$-type $F_p^{\lambda_+} \otimes F_q^{\lambda_-}$.

Key Point: Stable Range: If $n \geq p+q$, then $\widetilde{F}_{p,q}^{(\lambda_+,\lambda_-)}$ is a generalized Verma module in the sense of parabolic category $\mathcal{O}$. Moreover, as a K-module, $\widetilde{F}_{p,q}^{(\lambda_+,\lambda_-)}$ is particularly simple:

$$\widetilde{F}_{p,q}^{(\lambda_+,\lambda_-)} \cong M_{p,q}(\mathbb{C}) \otimes F_p^{\lambda_+} \otimes F_q^{\lambda_-}. \tag{5.6}$$

To compute $c^0_{(\underline{\mu_+},\underline{\mu_-})}$ we use the dual pair in (5.5) and see-saw reciprocity. The assumption that $n \geq \sum_{i=1}^r (p_i + q_i)$ in Theorem 5.3 is to ensure that we are in the stable range. In more detail, we have the see-saw pairs:

$$\begin{array}{ccc} \underbrace{GL(n,\mathbb{C}) \times \cdots \times GL(n,\mathbb{C})}_{r-\text{copies}} & -- & \mathfrak{gl}(p_1+q_1,\mathbb{C}) \oplus \cdots \oplus \mathfrak{gl}(p_r+q_r,\mathbb{C}) \\ \cup & & \cap \\ GL(n,\mathbb{C})_\Delta & -- & \mathfrak{gl}(p_1+\cdots+p_r+q_1+\cdots+q_r,\mathbb{C}) \end{array}, \tag{5.7}$$

where $GL(n,\mathbb{C})_\Delta \subset GL(n,\mathbb{C}) \times \cdots \times GL(n,\mathbb{C})$ indicates the diagonal copy of $GL(n,\mathbb{C})$ inside the product, and $\mathfrak{gl}(p_1+q_1,\mathbb{C}) \oplus \cdots \oplus \mathfrak{gl}(p_r+q_r,\mathbb{C})$ is embedded as block diagonal matrices in $\mathfrak{gl}(p_1+\cdots+p_r+q_1+\cdots+q_r,\mathbb{C})$. Using (5.5) and see-saw reciprocity (see for example [GW98], Section 9.2), we have

$$c^0_{(\underline{\mu_+},\underline{\mu_-})} := [\widetilde{F}_{p_1,q_1}^{(\mu_+^{(1)},\mu_-^{(1)})} \otimes \cdots \otimes \widetilde{F}_{p_n,q_n}^{(\mu_+^{(r)},\mu_-^{(r)})} : \widetilde{F}^0_{p_1+\cdots+p_r,q_1+\cdots+q_r}]. \tag{5.8}$$

To compute the branching multiplicity on the RHS of (5.8), it suffices to understand the modules $\tilde{F}_{p_i,q_i}^{(\mu_+^{(i)},\mu_-^{(i)})}$ as $GL(p_i,\mathbb{C}) \times GL(q_i,\mathbb{C})$-modules. If we stay in the stable range, i.e. $n \geq \sum p_i + q_i$, we can use (5.6). The RHS of (5.8) is then computable using contingency tables. This of course begs the questions of what happens outside the stable range, and this is where the technical difficulties arise in this approach. We conclude this section by suggesting the following problems for future research:

Problem 5.7.
(1) Compute general tensor product multiplicities for other classical groups via contingency tables, e.g. $O(n,\mathbb{C})_\Delta \subset O(n,\mathbb{C}) \times \cdots \times O(n,\mathbb{C})$, $Sp(2n,\mathbb{C})_\Delta \subset Sp(2n,\mathbb{C}) \times \cdots \times Sp(2n,\mathbb{C})$. (See Remark 5.4.)
(2) Compute generalized L-R coefficients outside of the stable range, i.e. $n < \sum_{i=1}^r p_i + q_i$.

Outside of the stable range the $(\mathfrak{gl}(p+q,\mathbb{C}), GL(p,\mathbb{C}) \times GL(q,\mathbb{C}))$-modules $\widetilde{F}_{p,q}^{(\lambda_+,\lambda_-)}$ that appear in (5.5) are no longer generalized Verma modules. However, they are resolved by sums of generalized Verma modules (see Theorem 2, [EW04]). In [EW04], the authors use these resolutions to compute L-R coefficients $[F_n^\lambda : F_n^\mu \otimes F_n^\gamma]$ outside of the stable range. We believe that we can use [EW04] and the methods outlined here to compute *generalized* L-R coeffecients outside of the stable range for different classical groups.

References

[AM07] A. Alekseev and E. Meinrenken, *Ginzburg-Weinstein via Gelfand-Zeitlin*, J. Differential Geom. **76** (2007), no. 1, 1–34.

[AR12] Jean Avan and Eric Ragoucy, *A new dynamical reflection algebra and related quantum integrable systems*, Lett. Math. Phys. **101** (2012), no. 1, 85–101.

[BB81] Alexandre Beĭlinson and Joseph Bernstein, *Localisation de $\mathfrak{g}$-modules*, C. R. Acad. Sci. Paris Sér. I Math. **292** (1981), no. 1, 15–18.

[BB85] W. Borho and J.-L. Brylinski, *Differential operators on homogeneous spaces. III. Characteristic varieties of Harish-Chandra modules and of primitive ideals*, Invent. Math. **80** (1985), no. 1, 1–68.

[CEa] Mark Colarusso and Sam Evens, *Eigenvalue coincidences and K-orbits, I*, to appear in the Journal of Algebra, 2014, 22 pages.

[CEb] Mark Colarusso and Sam Evens, *Eigenvalue coincidences and multiplicity free spherical pairs*, submitted to arxiv, October 15, 2014, 36 pages.

[CEc] Mark Colarusso and Sam Evens, *The Gelfand-Zeitlin integrable system and K-orbits on the flag variety*, to appear in: "Symmetry: Representation Theory and its Applications: ," Progr. Math. Birkauser, Boston, 2014, 36 pages.

[CE10] Mark Colarusso and Sam Evens, *On algebraic integrability of Gelfand-Zeitlin fields*, Transform. Groups **15** (2010), no. 1, 46–71.

[CE12] Mark Colarusso and Sam Evens, *K-orbits on the flag variety and strongly regular nilpotent matrices*, Selecta Math. (N.S.) **18** (2012), no. 1, 159–177.

[CG97] Neil Chriss and Victor Ginzburg, *Representation theory and complex geometry*, Birkhäuser Boston Inc., Boston, MA, 1997.

[Col09] Mark Colarusso, *The Gelfand-Zeitlin integrable system and its action on generic elements of $\mathfrak{gl}(n)$ and $\mathfrak{so}(n)$*, New Developments in Lie Theory and Geometry (Cruz Chica, Córdoba, Argentina, 2007), Contemp. Math., vol. 491, Amer. Math. Soc., Providence, RI, 2009, pp. 255–281.

[Col11] Mark Colarusso, *The orbit structure of the Gelfand-Zeitlin group on $n \times n$ matrices*, Pacific J. Math. **250** (2011), no. 1, 109–138.

[DCKP93] C. De Concini, V. G. Kac, and C. Procesi, *Some remarkable degenerations of quantum groups*, Comm. Math. Phys. **157** (1993), no. 2, 405–427.

[DEFK10] Anastasia Doikou, Stefano Evangelisti, Giovanni Feverati, and Nikos Karaiskos, *Introduction to quantum integrability*, Internat. J. Modern Phys. A **25** (2010), no. 17, 3307–3351.

[DFO94] Yu. A. Drozd, V. M. Futorny, and S. A. Ovsienko, *Harish-Chandra subalgebras and Gel′fand-Zetlin modules*, Finite-dimensional algebras and related topics (Ottawa, ON, 1992), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 424, Kluwer Acad. Publ., Dordrecht, 1994, pp. 79–93.

[DK85] Jiri Dadok and Victor Kac, *Polar representations*, J. Algebra **92** (1985), no. 2, 504–524.

[Dri87] V. G. Drinfel′d, *Quantum groups*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986) (Providence, RI), Amer. Math. Soc., 1987, pp. 798–820.

[EHW83] T. Enright, R. Howe, and N.R. Wallach, *A classification of unitary highest weight modules*, Representation theory of reductive groups (Park City, Utah, 1982), Birkhäuser Boston, Boston, Mass., 1983, pp. 97–143.

[Eis95] David Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995, With a view toward algebraic geometry.

[EL07] Sam Evens and Jiang-Hua Lu, *Poisson geometry of the Grothendieck resolution of a complex semisimple group*, Mosc. Math. J. **7** (2007), no. 4, 613–642.

[ES01] Pavel Etingof and Olivier Schiffmann, *Lectures on the dynamical Yang-Baxter equations*, Quantum groups and Lie theory (Durham, 1999), London Math. Soc. Lecture Note Ser., vol. 290, Cambridge Univ. Press, Cambridge, 2001, pp. 89–129.

[Eti07] Pavel Etingof, *Calogero-Moser systems and representation theory*, Zurich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich, 2007.

[EW04] Thomas J. Enright and Jeb F. Willenbring, *Hilbert series, Howe duality and branching for classical groups*, Ann. of Math. (2) **159** (2004), no. 1, 337–375.

[Fer90] S. L. Fernando, *Lie algebra modules with finite-dimensional weight spaces. I*, Trans. Amer. Math. Soc. **322** (1990), no. 2, 757–781.

[FGRa] Vyacheslav Futorny, Dimitar Grantcharov, and Luis Enrique Ramirez, *Generic irreducible gelfand-tsetlin modules for* $\mathfrak{gl}(n)$, preprint:arXiv:1409.8413, 13 pages.

[FGRb] Vyacheslav Futorny, Dimitar Grantcharov, and Luis Enrique Ramirez, *Singular gelfand-zeitlin modules of* $\mathfrak{gl}(n, \mathbb{C})$, preprint:arXiv:1409.0550, 23 pages.

[FO05] Vyacheslav Futorny and Serge Ovsienko, *Kostant's theorem for special filtered algebras*, Bull. London Math. Soc. **37** (2005), no. 2, 187–199.

[FO07] Vyacheslav Futorny and Serge Ovsienko, *Harish-Chandra categories*, Proceedings of the XVIth Latin American Algebra Colloquium (Spanish), Bibl. Rev. Mat. Iberoamericana, Rev. Mat. Iberoamericana, Madrid, 2007, pp. 271–286.

[FO14] Vyacheslav Futorny and Serge Ovsienko, *Fibers of characters in Gelfand-Tsetlin categories*, Trans. Amer. Math. Soc. **366** (2014), no. 8, 4173–4208.

[FOS11] Vyacheslav Futorny, Serge Ovsienko, and Manuel Saorin, *Gelfand-Tsetlin categories*, Groups, algebras and applications, Contemp. Math., vol. 537, Amer. Math. Soc., Providence, RI, 2011, pp. 193–203.

[FR96] Hermann Flaschka and Tudor Ratiu, *A convexity theorem for Poisson actions of compact Lie groups*, Ann. Sci. École Norm. Sup. (4) **29** (1996), no. 6, 787–809.

[GC50a] I. M. Gel′fand and M. L. Cetlin, *Finite-dimensional representations of groups of orthogonal matrices*, Doklady Akad. Nauk SSSR (N.S.) **71** (1950), 1017–1020.

[GC50b] I. M. Gel′fand and M. L. Cetlin, *Finite-dimensional representations of the group of unimodular matrices*, Doklady Akad. Nauk SSSR (N.S.) **71** (1950), 825–828.

[GS83] V. Guillemin and S. Sternberg, *The Gel′fand-Cetlin system and quantization of the complex flag manifolds*, J. Funct. Anal. **52** (1983), no. 1, 106–128.

[GW92] Viktor L. Ginzburg and Alan Weinstein, *Lie-Poisson structure on some Poisson Lie groups*, J. Amer. Math. Soc. **5** (1992), no. 2, 445–453.

[GW98] Roe Goodman and Nolan R. Wallach, *Representations and invariants of the classical groups*, Encyclopedia of Mathematics and its Applications, vol. 68, Cambridge University Press, Cambridge, 1998.

[GY09] Michael Gekhtman and Milen Yakimov, *Completeness of determinantal hamiltonian flows on the matrix affine poisson space*, Lett. Math. Phys. (2009), 13 pages.

[Har06] Megumi Harada, *The symplectic geometry of the Gel′fand-Cetlin-Molev basis for representations of* $\mathrm{Sp}(2n, \mathbb{C})$, J. Symplectic Geom. **4** (2006), no. 1, 1–41.

[Has04] Takashi Hashimoto, B_{n-1}*-orbits on the flag variety* GL_n/B_n, Geom. Dedicata **105** (2004), 13–27.

[HL93] Timothy J. Hodges and Thierry Levasseur, *Primitive ideals of* $\mathbf{C}_q[\mathrm{SL}(3)]$, Comm. Math. Phys. **156** (1993), no. 3, 581–605.

[How89] Roger Howe, *Remarks on classical invariant theory*, Trans. Amer. Math. Soc. **313** (1989), no. 2, 539–570.

[HTW05] Roger Howe, Eng-Chye Tan, and Jeb F. Willenbring, *Stable branching rules for classical symmetric pairs*, Trans. Amer. Math. Soc. **357** (2005), no. 4, 1601–1626 (electronic).

[Jan83] Jens Carsten Jantzen, *Einhüllende Algebren halbeinfacher Lie-Algebren*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 3, Springer-Verlag, Berlin, 1983.

[JL94] Anthony Joseph and Gail Letzter, *Separation of variables for quantized enveloping algebras*, Amer. J. Math. **116** (1994), no. 1, 127–177.

[Jos95] Anthony Joseph, *Quantum groups and their primitive ideals*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 29, Springer-Verlag, Berlin, 1995.

[Kho05] O. Khomenko, *Some applications of Gelfand-Zetlin modules*, Representations of algebras and related topics, Fields Inst. Commun., vol. 45, Amer. Math. Soc., Providence, RI, 2005, pp. 205–213.

[Kin71] R. C. King, *Modification rules and products of irreducible representations of the unitary, orthogonal, and symplectic groups*, J. Mathematical Phys. **12** (1971), 1588–1598.

[Kin90] Ronald C. King, *S-functions and characters of Lie algebras and Lie groups*, Topics in algebra, Part 2 (Warsaw, 1988), Banach Center Publ., vol. 26, PWN, Warsaw, 1990, pp. 327–344.

[Kno90] Friedrich Knop, *Weylgruppe und Momentabbildung*, Invent. Math. **99** (1990), no. 1, 1–23.

[Kos63] Bertram Kostant, *Lie group representations on polynomial rings*, Amer. J. Math. **85** (1963), 327–404.

[Kos79] Bertram Kostant, *The solution to a generalized Toda lattice and representation theory*, Adv. in Math. **34** (1979), no. 3, 195–338.

[Kos09] Bertram Kostant, *Fomenko-Mischenko theory, Hessenberg varieties, and polarizations*, Lett. Math. Phys. **90** (2009), no. 1-3, 253–285.

[KR71] B. Kostant and S. Rallis, *Orbits and representations associated with symmetric spaces*, Amer. J. Math. **93** (1971), 753–809.

[Kre07] A. Kresch, *Flag varieties and Schubert calculus*, Algebraic groups, Universitätsverlag Göttingen, Göttingen, 2007, pp. 73–86.

[KS98] Leonid I. Korogodski and Yan S. Soibelman, *Algebras of functions on quantum groups. Part I*, Mathematical Surveys and Monographs, vol. 56, American Mathematical Society, Providence, RI, 1998.

[KT99] Allen Knutson and Terence Tao, *The honeycomb model of* $\mathrm{GL}_n(\mathbf{C})$ *tensor products. I. Proof of the saturation conjecture*, J. Amer. Math. Soc. **12** (1999), no. 4, 1055–1090.

[KT01] Allen Knutson and Terence Tao, *Honeycombs and sums of Hermitian matrices*, Notices Amer. Math. Soc. **48** (2001), no. 2, 175–186.

[KTW04] Allen Knutson, Terence Tao, and Christopher Woodward, *The honeycomb model of* $\mathrm{GL}_n(\mathbb{C})$ *tensor products. II. Puzzles determine facets of the Littlewood-Richardson cone*, J. Amer. Math. Soc. **17** (2004), no. 1, 19–48.

[KV78] M. Kashiwara and M. Vergne, *On the Segal-Shale-Weil representations and harmonic polynomials*, Invent. Math. **44** (1978), no. 1, 1–47.

[KW06a] Bertram Kostant and Nolan Wallach, *Gelfand-Zeitlin theory from the perspective of classical mechanics. I*, Studies in Lie theory, Progr. Math., vol. 243, Birkhäuser Boston, Boston, MA, 2006, pp. 319–364.

[KW06b] Bertram Kostant and Nolan Wallach, *Gelfand-Zeitlin theory from the perspective of classical mechanics. II*, The unity of mathematics, Progr. Math., vol. 244, Birkhäuser Boston, Boston, MA, 2006, pp. 387–420.

[LW90] Jiang-Hua Lu and Alan Weinstein, *Poisson Lie groups, dressing transformations, and Bruhat decompositions*, J. Differential Geom. **31** (1990), no. 2, 501–526.

[Mat86] Hideyuki Matsumura, *Commutative ring theory*, Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1986, Translated from the Japanese by M. Reid.

[Mat00] Olivier Mathieu, *Classification of irreducible weight modules*, Ann. Inst. Fourier (Grenoble) **50** (2000), no. 2, 537–592.

[Maz01] Volodymyr Mazorchuk, *On Gelfand-Zetlin modules over orthogonal Lie algebras*, Algebra Colloq. **8** (2001), no. 3, 345–360.

[Mil93] Dragan Miličić, *Algebraic $\mathcal{D}$-modules and representation theory of semisimple Lie groups*, The Penrose transform and analytic cohomology in representation theory (South Hadley, MA, 1992), Contemp. Math., vol. 154, Amer. Math. Soc., Providence, RI, 1993, pp. 133–168.

[MŌ90] Toshihiko Matsuki and Toshio Ōshima, *Embeddings of discrete series into principal series*, The orbit method in representation theory (Copenhagen, 1988), Progr. Math., vol. 82, Birkhäuser Boston, Boston, MA, 1990, pp. 147–175.

[MO98] V. S. Mazorchuk and S. A. Ovsienko, *Submodule structure of generalized Verma modules induced from generic Gelfand-Zetlin modules*, Algebr. Represent. Theory **1** (1998), no. 1, 3–26.

[Mol07] Alexander Molev, *Yangians and classical Lie algebras*, Mathematical Surveys and Monographs, vol. 143, American Mathematical Society, Providence, RI, 2007.

[MT00] Volodymyr Mazorchuk and Lyudmyla Turowska, *On Gelfand-Zetlin modules over $U_q(\mathrm{gl}_n)$*, Czechoslovak J. Phys. **50** (2000), no. 1, 139–144, Quantum groups and integrable systems (Prague, 1999).

[Ovs02] Serge Ovsienko, *Finiteness statements for Gelfand-Zetlin modules*, Third International Algebraic Conference in the Ukraine (Ukrainian), Natsīonal. Akad. Nauk Ukraïni, Īnst. Mat., Kiev, 2002, pp. 323–338.

[Ovs03] Serge Ovsienko, *Strongly nilpotent matrices and Gelfand-Zetlin modules*, Linear Algebra Appl. **365** (2003), 349–367, Special issue on linear algebra methods in representation theory.

[Pan90] D. I. Panyushev, *Complexity and rank of homogeneous spaces*, Geom. Dedicata **34** (1990), no. 3, 249–269.

[PS08] Beresford Parlett and Gilbert Strang, *Matrices with prescribed Ritz values*, Linear Algebra Appl. **428** (2008), no. 7, 1725–1739.

[PS12] Ivan Penkov and Vera Serganova, *On bounded generalized Harish-Chandra modules*, Ann. Inst. Fourier (Grenoble) **62** (2012), no. 2, 477–496.

[PSZ04] Ivan Penkov, Vera Serganova, and Gregg Zuckerman, *On the existence of $(\mathfrak{g}, \mathfrak{l})$-modules of finite type*, Duke Math. J. **125** (2004), no. 2, 329–349.

[PZ04a] Ivan Penkov and Gregg Zuckerman, *Generalized Harish-Chandra modules: a new direction in the structure theory of representations*, Acta Appl. Math. **81** (2004), no. 1-3, 311–326.

[PZ04b] Ivan Penkov and Gregg Zuckerman, *Generalized Harish-Chandra modules with generic minimal $\mathfrak{k}$-type*, Asian J. Math. **8** (2004), no. 4, 795–811.

[PZ07] Ivan Penkov and Gregg Zuckerman, *Construction of generalized Harish-Chandra modules with arbitrary minimal $\mathfrak{t}$-type*, Canad. Math. Bull. **50** (2007), no. 4, 603–609.

[PZ12] Ivan Penkov and Gregg Zuckerman, *On the structure of the fundamental series of generalized Harish-Chandra modules*, Asian J. Math. **16** (2012), no. 3, 489–514.

[Ram12] Luis Enrique Ramirez, *Combinatorics of irreducible Gelfand-Tsetlin $\mathfrak{sl}(3)$-modules*, Algebra Discrete Math. **14** (2012), no. 2, 276–296.

[Ric79] R. W. Richardson, *The conjugating representation of a semisimple group*, Invent. Math. **54** (1979), no. 3, 229–245.

[RS90] R. W. Richardson and T. A. Springer, *The Bruhat order on symmetric varieties*, Geom. Dedicata **35** (1990), no. 1-3, 389–436.

[RS93] R. W. Richardson and T. A. Springer, *Combinatorics and geometry of K-orbits on the flag manifold*, Linear algebraic groups and their representations (Los Angeles, CA, 1992), Contemp. Math., vol. 153, Amer. Math. Soc., Providence, RI, 1993, pp. 109–142.

[Sch91] Wilfried Schmid, *Construction and classification of irreducible Harish-Chandra modules*, Harmonic analysis on reductive groups (Brunswick, ME, 1989), Progr. Math., vol. 101, Birkhäuser Boston, Boston, MA, 1991, pp. 235–275.

[Sch05] Wilfried Schmid, *Geometric methods in representation theory*, Poisson geometry, deformation quantisation and group representations, London Math. Soc. Lecture Note Ser., vol. 323, Cambridge Univ. Press, Cambridge, 2005, Lecture notes taken by Matvei Libine, pp. 273–323.

[SP09] Noam Shomron and Beresford N. Parlett, *Linear algebra meets Lie algebra: the Kostant-Wallach theory*, Linear Algebra Appl. **431** (2009), no. 10, 1745–1767.

[Vog83] David A. Vogan, *Irreducible characters of semisimple Lie groups. III. Proof of Kazhdan-Lusztig conjecture in the integral case*, Invent. Math. **71** (1983), no. 2, 381–417.

[Vog91] David A. Vogan, Jr., *Associated varieties and unipotent representations*, Harmonic analysis on reductive groups (Brunswick, ME, 1989), Progr. Math., vol. 101, Birkhäuser Boston, Boston, MA, 1991, pp. 315–388.

[Zuc12] Gregg Zuckerman, *Generalized Harish-Chandra modules*, Highlights in Lie algebraic methods, Progr. Math., vol. 295, Birkhäuser/Springer, New York, 2012, pp. 123–143.