EIGENVALUE COINCIDENCES AND MULTIPLICITY FREE SPHERICAL PAIRS

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Abstract. In recent work, we related the structure of subvarieties of $n \times n$ complex matrices defined by eigenvalue coincidences to $GL(n-1,\mathbb{C})$-orbits on the flag variety of $\mathfrak{gl}(n,\mathbb{C})$. In the first part of this paper, we extend these results to the complex orthogonal Lie algebra $\mathfrak{g} = \mathfrak{so}(n,\mathbb{C})$. In the second part of the paper, we use these results to study the geometry and invariant theory of the K-action on $\mathfrak{g}$, in the cases where $(\mathfrak{g}, K)$ is $(\mathfrak{gl}(n,\mathbb{C}), GL(n-1,\mathbb{C}))$ or $(\mathfrak{so}(n,\mathbb{C}), SO(n-1,\mathbb{C}))$. We study the geometric quotient $\mathfrak{g} \to \mathfrak{g}//K$ and describe the closed K-orbits on $\mathfrak{g}$ and the structure of the zero fibre. We also prove that for $x \in \mathfrak{g}$, the K-orbit $\mathrm{Ad}(K)\cdot x$ has maximal dimension if and only if the algebraically independent generators of the invariant ring $\mathbb{C}[\mathfrak{g}]^K$ are linearly independent at x, which extends a theorem of Kostant. We give applications of our results to the Gelfand-Zeitlin system.

1. Introduction

This paper studies two related questions. Let $x \in \mathfrak{gl}(n,\mathbb{C})$ be an $n \times n$ complex matrix, and let $x_{\mathfrak{k}} \in \mathfrak{gl}(n-1,\mathbb{C})$ be the $(n-1)\times(n-1)$ submatrix in the upper left corner of x. In [CE15], we studied the subvariety of $\mathfrak{gl}(n,\mathbb{C})$ consisting of matrices x such that x and $x_{\mathfrak{k}}$ have a specified number of eigenvalues in common. In the first part of the paper, we extend these results from $\mathfrak{gl}(n,\mathbb{C})$ to $\mathfrak{so}(n,\mathbb{C})$. In the second part of the paper, we use the results from [CE15] and the first part of the paper to study the action of K on $\mathfrak{g}$ by conjugation in the two cases ($K = GL(n-1,\mathbb{C}), \mathfrak{g} = \mathfrak{gl}(n,\mathbb{C})$) and ($K = SO(n-1,\mathbb{C}), \mathfrak{g} = \mathfrak{so}(n,\mathbb{C})$). By a theorem of Knop [Kno94], the algebra $\mathbb{C}[\mathfrak{g}]^K = \mathbb{C}[\mathfrak{g}]^G \otimes \mathbb{C}[\mathfrak{k}]^K$ is a polynomial algebra. It follows that the quotient morphism $\mathfrak{g} \to \mathfrak{g}//K$ can be identified with a morphism Φ_n from $\mathfrak{g}$ to affine space, which is a partial version of a morphism considered by Kostant and Wallach [KW06a, KW06b]. We study this morphism, and as a consequence, we determine explicitly the closed K-orbits on $\mathfrak{g}$ and the structure of the zero fibre. We also prove a variant of Kostant's theorem using linear independence of differentials to characterize regular elements [Kos63]. This variant of Kostant's theorem allows us to give a simpler definition of the strongly regular elements of $\mathfrak{g}$, which were introduced by Kostant and Wallach to construct the Gelfand-Zeitlin integrable system on $\mathfrak{g}$ ([KW06a], [Col09]). We use this simpler definition to establish new results about the Gelfand-Zeitlin system on $\mathfrak{so}(n,\mathbb{C})$.

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In more detail, let $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$ and let $\mathfrak{k} \subset \mathfrak{g}$ be the symmetric subalgebra $\mathfrak{k} = \mathfrak{so}(n-1, \mathbb{C})$ fixed by an involution θ of $\mathfrak{g}$. Let r_n and r_{n-1} be the ranks of $\mathfrak{g}$ and $\mathfrak{k}$ respectively. Recall that if a is an eigenvalue of $x \in \mathfrak{so}(n, \mathbb{C})$, then $-a$ is also an eigenvalue of x, and if n is odd, then the eigenvalue 0 of x occurs an odd number of times. Let

$$\sigma(x) = \{\pm b_1, \dots, \pm b_{r_n}\} \tag{1.1}$$

be the eigenvalues of x, listed with multiplicity, except that if n is odd, we only list the eigenvalue 0 $2j$ times if it appears with multiplicity $2j+1$. We call $\sigma(x)$ the spectrum of x. For $x \in \mathfrak{g}$, let $x = x_{\mathfrak{k}} + x_{\mathfrak{p}}$ where $x \in \mathfrak{k}$ and $x_{\mathfrak{p}} \in \mathfrak{g}^{-\theta}$, and let $\sigma(x_{\mathfrak{k}}) = \{\pm a_1, \dots, \pm a_{r_{n-1}}\}$ be the spectrum of $x_{\mathfrak{k}}$, regarded as an element of $\mathfrak{so}(n-1, \mathbb{C})$. We consider the *eigenvalue coincidence* varieties $\mathfrak{g}(\geq i)$ consisting of $x \in \mathfrak{g}$ such that $\sigma(x)$ and $\sigma(x_{\mathfrak{k}})$ share at least $2i$ elements, counting multiplicity. More precisely, for $i = 0, \dots, r_{n-1}$,

$$\mathfrak{g}(\geq i) := \{x \in \mathfrak{g} : b_{j_m} = \pm a_{k_m},\ m = 1, \dots, i \text{ with } j_r \neq j_s \text{ and } k_r \neq k_s \text{ for } r \neq s\}. \tag{1.2}$$

For $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ and $\mathfrak{k} = \mathfrak{gl}(n-1, \mathbb{C})$ thought of as the $(n-1) \times (n-1)$ upper left corner of $\mathfrak{g}$, the analogous varieties were studied in [CE15]. We let $\sigma(x)$ denote the eigenvalues of x, and let $\sigma(x_{\mathfrak{k}})$ denote the eigenvalues of $x_{\mathfrak{k}}$ regarded as an element of $\mathfrak{gl}(n-1, \mathbb{C})$. In this case, the variety $\mathfrak{g}(\geq i)$ consists of elements x such that $\sigma(x)$ and $\sigma(x_{\mathfrak{k}})$ share at least i elements, counted with multiplicity.

We make use of the following notation throughout. We denote the flag variety of a reductive Lie algebra $\mathfrak{g}$ by $\mathcal{B}_{\mathfrak{g}}$, or by $\mathcal{B}$ when $\mathfrak{g}$ is understood. For a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$, we denote its K-orbit in $\mathcal{B}$ by $Q = K \cdot \mathfrak{b}$. We denote the K-saturation of $\mathfrak{b}$ in $\mathfrak{g}$ by $Y_Q := \mathrm{Ad}(K)\mathfrak{b} = \{\mathrm{Ad}(k)x : k \in K, x \in \mathfrak{b}\}$. Note that the variety Y_Q depends only on the K-orbit Q in $\mathcal{B}$. We prove the following result.

Theorem 1.1. *The irreducible component decomposition of the variety* $\mathfrak{g}(\geq i)$ *is given by*

$$\mathfrak{g}(\geq i) = \bigcup_{codim(Q)=i} \overline{Y_Q}. \tag{1.3}$$

In particular, if $\mathfrak{g} = \mathfrak{so}(2l, \mathbb{C})$ *is type D then the varieties* $\mathfrak{g}(\geq i)$ *are all irreducible. If* $\mathfrak{g} = \mathfrak{so}(2l+1, \mathbb{C})$ *is type B then* $\mathfrak{g}(\geq i)$ *is irreducible for* $i = 0, \dots, l-1$ *and has exactly two irreducible components when* $i = l$.

Although this statement is similar to Theorem 1.1 of [CE15], the proof requires some significant new ideas, because computations analogous to those performed in [CE15] for $\mathfrak{gl}(n, \mathbb{C})$ are intractable for $\mathfrak{so}(n, \mathbb{C})$.

In the second part of the paper, we consider the pairs (G, K) given by $G = GL(n, \mathbb{C})$, $K = GL(n-1, \mathbb{C})$ and $G = SO(n, \mathbb{C})$, $K = SO(n-1, \mathbb{C})$, which are essentially the only symmetric pairs for which the branching rule for finite dimensional representations from G to K is multiplicity free. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the corresponding Lie algebras. For each pair, let $\tilde{G} = G \times K$ and let K_{Δ} be the diagonal embedding of K in $\tilde{G}$. It is standard that K_{Δ} is a spherical subgroup of $\tilde{G}$ (Proposition 4.6). We consider the coisotropy representation of K_{Δ} on $\tilde{\mathfrak{g}}/\mathfrak{k}_{\Delta}$, which coincides with the adjoint action of K on $\mathfrak{g}$. We say that $x \in \mathfrak{g}$ is

n-strongly regular if x is in a K-orbit of maximal dimension. We write the generators of $\mathbb{C}[\mathfrak{g}]^K$ as $\{f_{n-1,1}, \ldots, f_{n-1,r_{n-1}}; f_{n,1}, \ldots, f_{n,r_n}\}$ (for $\mathfrak{gl}(i,\mathbb{C})$, $r_i = i$). The following theorem extends a basic result of Kostant [Kos63].

Theorem 1.2. *[Theorem 4.8] An element $x \in \mathfrak{g}$ is n-strongly regular if and only if*

$$df_{n-1,1}(x) \wedge \cdots \wedge df_{n-1,r_{n-1}}(x) \wedge df_{n,1}(x) \wedge \cdots \wedge df_{n,r_n}(x) \neq 0. \tag{1.4}$$

Using Theorem 1.2, we show that the Zariski open set $\mathfrak{g}(0) = \{x \in \mathfrak{g} : \sigma(x) \cap \sigma(x_{\mathfrak{k}}) = \emptyset\}$ consists entirely of n-strongly regular elements (Theorem 4.16) and use it to show that for $x \in \mathfrak{g}(0)$, the fibre $\Phi_n^{-1}(\Phi_n(x))$ is a single K-orbit (Corollary 4.17). Using this result along with Theorem 1.1 and Theorem 3.7 of [CE15], we give explicit representatives for all closed K-orbits on $\mathfrak{g}$ (Theorem 4.20) and determine the nilfibre $\Phi_n^{-1}(0)$ (Theorem 4.23). In contrast to the case of $\mathfrak{gl}(n,\mathbb{C})$, we show that for $\mathfrak{so}(n,\mathbb{C})$ the fibre $\Phi_n^{-1}(0)$ contains no n-strongly regular elements (Proposition 4.24 and Corollary 4.25).

This work is motivated by our interest in the Gelfand-Zeitlin system, which is a maximal Poisson commutative family J_{GZ} in $\mathbb{C}[\mathfrak{g}]$ defined using a family of subalgebras $\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots \subset \mathfrak{g}_{n-1} \subset \mathfrak{g}_n$, where $\mathfrak{g}_i = \mathfrak{gl}(i,\mathbb{C})$ in the general linear case, and $\mathfrak{g}_i = \mathfrak{so}(i,\mathbb{C})$ in the orthogonal case. Elements of $\mathfrak{g}$ for which the differentials $\{x \in \mathfrak{g} : df(x),\ f \in J_{GZ}\}$ are linearly independent are called *strongly regular*. In [KW06a], Kostant and Wallach show that any regular adjoint orbit in $\mathfrak{gl}(n,\mathbb{C})$ contains strongly regular elements which implies that the Gelfand-Zeitlin system is completely integrable on any regular adjoint orbit. Using different techniques, the first author produced strongly regular elements in certain regular semisimple orbits of $\mathfrak{so}(n,\mathbb{C})$ and proved the integrability of the Gelfand-Zeitlin system on these orbits [Col09]. The case of $\mathfrak{gl}(n,\mathbb{C})$ is much better understood than the case of $\mathfrak{so}(n,\mathbb{C})$, because the $\mathfrak{so}(n,\mathbb{C})$ Gelfand-Zeitlin system is less amenable to computation. We hope that our methods will make the Gelfand-Zeitlin system for $\mathfrak{so}(n,\mathbb{C})$ more tractable to understand and will improve our understanding of the Gelfand-Zeitlin system for $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{C})$. As a step in this direction, we observe that Theorem 1.2 can be used to simplify the criterion for an element of $\mathfrak{gl}(n,\mathbb{C})$ and $\mathfrak{so}(n,\mathbb{C})$ to be strongly regular from [KW06a, Col09] (Proposition 4.12). This allows us to identify a previously unknown set of strongly regular elements of $\mathfrak{so}(n,\mathbb{C})$ (Proposition 4.18), and in later work, we will show that every regular adjoint of $\mathfrak{so}(n,\mathbb{C})$ orbit contains strongly regular elements, implying the integrability of the Gelfand-Zeitlin system on all regular adjoint orbits of $\mathfrak{so}(n,\mathbb{C})$. We can also show that in contrast to the case of $\mathfrak{gl}(n,\mathbb{C})$ there are no strongly regular elements $x \in \mathfrak{so}(n,\mathbb{C})$ with the property that $f(x) = 0$ for all $f \in J_{GZ}$ (Remark 4.28). These observations were previously inaccessible using the more computational methods of [Col09, Col11].

We also plan to apply results of this paper to study the Gelfand-Zeitlin modules introduced by Drozd, Futorny, and Ovsienko [DFO94], which are quantum analogues of the Gelfand-Zeitlin integrable systems. Our results in this paper develop parts of a Kostant-Rallis theory [KR71] for the spherical pair $(\tilde{\mathfrak{g}}, K_\Delta)$, and we expect it to play an important role in understanding a category of Harish-Chandra modules for these spherical pairs, especially through the study of associated varieties. Using an equivalence of categories analogous to the equivalence between category $\mathcal{O}$ and certain Harish-Chandra modules ([BB85], Section 3.4), we plan to use $(\tilde{\mathfrak{g}}, K_{\Delta})$-modules in our future work to produce examples of Gelfand-Zeitlin modules and other closely related modules.

This paper is organized as follows. The first part of the paper comprises Sections 2 and 3. In Section 2, we establish a number of preliminary results. In Section 3, we prove Theorem 1.1. The second part of the paper consists of Section 4, in which we prove Theorem 1.2, determine the closed K-orbits on $\mathfrak{g}$, and discuss applications to strongly regular elements. In the appendix, we give an alternative, simpler proof of a theorem of Knop in a special case.

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2. Preliminaries

We recall basic facts concerning orthogonal Lie algebras, and develop some basic framework for the study of eigenvalue coincidence varieties. We also classify the $K = SO(n-1, \mathbb{C})$-orbits on the flag variety $\mathcal{B}$ of $\mathfrak{so}(n, \mathbb{C})$ and give explicit representatives for each orbit.

2.1. **Realization of Orthogonal Lie algebras.** We give explicit descriptions of standard Cartan subalgebras and corresponding root systems of $\mathfrak{so}(n, \mathbb{C})$. Our exposition follows Chapters 1 and 2 of [GW98].

Let β be the non-degenerate, symmetric bilinear form on $\mathbb{C}^n$ given by

$$\beta(x, y) = x^T S_n y, \tag{2.1}$$

where x, y are $n \times 1$ column vectors and S_n is the $n \times n$ matrix:

$$S_n = \begin{bmatrix} 0 & \dots & \dots & 0 & 1 \\ \vdots & & & 1 & 0 \\ \vdots & & \iddots & & \vdots \\ 0 & 1 & \dots & 0 & \vdots \\ 1 & 0 & \dots & \dots & 0 \end{bmatrix} \tag{2.2}$$

with ones down the skew diagonal and zeroes elsewhere. The special orthogonal group is

$$SO(n, \mathbb{C}) := \{g \in SL(n, \mathbb{C}) : \beta(gx, gy) = \beta(x, y) \; \forall x, y \in \mathbb{C}^n\}.$$

Its Lie algebra is

$$\mathfrak{so}(n, \mathbb{C}) = \{Z \in \mathfrak{gl}(n, \mathbb{C}) : \beta(Zx, y) = -\beta(x, Zy) \, \forall \, x, y \in \mathbb{C}^n\}.$$

For our purposes, it will be convenient to have explicit matrix descriptions of $\mathfrak{so}(n, \mathbb{C})$. We consider the cases where n is odd and even separately. Throughout, we denote the standard basis of $\mathbb{C}^n$ by $\{e_1, \dots, e_n\}$.

2.1.1. *Realization of* $\mathfrak{so}(2l,\mathbb{C})$. Let $\mathfrak{g} = \mathfrak{so}(2l,\mathbb{C})$ be of type D. The subalgebra of diagonal matrices $\mathfrak{h} := \mathrm{diag}[a_1,\dots,a_l,-a_l,\dots,-a_1]$, $a_i \in \mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}$. We refer to $\mathfrak{h}$ as the *standard Cartan subalgebra*. Let $\epsilon_i \in \mathfrak{h}^*$ be the linear functional $\epsilon_i(\mathrm{diag}[a_1,\dots,a_l,-a_l,\dots,-a_1]) = a_i$, and let $\Phi(\mathfrak{g},\mathfrak{h})$ be the roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. It is well-known that

$$\Phi(\mathfrak{g},\mathfrak{h}) = \{\epsilon_i - \epsilon_j,\, \pm(\epsilon_i+\epsilon_j) :\ 1 \le i \ne j \le l\}. \tag{2.3}$$

We take as our *standard positive roots* the set:

$$\Phi^+(\mathfrak{g},\mathfrak{h}) := \{\epsilon_i - \epsilon_j,\, \epsilon_i+\epsilon_j :\ 1 \le i < j \le l\}. \tag{2.4}$$

with corresponding simple roots

$$\Pi := \{\alpha_1,\dots,\alpha_{l-1},\alpha_l\} \text{ where } \alpha_i = \epsilon_i - \epsilon_{i+1},\ i=1,\dots,l-1,\ \alpha_l = \epsilon_{l-1}+\epsilon_l. \tag{2.5}$$

The *standard* Borel subalgebra $\mathfrak{b}_+ := \bigoplus_{\alpha\in\Phi^+(\mathfrak{g},\mathfrak{h})} \mathfrak{g}_\alpha$ is easily seen to be the set of upper triangular matrices in $\mathfrak{g}$.

For the purposes of computations with $\mathfrak{so}(2l,\mathbb{C})$, it is convenient to relabel part of the standard basis of $\mathbb{C}^n$ as $e_{-j} := e_{2l+1-j}$ for $j = 1,\dots,l$.

2.1.2. *Realization of* $\mathfrak{so}(2l+1,\mathbb{C})$. Let $\mathfrak{g} = \mathfrak{so}(2l+1,\mathbb{C})$ be of type B. The subalgebra of diagonal matrices $\mathfrak{h} := \mathrm{diag}[a_1,\dots,a_l,0,-a_l,\dots,-a_1]$, $a_i \in \mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}$. We again refer to $\mathfrak{h}$ as the *standard Cartan subalgebra*. Let $\epsilon_i \in \mathfrak{h}^*$ be the linear functional $\epsilon_i(\mathrm{diag}[a_1,\dots,a_l,0,-a_l,\dots,-a_1]) = a_i$. In this case, we have

$$\Phi(\mathfrak{g},\mathfrak{h}) = \{\epsilon_i - \epsilon_j,\, \pm(\epsilon_i+\epsilon_j) :\ 1 \le i \ne j \le l\} \cup \{\pm\epsilon_k :\ 1 \le k \le l\}. \tag{2.6}$$

We take as our *standard positive roots* the set:

$$\Phi^+(\mathfrak{g},\mathfrak{h}) := \{\epsilon_i - \epsilon_j,\, \epsilon_i+\epsilon_j :\ 1 \le i < j \le l\} \cup \{\epsilon_k :\ 1 \le k \le l\}. \tag{2.7}$$

with corresponding simple roots

$$\Pi := \{\alpha_1,\dots,\alpha_{l-1},\alpha_l\} \text{ where } \alpha_i = \epsilon_i - \epsilon_{i+1},\ i=1,\dots,l-1,\ \alpha_l = \epsilon_l. \tag{2.8}$$

The *standard* Borel subalgebra $\mathfrak{b}_+ := \bigoplus_{\alpha\in\Phi^+(\mathfrak{g},\mathfrak{h})} \mathfrak{g}_\alpha$ is easily seen to be the set of upper triangular matrices in $\mathfrak{g}$.

As for $\mathfrak{so}(2l+1,\mathbb{C})$, we relabel the standard basis of $\mathbb{C}^n$ by letting $e_{-j} := e_{2l+2-j}$ for $j = 1,\dots,l$ and $e_0 := e_{l+1}$.

2.2. **Real Rank 1 symmetric subalgebras.** For later use, recall the realization of $\mathfrak{so}(n-1,\mathbb{C})$ as a symmetric subalgebra of $\mathfrak{so}(n,\mathbb{C})$. For $\mathfrak{g} = \mathfrak{so}(2l+1,\mathbb{C})$, let t be an element of the Cartan subgroup with Lie algebra $\mathfrak{h}$ with the property that $\mathrm{Ad}(t)|_{\mathfrak{g}_{\alpha_i}} = \mathrm{id}$ for $i = 1,\dots,l-1$ and $\mathrm{Ad}(t)|_{\mathfrak{g}_{\alpha_l}} = -\mathrm{id}$. Consider the involution $\theta_{2l+1} := \mathrm{Ad}(t)$. Then $\mathfrak{k} = \mathfrak{so}(2l,\mathbb{C}) = \mathfrak{g}^{\theta_{2l+1}}$ (see [Kna02], p. 700). Note that $\mathfrak{h} \subset \mathfrak{k}$. In the case $\mathfrak{g} = \mathfrak{so}(2l,\mathbb{C})$, $\mathfrak{k} = \mathfrak{so}(2l-1,\mathbb{C}) = \mathfrak{g}^{\theta_{2l}}$, where θ_{2l} is the involution induced by the diagram automorphism interchanging the simple roots α_{l-1} and α_l (see [Kna02], p. 703). Note that in this case, $\theta_{2l}(\epsilon_l) = -\epsilon_l$ and $\theta_{2l}(\epsilon_i) = \epsilon_i$ for $i = 1, \dots, l-1$. We will omit the subscripts $2l+1$ and $2l$ from θ when $\mathfrak{g}$ is understood.

We also denote the corresponding involution of $G = SO(n, \mathbb{C})$ by θ. The group $G^\theta = S(O(n-1, \mathbb{C}) \times O(1, \mathbb{C}))$ is disconnected. We let $K := (G^\theta)^0$ be the identity component of G^θ. Then $K = SO(n-1, \mathbb{C})$, and $\mathrm{Lie}(K) = \mathfrak{k} = \mathfrak{g}^\theta$.

2.3. **Notation.** We now lay out some of the notation that we will use throughout Sections 2 and 3.

Notation 2.1.
(1) We let $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$ and $\mathfrak{k} = \mathfrak{so}(n-1, \mathbb{C})$ be the symmetric subalgebra given in Section 2.2, unless otherwise mentioned. It will also be convenient at times to denote the Lie algebra $\mathfrak{so}(i, \mathbb{C})$ by $\mathfrak{g}_i$.
(2) We let r_i be the rank of $\mathfrak{g}_i$.
(3) For $x \in \mathfrak{so}(n, \mathbb{C})$, we let $x_{\mathfrak{k}}$ the projection of x onto $\mathfrak{k}$ off $\mathfrak{g}^{-\theta}$, the -1- eigenspace of θ.
(4) For any Lie algebra $\mathfrak{g}$, we denote by $\mathbb{C}[\mathfrak{g}]$, the ring of polynomial functions on $\mathfrak{g}$ and by $\mathbb{C}[\mathfrak{g}]^G$ the ring of adjoint invariant polynomial functions on $\mathfrak{g}$.

2.4. **The partial Kostant-Wallach map.** For $i = n-1, n$, let $\chi_i : \mathfrak{g}_i \to \mathbb{C}^{r_i}$ be the adjoint quotient. We define the *partial Kostant-Wallach map* to be

(2.9)
$$\Phi_n : \mathfrak{g} \to \mathbb{C}^{r_{n-1}} \oplus \mathbb{C}^{r_n},$$

$$\Phi_n(x) = (\chi_{n-1}(x_{\mathfrak{k}}), \chi_n(x)) = (f_{n-1,1}(x_{\mathfrak{k}}), \dots, f_{n-1,r_{n-1}}(x_{\mathfrak{k}}), f_{n,1}(x), \dots, f_{n,r_n}(x)),$$

where $\mathbb{C}[\mathfrak{g}_i]^{G_i} = \mathbb{C}[f_{i,1}, \dots, f_{i,r_i}]$.

Proposition 2.2.
(1) $\mathbb{C}[\mathfrak{g}]^K = \mathbb{C}[\mathfrak{g}]^G \otimes \mathbb{C}[\mathfrak{k}]^K$.
(2) Φ_n *coincides with the invariant theory quotient morphism* $\mathfrak{g} \to \mathfrak{g}//K$. *In particular,* Φ_n *is surjective.*
(3) *The morphism* Φ_n *is flat. In particular, its fibres are equidimensional varieties of dimension* $\dim \mathfrak{g} - r_n - r_{n-1}$.

Proof. Recall the well-known fact that the fixed point algebra $U(\mathfrak{g})^K$ of K in the enveloping algebra $U(\mathfrak{g})$ is commutative [Joh01]. Hence, $U(\mathfrak{g})^K$ coincides with its centre, $Z(U(\mathfrak{g})^K)$. In Theorem 10.1 of [Kno94], Knop shows that $Z(U(\mathfrak{g})^K) \cong U(\mathfrak{g})^G \otimes_{\mathbb{C}} U(\mathfrak{g})^K$. The first assertion now follows by taking the associated graded algebra with respect to the usual filtration of $U(\mathfrak{g})$. By the first assertion, Φ_n coincides with the invariant theory quotient $\mathfrak{g} \to \mathfrak{g}//K$, which gives the second assertion. Note that if we embed $\mathfrak{k}$ diagonally in $\mathfrak{g} \times \mathfrak{k}$, then $\mathfrak{k}^\perp \cong \mathfrak{g}$, and this isomorphism is K-equivariant. Then the flatness of Φ_n follows by Korollar 7.2 of [Kno90b], which gives a criterion for flatness of invariant theory quotients in the setting of spherical homogeneous spaces (see also [Pan90]).

Q.E.D.

Remark 2.3. For $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{C})$ and $\mathfrak{k} = \mathfrak{gl}(n-1,\mathbb{C})$ thought of as the subalgebra of $(n-1)\times(n-1)$ matrices in the left hand corner of $\mathfrak{g}$, Proposition 2.2 is also true by the same proof. Here $K = GL(n-1,\mathbb{C})$ is the algebraic subgroup of $GL(n,\mathbb{C})$ corresponding to $\mathfrak{k}$. In this case, we proved Proposition 2.2 (3) by more elementary means in [CE15]. In the appendix, we use conormal geometry to give a more elementary proof of Korollar 7.2 of [Kno90b] for a class of spherical varieties, which applies to our setting.

Corollary 2.4. $\mathbb{C}[\mathfrak{g}]$ *is a free* $\mathbb{C}[\mathfrak{g}]^K$*-module.*

Proof. This follows by Lemma 2.5 of [CE15] and the above Proposition 2.2 (3).

Q.E.D.

We proved the analogous result for $\mathfrak{gl}(n,\mathbb{C})$ in Proposition 2.6 of [CE15], and noted that it follows from a result of Futorny and Ovsienko, which states that $U(\mathfrak{gl}(n,\mathbb{C}))$ is free over the Gelfand-Zeitlin subalgebra [Ovs03, FO05]. It is not known whether the corresponding statement is true in the orthogonal case. Our result that $\mathbb{C}[\mathfrak{so}(n,\mathbb{C})]$ is free over $\mathbb{C}[\mathfrak{so}(n,\mathbb{C})]^{SO(n-1,\mathbb{C})}$ is a natural first step towards extending the result of Futorny and Ovsienko to the orthogonal setting and will be important in studying Gelfand-Zeitlin modules for the $\mathfrak{so}(n,\mathbb{C})$.

2.5. **General Properties of eigenvalue coincidence varieties $\mathfrak{g}(\geq i)$.** In this section, we develop some fundamental facts about the eigenvalue coincidence varieties discussed in the introduction (see (1.2)).

Let $\mathfrak{h}$ be the Cartan subalgebra of diagonal matrices in $\mathfrak{g}$, and let $\mathfrak{h}_{\mathfrak{k}} \subset \mathfrak{k}$ be the Cartan subalgebra of diagonal matrices in $\mathfrak{k}$. We denote elements of $\mathfrak{h}_{\mathfrak{k}} \times \mathfrak{h}$ by (a,b), where $a = (a_1,\ldots,a_{r_{n-1}})$ and $b = (b_1,\ldots,b_{r_n})$ represent the diagonal coordinates of $a \in \mathfrak{h}_{\mathfrak{k}}$ and $b \in \mathfrak{h}$ as in Section 2.1 above. Let $W = W(\mathfrak{g},\mathfrak{h})$ be the Weyl group of $\mathfrak{g}$, and let $W_K = W(\mathfrak{k},\mathfrak{h}_{\mathfrak{k}})$ be the Weyl group of $\mathfrak{k}$. For $i = 1,\ldots,r_{n-1}$ define:

$$(\mathfrak{h}_{\mathfrak{k}} \times \mathfrak{h})(\geq i) := \{(a,b) : \exists\, v \in W_K,\ u \in W \text{ such that } (v\cdot a)_j = (u\cdot b)_j,\ j = 1,\ldots,i\}.$$

We note that $(\mathfrak{h}_{\mathfrak{k}} \times \mathfrak{h})(\geq i)$ is a $W_K \times W$-invariant closed subvariety of $\mathfrak{h}_{\mathfrak{k}} \times \mathfrak{h}$ and is equidimensional of codimension i. Let $p_G : \mathfrak{h} \to \mathfrak{h}/W$ and $p_K : \mathfrak{h}_{\mathfrak{k}} \to \mathfrak{h}_{\mathfrak{k}}/W_K$ be the invariant theory quotients. Consider the finite morphism $p := p_K \times p_G : (\mathfrak{h}_{\mathfrak{k}} \times \mathfrak{h}) \to (\mathfrak{h}_{\mathfrak{k}} \times \mathfrak{h})/(W_K \times W)$. Let $F_G : \mathfrak{h}/W \to \mathbb{C}^{r_n}$ and $F_K : \mathfrak{h}_{\mathfrak{k}}/W_K \to \mathbb{C}^{r_{n-1}}$ be the Chevalley isomorphisms, and let $V^{r_{n-1},r_n} := \mathbb{C}^{r_{n-1}} \times \mathbb{C}^{r_n}$, so that $F_K \times F_G : (\mathfrak{h}_{\mathfrak{k}} \times \mathfrak{h})/(W_K \times W) \to V^{r_{n-1},r_n}$ is an isomorphism. The following varieties play a major role in our study of orthogonal eigenvalue coincidences.

Definition 2.5. For $i = 0,\ldots,r_{n-1}$, we let

$$V^{r_{n-1},r_n}(\geq i) := (F_K \times F_G)((\mathfrak{h}_{\mathfrak{k}} \times \mathfrak{h})(\geq i)/(W_K \times W)), \tag{2.10}$$

$$V^{r_{n-1},r_n}(i) := V^{r_{n-1},r_n}(\geq i) \setminus V^{r_{n-1},r_n}(\geq i+1). \tag{2.11}$$

For convenience, we let $V^{r_{n-1},r_n}(r_{n-1}+1) = \emptyset$.

Lemma 2.6. *The set* $V^{r_{n-1},r_n}(\geq i)$ *is an irreducible closed subvariety of* V^{r_{n-1},r_n} *of dimension* $r_n + r_{n-1} - i$*. Further,* $V^{r_{n-1},r_n}(i)$ *is open and dense in* $V^{r_{n-1},r_n}(\geq i)$*.*

Proof. Indeed, the set

$$Y := \{(a,b) \in \mathfrak{h}_{\mathfrak{k}} \times \mathfrak{h} : a_j = b_j \text{ for } j = 1,\ldots,i\} \tag{2.12}$$

is closed and irreducible of dimension $r_n + r_{n-1} - i$. The first assertion follows since $(F_K \times F_G) \circ p$ is a finite morphism and $(F_K \times F_G) \circ p(Y) = V^{r_{n-1},r_n}(\geq i)$. The last assertion of the lemma now follows from Equation (2.11).

Q.E.D.

We define

$$\mathfrak{g}(\geq i) := \Phi_n^{-1}(V^{r_{n-1},r_n}(\geq i)), \tag{2.13}$$

(recall Equation (2.9) for the definition Φ_n).

For $i = 0,\ldots,r_{n-1}$, we define

$$\mathfrak{g}(i) := \mathfrak{g}(\geq i) \setminus \mathfrak{g}(\geq i+1) = \Phi_n^{-1}(V^{r_{n-1},r_n}(i)). \tag{2.14}$$

Note that we have a partition of $\mathfrak{g}$ into disjoint locally closed sets:

$$\mathfrak{g} = \bigcup_{i=0}^{r_{n-1}} \mathfrak{g}(i). \tag{2.15}$$

Remark 2.7. We show that the definition of $\mathfrak{g}(\geq i)$ in (2.13) agrees with the one we gave in (1.2). We recall that $\Phi_n = (\chi_{n-1}, \chi_n)$, where $\chi_i : \mathfrak{so}(i,\mathbb{C}) \to \mathbb{C}^{r_i}$ is the adjoint quotient. Let $\mathfrak{g} = \mathfrak{so}(2l,\mathbb{C})$ and let $x \in \mathfrak{g}$ satisfy the property of Equation (1.2). Then there is $g \in G$ and $k \in K$ such that $\mathrm{Ad}(g)x$ and $\mathrm{Ad}(k)x_{\mathfrak{k}}$ are upper triangular, $\mathrm{Ad}(g)x$ has diagonal part $\mathrm{diag}[b_1,\ldots,b_l,-b_l,\ldots,-b_1]$, $\mathrm{Ad}(k)x$ has diagonal part $\mathrm{diag}[a_1,\ldots,a_{l-1},0,0,-a_{l-1},\ldots,-a_1]$, and $b_{j_1} = \pm a_{k_1},\ldots,b_{j_i} = \pm a_{k_i}$. We claim $\Phi_n(x) \in V^{r_{n-1},r_n}(\geq i)$. Note that $\Phi_n(x) = (F_K \times F_G) \circ p((a_1,\ldots,a_{l-1}),(b_1,\ldots,b_l))$. Since W contains the subgroup S_l, and W_K contains the subgroup S_{l-1}, we have

$$p((a_1,\ldots,a_{l-1}),(b_1,\ldots,b_l)) = p((a_1,\ldots,a_{l-1}),(\pm a_1,\ldots,\pm a_i,b_{i+1},\ldots,b_l)).$$

Since W_K contains all sign changes of the coordinates of $\mathfrak{h}_{\mathfrak{k}}$, it follows that that

$$p((a_1,\ldots,a_i,\ldots,a_{l-1}),(\pm a_1,\ldots,\pm a_i,b_{i+1},\ldots,b_l)) =$$
$$p((\pm a_1,\ldots,\pm a_i,\ldots,a_{l-1}),(\pm a_1,\ldots,\pm a_i,b_{i+1},\ldots,b_l)).$$

It now follows that $\Phi_n(x) \in (F_K \times F_G) \circ p(Y) = V^{r_{n-1},r_n}(\geq i)$, where Y is the variety defined in (2.12). Thus, $x \in \mathfrak{g}(\geq i)$. We leave the converse to the reader. The case of $\mathfrak{g} = \mathfrak{so}(2l+1,\mathbb{C})$ follows by similar reasoning.

We now use the flatness of the Kostant-Wallach morphism asserted in Proposition 2.2 to study the varieties $\mathfrak{g}(\geq i)$.

Proposition 2.8. (1) *The variety* $\mathfrak{g}(\geq i)$ *is equidimensional of dimension* $\dim \mathfrak{g} - i$*.*

(2) $\overline{\mathfrak{g}(i)} = \mathfrak{g}(\geq i) = \bigcup_{k\geq i} \mathfrak{g}(k)$.

Proof. By Proposition 2.2, the morphism Φ_n is flat. By Proposition III.9.5 and Corollary III.9.6 of [Har77], the variety $\mathfrak{g}(\geq i)$ is equidimensional of dimension $\dim(V^{r_n-1,r_n}(\geq i)) + \dim\mathfrak{g} - r_n - r_{n-1}$, which gives the first assertion by Lemma 2.6. For the second assertion, by the flatness of Φ_n, Theorem VIII.4.1 of [Gro03], and Lemma 2.6,

$$\overline{\mathfrak{g}(i)} = \overline{\Phi_n^{-1}(V^{r_{n-1},r_n}(i))} = \Phi_n^{-1}(\overline{V^{r_{n-1},r_n}(i)}) = \Phi_n^{-1}(V^{r_{n-1},r_n}(\geq i)) = \mathfrak{g}(\geq i). \tag{2.16}$$

The remaining equality follows since $V^{r_{n-1},r_n}(\geq i) = \cup_{k\geq i} V^{r_{n-1},r_n}(k)$.

Q.E.D.

2.6. **The varieties Y_Q.** We now study the geometry of the varieties $Y_Q = \mathrm{Ad}(K)\mathfrak{b}$ for a K-orbit $Q = K\cdot\mathfrak{b}$ in $\mathcal{B}$. We begin by studying more general objects $Y_{Q_{\mathfrak{r}}}$, where $Q_{\mathfrak{r}}$ is a K-orbit in a partial flag variety.

For a parabolic subgroup $P \subset G$ with Lie algebra $\mathfrak{p} \subset \mathfrak{g}$, consider the partial Grothendieck resolution $\tilde{\mathfrak{g}}^{\mathfrak{p}} = \{(x,\mathfrak{r}) \in \mathfrak{g}\times G/P \mid x\in\mathfrak{r}\}$, as well as the morphisms $\mu : \tilde{\mathfrak{g}}^{\mathfrak{p}} \to \mathfrak{g}$, $\mu(x,\mathfrak{r}) = x$, and $\pi : \tilde{\mathfrak{g}}^{\mathfrak{p}} \to G/P$, $\pi(x,\mathfrak{r}) = \mathfrak{r}$. For $\mathfrak{r}\in G/P$, let $Q_{\mathfrak{r}} = K\cdot\mathfrak{r} \subset G/P$. It is well-known that π is a smooth morphism of relative dimension $\dim\mathfrak{p}$, and μ is proper with generically finite restriction to $\pi^{-1}(Q_{\mathfrak{r}})$ (see p. 622 of [CE15]). Thus, $\pi^{-1}(Q_{\mathfrak{r}})$ has dimension $\dim(Q_{\mathfrak{r}}) + \dim(\mathfrak{r})$.

Notation 2.9. For a parabolic subalgebra $\mathfrak{r}$ with K-orbit $Q_{\mathfrak{r}} \subset G/P$, we consider the irreducible subset

$$Y_{\mathfrak{r}} := \mu(\pi^{-1}(Q_{\mathfrak{r}})) = \mathrm{Ad}(K)\mathfrak{r}. \tag{2.17}$$

$Y_{\mathfrak{r}}$ depends only on $Q_{\mathfrak{r}}$, and we will also denote this set as

$$Y_{Q_{\mathfrak{r}}} := Y_{\mathfrak{r}}. \tag{2.18}$$

It follows from generic finiteness of μ that $Y_{Q_{\mathfrak{r}}}$ contains an open subset of dimension

$$\dim(Y_{Q_{\mathfrak{r}}}) := \dim\pi^{-1}(Q_{\mathfrak{r}}) = \dim\mathfrak{r} + \dim(Q_{\mathfrak{r}}) = \dim\mathfrak{r} + \dim(\mathfrak{k}/\mathfrak{k}\cap\mathfrak{r}). \tag{2.19}$$

Remark 2.10. Since μ is proper, the set $Y_{Q_{\mathfrak{r}}}$ is closed when $Q_{\mathfrak{r}} = K\cdot\mathfrak{r}$ is closed in G/P.

Remark 2.11. Note that

$$\mathfrak{g} = \bigcup_{Q\subset G/P} Y_Q,$$

where the union is taken over the finitely many K-orbits in G/P.

Lemma 2.12. *Let $Q \subset G/P$ be a K-orbit. Then*

$$\overline{Y_Q} = \bigcup_{Q'\subset\overline{Q}} Y_{Q'}. \tag{2.20}$$

Proof. Since π is a smooth morphism, it is flat by Theorem III.10.2 of [Har77]. Thus, by Theorem VIII.4.1 of [Gro03], $\pi^{-1}(\overline{Q}) = \overline{\pi^{-1}(Q)}$. The result follows since μ is proper.

Q.E.D.

Proposition 2.13. *Let $Q = K \cdot \mathfrak{b}$ be a K-orbit in $\mathcal{B}$ with codim$(Q) = i$. Then*

$$\dim Y_Q = \dim \mathfrak{g}(\geq i). \tag{2.21}$$

Proof. By Equation (2.19), it follows that

$$\dim Y_Q = \dim Q + \dim \mathfrak{b} = \dim(\mathcal{B}) - i + \dim(\mathfrak{b}) = \dim(\mathfrak{g}) - i.$$

The assertion follows by part (1) of Proposition 2.8.

Q.E.D.

Let codim$(Q) = i$. To see that $\overline{Y_Q}$ is an irreducible component of $\mathfrak{g}(\geq i)$, it remains to show that $\overline{Y_Q} \subset \mathfrak{g}(\geq i)$. For this, it is convenient to replace the K-orbit Q in $\mathcal{B}$ with a K-orbit $Q_{\mathfrak{r}}$ of a θ-stable parabolic subalgebra $\mathfrak{r} \supset \mathfrak{b}$ in a partial flag variety G/P. We will show that $\mathfrak{r} \in G/P$ can be chosen so that $\overline{Y_Q} = Y_{Q_{\mathfrak{r}}}$, and $Y_{Q_{\mathfrak{r}}} \subset \mathfrak{g}(\geq i)$. The first step is to relate the geometry of Q and $Q_{\mathfrak{r}}$ for a general θ-stable $\mathfrak{r}$ with $\mathfrak{r} \supset \mathfrak{b}$ and develop a necessary condition for $\overline{Y_Q} = Y_{Q_{\mathfrak{r}}}$. Let R be the parabolic subgroup of G with Lie algebra $\mathfrak{r}$. Consider the canonical fibre bundle:

$$R/B \to \mathcal{B} \xrightarrow{p} G/R,$$

which induces a bundle

$$Q \cap p^{-1}(Q_{\mathfrak{r}}) \to Q \to Q_{\mathfrak{r}} \tag{2.22}$$

over the K-orbit $Q_{\mathfrak{r}}$. To study the fibre bundle (2.22), we consider the θ-stable parabolic subalgebra $\mathfrak{r}$ in more detail. It follows from Theorem 2 of [BH00] that $\mathfrak{r}$ has a θ-stable Levi decomposition $\mathfrak{r} = \mathfrak{l} \oplus \mathfrak{u}$. The Levi subalgebra decomposes as $\mathfrak{l} = \mathfrak{z} \oplus \mathfrak{l}_{ss}$, with the centre $\mathfrak{z}$ and the semisimple part $\mathfrak{l}_{ss} = [\mathfrak{l}, \mathfrak{l}]$ both θ-stable. Further, $\mathfrak{k} \cap \mathfrak{r}$ is a parabolic subalgebra of $\mathfrak{k}$ with Levi decomposition

$$\mathfrak{k} \cap \mathfrak{r} = \mathfrak{k} \cap \mathfrak{l} \oplus \mathfrak{k} \cap \mathfrak{u} = \mathfrak{k} \cap \mathfrak{z} \oplus \mathfrak{k} \cap \mathfrak{l}_{ss} \oplus \mathfrak{k} \cap \mathfrak{u}.$$

The corresponding parabolic subgroup R is also θ-stable, and $K \cap R$ is a parabolic subgroup of K with Levi decomposition $(K \cap Z) \cdot (K \cap L_{ss}) \cdot K \cap U$ (see Theorem 2, [BH00]). In particular, $Q_{\mathfrak{r}} \cong K/(K \cap R)$ is closed. Recall that $R/B \cong \mathcal{B}_{\mathfrak{l}_{ss}}$. Thus, the fibre bundle (2.22) gives the K-orbit Q on $\mathcal{B}$ the structure of a K-homogeneous fibre bundle over the closed K-orbit $Q_{\mathfrak{r}} = K \cdot \mathfrak{r}$ in G/R with fibre the $K \cap L_{ss}$-orbit of $\mathfrak{b}$ in $R/B \cong \mathcal{B}_{\mathfrak{l}_{ss}}$, i.e.:

$$Q \cong K \times_{K \cap R} (K \cap L_{ss}) \cdot \mathfrak{b}. \tag{2.23}$$

Proposition 2.14. *Suppose that the orbit $(K \cap L_{ss}) \cdot \mathfrak{b}$ in (2.23) is open in $R/B \cong \mathcal{B}_{\mathfrak{l}_{ss}}$. Then $\dim Y_{\mathfrak{b}} = \dim Y_{\mathfrak{r}}$. Further, $Y_{\mathfrak{r}}$ is a closed, irreducible subvariety of $\mathfrak{g}$, so that $\overline{Y_{\mathfrak{b}}} = Y_{\mathfrak{r}}$.*

Proof. Indeed,

$$
\begin{aligned}
\dim Y_{\mathfrak{r}} &= \dim Q_{\mathfrak{r}} + \dim \mathfrak{r} \text{ (by (2.19))} \\
&= \dim Q - \dim \mathcal{B}_{\mathfrak{l}_{ss}} + \dim \mathfrak{r} \text{ (by (2.23))} \\
&= \dim Q - \dim \mathcal{B}_{\mathfrak{l}_{ss}} + \dim \mathcal{B}_{\mathfrak{l}_{ss}} + \dim \mathfrak{b} \\
&= \dim Q + \dim \mathfrak{b} \\
&= \dim Y_{\mathfrak{b}} \text{ (by (2.19))}.
\end{aligned} \tag{2.24}
$$

It follows from definitions that $Y_{\mathfrak{b}} \subset Y_{\mathfrak{r}}$. Since $Q_{\mathfrak{r}}$ is closed, $Y_{\mathfrak{r}}$ is closed by Remark 2.10. Thus, $\overline{Y_{\mathfrak{b}}} = Y_{\mathfrak{r}}$ since $Y_{\mathfrak{r}}$ is irreducible.

Q.E.D.

In Theorems 3.1 and 3.2, we show that for any Borel subalgebra $\mathfrak{b} \in \mathcal{B}$ whose orbit $K \cdot \mathfrak{b}$ has codimension i, there is a θ-stable parabolic subalgebra $\mathfrak{r}$ with $\mathfrak{b} \subset \mathfrak{r}$ such that the hypothesis of Proposition 2.14 is satisfied, and $Y_{Q_{\mathfrak{r}}} \subset \mathfrak{g}(\geq i)$. To do this, we need to classify the K-orbits on $\mathcal{B}$ and develop explicit descriptions of representatives of the K-orbits on $\mathcal{B}$.

Remark 2.15. When $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ and $\mathfrak{k} = \mathfrak{gl}(n-1, \mathbb{C}) \oplus \mathfrak{gl}(1, \mathbb{C})$ is the symmetric subalgebra of block diagonal matrices, we have shown that for any Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$, there is a θ-stable parabolic subalgebra $\mathfrak{r}$ with $\mathfrak{b} \subset \mathfrak{r}$ so that $(K \cap L_{ss}) \cdot \mathfrak{b}$ is open in $\mathcal{B}_{\mathfrak{l}_{ss}}$. This is implicit in Lemma 3.5 of [CE12] and in the computations of Proposition 2.15 of [CE15].

2.7. **Description of K-orbits on $\mathcal{B}$ in the orthogonal case.** We classify the K-orbits on $\mathcal{B}$ for $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$ and $\mathfrak{k} = \mathfrak{so}(n-1, \mathbb{C})$. In particular, we explain how to recover the orbit diagrams from Figure 4.3 of [Col85], but also give explicit representatives of each orbit for later use.

We begin by recalling some generalities regarding an involution θ of a semisimple Lie algebra $\mathfrak{g}$ and orbits of $K = G^\theta$ on the flag variety $\mathcal{B}$ of $\mathfrak{g}$ (see [Mat79, RS90, Vog83, CE] for more details). Each Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ contains a θ-stable Cartan subalgebra $\mathfrak{t}$. Let $\Phi(\mathfrak{g}, \mathfrak{t})$ denote the roots of $\mathfrak{t}$ in $\mathfrak{g}$, let $\Phi^+_{\mathfrak{b}}$ denote the roots of $\mathfrak{t}$ in $\mathfrak{b}$, which we take to be the positive roots. Let $\mathfrak{g}_\alpha$ denote the root space for a root α. Then the θ-action on $\mathfrak{t}$ induces an action on $\Phi(\mathfrak{g}, \mathfrak{t})$. Using this action, we define the *type* of a root $\alpha \in \Phi^+_{\mathfrak{b}}$ as follows. A root α is called *real* if $\theta(\alpha) = -\alpha$, *imaginary* if $\theta(\alpha) = \alpha$, and *complex* if $\theta(\alpha) \neq \pm\alpha$. If α is imaginary, then α is called *compact* if $\theta|_{\mathfrak{g}_\alpha} = \text{id}$ and *noncompact* if $\theta|_{\mathfrak{g}_\alpha} = -\text{id}$. If α is complex, then α is called *complex θ-stable* if $\theta(\alpha)$ is positive, and otherwise is called *complex θ-unstable*. These notions do not depend on the choice of θ-stable Cartan subalgebra $\mathfrak{t} \subset \mathfrak{b}$, nor on the choice of $\mathfrak{b}$ in the K-orbit $K \cdot \mathfrak{b}$.

Example 2.16. Let $\mathfrak{g} = \mathfrak{so}(2l+1, \mathbb{C})$ and $\mathfrak{k} = \mathfrak{so}(2l, \mathbb{C})$, and let $\theta = \text{Ad}(t)$ be as in Section 2.2. Let $\Phi(\mathfrak{g}, \mathfrak{h})$ be the set of standard roots of $\mathfrak{g}$ as in (2.6). The roots

$\{\pm(\epsilon_i - \epsilon_j),\ \pm(\epsilon_i + \epsilon_j),\ 1 \leq i < j \leq l\}$ are compact imaginary, and the roots $\{\pm\epsilon_i\ i = 1, \ldots, l\}$ are non-compact imaginary.

Now let $\mathfrak{g} = \mathfrak{so}(2l, \mathbb{C})$ and $\mathfrak{k} = \mathfrak{so}(2l-1, \mathbb{C})$ and θ is as in Section 2.2. Then the simple roots $\alpha_{l-1} = \epsilon_{l-1} - \epsilon_l$ and $\alpha_l = \epsilon_{l-1} + \epsilon_l$ are complex θ-stable with $\theta(\alpha_{l-1}) = \alpha_l$. Note that we can choose as a representative for θ a nontrivial element in the Weyl group of $GL(\mathbb{C}e_l + \mathbb{C}e_{-l})$. Therefore, the roots $\{\pm(\epsilon_i + \epsilon_j), \pm(\epsilon_i - \epsilon_j), 1 \leq i < j \leq l-1\}$ are compact imaginary, whereas the roots $\{\pm(\epsilon_i + \epsilon_l), \pm(\epsilon_i - \epsilon_l), 1 \leq i \leq l-1\}$ are complex θ-stable with $\theta(\epsilon_i \pm \epsilon_l) = \epsilon_i \mp \epsilon_l$. The θ-stable subspace $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{\theta(\alpha)}$ decomposes as $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{\theta(\alpha)} = ((\mathfrak{g}_\alpha \oplus \mathfrak{g}_{\theta(\alpha)}) \cap \mathfrak{k}) \oplus ((\mathfrak{g}_\alpha \oplus \mathfrak{g}_{\theta(\alpha)}) \cap \mathfrak{g}^{-\theta})$.

We make use of the following notation throughout the paper.

Notation 2.17. Let T be the maximal torus with Lie algebra $\mathfrak{t}$, and let W be the Weyl group with respect to T. For an element $w \in W$, let $\dot{w} \in N_G(T)$ be a representative of w. If $\mathfrak{t} \subset \mathfrak{b}$, with $\mathfrak{b} \in \mathcal{B}$, then $\mathrm{Ad}(\dot{w})\mathfrak{b}$ is independent of the choice of representative $\dot{w}$ of w, and we denote it by $w(\mathfrak{b})$.

Let $Q = K \cdot \mathfrak{b}$ and suppose that $\alpha \in \Phi^+_{\mathfrak{b}}$ is a simple root for $\mathfrak{b}$. Let $\mathcal{P}_\alpha$ be the variety of parabolic subalgebras of type α, and consider the projection $\pi_\alpha : \mathcal{B} \to \mathcal{P}_\alpha$. Let $m(s_\alpha) \cdot Q$ be the unique K-orbit of maximal dimension in $\pi_\alpha^{-1}(\pi_\alpha(Q))$. For each simple root α, choose root vectors $e_\alpha \in \mathfrak{g}_\alpha$, $f_\alpha \in \mathfrak{g}_{-\alpha}$, and $h_\alpha = [e_\alpha, f_\alpha]$ such that $\mathrm{span}\{e_\alpha, f_\alpha, h_\alpha\}$ forms a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2, \mathbb{C})$. Choose a Lie algebra homomorphism $\phi_\alpha : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{g}$ such that:

$$\phi_\alpha : \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \to e_\alpha,\ \phi_\alpha : \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \to f_\alpha,\ \phi_\alpha : \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \to h_\alpha \tag{2.25}$$

Also denote by $\phi_\alpha : SL(2, \mathbb{C}) \to G$ the induced Lie group homomorphism, and let

$$u_\alpha = \phi_\alpha \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & \imath \\ \imath & 1 \end{bmatrix} \right). \tag{2.26}$$

Lemma 2.18. *[*[RS90]*, 4.3]*
Let $Q = K \cdot \mathfrak{b}$ be a K-orbit on $\mathcal{B}$.

(1) *$m(s_\alpha) \cdot Q = Q$ unless α is either noncompact or complex θ-stable, and when $m(s_\alpha) \cdot Q \neq Q$, then $\dim(m(s_\alpha) \cdot Q) = \dim(Q) + 1$.*
(2) *If α is noncompact for Q, then $m(s_\alpha) \cdot Q = K \cdot \mathrm{Ad}(u_\alpha)\mathfrak{b}$ and the K-orbits in $\pi_\alpha^{-1}(\pi_\alpha(Q))$ are $Q, m(s_\alpha) \cdot Q$, and $K \cdot s_\alpha(\mathfrak{b})$. Further, $m(s_\alpha) \cdot K \cdot s_\alpha(\mathfrak{b}) = m(s_\alpha) \cdot Q$.*
(3) *If α is complex θ-stable for Q, then $m(s_\alpha) \cdot \mathfrak{b} = K \cdot s_\alpha(\mathfrak{b})$, and $\pi_\alpha^{-1}(\pi_\alpha(Q))$ consists of Q and $m(s_\alpha) \cdot Q$.*

The action by operators $m(s_\alpha)$ on K-orbits is called the monoidal action [RS90].

Lemma 2.19. *[*[RS90]*, Theorem 4.6]*
Every K-orbit on $\mathcal{B}$ is of the form $m(s_{\beta_1}) \cdots m(s_{\beta_k}) \cdot \mathfrak{b}_1$, where $K \cdot \mathfrak{b}_1$ is a closed K-orbit on $\mathcal{B}$, $k \geq 0$, and $\beta_1, \ldots, \beta_k$ are simple roots.

We now briefly recall the classification of closed K-orbits on $\mathcal{B}$ from Section 4.3 of [CE]. Let $K \cdot \mathfrak{b}_0$ be a closed K-orbit with $\mathfrak{b}_0$ containing a θ-stable Cartan subalgebra $\mathfrak{t}$ corresponding to a maximal torus T. Since T is θ-stable, θ acts naturally on the Weyl group $W = N_G(T)/T$. Further, the subgroup $T \cap K$ is a Cartan subgroup of K and the Weyl group $W_K = N_K(T \cap K)/(T \cap K)$ embeds into W (Lemmas 5.1 and 5.3, [Ric82]), and is contained in W^θ, the fixed points of θ on W.

Lemma 2.20. *[[CE], Theorem 4.10] The map $W^\theta/W_K \to K\backslash\mathcal{B}$ given by $wW_K \mapsto K \cdot w^{-1} \cdot (\mathfrak{b}_0)$ is a bijection to the closed K-orbits on $\mathcal{B}$.*

Remark 2.21. If the K-orbit $Q = K \cdot \mathfrak{b}$ is closed, then $\mathfrak{b} \cap \mathfrak{k}$ is a Borel subalgebra in $\mathfrak{k}$ (see Lemma 5.1 of [Ric82]). Thus, $\dim(Q) = \dim(\mathcal{B}_{\mathfrak{k}})$.

We now return to the case where $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$ and $\mathfrak{k} = \mathfrak{so}(n-1, \mathbb{C})$ and use Lemma 2.20 to determine the closed K-orbits on $\mathcal{B}$. Before doing that, we state a result on the relation between W and W_K, which we will also need later. Recall that when $\mathfrak{g} = \mathfrak{so}(2l+1, \mathbb{C})$, $W = S_l \rtimes U_l$, where U_l is the group generated by sign changes $\tau_i, 1 \leq i \leq l$ in the root system, with $\tau_i(\epsilon_i) = -\epsilon_i$ and $\tau_i(\epsilon_j) = \epsilon_j$ for $j \neq i$. When $\mathfrak{g} = \mathfrak{so}(2l, \mathbb{C})$, $W = S_l \rtimes T_l$, where T_l is the subgroup of U_l generated by products $\tau_i\tau_j, 1 \leq i < j \leq l$.

Proposition 2.22.
(1) *Let $\mathfrak{g} = \mathfrak{so}(2l+1, \mathbb{C})$ and let $\mathfrak{k} = \mathfrak{so}(2l, \mathbb{C})$. Then $W = W^\theta$ and $W/W_K = \{eW_K, s_{\alpha_l}W_K\}$, where e denotes the identity element in W. In particular, W_K has index 2 in W. Further, W_K is the subgroup $S_l \rtimes T_l$ of W.*
(2) *Let $\mathfrak{g} = \mathfrak{so}(2l, \mathbb{C})$ and $\mathfrak{k} = \mathfrak{so}(2l-1, \mathbb{C})$. Then W^θ is the subgroup of W generated by the elements $s_{\alpha_1}, \dots, s_{\alpha_{l-2}}, s_{\alpha_{l-1}} \cdot s_{\alpha_l}$, and $W^\theta = W_K$.*

Proof. For (1), since θ is inner, we know $W^\theta = W$, and the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is also a Cartan subalgebra of $\mathfrak{k}$. By Example 2.16, the roots $\epsilon_i \pm \epsilon_j$ are exactly the roots of $\mathfrak{k}$ with respect to $\mathfrak{h}$. The assertion about W/W_K now follows by the above remarks on Weyl groups, and the observation that $s_{\alpha_l} = \tau_l \notin W_K$. The rest of (1) follows easily.

For (2), the first statement follows from 1.32(b) in [Ste68]. The second statement can be deduced from the proof of (5) in Theorem 8.2 of [Ste68], but we provide a more direct proof. An easy calculation shows that for $\sigma \in S_l$, $\theta\sigma\theta^{-1} = \sigma \cdot \tau_{\sigma^{-1}(l)} \cdot \tau_l$. Note that T_l is commutative and θ acts trivially on T_l. It follows that W^θ is identified with the semi-direct product of S_{l-1} with T_l. Since $W_K \subset W^\theta$, the second statement follows.

Q.E.D.

Now it remains to describe the monoidal action. To determine the type of a root for a K-orbit $Q = K \cdot \mathfrak{b}$, it is convenient to replace the involution θ by another involution θ_Q, which preserves the standard Cartan subalgebra of diagonal matrices $\mathfrak{h}$ and thus acts on the standard root system $\Phi(\mathfrak{g}, \mathfrak{h})$. Suppose that $\mathfrak{b} = \mathrm{Ad}(v)\mathfrak{b}_+$, where $\mathfrak{b}_+$ is the standard Borel subalgebra of upper triangular matrices. Then

$$\theta_Q := \mathrm{Ad}(v^{-1}) \circ \theta \circ \mathrm{Ad}(v).$$

It is easy to check that the type of a standard positive root $\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{h})$ with respect to θ_Q is the same as the type of the positive root $\mathrm{Ad}(v)\alpha := \alpha \circ \mathrm{Ad}(v^{-1})$ for $\mathfrak{b}$ with respect to θ (see Definition 4.6 and Proposition 4.7 of [CE]). In Section 4.4 of [CE], we give an inductive method of constructing the involution $\theta_{m(s_\alpha)\cdot Q}$ from the involution θ_Q (Propositions 4.27 and 4.28).

Proposition 2.23. *Let $\mathfrak{g} = \mathfrak{so}(2l+1, \mathbb{C})$ and $\mathfrak{k} = \mathfrak{so}(2l, \mathbb{C})$.*

(1) *There are exactly $l+2$ K-orbits on the flag variety $\mathcal{B}$ of $\mathfrak{g}$.*
(2) *We let $\mathfrak{b}_+$ be the upper triangular matrices in $\mathfrak{g}$, and let $\mathfrak{b}_- := s_{\alpha_l}(\mathfrak{b}_+)$. Exactly two K-orbits on $\mathcal{B}$ are closed, and they are $Q_+ := K \cdot \mathfrak{b}_+$ and $Q_- := K \cdot \mathfrak{b}_-$. Further, $m(s_{\alpha_l}) \cdot Q_+ = m(s_{\alpha_l}) \cdot Q_- = K \cdot \mathrm{Ad}(u_{\alpha_l})\mathfrak{b}_+$.*
(3) *The non-closed orbits are of the form*

$$Q_i := m(s_{\alpha_{i+1}}) \cdot m(s_{\alpha_{i+2}}) \cdots m(s_{\alpha_{l-1}}) \cdot m(s_{\alpha_l}) \cdot Q_+$$

for $i = 0, \ldots, l-1$. Moreover, the codimension of Q_i in $\mathcal{B}$ is i. Further,

$$\mathfrak{b}_i := \mathrm{Ad}(u_{\alpha_l}) s_{\alpha_{l-1}} s_{\alpha_{l-2}} \ldots s_{\alpha_{i+1}}(\mathfrak{b}_+) \in Q_i.$$

In particular, the unique open K-orbit contains the Borel subalgebra

$$\mathfrak{b}_0 = \mathrm{Ad}(u_{\alpha_l}) s_{\alpha_{l-1}} s_{\alpha_{l-2}} \ldots s_{\alpha_1}(\mathfrak{b}_+). \tag{2.27}$$

Proof. For (2), since $\mathfrak{b}_+$ is θ-stable, $K \cdot \mathfrak{b}_+$ is closed by Proposition 4.12 of [CE]. By Lemma 2.20 and part (1) of Proposition 2.22, there are two closed orbits, and they are $\mathfrak{b}_+$ and $\mathfrak{b}_-$. The assertion that $\mathfrak{b}_{l-1} = \mathrm{Ad}(u_{\alpha_l})\mathfrak{b}_+ \in Q_{l-1}$ follows from part (2) of Lemma 2.18 and Example 2.16. The second statement of part (2) of Lemma 2.18 implies that $m(s_{\alpha_l}) \cdot Q_- = m(s_{\alpha_l}) \cdot Q_+ = Q_{l-1}$. By Remark 2.21, $\mathrm{codim}(Q_+) = \mathrm{codim}(Q_-) = \dim(\mathcal{B}) - \dim(\mathcal{B}_{\mathfrak{k}}) = l$. Thus, $\mathrm{codim}(Q_{l-1}) = l-1$ by part (1) of Lemma 2.18.

For (3), first recall that by Proposition 4.27 of [CE], the involution $\theta_{Q_{l-1}}$ associated to $Q_{l-1} = K \cdot \mathfrak{b}_{l-1}$ is $\mathrm{Ad}(\dot{s}_{\alpha_l}^{-1}) \circ \theta$, so the involution on the standard root system is s_{α_l}, which is a sign change in the last variable. Since $\theta = \mathrm{Ad}(t)$, it follows from a calculation in $SO(3, \mathbb{C})$ that we can choose the representative $\dot{s}_{\alpha_l}$ so that $\theta_{Q_{l-1}} \in GL(\mathbb{C}e_l + \mathbb{C}e_{-l})$. It follows easily that $\alpha_1, \ldots, \alpha_{l-2}$ are compact for $\theta_{Q_{l-1}}$, while α_{l-1} is complex $\theta_{Q_{l-1}}$-stable, and α_l is real. Hence, the only monoidal action which gives us a new orbit is $m(s_{\alpha_{l-1}}) \cdot Q_{l-1} = Q_{l-2}$. By part (3) of Lemma 2.18, the Borel subalgebra $\mathfrak{b}_{l-2} = \mathrm{Ad}(\tilde{\dot{s}}_{\alpha_{l-1}})(\mathfrak{b}_{l-1})$, where $\tilde{\dot{s}}_{\alpha_{l-1}}$ is a representative of the simple reflection $s_{\alpha_{l-1}}$ defined with respect to $\mathfrak{b}_{l-1}$. Since $\mathfrak{b}_{l-1} = \mathrm{Ad}(u_{\alpha_l})\mathfrak{b}_+$, it follows that $\tilde{\dot{s}}_{\alpha_{l-1}} = u_{\alpha_l} \dot{s}_{\alpha_{l-1}} u_{\alpha_l}^{-1}$. Hence, $\mathfrak{b}_{l-2} = \mathrm{Ad}(u_{\alpha_l} \dot{s}_{\alpha_l} u_{\alpha_l}^{-1} u_{\alpha_l})(\mathfrak{b}_+)$, which verifies the last part of (3) for $i = l-2$. By Proposition 4.28 of [CE], the involution $\theta_{Q_{l-2}}$ associated to Q_{l-2} is $\mathrm{Ad}(\dot{s}_{\alpha_{l-1}})^{-1} \circ \theta_{Q_{l-1}} \circ \mathrm{Ad}(\dot{s}_{\alpha_{l-1}})$. We can choose $\dot{s}_{\alpha_{l-1}}$, so that $\theta_{Q_{l-2}} \in GL(\mathbb{C}e_{l-1} + \mathbb{C}e_{-(l-1)})$. Thus, $\alpha_1, \ldots, \alpha_{l-3}$, and α_l are compact, while α_{l-2} is complex $\theta_{Q_{l-2}}$-stable and α_{l-1} is complex $\theta_{Q_{l-2}}$-unstable. Now an inductive argument, which we leave to the reader, shows that if we define $\mathfrak{b}_i$ as in assertion (3), and let θ_{Q_i} be the involution relative to Q_i, the roots $\alpha_1, \ldots, \alpha_{i-1}$ are compact for θ_{Q_i}, α_i is θ_{Q_i}-stable, α_{i+1} is θ_{Q_i}-unstable, and $\alpha_{i+2}, \ldots, \alpha_l$ are compact for θ_{Q_i}. Hence, from Q_i, the only monoidal action which gives a new orbit is $m(s_{\alpha_i}) \cdot Q_i = Q_{i-1}$ and $Q_{i-1} = K \cdot \mathfrak{b}_{i-1}$, by using the same argument as in the case $i = l-2$. As a consequence, the codimension of Q_{i-1} in $\mathcal{B}$ is $i-1$. It now follows that $Q_0, \ldots, Q_{l-1}$ are distinct orbits. The induction argument implies that no monoidal actions change Q_0, and it follows by Lemma 2.19 that $Q_+, Q_-, Q_{l-1}, \ldots, Q_0$ are all the K-orbits. This completes the proof of (3), and (1) is an easy consequence.

Q.E.D.

Proposition 2.24. *Let* $\mathfrak{g} = \mathfrak{so}(2l, \mathbb{C})$ *and* $\mathfrak{k} = \mathfrak{so}(2l-1, \mathbb{C})$.

(1) *There are exactly* l K*-orbits in the flag variety* $\mathcal{B}$ *of* $\mathfrak{g}$.
(2) *Let* $\mathfrak{b}_+$ *be the set of upper triangular matrices in* $\mathfrak{g}$. *Then* $Q_+ := K \cdot \mathfrak{b}_+$ *is the only closed* K*-orbit.*
(3) *Let*
$$Q_i := m(s_{\alpha_i}) \ldots m(s_{\alpha_{l-1}}) \cdot Q_+$$
and let
$$\mathfrak{b}_i := s_{\alpha_{l-1}} s_{\alpha_{l-2}} \ldots s_{\alpha_i}(\mathfrak{b}_+) \text{ for } i = 1, \ldots, l-1$$
Then $Q_i = K \cdot \mathfrak{b}_i$ *has codimension* $i-1$ *in* $\mathcal{B}$. *The distinct* K*-orbits are* $Q_+, Q_{l-1}, \ldots, Q_1$. *In particular, the unique open orbit is* Q_1 *and contains the Borel subalgebra*

$$\mathfrak{b}_1 = s_{\alpha_{l-1}} s_{\alpha_{l-2}} \ldots s_{\alpha_1}(\mathfrak{b}_+). \tag{2.28}$$

Proof. For (2), since $\mathfrak{b}_+$ is preserved by θ, $Q_+ = K \cdot \mathfrak{b}_+$ is closed by Proposition 4.12 of [CE]. Thus, Q_+ is the unique closed K-orbit by part (2) of Proposition 2.22 and Lemma 2.20. By Remark 2.21, we have $\operatorname{codim}(Q_+) = \dim(\mathcal{B}) - \dim(\mathcal{B}_{\mathfrak{k}}) = l - 1$.

For (3), we saw in Example 2.16 that α_{l-1} and α_l are complex θ-stable, and that all other simple roots are compact. Hence, $m(s_{\alpha_l}) \cdot Q_+$ and $m(s_{\alpha_{l-1}}) \cdot Q_+$ are the orbits of dimension $\dim Q_+ + 1$. We claim that they coincide. Indeed, by part (3) of Lemma 2.18, $m(s_{\alpha_l}) \cdot Q_+ = K \cdot s_{\alpha_l}(\mathfrak{b}_+)$ and $m(s_{\alpha_{l-1}}) \cdot Q_+ = K \cdot s_{\alpha_{l-1}}(\mathfrak{b}_+)$. We may choose the representatives for $s_{\alpha_{l-1}}$ and s_{α_l} in W so that $\theta(\dot{s}_{\alpha_{l-1}}) = \dot{s}_{\alpha_l}$, and since α_{l-1} and α_l are perpendicular, we may assume $\dot{s}_{\alpha_{l-1}}$ and $\dot{s}_{\alpha_l}$ commute. Note that $s_{\alpha_l}(\mathfrak{b}_+) = s_{\alpha_l} s_{\alpha_{l-1}} s_{\alpha_{l-1}}(\mathfrak{b}_+)$. It follows that

$$\theta(\dot{s}_{\alpha_l} \dot{s}_{\alpha_{l-1}}) = \dot{s}_{\alpha_{l-1}} \dot{s}_{\alpha_l} = \dot{s}_{\alpha_l} \dot{s}_{\alpha_{l-1}}.$$

Thus, $\dot{s}_{\alpha_l} \dot{s}_{\alpha_{l-1}} \in K$, and hence $K \cdot s_{\alpha_{l-1}}(\mathfrak{b}_+) = K \cdot s_{\alpha_l}(\mathfrak{b}_+)$, which establishes the claim. The orbit $Q_{l-1} = m(s_{\alpha_{l-1}}) \cdot Q_+$ has involution $\theta_{Q_{l-1}} = \operatorname{Ad}(\dot{s}_{\alpha_{l-1}}^{-1}) \circ \theta \circ \operatorname{Ad}(\dot{s}_{\alpha_{l-1}})$, which changes the sign of the $l-1$ coordinate of $\mathfrak{h}$, and no other coordinates. It now follows that α_{l-2} is complex $\theta_{Q_{l-1}}$-stable, while $\alpha_1, \ldots, \alpha_{l-3}$ are compact, and α_{l-1} and α_l are complex $\theta_{Q_{l-1}}$-unstable. The remainder of the argument follows by an easy induction similar to the proof of part (3) of Proposition 2.23. Part (1) is an easy consequence of (2) and (3).

Q.E.D.

3. Orthogonal Eigenvalue coincidence varieties and K-orbits

We prove Theorem 1.1 in this section.

3.1. The varieties $\overline{Y_Q}$ as irreducible components of $\mathfrak{g}(\geq i)$.

Using our work in Sections 2.6 and 2.7, we show that the Zariski closure of the varieties $\overline{Y_Q}$ with $\mathrm{codim}(Q) = i$ are irreducible components of the eigenvalue coincidence varieties $\mathfrak{g}(\geq i)$. We consider the case where $\mathfrak{g}$ is type D and type B separately. We first consider Y_Q, where the K-orbit Q not closed.

Case I: $\mathfrak{g} = \mathfrak{so}(2l+1, \mathbb{C})$, $\mathfrak{k} = \mathfrak{so}(2l, \mathbb{C})$

Theorem 3.1. *Let $\mathfrak{g}(\geq i)$, $i = 0, \ldots, l-1$ be the orthogonal eigenvalue coincidence variety defined in (2.13). Let $Q = K \cdot \mathfrak{b} \subset \mathcal{B}$ be a K-orbit with codim(Q) $= i$.*

(1) *There exists a θ-stable parabolic subalgebra $\mathfrak{r}$ with $\mathfrak{b} \subset \mathfrak{r}$ such that the hypothesis of Proposition 2.14 is satisfied. The parabolic subalgebra $\mathfrak{r}$ has θ-stable Levi decomposition*

$$\mathfrak{r} = \mathfrak{l} \oplus \mathfrak{u} \text{ with } \mathfrak{l}_{ss} \cong \mathfrak{so}(2(l-i)+1, \mathbb{C}) \text{ and } \mathfrak{z} \cong (\mathfrak{gl}(1, \mathbb{C}))^i. \tag{3.1}$$

Let $L_{ss} \cong SO(2(l-i)+1, \mathbb{C}) \subset G$ be the connected algebraic subgroup with Lie algebra $\mathfrak{l}_{ss}$. The restriction $\theta|_{\mathfrak{l}_{ss}} = \theta_{2(l-i)+1}$ is the involution on $\mathfrak{so}(2(l-i)+1, \mathbb{C})$ defining $\mathfrak{so}(2(l-i), \mathbb{C})$, so that

$$\mathfrak{l}_{ss}^{\theta} = \mathfrak{l}_{ss} \cap \mathfrak{k} \cong \mathfrak{so}(2(l-i), \mathbb{C}) \text{ and thus } (L_{ss}^{\theta})^0 = K \cap L_{ss} \cong SO(2(l-i), \mathbb{C}). \tag{3.2}$$

Furthermore, $SO(2(l-i), \mathbb{C}) \cdot (\mathfrak{b} \cap \mathfrak{l}_{ss})$ is open in $\mathcal{B}_{\mathfrak{so}(2(l-i)+1, \mathbb{C})}$.

(2) *We have $\overline{Y_Q} = Y_{Q_{\mathfrak{r}}}$, and the variety $Y_{Q_{\mathfrak{r}}}$ is an irreducible component of $\mathfrak{g}(\geq i)$.*

Proof. We first prove (1). By K-equivariance, it suffices to prove the statement for any representative $\mathfrak{b}$ of the K-orbit Q of codimension i. By part (3) of Proposition 2.23, we can take $\mathfrak{b} = \mathfrak{b}_i = \mathrm{Ad}(u_{\alpha_l}) s_{\alpha_{l-1}} \ldots s_{\alpha_{i+1}}(\mathfrak{b}_+)$. Let $\mathfrak{r} \subset \mathfrak{g}$ be the standard parabolic subalgebra generated by $\mathfrak{b}_+$ and the negative simple root spaces $\mathfrak{g}_{-\alpha_l}, \mathfrak{g}_{-\alpha_{l-1}}, \cdots, \mathfrak{g}_{-\alpha_{i+1}}$. Note that $\mathfrak{r}$ is θ-stable with Levi decomposition (3.1) and also $\theta|_{\mathfrak{l}_{ss}} = \theta_{2(l-i)+1}$. Equation (3.2) follows. To see that $\mathfrak{b}_i \subset \mathfrak{r}$, note that we can choose the representative $\dot{s}_{\alpha_j}$ of s_{α_j} so that $\dot{s}_{\alpha_j} \in L_{ss}$ for $j = i+1, \ldots, l$, and $u_{\alpha_l} \in L_{ss}$ by Equation (2.26). Thus, the element

$$v := u_{\alpha_l} \dot{s}_{\alpha_{l-1}} \ldots \dot{s}_{\alpha_{i+1}} \in L_{ss} \subset R. \tag{3.3}$$

Hence, $\mathfrak{b}_i = \mathrm{Ad}(v)\mathfrak{b}_+ \subset \mathrm{Ad}(v)\mathfrak{r} = \mathfrak{r}$.

It remains to show that $(K \cap L_{ss}) \cdot (\mathfrak{b} \cap \mathfrak{l}_{ss})$ can be identified with the open $SO(2(l-i), \mathbb{C})$-orbit in the flag variety $\mathcal{B}_{\mathfrak{so}(2(l-i)+1, \mathbb{C})}$. Note that $\mathfrak{b}_+ \cap \mathfrak{l}_{ss}$ can be identified with the standard Borel subalgebra $\mathfrak{b}_{+, \mathfrak{so}(2(l-i)+1, \mathbb{C})}$ of upper triangular matrices in $\mathfrak{so}(2(l-i)+1, \mathbb{C})$. Since the element v in Equation (3.3) is in L_{ss}, we have:

$$\mathfrak{b} \cap \mathfrak{l}_{ss} = (\mathrm{Ad}(v)\mathfrak{b}_+) \cap \mathfrak{l}_{ss} = \mathrm{Ad}(v)(\mathfrak{b}_+ \cap \mathfrak{l}_{ss}) = \mathrm{Ad}(v)\mathfrak{b}_{+, \mathfrak{so}(2(l-i)+1, \mathbb{C})}.$$

It follows from Equations (2.27) and (3.3) that $\mathrm{Ad}(v)\mathfrak{b}_{+,\mathfrak{so}(2(l-i)+1,\mathbb{C})} \subset \mathcal{B}_{\mathfrak{so}(2(l-i)+1,\mathbb{C})}$ is a representative of the open $SO(2(l-i),\mathbb{C})$-orbit on $\mathfrak{b}_{\mathfrak{so}(2(l-i)+1,\mathbb{C})}$.

We now prove (2). The first statement of (2) follows immediately from part (1) and Proposition 2.14. By Proposition 2.13, to see that $Y_{\mathfrak{r}}$ is an irreducible component of $\mathfrak{g}(\geq i)$, it suffices to show that $Y_{\mathfrak{r}} \subset \mathfrak{g}(\geq i)$. Consider the partial Kostant-Wallach map Φ_n defined in Equation (2.9). Let $\mathfrak{q}$ be a parabolic subalgebra of $\mathfrak{g}$ with $\mathfrak{q} \in Q_{\mathfrak{r}}$, and let $y \in \mathfrak{q}$. We need to show that $\Phi_n(y) \in V^{r_{n-1},r_n}(\geq i)$. Since the map Φ_n is K-invariant, it is enough to show that $\Phi_n(x) \in V^{r_{n-1},r_n}(\geq i)$ for $x \in \mathfrak{r}$. Recall that $\Phi_n(x) = (\chi_{n-1}(x_{\mathfrak{k}}), \chi_n(x))$, where $\chi_i : \mathfrak{so}(i,\mathbb{C}) \to \mathfrak{so}(i,\mathbb{C})//SO(i,\mathbb{C})$ is the adjoint quotient. For $x \in \mathfrak{r}$, let $x_{\mathfrak{l}}$ be the projection of x onto $\mathfrak{l}$ off of $\mathfrak{u}$. It is well-known that $\chi_n(x) = \chi_n(x_{\mathfrak{l}})$. Using the decomposition in (3.1), we can write $x_{\mathfrak{l}}$ as $x_{\mathfrak{l}} = x_{\mathfrak{z}} \oplus x_{\mathfrak{l}_{ss}}$ with $x_{\mathfrak{z}} \in \mathfrak{z} \cong (\mathfrak{gl}(1,\mathbb{C}))^i$ and $x_{\mathfrak{l}_{ss}} \in \mathfrak{l}_{ss} = \mathfrak{so}(2(l-i)+1,\mathbb{C})$. It is easy to see that the coordinates of $x_{\mathfrak{z}}$ are in the spectrum of x. Since $\mathfrak{r}$ is θ-stable, $\mathfrak{k} \cap \mathfrak{r}$ is a parabolic subalgebra of $\mathfrak{k}$ with Levi decomposition:

$$\mathfrak{k} \cap \mathfrak{r} = \mathfrak{k} \cap \mathfrak{l} \oplus \mathfrak{k} \cap \mathfrak{u} \text{ and } \mathfrak{k} \cap \mathfrak{l} = \mathfrak{k} \cap \mathfrak{z} \oplus \mathfrak{k} \cap \mathfrak{l}_{ss} \cong \mathfrak{z} \oplus \mathfrak{so}(2(l-i),\mathbb{C}),$$

where the isomorphism follows from (3.2) and the observation that $\mathfrak{z} \subset \mathfrak{h} \subset \mathfrak{k}$ (see Section 2.2). Since $x_{\mathfrak{k}} \in \mathfrak{k} \cap \mathfrak{r}$, we know $\chi_{n-1}(x_{\mathfrak{k}}) = \chi_{n-1}((x_{\mathfrak{k}})_{\mathfrak{l}\cap\mathfrak{k}})$. Now, $(x_{\mathfrak{k}})_{\mathfrak{l}\cap\mathfrak{k}} = x_{\mathfrak{z}} + x_{\mathfrak{so}(2(l-i),\mathbb{C})}$, and the coordinates of $x_{\mathfrak{z}}$ are in the spectrum of $(x_{\mathfrak{k}})_{\mathfrak{l}\cap\mathfrak{k}}$. Thus, Remark 2.7 implies that $\Phi_n(x) = (\chi_{n-1}(x_{\mathfrak{k}}), \chi_n(x)) \in V^{r_{n-1},r_n}(\geq i)$, and it follows that $Y_{\mathfrak{r}} \subset \mathfrak{g}(\geq i)$.

Q.E.D.

Case II: $\mathfrak{g} = \mathfrak{so}(2l,\mathbb{C})$, $\mathfrak{k} = \mathfrak{so}(2l-1,\mathbb{C})$.

Theorem 3.2. *Let $\mathfrak{g}(\geq i-1)$ for $i = 1,\ldots,l-1$ be the orthogonal eigenvalue coincidence variety defined in (2.13). Let $Q = K \cdot \mathfrak{b} \subset \mathcal{B}$ be a K-orbit with codim$(Q) = i-1$.*

(1) *There exists a θ-stable parabolic subalgebra $\mathfrak{r}$ with $\mathfrak{b} \subset \mathfrak{r}$, and $\mathfrak{r}$ satisfies the hypothesis of Proposition 2.14. The parabolic subalgebra $\mathfrak{r}$ has θ-stable Levi decomposition*

$$\mathfrak{r} = \mathfrak{l} \oplus \mathfrak{u} \text{ with } \mathfrak{l}_{ss} \cong \mathfrak{so}(2(l-i)+2,\mathbb{C}) \text{ and } \mathfrak{z} \cong (\mathfrak{gl}(1,\mathbb{C}))^{i-1}. \quad (3.4)$$

Let $L_{ss} \cong SO(2(l-i)+2,\mathbb{C}) \subset G$ be the connected algebraic subgroup with Lie algebra $\mathfrak{l}_{ss}$. Then $\theta|_{\mathfrak{l}_{ss}} = \theta_{2l-2i+2}$, so that

$$\mathfrak{l}_{ss}^{\theta} = \mathfrak{l}_{ss} \cap \mathfrak{k} \cong \mathfrak{so}(2(l-i)+1,\mathbb{C}) \text{ and thus } (L_{ss}^{\theta})^0 = K \cap L_{ss} \cong SO(2(l-i)+1,\mathbb{C}).$$

Furthermore, $SO(2(l-i)+1,\mathbb{C}) \cdot (\mathfrak{b} \cap \mathfrak{l}_{ss})$ is open in $\mathcal{B}_{\mathfrak{so}(2(l-i)+2,\mathbb{C})}$.

(2) *We have $\overline{Y_Q} = Y_{Q_{\mathfrak{r}}}$, and the variety $Y_{Q_{\mathfrak{r}}}$ is an irreducible component of $\mathfrak{g}(\geq i-1)$.*

Proof. The proof is very similar to the proof of Theorem 3.1. We begin with the proof of part (1). Again, by K-equivariance, it suffices to prove the statement for any representative $\mathfrak{b}$ of the K-orbit Q of codimension $i-1$. By part (3) of Proposition 2.24, we can take $\mathfrak{b} = \mathfrak{b}_i = s_{\alpha_{l-1}} s_{\alpha_{l-2}} \ldots s_{\alpha_i}(\mathfrak{b}_+)$. Let $\mathfrak{r} \subset \mathfrak{g}$ be the standard parabolic subalgebra generated by $\mathfrak{b}_+$ and the negative simple root spaces $\mathfrak{g}_{-\alpha_l}, \mathfrak{g}_{-\alpha_{l-1}}, \ldots, \mathfrak{g}_{-\alpha_i}$. We claim that $\mathfrak{r}$ is θ-stable. Indeed, we saw in Example 2.16 that the roots α_i are compact imaginary for $i = 1, \ldots, l-2$ and that α_{l-1} and α_l are complex θ-stable with $\theta(\alpha_{l-1}) = \alpha_l$. It then follows easily that $\mathfrak{r}$ has Levi decomposition (3.4) and that $\theta|_{\mathfrak{l}_{ss}} = \theta_{2(l-i)+2}$, whence $\mathfrak{l}_{ss}^{\theta} = \mathfrak{k} \cap \mathfrak{l}_{ss} \cong \mathfrak{so}(2(l-i)+1, \mathbb{C})$, and $(L_{ss}^{\theta})^0 = K \cap L_{ss} \cong SO(2(l-i)+1, \mathbb{C})$. The remainder of the proof proceeds exactly as in the proof of part (1) of Theorem 3.1, using Equation (2.28) instead of Equation (2.27).

The proof of (2) is also analogous to the proof of part (2) of Theorem 3.1. The key observation is that for $x \in \mathfrak{r}$ with $x_{\mathfrak{l}} = x_{\mathfrak{z}} \oplus x_{\mathfrak{l}_{ss}}$ the coordinates of $x_{\mathfrak{z}} \in \mathfrak{z} \cong (\mathfrak{gl}(1, \mathbb{C}))^{i-1}$ are in the spectrum of both $x \in \mathfrak{g}$ and $x_{\mathfrak{k}} \in \mathfrak{k}$. To show this, one observes that $\mathfrak{z} \subset \mathfrak{k}$, which follows since θ permutes the simple roots of $\mathfrak{l}$. We leave the remaining details to the reader.

Q.E.D.

Remark 3.3. Note that $\mathfrak{z} \subset \mathfrak{k}$, where $\mathfrak{z}$ is the centre of the Levi subalgebras $\mathfrak{l}$ in Theorems 3.1 and 3.2.

We now consider the case where Q is a closed K-orbit.

Theorem 3.4. *Let Q be a closed K-orbit on $\mathcal{B}$. Then Y_Q is an irreducible component of $\mathfrak{g}(\geq r_{n-1}) = \mathfrak{g}(r_{n-1})$.*

Proof. We show that for a closed K-orbit $Q = K \cdot \mathfrak{b}$, the subvariety $Y_Q \subset \mathfrak{g}(r_{n-1})$. It then follows from Proposition 2.13 that Y_Q is an irreducible component of $\mathfrak{g}(r_{n-1})$. By K-equivariance, it suffices to show that $\mathfrak{b} \subset \mathfrak{g}(r_{n-1})$. If $\mathfrak{g}$ is of type B, then part (2) of Proposition 2.23 implies that $\mathfrak{b} = \mathfrak{b}_+$ or $\mathfrak{b} = s_{\alpha_l}(\mathfrak{b}_+)$. In either case, $\mathfrak{b}$ contains the standard diagonal Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Now by Remark 2.21, $\mathfrak{b} \cap \mathfrak{k}$ is a Borel subalgebra of $\mathfrak{k}$ with Levi decomposition

$$\mathfrak{b} \cap \mathfrak{k} = \mathfrak{h} \oplus (\mathfrak{n} \cap \mathfrak{k}),$$

where $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ is the nilradical of $\mathfrak{b}$. Thus, for $x \in \mathfrak{b}$ with $x = x_{\mathfrak{h}} + x_{\mathfrak{n}}$, with $x_{\mathfrak{h}} \in \mathfrak{h}$ and $x_{\mathfrak{n}} \in \mathfrak{n}$, the coordinates of $x_{\mathfrak{h}}$ are in the spectrum of both x and $x_{\mathfrak{k}}$. It follows that $\mathfrak{b} \subset \mathfrak{g}(r_{n-1})$.

If $\mathfrak{g}$ is of type D, then part (2) of Proposition 2.24 states that $Q_+ = K \cdot \mathfrak{b}_+$ is the only closed K-orbit. We recall that $\theta(\epsilon_{r_n}) = -\epsilon_{r_n}$, and $\theta(\epsilon_i) = \epsilon_i$ for all $i \neq r_n$ (see Section 2.2). Therefore, $\mathfrak{h} \cap \mathfrak{k} = \mathrm{diag}[b_1, \ldots, b_{r_{n-1}}, 0, 0, -b_{r_{n-1}}, \ldots, -b_1]$, and $\mathfrak{b} \cap \mathfrak{k}$ is a Borel subalgebra of $\mathfrak{k}$ with Levi decomposition $\mathfrak{b} \cap \mathfrak{k} = \mathfrak{h} \cap \mathfrak{k} \oplus \mathfrak{n} \cap \mathfrak{k}$. Thus, for any $x \in \mathfrak{b}$, $x = x_{\mathfrak{h}} + x_{\mathfrak{n}}$, and $x_{\mathfrak{h}} = x_{\mathfrak{h} \cap \mathfrak{k}} + x_{\mathfrak{h} \cap \mathfrak{g}^{-\theta}}$, with $x_{\mathfrak{k}} = x_{\mathfrak{h} \cap \mathfrak{k}} + x_{\mathfrak{n} \cap \mathfrak{k}} \in \mathfrak{k} \cap \mathfrak{b}$. Thus, $x_{\mathfrak{h} \cap \mathfrak{k}} \in \mathfrak{h}$ is in the spectrum of both x and $x_{\mathfrak{k}}$. It follows that $\mathfrak{b}_+ \subset \mathfrak{g}(r_{n-1})$.

Q.E.D.

Remark 3.5. As we noted in Remark 2.15, the hypothesis of Proposition 2.14 is true for the real rank one symmetric pair $(\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C}), \mathfrak{k} = \mathfrak{gl}(n-1, \mathbb{C}) \oplus \mathfrak{gl}(1, \mathbb{C}))$ and for any K-orbit Q. The analogue of part (2) of Theorems 3.1 and 3.2 also holds in this setting (see Theorems 3.6 and 3.7, [CE15]).

Let $(\mathfrak{g}, \mathfrak{k})$ be a symmetric pair, and let $Q = K \cdot \mathfrak{b} \subset \mathcal{B}$ be an arbitrary K-orbit in the flag variety $\mathcal{B}$ of $\mathfrak{g}$. Then if $\mathfrak{r} \subset \mathfrak{g}$ is a θ-stable parabolic subalgebra with $\mathfrak{b} \subset \mathfrak{r}$, the K-orbit Q has the structure of a fibre bundle as in (2.23). However, $(K \cap L_{ss}) \cdot (\mathfrak{b} \cap \mathfrak{l}_{ss})$ need not be the open K-orbit in $\mathcal{B}_{\mathfrak{l}_{ss}}$.

The hypothesis of Proposition 2.14 does not hold for the real rank one symmetric pair with $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$ and $\mathfrak{k} = \mathfrak{sp}(2n-2, \mathbb{C}) \oplus \mathfrak{sp}(2, \mathbb{C})$. Further, one can show that for this case, the varieties Y_Q are not irreducible components of the natural eigenvalue coincidence varieties. It would be interesting to further analyze objects analogous to those studied in this paper in that example.

3.2. **Every irreducible component of $\mathfrak{g}(\geq i)$ is of the form $\overline{Y_Q}$.** In this section, we complete the last step of the proof of Theorem 1.1. Consider the regular semisimple elements $\mathfrak{k}^{rs}$ of $\mathfrak{k}$, and let $\mathfrak{h}_{\mathfrak{k}}^{reg} = \mathfrak{k}^{rs} \cap \mathfrak{h}_{\mathfrak{k}}$. For x in $\mathfrak{g}$, consider the spectrum $\sigma(x_{\mathfrak{k}}) = \{\pm a_1, \ldots, \pm a_{r_{n-1}}\}$ of $x_{\mathfrak{k}}$. If $\mathfrak{k}$ is type D, $x_{\mathfrak{k}} \in \mathfrak{k}^{rs}$ if and only if $a_i \neq \pm a_j$ for $i \neq j$. If $\mathfrak{k}$ is type B, $x_{\mathfrak{k}} \in \mathfrak{k}^{rs}$ if and only if $a_i \neq \pm a_j$ for $i \neq j$, and all $a_i \neq 0$.

Theorem 3.6. *Every irreducible component of the variety $\mathfrak{g}(\geq i)$, $i = 0, \ldots, r_{n-1}$ is of the form $\overline{Y_Q}$ for some K-orbit Q on $\mathcal{B}$ with codim$(Q) = i$.*

Proof. Consider the set:

$$U := \{x \in \mathfrak{g} : \ x_{\mathfrak{k}} \in \mathfrak{k}^{rs} \text{ and } 0 \notin \sigma(x_{\mathfrak{k}})\}. \tag{3.5}$$

Let $U(\geq i) := U \cap \mathfrak{g}(\geq i)$. Note that $\mathfrak{h} \cap U(\geq i) \neq \emptyset$, so that U and $U(\geq i)$ are non-empty Zariski open subsets of $\mathfrak{g}$ and $\mathfrak{g}(\geq i)$ respectively. By Proposition 2.2 and Exercise III.9.1 of [Har77], $\Phi_n(U) \subset V^{r_{n-1}, r_n}$ is open. Thus, $V^{r_{n-1}, r_n}(\geq i) \setminus \Phi_n(U)$ is a proper, closed subvariety of $V^{r_{n-1}, r_n}(\geq i)$ and therefore has positive codimension by Lemma 2.6. It follows by Propositions 2.2 and 2.8 and Corollary III.9.6 of [Har77] that $\mathfrak{g}(\geq i) \setminus U(\geq i) = \Phi_n^{-1}(V^{r_{n-1}, r_n}(\geq i) \setminus \Phi_n(U))$ is a proper, closed subvariety of $\mathfrak{g}(\geq i)$ of positive codimension. Since $\mathfrak{g}(\geq i)$ is equidimensional, it follows that $Z \cap U(\geq i)$ is nonempty for any irreducible component Z of $\mathfrak{g}(\geq i)$. Thus, it suffices to show that

$$U(\geq i) \subset \bigcup_{\text{codim}(Q)=i} \overline{Y_Q}. \tag{3.6}$$

To prove Equation (3.6), we consider the following subvariety of $U(\geq i)$:

$$\Xi := \{x \in U(\geq i) : \ x_{\mathfrak{k}} = (a_1, \ldots, a_{r_{n-1}}) \in \mathfrak{h}_{\mathfrak{k}}^{reg}, \text{ and } \sigma(x_{\mathfrak{k}}) \cap \sigma(x) \supset \{\pm a_1, \ldots, \pm a_i\},\ a_j \neq 0\, \forall j\}. \tag{3.7}$$

It is easy to check that any element of $U(\geq i)$ is K-conjugate to an element in Ξ. Thus, by the K-equivariance of the varieties $\overline{Y_Q}$, it is enough to show that

$$\Xi \subset \bigcup_{\text{codim}(Q)=i} \overline{Y_Q}. \tag{3.8}$$

We consider the cases where $\mathfrak{g}$ is type B and type D separately. First, we assume that $\mathfrak{g} = \mathfrak{so}(2l+1, \mathbb{C})$. By Theorem 3.1, it suffices to show that

$$\Xi \subset Y_{Q_{\mathfrak{r}}} \text{ for } i < l, \tag{3.9}$$

where $\mathfrak{r}$ is the parabolic subalgebra generated by $\mathfrak{b}_+$ and the negative simple root spaces $\mathfrak{g}_{-\alpha_{i+1}}, \dots, \mathfrak{g}_{-\alpha_l}$. For $i = l$ we need to show that

$$\Xi \subset Y_{Q_+} \cup Y_{Q_-}, \tag{3.10}$$

where $Q_+ = K \cdot \mathfrak{b}_+$ and $Q_- = K \cdot \mathfrak{b}_-$ are the distinct closed K-orbits on $\mathcal{B}$ (see part (2) of Proposition 2.23). To prove Equations (3.9) and (3.10), we need to describe the variety Ξ in more detail. Recall from Example 2.16 that

$$\mathfrak{g}^{-\theta} = \bigoplus_{j=1}^{l} \mathfrak{g}_{\epsilon_j} \oplus \mathfrak{g}_{-\epsilon_j}.$$

Let $e_{\pm\epsilon_j} \in \mathfrak{g}_{\pm\epsilon_j}$ be a nonzero root vector. Consider elements of the form:

$$\underline{a} \oplus_{j=1}^{l} u_j e_{\epsilon_j} \oplus_{j=1}^{l} v_j e_{-\epsilon_j}, \tag{3.11}$$

where $\underline{a} = \text{diag}[a_1, \dots, a_l, 0, -a_l, \dots, -a_1] \in \mathfrak{h}$, $a_i \neq \pm a_j$ if $i \neq j$, and each $a_i \neq 0$. We choose the root vectors $e_{\pm\epsilon_j}$ so that Ξ consists of matrices of the form:

$$X := \begin{bmatrix} a_1 & \dots & 0 & u_1 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & \dots & a_l & u_l & 0 & \dots & 0 \\ v_1 & \dots & v_l & 0 & -u_l & \dots & -u_1 \\ 0 & \dots & 0 & -v_l & -a_l & \dots & 0 \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & -v_1 & 0 & \dots & -a_1 \end{bmatrix} \tag{3.12}$$

with $a_k \neq \pm a_j$ for $k \neq j$, $a_j \neq 0$ for $j = 1, \dots, l$, and $\pm a_j$ is an eigenvalue of X for $j = 1, \dots, i$. It is easy to see that the elements $\pm a_j$ for $j = 1, \dots, i$ are eigenvalues of X if and only if

$$u_j v_j = 0 \text{ for } j = 1, \dots, i. \tag{3.13}$$

This follows easily from the fact that a_j is an eigenvalue of X if and only if the matrix $X - a_j Id_{2l+1}$ is singular, where Id_{2l+1} denotes the $(2l+1) \times (2l+1)$ identity matrix.

We can now describe the irreducible components of Ξ using (3.13). For $k = 1, \dots, i$, we define an index j_k equal to either $j_k = U$ (U for upper) or $j_k = L$ (L for lower). Consider the subvariety $\Xi_{j_1,\dots,j_i} \subset \Xi$ defined by:

$$\Xi_{j_1,\dots,j_i} := \{x \in \Xi : v_k = 0 \text{ if } j_k = U, u_k = 0 \text{ if } j_k = L\}. \tag{3.14}$$

Then

$$\Xi = \bigcup_{j_k=U,\,L} \Xi_{j_1,\dots,j_i} \tag{3.15}$$

is the irreducible component decomposition of Ξ. Notice that in the case $j_k = U$ for all $k = 1, \dots, i$, then

$$\Xi_{U,\dots,U} \subset \mathfrak{r}. \tag{3.16}$$

This follows from the observation that $\epsilon_j = \alpha_j + \cdots + \alpha_l$ for any $j = 1, \dots, l$. Thus, for $j = i+1, \dots, l$, $\mathfrak{g}_{\pm\epsilon_j} \subset \mathfrak{l}_{ss} \subset \mathfrak{r}$, where $\mathfrak{l}_{ss}$ is the semisimple part of the Levi factor of $\mathfrak{r}$, and $\mathfrak{g}_{\epsilon_j} \subset \mathfrak{u}$ for $j = 1, \dots, i$ (see (3.1)). Observe also that if $j_k = L$ for some $k = 1, \dots, i$, then

$$\mathrm{Ad}(\dot{s}_{\epsilon_k})\Xi_{j_1,\dots,j_{k-1},L,\dots,j_i} = \Xi_{j_1,\dots,j_{k-1},U,\dots,j_i}. \tag{3.17}$$

This follows immediately from the fact that $\mathrm{Ad}(\dot{s}_{\epsilon_i})\mathfrak{g}_{\epsilon_j} = \mathfrak{g}_{\epsilon_j}$ for $j \neq i$, and $\mathrm{Ad}(\dot{s}_{\epsilon_i})\mathfrak{g}_{\pm\epsilon_i} = \mathfrak{g}_{\mp\epsilon_i}$.

We now analyze the irreducible variety $\Xi_{j_1,\dots,j_i}$. Suppose that for the subsequence $1 \leq k_1 < \cdots < k_{m-1} \leq i$ we have $j_{k_1} = j_{k_2} = \cdots = j_{k_{m-1}} = L$ and that for the complementary subsequence $k_m < \cdots < k_i$ we have $j_{k_m} = j_{k_{m+1}} = \cdots = j_{k_i} = U$. First, suppose that $i < l$. Consider the element

$$\sigma := s_{\epsilon_{k_1}} s_{\epsilon_{k_2}} \dots s_{\epsilon_{k_{m-1}}} \in W. \tag{3.18}$$

It follows from Equations (3.16) and (3.17) that

$$\mathrm{Ad}(\dot{\sigma})\Xi_{j_1,\dots,j_i} \subset \mathfrak{r}. \tag{3.19}$$

Note that s_{ϵ_j} acts on the coordinates of $\mathfrak{h}$ by sign change in the j-th coordinate. Thus, if $m-1$ is even, it follows from part (1) of Proposition 2.22 that $\sigma \in W_K$, and we can choose its representative $\dot{\sigma} \in K$. If $m-1$ is odd, then replace σ by $\tau := s_{\epsilon_l} s_{\epsilon_{k_1}} s_{\epsilon_{k_2}} \dots s_{\epsilon_{k_{m-1}}}$. Then we can choose $\dot{\tau} \in K$, and since we can choose $\dot{s}_{\epsilon_l} \in L_{ss}$, Equation (3.19) implies

$$\mathrm{Ad}(\dot{\tau})\Xi_{j_1,\dots,j_i} \subset \mathfrak{r}.$$

In either case, the component $\Xi_{j_1,\dots,j_i}$ is W_K-conjugate to a subvariety of $\mathfrak{r}$, and Equation (3.9) follows from (3.15). Now consider the case where $i = l$. Choose σ as in (3.18). Then it follows from (3.16) that $\mathrm{Ad}(\dot{\sigma})\Xi_{j_1,\dots,j_l} \subset \mathfrak{b}_+$. Now if $m-1$ is even, then $\Xi_{j_1,\dots,j_l} \subset Y_{Q_+} = \mathrm{Ad}(K)\mathfrak{b}_+$. However, if $m-1$ is odd, then $\mathrm{Ad}(\dot{\tau})\Xi_{j_1,\dots,j_l} \subset s_{\epsilon_l}(\mathfrak{b}_+)$, whence $\Xi_{j_1,\dots,j_l} \subset Y_{Q_-} = \mathrm{Ad}(K)s_{\epsilon_l}(\mathfrak{b}_+)$. Thus, Equation (3.10) is proven.

We now prove (3.8) when $\mathfrak{g} = \mathfrak{so}(2l, \mathbb{C})$. By Theorem 3.2, it suffices to prove

$$\Xi \subset Y_{Q_{\mathfrak{r}}}, \tag{3.20}$$

where $\mathfrak{r}$ is the parabolic subalgebra generated by $\mathfrak{b}_+$ and the negative simple root spaces $\mathfrak{g}_{-\alpha_{i+1}}, \ldots, \mathfrak{g}_{-\alpha_l}$ for $i < l-1$, and $\mathfrak{r} = \mathfrak{b}_+$ for $i = l-1$. Recall from Example 2.16 that

$$\mathfrak{g}^{-\theta} = \bigoplus_{j=1}^{l-1} (\mathfrak{g}_{\epsilon_j - \epsilon_l} \oplus \mathfrak{g}_{\epsilon_j + \epsilon_l})^{-\theta} \oplus (\mathfrak{g}_{-(\epsilon_j - \epsilon_l)} \oplus \mathfrak{g}_{-(\epsilon_j + \epsilon_l)})^{-\theta}.$$

Let $e_{\pm j}$ be a basis for $(\mathfrak{g}_{\pm(\epsilon_j - \epsilon_l)} \oplus \mathfrak{g}_{\pm(\epsilon_j + \epsilon_l)})^{-\theta}$ respectively. Consider elements of the form

$$\underline{a} \oplus_{j=1}^{l-1} u_j e_j \oplus_{j=1}^{l-1} v_j e_{-j}, \tag{3.21}$$

where $\underline{a} = \mathrm{diag}[a_1, \ldots, a_l, -a_l, \ldots, -a_1]$, $a_i \neq \pm a_j$, $a_i \neq 0$ for $i, j \leq l-1$, and $u_j, v_j \in \mathbb{C}$. Arguing as in the previous case, we see Ξ consists of elements of the form (3.21) satisfying

$$u_j v_j = 0 \text{ for } j = 1, \ldots, i. \tag{3.22}$$

We define the varieties $\Xi_{j_1,\ldots,j_i}$ with $j_k = L, U$ analogously to (3.14). We have $\Xi = \bigcup_{j_k = L,U} \Xi_{j_1,\ldots,j_i}$ (cf. (3.15)). Now we observe that if $j_k = U$ for all k, then

$$\Xi_{U,\ldots,U} \subset \mathfrak{r}. \tag{3.23}$$

This follows from the observation that $\epsilon_j - \epsilon_l = \alpha_j + \cdots + \alpha_{l-1}$ and $\epsilon_j + \epsilon_l = \alpha_j + \cdots + \alpha_{l-2} + \alpha_l$. Thus, for $j = i+1, \ldots, l$, the root spaces $\mathfrak{g}_{\pm(\epsilon_j - \epsilon_l)}$ and $\mathfrak{g}_{\pm(\epsilon_j + \epsilon_l)}$ are in $\mathfrak{l}_{ss} \subset \mathfrak{r}$. Further, for $j = 1, \ldots i$, the root spaces $\mathfrak{g}_{\epsilon_j - \epsilon_l}$ and $\mathfrak{g}_{\epsilon_j + \epsilon_l} \subset \mathfrak{u} \subset \mathfrak{r}$ (see (3.4)). We now show that any $\Xi_{j_1,\ldots,j_i} \subset Y_{Q_{\mathfrak{r}}}$. Recall from part (2) of Proposition 2.22 that

$$W^\theta = W_K = \langle s_{\alpha_1}, \ldots, s_{\alpha_{l-2}}, s_{\alpha_{l-1}} \cdot s_{\alpha_l} \rangle.$$

For $j = 1, \ldots, i$, define $w_j := s_{\epsilon_j - \epsilon_{l-1}} s_{\alpha_{l-1}} s_{\alpha_l} s_{\epsilon_j - \epsilon_{l-1}}$. Then $w_j \in W_K$, and w_j has order 2. In fact, w_j acts on $\mathfrak{h}$ via

$$w_j : (a_1, \ldots, a_j, \ldots, a_l) \to (a_1, \ldots, -a_j, \ldots, -a_l). \tag{3.24}$$

We claim that

$$\mathrm{Ad}(\dot{w}_j)\Xi_{j_1,\ldots,j_{k-1},L,\ldots,j_i} \subset \Xi_{j_1,\ldots,U,\ldots,j_i}. \tag{3.25}$$

Indeed, (3.24) implies that

$$w_j \cdot (\epsilon_j + \epsilon_l) = -(\epsilon_j + \epsilon_l),\ w_j \cdot (\epsilon_j - \epsilon_l) = -(\epsilon_j - \epsilon_l), \text{ and } w_j \cdot (\epsilon_k + \epsilon_l) = \epsilon_k - \epsilon_l \text{ for } k \neq j.$$

Further, since $w_j \in W_K$,

$$\mathrm{Ad}(\dot{w}_j) : (\mathfrak{g}_{\pm(\epsilon_j - \epsilon_l)} \oplus \mathfrak{g}_{\pm(\epsilon_j + \epsilon_l)})^{-\theta} \mapsto (\mathfrak{g}_{\mp(\epsilon_j - \epsilon_l)} \oplus \mathfrak{g}_{\mp(\epsilon_j + \epsilon_l)})^{-\theta}$$

and $\mathrm{Ad}(\dot{w}_j)$ stabilizes the space $(\mathfrak{g}_{\pm(\epsilon_k - \epsilon_l)} \oplus \mathfrak{g}_{\pm(\epsilon_k + \epsilon_l)})^{-\theta}$ for $k \neq j$. Equation (3.25) now follows from the definition of the varieties $\Xi_{j_1,\ldots,j_i}$. Thus, if we are given a variety $\Xi_{j_1,\ldots,j_i}$ with $j_{k_1} = j_{k_2} = \cdots = j_{k_{m-1}} = L$, it follows from (3.25) and (3.23) that

$$\mathrm{Ad}(\dot{w}_{k_1} \ldots \dot{w}_{k_{m-1}})\Xi_{j_1,\ldots,j_i} \subset \mathfrak{r}.$$

Since $\Xi = \bigcup_{j_k = L,U} \Xi_{j_1,\ldots,j_i}$, Equation (3.20) follows. This completes the proof.

Q.E.D.

Proof of Theorem 1.1. Equation (1.3) follows from Theorems 3.1 and 3.2 along with Theorem 3.6. The statement about the number of irreducible components of $\mathfrak{g}(\geq i)$ follows from Parts 2 and 3 of Propositions 2.23 and 2.24.

Q.E.D.

Corollary 3.7. *Recall the variety $\mathfrak{g}(i)$ defined in Equation (2.14). The irreducible component decomposition of $\mathfrak{g}(i)$ is*

$$\mathfrak{g}(i) = \bigcup_{\mathit{codim}(Q)=i} Y_Q \cap \mathfrak{g}(i). \tag{3.26}$$

Proof. Theorem 1.1 and Equation (1.3) imply that the irreducible component decomposition of the variety $\mathfrak{g}(i)$ is

$$\mathfrak{g}(i) = \bigcup_{\mathrm{codim}(Q)=i} \overline{Y_Q} \cap \mathfrak{g}(i), \tag{3.27}$$

By Propositions 2.8 (1) and 2.13, we have $\overline{Y_Q} \cap \mathfrak{g}(i) \neq \emptyset$ for all Q with $\mathrm{codim}(Q) = i$. For each K-orbit Q with $\mathrm{codim}(Q) = i$, we claim that

$$\overline{Y_Q} \cap \mathfrak{g}(i) = Y_Q \cap \mathfrak{g}(i). \tag{3.28}$$

Indeed, suppose that (3.28) were false. Then since $\overline{Y_Q} = \bigcup_{Q' \subset \overline{Q}} Y_{Q'}$ by Lemma 2.12, there exists a K-orbit Q' with $\mathrm{codim}(Q') > \mathrm{codim}(Q)$ such that $Y_{Q'} \cap \mathfrak{g}(i) \neq \emptyset$. But this contradicts Theorem 1.1 which asserts that $Y_{Q'} \subset \mathfrak{g}(\geq i+1)$. Equation (3.26) now follows from (3.28) and (3.27).

Q.E.D.

The following corollary will be useful in Sections 4.3 and 4.4.

Corollary 3.8. *For $i = 0, \ldots, r_{n-1} - 1$, the irreducible component decomposition of $\mathfrak{g}(i)$ is*

$$\mathfrak{g}(i) = Y_{Q_{\mathfrak{r}}} \cap \mathfrak{g}(i), \tag{3.29}$$

where $\mathfrak{r}$ is the θ-stable parabolic subalgebra of Theorems 3.1 and 3.2. For $i = r_{n-1}$ and $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$,

$$\mathfrak{g}(r_{n-1}) = \mathfrak{g}(\geq r_{n-1}) = Y_{Q_+}, \tag{3.30}$$

where $Q_+ = K \cdot \mathfrak{b}_+$ is the unique closed K-orbit on $\mathcal{B}$ (see part (2) of Proposition 2.24). For $\mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C})$ the irreducible component decomposition of $\mathfrak{g}(r_{n-1})$ is

$$\mathfrak{g}(r_{n-1}) = \mathfrak{g}(\geq r_{n-1}) = Y_{Q_+} \cup Y_{Q_-}, \tag{3.31}$$

where Q_+ and Q_- are the distinct closed K-orbits on $\mathcal{B}$ (see part (2) of Proposition 2.23).

Proof. The result follows immediately from Equation (3.27) and part (2) of Theorems 3.1 and 3.2.

Q.E.D.

4. The geometric invariant theory of multiplicity free spherical pairs

In this section, we study the K-action on $\mathfrak{g}$ in the cases $(K, \mathfrak{g}) = (GL(n-1, \mathbb{C}), \mathfrak{gl}(n, \mathbb{C}))$ and $(SO(n-1, \mathbb{C}), \mathfrak{so}(n, \mathbb{C}))$. We extend a result of Kostant characterizing regular elements using differentials in Theorem 4.8. We then analyze the K-action on the subvariety $\mathfrak{g}(0)$, and show that all the K-orbits in $\mathfrak{g}(0)$ are closed. We use the above analysis to give representatives of the closed K-orbits in $\mathfrak{g}$, and discuss some applications to strongly regular elements.

Definition 4.1. Let G be a reductive, algebraic group, and let $H \subset G$ be a reductive algebraic subgroup. The pair (G, H) is called spherical if H acts on the flag variety $\mathcal{B}$ of $\mathfrak{g}$ with finitely many orbits.

Remark 4.2. Let V be a rational G-representation, and let V^H be the set of H-fixed vectors in V. It is well-known that Definition 4.1 is equivalent to the statement that $\dim V^H \leq 1$ for every irreducible, rational G-representation V (see [VK78], [Bri87]).

Let (G, H) be a spherical pair. Let $\mathfrak{g} = \mathrm{Lie}(G)$ and let $\mathfrak{h} = \mathrm{Lie}(H)$. Let $\langle\langle \cdot, \cdot \rangle\rangle$ denote the Killing form on $\mathfrak{g}$, and let $\mathfrak{h}^{\perp}$ be the annihilator of $\mathfrak{h}$ with respect to $\langle\langle \cdot, \cdot \rangle\rangle$. Then the adjoint action of G on $\mathfrak{g}$ restricts to an action of H on $\mathfrak{h}^{\perp}$, which is referred to in the literature as the coisotropy representation of H (see [Pan90]). Let $\mathbb{C}[\mathfrak{h}^{\perp}]^H$ be the ring of H-invariant polynomials on $\mathfrak{h}^{\perp}$. Then it is well-known that $\mathbb{C}[\mathfrak{h}^{\perp}]^H$ is a polynomial algebra (Kor 7.2 of [Kno90b] or Corollary 5 of [Pan90]). Consider the geometric invariant theory quotient $\Psi : \mathfrak{h}^{\perp} \to \mathfrak{h}^{\perp}//H$. In Korollar 7.2 of [Kno90b], Knop proved that Ψ is flat. We consider spherical pairs satisfying:

$$\dim \mathcal{B} = \dim \mathfrak{h}^{\perp} - \dim \mathfrak{h}^{\perp}//H. \tag{4.1}$$

In the appendix, we give a different and simpler proof of Knop's result for spherical pairs satisfying (4.1) by using conormal geometry.

We now analyze further what the condition in Equation (4.1) means for the coisotropy representation. If an algebraic group A acts on an irreducible variety Y, we say $y \in Y$ is A-regular if $\dim(A \cdot y) \geq \dim(A \cdot z)$ for all $z \in Y$. When the group A is clear, we let Y_{reg} denote its A-regular elements. Recall that an element $x \in \mathfrak{g}$ is $\mathrm{Ad}(G)$-regular if $\dim(\mathrm{Ad}(G) \cdot x) = \dim(\mathfrak{g}) - \mathrm{rank}(\mathfrak{g})$. A basic result of Kostant (Theorem 9,[Kos63]) states that if $\mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[\psi_1, \ldots, \psi_r]$ is the ring of $\mathrm{Ad}(G)$-invariant polynomials on $\mathfrak{g}$, then

$$x \in \mathfrak{g}_{reg} \text{ if and only if } d\psi_1(x) \wedge \cdots \wedge d\psi_r(x) \neq 0. \tag{4.2}$$

If $x \in \mathfrak{g}_{reg}$, and we identify $T_x^*(\mathfrak{g})$ with $\mathfrak{g}$ using the non-degenerate form on $\mathfrak{g}$, then

$$\mathrm{span}\{d\psi_i(x) : i = 1 \ldots, r\} = \mathfrak{z}_{\mathfrak{g}}(x), \tag{4.3}$$

where $\mathfrak{z}_{\mathfrak{g}}(x)$ denotes the centralizer of x in $\mathfrak{g}$. We study the set of H-regular elements:

$$\mathfrak{h}^{\perp}_{reg} = \{x \in \mathfrak{h}^{\perp} : \dim H \cdot x \text{ is maximal}\}. \tag{4.4}$$

The following result relates the sets $\mathfrak{h}^{\perp}_{reg}$ and $\mathfrak{g}_{reg}$.

Theorem 4.3. *Let (G,H) be a spherical pair. Then the following conditions are equivalent.*

(1) *Equation (4.1) holds.*
(2) *We have $\mathfrak{h}^{\perp}_{reg} \subset \mathfrak{g}_{reg}$.*

Proof. We first show that (1) implies (2). Let $x \in \mathfrak{h}^{\perp}_{reg}$. By Theorems 3 and 6 and Equation (15) of [Pan90],

$$\dim \mathfrak{h}^{\perp}//H = \operatorname{codim}_{\mathfrak{h}^{\perp}} H \cdot x. \tag{4.5}$$

By (1),

$$\dim H \cdot x = \dim \mathcal{B}. \tag{4.6}$$

By Proposition 1 of [Pan90], we know that $\dim(\mathrm{Ad}(G)x) \geq 2\dim(H \cdot x) = 2\dim(\mathcal{B})$. It follows that $x \in \mathfrak{g}_{reg}$. For the converse, by Theorem 3 of [Pan90], there is a dense open subset U of $\mathfrak{h}^{\perp}_{reg}$ such that if $y \in U$, then $\dim(\mathrm{Ad}(G)y) = 2\dim(H \cdot y)$. Let $x \in U \subset \mathfrak{g}_{reg}$. Then $\dim(H \cdot x) = \frac{1}{2}\dim(\mathrm{Ad}(G)x) = \dim(\mathcal{B})$. The assertion now follows by Theorems 3 and 6 of [Pan90].

Q.E.D.

Let θ be an involution of $\mathfrak{g}$. It is well-known that the pair $(\mathfrak{g}, \mathfrak{k} := \mathfrak{g}^{\theta})$ is spherical [Mat79, Spr85]. Recall that an involution θ of $\mathfrak{g}$ is called quasi-split if there is a Borel subalgebra $\mathfrak{b} \in \mathcal{B}$ such that $\mathfrak{b} \cap \theta(\mathfrak{b})$ is a Cartan subalgebra of $\mathfrak{g}$. Let K be the connected subgroup of G with Lie algebra $\mathfrak{k}$. Let $\mathfrak{p} := \mathfrak{g}^{-\theta} \cong \mathfrak{k}^{\perp}$. Let $\mathfrak{h}$ be a θ-stable Cartan subalgebra of $\mathfrak{g}$ such that $\dim(\mathfrak{h}^{-\theta})$ is maximal among all θ-stable Cartan subalgebras of $\mathfrak{g}$. We let $\mathfrak{a} = \mathfrak{h}^{-\theta}$, and let $\mathfrak{m} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})$. We let Φ_c be the compact roots for $\mathfrak{h}$ in $\mathfrak{g}$.

Proposition 4.4. *Let θ be an involution of $\mathfrak{g}$. Then the spherical pair $(\mathfrak{g}, \mathfrak{k})$ satisfies Equation (4.1) if and only if θ is quasi-split.*

Proof. Let $x \in \mathfrak{p}$ be K-regular. By Equation (4.5), we know $\dim(\mathfrak{p}) - \dim(\mathfrak{p}//K) = \dim(K) - \dim(K_x)$. If $K \cdot \mathfrak{b}$ denotes the open K-orbit on $\mathcal{B}$ and $K_{\mathfrak{b}}$ is the stabilizer of $\mathfrak{b}$, then $\dim(\mathcal{B}) = \dim(K) - \dim(K_{\mathfrak{b}})$, so Equation (4.1) holds if and only if $\dim(K_{\mathfrak{b}}) = \dim(K_x)$. By Proposition 8 of [KR71], $\dim(K_x) = \dim(\mathfrak{m})$. By Proposition 6.70 and page 394 of [Kna02], it follows that $\dim(\mathfrak{m}) = \dim(\mathfrak{h}^{\theta}) + |\Phi_c|$. By Corollary 2 of [BH00], we know $\dim(K_{\mathfrak{b}}) = \dim(\mathfrak{h}^{\theta}) + \frac{1}{2}|\Phi_c|$. It follows that Equation (4.1) is equivalent to the assertion that Φ_c is empty, which happens if and only if θ is quasi-split by Lemma 8.3 of [RS90].

Q.E.D.

4.1. **Kostant's Theorem for the K-action on $\mathfrak{g}$.** We now apply Theorem 4.3 to study the K-action on $\mathfrak{g}$ in the cases where $(K, \mathfrak{g}) = (GL(n-1, \mathbb{C}), \mathfrak{gl}(n, \mathbb{C}))$ or $(SO(n-1, \mathbb{C}), \mathfrak{so}(n, \mathbb{C}))$. For $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$, we view $\mathfrak{k} = \mathfrak{gl}(n-1, \mathbb{C})$ as the top left hand corner of $\mathfrak{g}$. Then K is the corresponding algebraic subgroup of $G = GL(n, \mathbb{C})$.

Notation 4.5. The following extends notation from the $\mathfrak{so}(n, \mathbb{C})$ case to the $\mathfrak{gl}(n, \mathbb{C})$ case. We continue to use the notation $r_i = \text{rank}(\mathfrak{g}_i)$, so that $r_i = i$ for $\mathfrak{g}_i = \mathfrak{gl}(i, \mathbb{C})$. The partial Kostant-Wallach map $\Phi_n : \mathfrak{gl}(n, \mathbb{C}) \to \mathbb{C}^{n-1} \times \mathbb{C}^n$ is $\Phi_n(x) = (f_{i,j}(x))_{i=n-1,n; j=1,\ldots,i}$, where $\mathbb{C}[\mathfrak{g}_i]^{G_i} = \mathbb{C}[f_{i,1}, \ldots, f_{i,i}]$. Proposition 2.2 also holds in this case as was noted in Remark 2.3, and Φ_n is identified with the quotient morphism $\mathfrak{g} \to \mathfrak{g}//K$. The varieties $\mathfrak{g}(\geq i)$, $\mathfrak{g}(i)$, $V^{r_{n-1}, r_n}(\geq i)$, and $V^{r_{n-1}, r_n}(i)$ are all defined analogously. See Section 3 of [CE15] for details.

To apply Theorem 4.3 and the theory of spherical varieties to this situation, we consider the following setup. Let G be a connected, reductive algebraic group, let K be a reductive, connected algebraic subgroup, and let $\mathfrak{g}$ and $\mathfrak{k}$ be their Lie algebras. We say that the branching from G to K is multiplicity free if for every irreducible, finite dimensional rational G-representation V, and every irreducible, finite dimensional rational K-representation W we have $\dim \text{Hom}_K(W, V) \leq 1$. Now let $\tilde{G} = G \times K$ and $K_\Delta \subset \tilde{G}$ be the diagonal copy of K in $G \times K$, i.e.

$$K_\Delta := \{(g, g) : g \in K\}.$$

Consider the pair $(\tilde{G}, K_\Delta)$ with Lie algebra pair $(\tilde{\mathfrak{g}}, \mathfrak{k}_\Delta)$. The following result is well-known.

Proposition 4.6.
(1) *The pair $(\tilde{G}, K_\Delta)$ is spherical if and only if the branching rule from G to K is multiplicity free.*
(2) *For the pairs $(G, K) = (GL(n, \mathbb{C}), GL(n-1, \mathbb{C}))$ and $(SO(n, \mathbb{C}), SO(n-1, \mathbb{C}))$, $(\tilde{G}, K_\Delta)$ is spherical.*

Proof. The first statement follows by Theorem B of [Bru97], together with the easy observaton that a Borel subgroup B_K of K has an open orbit on the flag variety G/B of G if and only if K_Δ has an open orbit on $G/B \times K/B_K$. The second statement follows from the first statement and well-known branching laws (see [Joh01]).

Q.E.D.

A spherical pair $(\tilde{G}, K_\Delta)$ satisfying the above property is called a *multiplicity free spherical pair*. A result of Knop shows that up to isogeny, these are essentially the only two multiplicity free spherical pairs [Kno90a]. In the sequel, unless otherwise specified, we assume that that $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{gl}(n, \mathbb{C}), \mathfrak{gl}(n-1, \mathbb{C}))$ or $(\mathfrak{so}(n, \mathbb{C}), \mathfrak{so}(n-1, \mathbb{C}))$.

It is easy to see that the restriction of $\langle\langle \cdot, \cdot \rangle\rangle$ to $\mathfrak{k}$ is non-degenerate. Equip $\tilde{\mathfrak{g}} = \text{Lie}(\tilde{G}) = \mathfrak{g} \oplus \mathfrak{k}$ with the non-degenerate invariant form $\langle \cdot, \cdot \rangle = \langle\langle \cdot, \cdot \rangle\rangle + (\langle\langle \cdot, \cdot \rangle\rangle)|_{\mathfrak{k}}$. For $x \in \mathfrak{g}$, let $x = x_{\mathfrak{k}} + x_{\mathfrak{p}}$ with $x_{\mathfrak{k}} \in \mathfrak{k}$ and $x_{\mathfrak{p}} \in \mathfrak{k}^\perp$. An easy calculation shows that

$$\mathfrak{k}_\Delta^\perp = \{(x, -x_{\mathfrak{k}}) : x \in \mathfrak{g},\, x_{\mathfrak{k}} \in \mathfrak{k}\}.$$

Note that $\mathfrak{k}_\Delta^\perp \cong \mathfrak{g}$ via the map $(x, -x_\mathfrak{k}) \mapsto x$. This isomorphism intertwines the coisotropy representation of K_Δ on $\mathfrak{k}_\Delta^\perp$ with the action of K on $\mathfrak{g}$ via conjugation. We can now use the geometry of spherical varieties, in particular Theorem 4.3, to study the geometry of the K-conjugation on $\mathfrak{g}$ and the partial Kostant-Wallach map Φ_n (see (2.9)).

Lemma 4.7. *Consider the multiplicity-free spherical pairs* $(\tilde{G}, K_\Delta)$.

(1) *Equation (4.1) holds.*
(2) $\dim(K) = \dim(K \cdot x)$ *if and only if* $x \in (\mathfrak{k}_\Delta^\perp)_{reg}$.
(3)

$$(\mathfrak{k}_\Delta^\perp)_{reg} \cong \{x \in \mathfrak{g} : \mathfrak{z}_\mathfrak{k}(x_\mathfrak{k}) \cap \mathfrak{z}_\mathfrak{g}(x) = 0\}. \tag{4.7}$$

Proof. Equation (4.1) is equivalent to the routine identity

$$\dim(\mathcal{B}_\mathfrak{g}) + \dim(\mathcal{B}_\mathfrak{k}) = \dim(\mathfrak{g}) - r_n - r_{n-1} = \dim(\mathfrak{g}) - \dim(\mathfrak{g}//K). \tag{4.8}$$

To prove the second assertion, let $x \in (\mathfrak{k}_\Delta)^\perp_{reg}$. Since (1) holds, we can apply Equation (4.6) to conclude that $\dim(K \cdot x) = \dim(\mathcal{B}_\mathfrak{g}) + \dim(\mathcal{B}_\mathfrak{k})$. The assertion now follows from (4.8) and the simple observation

$$\dim(\mathfrak{g}) - r_n - r_{n-1} = \dim K. \tag{4.9}$$

The second assertion implies that $(x, -x_\mathfrak{k}) \in (\mathfrak{k}_\Delta^\perp)_{reg}$ if and only if $\mathfrak{z}_\mathfrak{k}(x_\mathfrak{k}) \cap \mathfrak{z}_\mathfrak{k}(x) = 0$. The third assertion now follows since $\mathfrak{z}_\mathfrak{k}(x_\mathfrak{k}) \cap \mathfrak{z}_\mathfrak{k}(x) = \mathfrak{z}_\mathfrak{k}(x_\mathfrak{k}) \cap \mathfrak{z}_\mathfrak{g}(x)$.

Q.E.D.

We now describe the regular elements of the coisotropy representation of the spherical pairs $(\tilde{G}, K_\Delta)$, which establishes an analogue of Kostant's theorem. Denote the generators of $\mathbb{C}[\mathfrak{g}]^K$ by $\{f_{n-1,1}, \ldots, f_{n-1,r_{n-1}}; f_{n,1}, \ldots, f_{n,r_n}\}$. Let

$$\omega_{\mathfrak{g}//K} := df_{n-1,1} \wedge \cdots \wedge df_{n-1,r_{n-1}} \wedge df_{n,1} \wedge \cdots \wedge df_{n,r_n} \in \Omega^{r_{n-1}+r_n}(\mathfrak{g}).$$

Theorem 4.8.

$$x \in (\mathfrak{k}_\Delta^\perp)_{reg} \text{ if and only if } \omega_{\mathfrak{g}//K}(x) \neq 0.$$

Proof. We first suppose that $\omega_{\mathfrak{g}//K}(x) \neq 0$. By Equation (4.2), it follows that x is regular in $\mathfrak{g}$ and $x_\mathfrak{k}$ is regular in $\mathfrak{k}$. Equation (4.3) then implies that $\mathfrak{z}_\mathfrak{k}(x_\mathfrak{k}) \cap \mathfrak{z}_\mathfrak{g}(x) = 0$, so $x \in (\mathfrak{k}_\Delta^\perp)_{reg}$ by Equation (4.7).

Conversely, suppose $x \in (\mathfrak{k}_\Delta^\perp)_{reg}$. Then by Theorem 4.3 and part (1) of Lemma 4.7, $(x, -x_\mathfrak{k}) \in \tilde{\mathfrak{g}}_{reg}$. Thus, both $x \in \mathfrak{g}$ and $x_\mathfrak{k} \in \mathfrak{k}$ are regular. Hence by Equation (4.2),

$$df_{n-1,1}(x_\mathfrak{k}) \wedge \cdots \wedge df_{n-1,r_{n-1}}(x_\mathfrak{k}) \neq 0 \text{ and } df_{n,1}(x) \wedge \cdots \wedge df_{n,r_n}(x) \neq 0. \tag{4.10}$$

Since $x \in (\mathfrak{k}_\Delta^\perp)_{reg}$, $\mathfrak{z}_\mathfrak{k}(x_\mathfrak{k}) \cap \mathfrak{z}_\mathfrak{g}(x) = 0$ by Equation (4.7). It now follows from (4.10) and (4.3) that $\omega_{\mathfrak{g}//K}(x) \neq 0$.

Q.E.D.

Theorem 4.8 has an immediate corollary which is of interest in linear algebra.

Corollary 4.9. *Let $x \in \mathfrak{g}$ and suppose that $\mathfrak{z}_{\mathfrak{k}}(x_{\mathfrak{k}}) \cap \mathfrak{z}_{\mathfrak{g}}(x) = 0$. Then $x \in \mathfrak{g}$ and $x_{\mathfrak{k}} \in \mathfrak{k}$ are both regular.*

Proof. This follows by Equation (4.7) and Theorem 4.8.

Q.E.D.

Elements of $(\mathfrak{k}_{\Delta}^{\perp})_{reg}$ regarded as elements of $\mathfrak{g}$ play a major role in our study of the K-action on $\mathfrak{g}$, so we give them a special name.

Definition 4.10. An element $x \in \mathfrak{g}$ such that $\mathfrak{z}_{\mathfrak{k}}(x_{\mathfrak{k}}) \cap \mathfrak{z}_{\mathfrak{g}}(x) = 0$ is said to be n-strongly regular. We denote the set of n-strongly regular elements by $\mathfrak{g}_{nsreg}$.

Remark 4.11. In [CE15], we defined the set of n-strongly regular elements for $\mathfrak{g}$ to be the set of elements $x \in \mathfrak{g}$ for which $\omega_{\mathfrak{g}//K}(x) \neq 0$. It follows from Theorem 4.8 and Equation (4.7) that our new definition is consistent with the previous one and $\mathfrak{g}_{nsreg} \cong (\mathfrak{k}_{\Delta}^{\perp})_{reg}$.

We end this section by explaining how Corollary 4.9 can be used to simplify a crucial definition of Kostant and Wallach in the construction of the Gelfand-Zeitlin integrable system. We consider the chain of subalgebras

$$\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots \subset \mathfrak{g}_{n-1} \subset \mathfrak{g},$$

where $\mathfrak{g}_i = \mathfrak{gl}(i, \mathbb{C})$ (resp. $\mathfrak{so}(i, \mathbb{C})$) when $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ (resp. $\mathfrak{so}(n, \mathbb{C})$). Let $G_i \subset G$ be the corresponding, connected algebraic group. Let $\mathbb{C}[\mathfrak{g}_i]^{G_i} = \mathbb{C}[\psi_{i,1}, \ldots, \psi_{i,r_i}]$ be the ring of $\mathrm{Ad}(G_i)$-invariant polynomials on $\mathfrak{g}_i$, and let $\pi_i : \mathfrak{g} \to \mathfrak{g}_i$ be the projection off of $\mathfrak{g}_i^{\perp}$. For $j = 1, \ldots, r_i$, define $f_{i,j} := \pi_i^* \psi_{i,j}$. Then the Gelfand-Zeitlin collection of functions is

$$J_{GZ} := \{f_{i,j} : i = 1, \ldots, n;\ j = 1, \ldots, r_i\}. \tag{4.11}$$

For $x \in \mathfrak{g}$, consider the subset of $T_x^*(\mathfrak{g})$,

$$dJ_{GZ}(x) := \{df_{i,j}(x) : i = 1, \ldots, n;\ j = 1, \ldots, r_i\}.$$

The set $\mathfrak{g}_{sreg}$ of strongly regular elements was defined by Kostant and Wallach to be the open set

$$\mathfrak{g}_{sreg} := \{x \in \mathfrak{g} : dJ_{GZ}(x) \text{ is a linearly independent set}\}. \tag{4.12}$$

Then $\mathfrak{g}_{sreg}$ is nonempty (Theorem 2.3 of [KW06a] for $\mathfrak{gl}(n, \mathbb{C})$ and Theorem 3.2 of [Col09] for $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$). The existence of a strongly regular element in a regular G-adjoint orbit implies that the functions J_{GZ} in (4.11) form an integrable system on the orbit.

To characterize strongly regular elements, we need a little more notation. For an $n \times n$ matrix x, let $x_i := \pi_i(x)$, and let $\mathfrak{z}_{\mathfrak{g}_i}(x_i)$ denote the centralizer of x_i in $\mathfrak{g}_i$ thought of as a subalgebra of $\mathfrak{g}$.

Proposition 4.12. *An element $x \in \mathfrak{g}$ is strongly regular if and only if*

$$\mathfrak{z}_{\mathfrak{g}_i}(x_i) \cap \mathfrak{z}_{\mathfrak{g}_{i+1}}(x_{i+1}) = 0 \text{ for } i = 1, \ldots, n-1.$$

Proof. An element $x \in \mathfrak{g}$ is strongly regular if and only if the following two conditions hold:

$$\begin{aligned} &(1)\ x_i \in \mathfrak{g}_i, x_{i+1} \in \mathfrak{g}_{i+1} \text{ are regular for all } i = 1, \dots, n-1. \\ &(2)\ \mathfrak{z}_{\mathfrak{g}_i}(x_i) \cap \mathfrak{z}_{\mathfrak{g}_{i+1}}(x_{i+1}) = 0 \text{ for } i = 1, \dots, n-1. \end{aligned} \tag{4.13}$$

For the case $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ this is the content of Theorem 2.14 of [KW06a], and for $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$ it is Proposition 2.11 of [Col09]. It follows from Corollary 4.9 that if $x_{i+1} \in \mathfrak{g}_{i+1}$ satisfies (2) in (4.13), then it automatically satisfies (1).

Q.E.D.

4.2. **The K-orbit structure of $\mathfrak{g}(0)$.** We now study the K-orbit structure of the Zariski open subset

$$\mathfrak{g}(0) = \{x \in \mathfrak{g} : \sigma(x_{\mathfrak{k}}) \cap \sigma(x) = \emptyset\}.$$

We show that $\mathfrak{g}(0) \subset \mathfrak{g}_{nsreg}$, and that each K-orbit in $\mathfrak{g}(0)$ is closed in $\mathfrak{g}$. The fact that $\mathfrak{g}(0) \subset \mathfrak{g}_{nsreg}$ follows from the following result in linear algebra.

Lemma 4.13. *Let V be a finite dimensional complex vector space. Suppose we are given a direct sum decomposition of V*

$$V = V_1 \oplus V_2.$$

Let $X \in End(V)$, and let $Y \neq 0 \in End(V)$ such that $Y : V_1 \to V_1$ and $Y|_{V_2} = 0$. Suppose that $[Y, X] = 0$. Then X has a nonzero eigenvector $u \in V_1$.

Proof. The assumptions imply that the image $\mathrm{Im}(Y)$ of Y is nonzero, contained in V_1, and stable under the action of X. The result follows.

Q.E.D.

The following consequence plays a crucial role in our study of $\mathfrak{g}(0)$.

Proposition 4.14. *Let X, Y, and $V = V_1 \oplus V_2$ be as in the statement of Lemma 4.13. Define $X_1 : V_1 \to V_1$ by $X_1 := \pi_{V_1} \circ X|_{V_1}$, where $\pi_{V_1} : V \to V_1$ is the projection onto V_1 off V_2. Then $\sigma(X_1) \cap \sigma(X) \neq \emptyset$.*

Proof. Let $u \in V_1$ be an eigenvector of X of eigenvalue λ. It follows from definitions that

$$X_1 u = \pi_{V_1}(Xu) = \pi_{V_1}(\lambda u) = \lambda u.$$

Thus, $\lambda \in \sigma(X_1) \cap \sigma(X)$.

Q.E.D.

Example 4.15. Let $V = \mathbb{C}^n$, and let $e_1, \dots, e_n$ be the standard basis of $\mathbb{C}^n$. Let $V_1 = \mathrm{span}\{e_1, \dots, e_k\}$, and let $V_2 = \mathrm{span}\{e_{k+1}, \dots, e_n\}$. Let $x \in \mathfrak{gl}(n, \mathbb{C})$, and let x_k be the $k \times k$ submatrix in the upper lefthand corner of x. We embed $\mathfrak{gl}(k, \mathbb{C})$ in $\mathfrak{gl}(n, \mathbb{C})$ in the upper left corner. Suppose there exists nonzero $Y \in \mathfrak{gl}(k, \mathbb{C})$ with $[Y, x] = 0$. Then Proposition 4.14 implies that $\sigma(x_k) \cap \sigma(x) \neq 0$.

We now return to the pairs $(\mathfrak{gl}(n,\mathbb{C}), \mathfrak{gl}(n-1,\mathbb{C}))$ and $(\mathfrak{so}(n,\mathbb{C}), \mathfrak{so}(n-1,\mathbb{C}))$. Using Proposition 4.14, we can prove a fundamental result regarding the structure of $\mathfrak{g}(0)$.

Theorem 4.16. *Let $x \in \mathfrak{g}(0)$. Then $x \in \mathfrak{g}_{nsreg}$.*

Proof. Let $x \in \mathfrak{g}$ and suppose that $\mathfrak{z}_{\mathfrak{k}}(x_{\mathfrak{k}}) \cap \mathfrak{z}_{\mathfrak{g}}(x) \neq 0$. We show that $x \in \mathfrak{g}(\geq 1)$ by considering the types A, B, D separately. First, suppose that $\mathfrak{g}$ is type A. Then decompose $V = \mathbb{C}^n$ as $V = V_1 \oplus V_2$ where $V_1 = \operatorname{span}\{e_1, \dots, e_{n-1}\}$, and $V_2 = \operatorname{span}\{e_n\}$. Now apply Example 4.15. Similarly, when $\mathfrak{g} = \mathfrak{so}(2l,\mathbb{C})$, we decompose $\mathbb{C}^{2l}$ as $V = V_1 \oplus V_2$, where $V_1 = \operatorname{span}\{e_{\pm 1}, \dots, e_{\pm(l-1)}, e_l + e_{-l}\}$, and $V_2 = \operatorname{span}\{e_l - e_{-l}\}$. The reader can check that $\mathfrak{k}$ annihilates V_2. Since the involution θ acts on $e_{\pm i}$ via $\theta(e_{\pm i}) = e_{\pm i}$ for $i \neq l$ and $\theta(e_l) = e_{-l}$ (see Section 2.2), we know θ acts on V_1 as the identity and on V_2 as the negative of the identity. Therefore $x_{\mathfrak{k}} : V_1 \to V_1$ and $x_{\mathfrak{g}^{-\theta}} : V_1 \to V_2$, and it follows that $x_1 = x_{\mathfrak{k}}$, where $x_1 = \pi_{V_1}(x|_{V_1})$. The result now follows from Proposition 4.14. The case of $\mathfrak{so}(2l+1,\mathbb{C})$ follows by taking $V_1 = \operatorname{span}\{e_{\pm 1}, \dots, e_{\pm l}\}$, $V_2 = \operatorname{span}\{e_0\}$, and using Proposition 4.14.

Q.E.D.

Let $c = (c_{r_{n-1}}, c_{r_n}) \in V^{r_{n-1}, r_n}$ and write $c_{r_i} = (c_{i,1}, \dots, c_{i,r_i}) \in \mathbb{C}^{r_i}$ for $i = n-1, n$. Let $I_{n,c}$ be the ideal of $\mathbb{C}[\mathfrak{g}]$ generated by the functions $f_{i,j} - c_{i,j}$ for $i = n-1, n$ and $j = 1, \dots, r_i$.

Corollary 4.17. *Let $c = (c_{r_{n-1}}, c_{r_n}) \in V^{r_{n-1}, r_n}(0)$, so that $\Phi_n^{-1}(c) \subset \mathfrak{g}(0)$.*

1. *Then $I_{n,c}$ is radical, so that $I_{n,c}$ is the ideal of the fibre $\Phi_n^{-1}(c)$. Further, the variety $\Phi_n^{-1}(c)$ is smooth.*
2. *The fibre $\Phi_n^{-1}(c)$ is a single closed K-orbit.*

Proof. By Theorem 4.16 every element of the fibre $\Phi_n^{-1}(c)$ is n-strongly regular. It follows from Theorem 4.8 and Remark 4.11 that the differentials $\{df_{i,j}(x) : i = n-1, n;\ j = 1, \dots, r_i\}$ are independent for all $x \in \Phi_n^{-1}(c)$. By Theorem 18.15 (a) of [Eis95], the ideal $I_{n,c}$ is radical, so $I_{n,c}$ is the ideal of $\Phi_n^{-1}(c)$. The smoothness of $\Phi_n^{-1}(c)$ now follows since the differentials of the generators of $I_{n,c}$ are independent at every point of $\Phi_n^{-1}(c)$. For the second assertion, note first that

$$\dim(K) = \dim(\mathfrak{g}) - \dim(\mathfrak{g}//K) = \dim(\Phi_n^{-1}(c)),$$

where the first equality follows from Equations (4.8) and (4.9), and the second equality follows from Proposition 2.2 (3). By Lemma 4.7, $\dim(K \cdot x) = \dim(K)$ for all $x \in \Phi_n^{-1}(c)$. By Proposition 2.2 (2), each fibre $\Phi_n^{-1}(c)$ has a unique closed K-orbit, which implies the assertion.

Q.E.D.

Using Theorem 4.16, we can generalize a result of the first author to the orthogonal setting (cf the first statement of Theorem 5.15, [Col11]). Consider the Zariski open subvariety of $\mathfrak{so}(n,\mathbb{C})$

$$\mathfrak{so}(n,\mathbb{C})_\Theta := \{x \in \mathfrak{so}(n,\mathbb{C}) : \sigma(x_i)\cap\sigma(x_{i+1}) = \emptyset \text{ for } i = 2,\dots,n-1\}.$$

Proposition 4.18. *The elements of $\mathfrak{so}(n,\mathbb{C})_\Theta$ are strongly regular.*

Proof. If $x \in \mathfrak{so}(n,\mathbb{C})_\Theta$, then $x_i \in \mathfrak{so}(i,\mathbb{C})(0)$ for $i = 2,\dots,n$. It follows from Theorem 4.16 that $\mathfrak{z}_{\mathfrak{g}_{i-1}}(x_{i-1}) \cap \mathfrak{z}_{\mathfrak{g}_i}(x_i) = 0$. The result now follows from Proposition 4.12.

Q.E.D.

Remark 4.19. The Gelfand-Zeitlin system for $\mathfrak{gl}(n,\mathbb{C})$ is much better understood than the Gelfand-Zeitlin system for $\mathfrak{so}(n,\mathbb{C})$. Let J_{GZ} be the Gelfand-Zeitlin functions for either $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{C})$ or $\mathfrak{g} = \mathfrak{so}(n,\mathbb{C})$ defined in (4.11). Consider the Kostant-Wallach morphism:

$$\Phi : \mathfrak{g} \to \mathbb{C}^{r_1} \times \mathbb{C}^{r_2} \times \cdots \times \mathbb{C}^{r_{n-1}} \times \mathbb{C}^{r_n} \text{ given by } \Phi(x) = (f_{i,j}(x))_{f_{i,j}\in J_{GZ}} \tag{4.14}$$

(for $\mathfrak{g} = \mathfrak{so}(n,\mathbb{C})$, $\mathbb{C}^{r_1}$ is a point). In [KW06a], Kostant and Wallach prove that Φ is surjective, and in Theorem 5.15 of [Col11], the first author shows that for all x in the Zariski open set

$$\mathfrak{gl}(n,\mathbb{C})_\Theta := \{x \in \mathfrak{gl}(n,\mathbb{C}) : \sigma(x_i)\cap\sigma(x_{i+1}) = \emptyset \text{ for } i = 2,\dots,n-1\},$$

the fibre $\Phi^{-1}(\Phi(x))$ is irreducible. In later work, we will use the flatness assertion of Proposition 2.2 to show that Φ is surjective in the orthogonal case, which together with the preceding proposition, shows that every regular adjoint orbit contains strongly regular elements. This implies that the Gelfand-Zeitlin functions in (4.11) form an integrable system on every regular adjoint orbit in $\mathfrak{so}(n,\mathbb{C})$. We will also use Proposition 4.18 and part (2) of Corollary 4.17 to show that $\Phi^{-1}(\Phi(x))$ is irreducible for $x \in \mathfrak{so}(n,\mathbb{C})_\Theta$. The proofs of both these results for $\mathfrak{so}(n,\mathbb{C})$ are different and more conceptual than the analogous proofs for $\mathfrak{gl}(n,\mathbb{C})$, and we will develop these ideas in further work on the orthogonal Gelfand-Zeitlin system.

4.3. **Classification of closed K-orbits on $\mathfrak{g}$.** In Section 4.2, we showed that K-orbits in $\mathfrak{g}(0)$ are closed. In this section, we describe the other closed K-orbits in $\mathfrak{g}$. Our main tool is Theorem 1.1 when $(\mathfrak{g},K) = (\mathfrak{so}(n,\mathbb{C}), SO(n-1,\mathbb{C}))$ and Theorem 3.7 of [CE15] when $(\mathfrak{g},K) = (\mathfrak{gl}(n,\mathbb{C}), K = GL(n-1,\mathbb{C}))$. Recall the varieties $\mathfrak{g}(i) = \mathfrak{g}(\geq i) \setminus \mathfrak{g}(\geq i+1)$ defined in (2.14) and the partition $\mathfrak{g} = \bigcup_{i=0}^{r_{n-1}} \mathfrak{g}(i)$ of $\mathfrak{g}$ in (2.15). For the analogous definition in type A, see Equation (3.3) of [CE15].)

Theorem 4.20. *Let $x \in \mathfrak{g}(i)$, $i = 0,\dots,r_{n-1}$. Then $K\cdot x$ is closed if and only if $K\cdot x \cap \mathfrak{l}(i) \neq \emptyset$, where $\mathfrak{l}(i) := \mathfrak{g}(i)\cap\mathfrak{l}$ and $\mathfrak{l}$ is a θ-stable Levi subalgebra of the following form:*

(1) *If $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{C})$, then $\mathfrak{l}$ is the Levi subalgebra of block diagonal matrices*

$$\mathfrak{l} = \mathfrak{gl}(n-i,\mathbb{C}) \oplus \mathfrak{gl}(1,\mathbb{C})^i.$$

(2) *If $\mathfrak{g} = \mathfrak{so}(2l+1,\mathbb{C})$, then $\mathfrak{l}$ is the θ-stable Levi subalgebra defined in Theorem 3.1.*

(3) *If* $\mathfrak{g} = \mathfrak{so}(2l, \mathbb{C})$, *then* $\mathfrak{l}$ *is the* θ*-stable Levi subalgebra defined in Theorem 3.2.*

Theorem 4.20 will follow from two lemmas.

Lemma 4.21. *Let* $x \in \mathfrak{g}(i)$, *and let* $\mathfrak{l}$ *be the corresponding Levi subalgebra in Theorem 4.20. Then* $\overline{K \cdot x}$ *contains an element of* $\mathfrak{l}(i)$.

Proof. Any element $x \in \mathfrak{g}(i)$ is K-conjugate to an element in a θ-stable parabolic subalgebra $\mathfrak{r}$ with Levi factor $\mathfrak{l}$. This follows from Corollary 3.8 when $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$ and from Theorem 3.7 of [CE15] when $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$. Thus, we can assume that $x \in \mathfrak{r}$. We choose an element z in the centre $\mathfrak{z}$ of $\mathfrak{l}$ such that $\alpha(z) > 0$ for every root α of $\mathfrak{u}$, the nilradical of $\mathfrak{r}$. Note that $\mathfrak{z} \subset \mathfrak{k}$, which is clear for type A, and follows by Remark 3.3 for the orthogonal cases. Then

$$\lim_{t \to -\infty} \mathrm{Ad}(\exp tz)x \in \mathfrak{l} \cap \overline{K \cdot x} = \mathfrak{l}(i) \cap \overline{K \cdot x}, \tag{4.15}$$

where the last equality follows since $\overline{K \cdot x} \subset \Phi^{-1}(\Phi(x)) \subset \mathfrak{g}(i)$.

Q.E.D.

We now study the K-orbits of elements in $\mathfrak{l}$.

Lemma 4.22. *Let* $\mathfrak{l}$ *be one of the Levi subalgebras in Theorem 4.20. Any two elements in* $\mathfrak{l}(i)$ *which lie in the same fibre of the partial Kostant-Wallach map* Φ_n *are* K*-conjugate.*

Proof. Suppose that $x, y \in \Phi_n^{-1}(c)$, with $c \in V^{r_{n-1}, r_n}(0)$. Then Corollary 4.17 implies that x and y are K-conjugate.

Now suppose that $x, y \in \Phi_n^{-1}(c) \cap \mathfrak{l}$ with $c \in V^{r_{n-1}, r_n}(i)$ with $i > 0$. Decompose x and y as $x = x_{\mathfrak{z}} + x_{\mathfrak{l}_{ss}}$ and $y = y_{\mathfrak{z}} + y_{\mathfrak{l}_{ss}}$, with $x_{\mathfrak{z}}, y_{\mathfrak{z}} \in \mathfrak{z}$ and $x_{\mathfrak{l}_{ss}}, y_{\mathfrak{l}_{ss}} \in \mathfrak{l}_{ss}$. Then $\sigma(x) \cap \sigma(x_{\mathfrak{k}})$ are the coordinates of $x_{\mathfrak{z}}$ and similarly for y. Since $\Phi_n(x) = \Phi_n(y) \in V^{r_{n-1}, r_n}(i)$, we know $\sigma(x) \cap \sigma(x_{\mathfrak{k}}) = \sigma(y) \cap \sigma(y_{\mathfrak{k}})$. It follows that there exists $\dot{w} \in N_K(L \cap K)$ such that $\mathrm{Ad}(\dot{w})x_{\mathfrak{z}} = y_{\mathfrak{z}}$. So without loss of generality, we may assume that $x_{\mathfrak{z}} = y_{\mathfrak{z}}$. Since $x, y \in \mathfrak{g}(i) \cap \mathfrak{l}$, then $x_{\mathfrak{l}_{ss}}, y_{\mathfrak{l}_{ss}} \in \mathfrak{l}_{ss}(0)$, where

$$\mathfrak{l}_{ss}(0) := \{z \in \mathfrak{l}_{ss} : \ \sigma(z) \cap \sigma(z_{\mathfrak{k}}) = \emptyset\}.$$

Let $\Phi_{\mathfrak{l}_{ss}} : \mathfrak{l}_{ss} \to \mathbb{C}^{\mathrm{rk}(\mathfrak{l}_{ss} \cap \mathfrak{k})} \times \mathbb{C}^{\mathrm{rk}(\mathfrak{l}_{ss})}$ be the partial Kostant-Wallach map for $\mathfrak{l}_{ss}$. Then $\Phi_n(x) = \Phi_n(y)$ implies that $\Phi_{\mathfrak{l}_{ss}}(x_{\mathfrak{l}_{ss}}) = \Phi_{\mathfrak{l}_{ss}}(y_{\mathfrak{l}_{ss}})$. But then Corollary 4.17 applied to $\mathfrak{l}_{ss}$ forces $x_{\mathfrak{l}_{ss}}$ and $y_{\mathfrak{l}_{ss}}$ to lie in the same $K \cap L_{ss}$-orbit. This completes the proof.

Q.E.D.

We now prove Theorem 4.20.

Proof of Theorem 4.20. Suppose that $x \in \mathfrak{g}(i)$ with $\mathrm{Ad}(K) \cdot x$ closed. Then by Lemma 4.21, there exists an element $z \in \mathfrak{l}(i) \cap \overline{\mathrm{Ad}(K) \cdot x}$. But since $\mathrm{Ad}(K) \cdot x$ is closed, we conclude that $\mathrm{Ad}(K) \cdot x = \mathrm{Ad}(K) \cdot z$.

Conversely, suppose that $x \in \mathfrak{l}(i)$ and consider $\overline{\mathrm{Ad}(K) \cdot x}$. By Lemma 4.21, there exists $z \in \mathfrak{l}(i)$ with $K \cdot z$ closed and $K \cdot z \subset \overline{\mathrm{Ad}(K) \cdot x} \subset \Phi^{-1}(\Phi(x))$. Therefore $\mathrm{Ad}(K) \cdot x = \mathrm{Ad}(K) \cdot z$ by Lemma 4.22. Thus, $\mathrm{Ad}(K) \cdot x$ is closed.

Q.E.D.

4.4. **The nilfibre of the partial Kostant-Wallach map in the orthogonal case.** Though there are many similarities between the $GL(n-1, \mathbb{C})$-action on $\mathfrak{gl}(n, \mathbb{C})$ and the $SO(n-1, \mathbb{C})$-action on $\mathfrak{so}(n, \mathbb{C})$, in this subsection we show that their n-strongly regular sets are different. In the case of $\mathfrak{gl}(n, \mathbb{C})$, every fibre of the partial Kostant-Wallach map contains n-strongly regular elements. This follows easily from Theorem 2.3 of [KW06a]. However, this is not the case for $\mathfrak{so}(n, \mathbb{C})$. To see this, we need to study the nilfibre $\Phi_n^{-1}(0, 0)$ of the orthogonal partial Kostant-Wallach map in more detail using Theorems 1.1 and 4.8.

Theorem 4.23. *Let $\Phi_n : \mathfrak{so}(n, \mathbb{C}) \to \mathbb{C}^{r_{n-1}} \oplus \mathbb{C}^{r_n}$ be the orthogonal partial Kostant-Wallach map Φ_n defined in Equation (2.9).*

Case I: Suppose $n = 2l$. Then $\Phi_n^{-1}(0, 0) = K \cdot \mathfrak{n}_+$ is irreducible, where $\mathfrak{n}_+ = [\mathfrak{b}_+, \mathfrak{b}_+]$ and $Q_+ = K \cdot \mathfrak{b}_+$ is the unique closed K-orbit in $\mathcal{B}$ (see part (2) of Proposition 2.24).

Case II: Suppose $n = 2l + 1$. Then $\Phi_n^{-1}(0, 0) = K \cdot \mathfrak{n}_+ \cup K \cdot \mathfrak{n}_-$ has two irreducible components, where $\mathfrak{n}_\pm = [\mathfrak{b}_\pm, \mathfrak{b}_\pm]$ and $Q_\pm = K \cdot \mathfrak{b}_\pm$, are the two closed K-orbits in $\mathcal{B}$ (see part (2) of Proposition 2.23).

Proof. Let $Q = K \cdot \mathfrak{b}$ be a closed K-orbit. Let $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ be the nilradical of $\mathfrak{b}$. We first show that $\mathrm{Ad}(K) \cdot \mathfrak{n}$ is an irreducible component of $\Phi_n^{-1}(0, 0)$. Since Q is closed, $\mathfrak{b}$ is θ-stable by Proposition 4.12 of [CE]. Thus, $\mathfrak{b} \cap \mathfrak{k}$ is a Borel subalgebra of $\mathfrak{k}$ with nilradical $\mathfrak{n} \cap \mathfrak{k}$. It follows that for any $x \in \mathfrak{n}$, we have $\Phi_n(x) = (0, 0)$. By the K-equivariance of Φ_n, $\mathrm{Ad}(K) \cdot \mathfrak{n} \subset \Phi_n^{-1}(0, 0)$.

Recall the Grothendieck resolution $\widetilde{\mathfrak{g}} = \{(x, \mathfrak{b}) : x \in \mathfrak{b}\} \subset \mathfrak{g} \times \mathcal{B}$ and the morphisms $\pi : \widetilde{\mathfrak{g}} \to \mathcal{B}$, $\pi(x, \mathfrak{b}) = \mathfrak{b}$ and $\mu : \widetilde{\mathfrak{g}} \to \mathfrak{g}$, $\mu(x, \mathfrak{b}) = x$. Corollary 3.1.33 of [CG97] gives a G-equivariant isomorphism $\widetilde{\mathfrak{g}} \cong G \times_B \mathfrak{b}$. Under this isomorphism $\pi^{-1}(Q)$ is identified with the closed subvariety $K \times_{K \cap B} \mathfrak{b} \subset G \times_B \mathfrak{b}$. The closed subvariety $K \times_{K \cap B} \mathfrak{n} \subset K \times_{K \cap B} \mathfrak{b}$ maps surjectively under μ to $\mathrm{Ad}(K)\mathfrak{n}$. Since μ is proper, $\mathrm{Ad}(K)\mathfrak{n}$ is closed and irreducible. By Proposition 3.2.14 of [CG97], the restriction of μ to $K \times_{K \cap B} \mathfrak{n}$ generically has finite fibres. Thus, the same reasoning that we used in Equation (2.19) along with Propositions 2.13 and 2.8 shows that

$$\begin{aligned}
\dim \mathrm{Ad}(K)\mathfrak{n} &= \dim(K \times_{K \cap B} \mathfrak{n}) \\
&= \dim(Y_Q) - r_n \\
&= \dim(\mathfrak{g}(\geq r_{n-1})) - r_n \\
&= \dim(\mathfrak{g}) - r_{n-1} - r_n \\
&= \dim \Phi_n^{-1}(0, 0).
\end{aligned}$$

Thus, $\mathrm{Ad}(K) \cdot \mathfrak{n}$ is an irreducible component of $\Phi_n^{-1}(0,0)$.

We now show that every irreducible component of $\Phi_n^{-1}(0,0)$ is of the form $\mathrm{Ad}(K)\mathfrak{n}$. It follows from definitions that $\Phi_n^{-1}(0,0) \subset \mathfrak{g}(r_{n-1}) \cap \mathcal{N}$, where $\mathcal{N} \subset \mathfrak{g}$ is the nilpotent cone in $\mathfrak{g}$. Thus, if $\mathfrak{X}$ is an irreducible component of $\Phi_n^{-1}(0,0)$, then $\mathfrak{X} \subset \mathrm{Ad}(K)\mathfrak{n}$ by Equations (3.30) and (3.31) from Corollary 3.8. But then $\mathfrak{X} = \mathrm{Ad}(K)\mathfrak{n}$ by Proposition 2.2.

Q.E.D.

We use Theorem 4.23 to study $\Phi_n^{-1}(0,0)$ in more detail. In [CE15], we studied the interaction between the set of n-strongly regular elements for the pair $(\mathfrak{g} = \mathfrak{gl}(n,\mathbb{C}), \mathfrak{k} = \mathfrak{gl}(n-1,\mathbb{C}))$ and the nilfibre of the corresponding partial Kostant-Wallach map. We now show that unlike in the case of $\mathfrak{gl}(n,\mathbb{C})$, there are no n-strongly regular elements in the nilfibre of the partial Kostant-Wallach map for the orthogonal Lie algebra $\mathfrak{so}(n,\mathbb{C})$. The key observation is the following proposition, which can be viewed as an extension of Proposition 3.8 in [CE12].

Proposition 4.24. *Let $n > 3$, let $\mathfrak{g} = \mathfrak{so}(n,\mathbb{C})$, and let $K = SO(n-1,\mathbb{C})$. Let $\mathfrak{b} \subset \mathfrak{g}$ be a Borel subalgebra and suppose that the K-orbit $K \cdot \mathfrak{b}$ is closed in $\mathcal{B}$. Let $\mathfrak{n} = [\mathfrak{b},\mathfrak{b}]$ be the nilradical of $\mathfrak{b}$. Then*

$$\mathfrak{z}_{\mathfrak{k}}(\mathfrak{n} \cap \mathfrak{k}) \cap \mathfrak{z}_{\mathfrak{g}}(\mathfrak{n}) \neq 0, \tag{4.16}$$

where $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{n})$ is the centralizer of $\mathfrak{n}$ in $\mathfrak{g}$, and $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{n} \cap \mathfrak{k})$ is the centralizer of $\mathfrak{n} \cap \mathfrak{k}$ in $\mathfrak{k}$.

Proof. Consider a closed K-orbit Q in $\mathcal{B}$. By K-equivariance, it suffices to show Equation (4.16) for a representative $\mathfrak{b}$ of Q. By part (2) of Propositions 2.23 and 2.24, we can assume that the standard diagonal Cartan subalgebra $\mathfrak{h}$ is in $\mathfrak{b}$. Let $\phi \in \Phi^+(\mathfrak{g},\mathfrak{h})$ be the highest root of $\mathfrak{b}$. We claim for $n > 4$ that ϕ is compact imaginary. It then follows that the root space

$$\mathfrak{g}_\phi \subset \mathfrak{z}_{\mathfrak{k}}(\mathfrak{n} \cap \mathfrak{k}) \cap \mathfrak{z}_{\mathfrak{g}}(\mathfrak{n}).$$

Suppose first that $\mathfrak{g} = \mathfrak{so}(2l,\mathbb{C})$. By part (2) of Proposition 2.24, we can assume that $\mathfrak{b} = \mathfrak{b}_+$. The highest root is then $\epsilon_1 + \epsilon_2$, which is compact imaginary for $l > 2$ (Example 2.16.) If $\mathfrak{g} = \mathfrak{so}(2l+1,\mathbb{C})$, then by part (2) of Proposition 2.23, we can assume that $\mathfrak{b} = \mathfrak{b}_+$ or $\mathfrak{b} = \mathfrak{b}_- = s_{\alpha_l}(\mathfrak{b}_+)$. In both cases, the highest root is $\epsilon_1 + \epsilon_2$, which is compact imaginary (Example 2.16).

If $\mathfrak{g} = \mathfrak{so}(4,\mathbb{C})$, then $\phi = \epsilon_1 + \epsilon_2$ is complex θ-stable. Since $\mathfrak{n}$ is abelian in this case, $(\mathfrak{g}_\phi \oplus \mathfrak{g}_{\theta(\phi)})^\theta \subset \mathfrak{z}_{\mathfrak{k}}(\mathfrak{n} \cap \mathfrak{k}) \cap \mathfrak{z}_{\mathfrak{g}}(\mathfrak{n})$.

Q.E.D.

Corollary 4.25. *Let $n > 3$, and let $\Phi_n : \mathfrak{so}(n,\mathbb{C}) \to \mathbb{C}^{r_{n-1}} \oplus \mathbb{C}^{r_n}$ be the orthogonal partial Kostant-Wallach map. Then $\Phi_n^{-1}(0,0)$ contains no n-strongly regular elements.*

Proof. Suppose $x \in \Phi_n^{-1}(0,0)$, so by Theorem 4.23, x is contained in $\mathfrak{n}$, the nilradical of a Borel subalgebra $\mathfrak{b}$ with $K \cdot \mathfrak{b}$ closed. By Proposition 4.24, there is a nonzero element y of $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{n} \cap \mathfrak{k}) \cap \mathfrak{z}_{\mathfrak{g}}(\mathfrak{n})$. Then $y \in \mathfrak{z}_{\mathfrak{k}}(x_{\mathfrak{k}}) \cap \mathfrak{z}_{\mathfrak{g}}(x)$, so x is not n-strongly regular.

Q.E.D.

Remark 4.26. The assertion of the corollary is false for $n = 3$. In this case, $\mathfrak{so}(3,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{k} = \mathfrak{h} \subset \mathfrak{sl}(2,\mathbb{C})$, where $\mathfrak{h}$ is the standard Cartan subalgebra of $\mathfrak{sl}(2,\mathbb{C})$. In this case, it follows by Proposition 3.11 from [CE12] that each irreducible component contains strongly regular elements.

The following result is analogous to Proposition 3.11 of [CE15]. We let I_n be the ideal of $\mathbb{C}[\mathfrak{so}(n,\mathbb{C})]$ generated by elements of $\mathbb{C}[\mathfrak{so}(n,\mathbb{C})]^{SO(n-1,\mathbb{C})}$ of positive degree.

Corollary 4.27. *The ideal I_n is radical if and only if $n = 3$.*

Proof. By Theorem 18.15 (a) of [Eis95], the ideal I_n is radical if and only if the set of differentials $\{df_{i,j}(x) : j = 1, \ldots, r_i,\ i = n-1, n\}$ is linearly independent on an open, dense subset of each irreducible component of $\Phi_n^{-1}(0,0)$. It follows from Definition 4.10 and Theorem 4.8 that I_n is radical if and only if each irreducible component of $\Phi_n^{-1}(0,0)$ contains n-strongly regular elements. But it follows from Corollary 4.25 and the case of $\mathfrak{so}(3,\mathbb{C})$ in Remark 4.26 that each irreducible component of $\Phi_n^{-1}(0,0)$ contains n-strongly regular elements if and only if $n = 3$.

Q.E.D.

Note that we have derived results concerning the n-strongly regular set without using a slice, in contrast to the case of $\mathfrak{gl}(n,\mathbb{C})$ studied by Kostant and Wallach [KW06a], Theorem 2.3.

Remark 4.28. Consider the orthogonal Kostant-Wallach map Φ defined in (4.14). It follows from Corollary 4.25 that the nilfibre $\Phi^{-1}(0,\ldots,0)$ contains no strongly regular elements. This is very different than the case of $\mathfrak{gl}(n,\mathbb{C})$ studied extensively in [CE12].

5. Appendix

In the appendix, we prove a general result which implies Proposition 2.2. The proof is an adaptation of the proof of Proposition 2.3 from [CE15].

Theorem 5.1. *Let (G, H) be a spherical pair such that*

$$\dim \mathcal{B} = \dim \mathfrak{h}^{\perp} - \dim \mathfrak{h}^{\perp}//H.$$

(cf Equation (4.1)). Then $\Psi : \mathfrak{h}^{\perp} \to \mathfrak{h}^{\perp}//H$ is flat.

Proof. We first show that $\Psi^{-1}(0)$ is equidimensional of dimension $\dim \mathfrak{h}^{\perp} - \dim \mathfrak{h}^{\perp}//H$. Let C be an irreducible component of $\Psi^{-1}(0)$. By standard results, $\dim(C) \geq \dim \mathfrak{h}^{\perp} - \dim \mathfrak{h}^{\perp}//H$. Label the finite number of H-orbits on $\mathcal{B}$ by $Q_1, \ldots, Q_s$. Let $Z = \cup_{i=1}^{s} \overline{T^*_{Q_i}(\mathcal{B})}$ and note that the irreducicible components of Z are the subvarieties $Z_i := \overline{T^*_{Q_i}(\mathcal{B})}$, and also that $\dim Z_i = \dim \mathcal{B}$ for $i = 1, \ldots, s$. Recall the standard identification $T^*\mathcal{B} = \{(\mathfrak{b}, x) \in \mathcal{B} \times \mathfrak{g} : x \in [\mathfrak{b}, \mathfrak{b}]\}$ and let $\mu : T^*\mathcal{B} \to \mathfrak{g}$ be the moment map, $\mu(\mathfrak{b}, x) = x$.

We claim that $\Psi^{-1}(0) \subset \mu(Z)$. Indeed, by Theorem 6 of [Pan90] , $\mathbb{C}[\mathfrak{h}^{\perp}]^H = \mathbb{C}[g_1, \ldots, g_k]$ is a polynomial ring in k generators, which can be taken to be homogeneous. Further, the morphism $\Psi : \mathfrak{h}^{\perp} \to \mathfrak{h}^{\perp}//H$ may be identified with $(g_1, \ldots, g_k) : \mathfrak{h}^{\perp} \to \mathbb{C}^k$. For $f \in \mathbb{C}[\mathfrak{g}]^G$ of positive degree, note that $f|_{\mathfrak{h}^{\perp}} \in \mathbb{C}[\mathfrak{h}^{\perp}]^H$, so $f|_{\mathfrak{h}^{\perp}}$ is a polynomial of strictly positive degree in the variables $g_1, \ldots, g_k$. By the above identification, $g_i(x) = 0$ for each $x \in \Psi^{-1}(0)$, so $f(x) = 0$. By Proposition 16 of [Kos63], it follows that x is nilpotent, and hence lies in $\mathfrak{n} := [\mathfrak{b}, \mathfrak{b}]$, the nilradical of a Borel subalgebra $\mathfrak{b}$. Thus, if $Q_i = H \cdot \mathfrak{b}$, then $(\mathfrak{b}, x) \in Z_i$, and $x = \mu(\mathfrak{b}, x) \in \mu(Z)$.

Since μ is proper, it follows that $C \subset \mu(Z_i)$ for some i. Hence, $\dim C \leq \dim Z_i = \dim \mathcal{B} = \dim \mathfrak{h}^{\perp} - \dim \mathfrak{h}^{\perp}//H$, so $\dim C = \dim \mathfrak{h}^{\perp} - \dim \mathfrak{h}^{\perp}//H$.

Now for $x \in \mathfrak{h}^{\perp}$, let d_x be the maximum of the dimension of the irreducible components of $\Psi^{-1}(\Psi(x))$. Since the functions $g_1, \ldots, g_k$ are homogeneous, scalar multiplication by $\lambda \in \mathbb{C}^*$ induces an isomorphism $\Psi^{-1}(\Psi(x)) \cong \Psi^{-1}(\Psi(\lambda x))$, so $d_x = d_{\lambda x}$. By upper semi-continuity of dimension, the set $\{y \in \mathfrak{h}^{\perp} : d_y \geq d\}$ is closed for each integer d (Proposition 4.4 of [Hum75]). Hence, $d_y \leq d_0 = \dim \mathfrak{h}^{\perp} - \dim \mathfrak{h}^{\perp}//H$ for all $y \in \mathfrak{h}^{\perp}$. It follows that $d_y = \dim \mathfrak{h}^{\perp} - \dim \mathfrak{h}^{\perp}//H$ for all $y \in \mathfrak{h}^{\perp}$. Hence, Ψ is flat by the corollary to Theorem 23.1 of [Mat86].

Q.E.D.

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