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Dedicated to Bertram Kostant on the occasion of his 80th birthday.

Recently, Kostant and Wallach constructed an action of a simply connected Lie group $A \cong \mathbb{C}^{n(n-1)/2}$ on $\mathfrak{gl}(n)$ using a completely integrable system derived from the Poisson analogue of the Gelfand–Zeitlin subalgebra of the enveloping algebra. They show that A-orbits of dimension $n(n-1)/2$ form Lagrangian submanifolds of regular adjoint orbits in $\mathfrak{gl}(n)$ and describe the orbits of A on a certain Zariski open subset of regular semisimple elements. In this paper, we describe all A-orbits of dimension $n(n-1)/2$ and thus all polarizations of regular adjoint orbits obtained using Gelfand–Zeitlin theory.

1. Introduction

For $n \in \mathbb{N}$, let $\Delta^n_{i,j}$ be the set of ordered pairs of indices (i, j) such that $1 \le j \le i \le n$.

In [2006a; 2006b], Bertram Kostant and Nolan Wallach constructed an action of a complex, commutative, simply connected Lie group $A \cong \mathbb{C}^{n(n-1)/2}$ on the Lie algebra of $n \times n$ complex matrices $\mathfrak{gl}(n)$. The dimension of this group is exactly half the dimension of a regular adjoint orbit in $\mathfrak{gl}(n)$, and orbits of A of dimension $n(n-1)/2$ are Lagrangian submanifolds of regular adjoint orbits. We refer to the group A introduced by Kostant and Wallach as the Gelfand–Zeitlin group because of its connection with the Gelfand–Zeitlin algebra, as we will explain in Section 2.

The group A and its action are constructed as follows. Given $i < n$, we can think of $\mathfrak{gl}(i) \hookrightarrow \mathfrak{gl}(n)$ as a subalgebra by embedding an $i \times i$ matrix into the upper left corner of an $n \times n$ matrix. For $(i, j) \in \Delta^n_{i,j}$, let $f_{i,j}(x)$ be the polynomial on $\mathfrak{gl}(n)$ defined by $f_{i,j}(x) = \mathrm{tr}(x_i^j)$, where x_i denotes the $i \times i$ submatrix in the upper left corner of x. In [2006a], Konstant and Wallach showed that the functions $\{f_{i,j} \mid (i, j) \in \Delta^n_{i,j}\}$ are algebraically independent and Poisson commute with respect to the Lie–Poisson structure on $\mathfrak{gl}(n) \cong \mathfrak{gl}(n)^*$. The corresponding

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Hamiltonian vector fields $\xi_{f_{i,j}}$ generate a commutative Lie algebra $\mathfrak{a}$ of dimension $n(n-1)/2$. The group A is defined to be the simply connected, complex Lie group that corresponds to the Lie algebra $\mathfrak{a}$. The vector fields $\xi_{f_{i,j}}$ are complete [Kostant and Wallach 2006a, Theorem 3.5], and therefore $\mathfrak{a}$ integrates to a global action of $\mathbb{C}^{n(n-1)/2}$ on $\mathfrak{gl}(n)$, thus defining the action of the group A on $\mathfrak{gl}(n)$.

Our goal in this paper is to describe all A-orbits of dimension $n(n-1)/2$. An element $x \in \mathfrak{gl}(n)$ is called strongly regular if and only if its A-orbit is of dimension $n(n-1)/2$. We denote the set of strongly regular elements by $\mathfrak{gl}(n)^{\mathrm{sreg}}$. One way of studying such orbits is to study the action of A on fibers of the map $\Phi : \mathfrak{gl}(n) \to \mathbb{C}^{n(n+1)/2}$

$$\Phi(x) = (p_{1,1}(x_1), p_{2,1}(x_2), \dots, p_{n,n}(x)), \tag{1-1}$$

where $p_{i,j}(x_i)$ is the coefficient of t^{j-1} in the characteristic polynomial of x_i.

In [2006a, Theorem 2.3], Kostant and Wallach show that this map is surjective and that every fiber of this map $\Phi^{-1}(c) = \mathfrak{gl}(n)_c$ contains strongly regular elements. Following them, we denote the strongly regular elements in the fiber $\mathfrak{gl}(n)_c$ by $\mathfrak{gl}(n)_c^{\mathrm{sreg}}$. By [2006a, Theorem 3.12], the A-orbits in $\mathfrak{gl}(n)^{\mathrm{sreg}}$ are precisely the irreducible components of the fibers $\mathfrak{gl}(n)_c^{\mathrm{sreg}}$. Thus, our study of the action of A on $\mathfrak{gl}(n)^{\mathrm{sreg}}$ is reduced to studying the A-orbit structure of the fibers $\mathfrak{gl}(n)_c^{\mathrm{sreg}}$. In [2006a], the authors also describe the A-orbit structure on a special class of fibers that consist of certain regular semisimple elements. In this paper, we describe the A-orbit structure of $\mathfrak{gl}(n)_c^{\mathrm{sreg}}$ for any $c \in \mathbb{C}^{n(n+1)/2}$.

In Section 2, we describe the construction in [Kostant and Wallach 2006a] of the group A in more detail, and in Section 3, we describe their results about the A-orbits. We summarize these results briefly here. For any $x \in \mathfrak{gl}(i)$, let $\sigma(x)$ denote the spectrum of x. Kostant and Wallach describe the action of the group A on a Zariski open subset of regular semisimple elements defined by

$$\mathfrak{gl}(n)_\Omega = \{x \in \mathfrak{gl}(n) \mid x_i \text{ is regular semisimple, } \sigma(x_{i-1}) \cap \sigma(x_i) = \varnothing,\ 2 \le i \le n\}.$$

Let $c_i \in \mathbb{C}^i$ and consider $c = (c_1, c_2, \dots, c_n) \in \mathbb{C}^1 \times \mathbb{C}^2 \times \cdots \times \mathbb{C}^n = \mathbb{C}^{n(n+1)/2}$. Regard $c_i = (z_1, \dots, z_i)$ as the coefficients of the degree i monic polynomial

$$p_{c_i}(t) = z_1 + z_2 t + \cdots + z_i t^{i-1} + t^i. \tag{1-2}$$

Let Ω_n denote the Zariski open subset of $\mathbb{C}^{n(n+1)/2}$ given by the tuples c such that $p_{c_i}(t)$ has distinct roots and $p_{c_i}(t)$ and $p_{c_{i+1}}(t)$ have no roots in common. Clearly, $\mathfrak{gl}(n)_\Omega = \bigcup_{c \in \Omega_n} \mathfrak{gl}(n)_c$. The action of A on $\mathfrak{gl}(n)_\Omega$ is described as follows.

Theorem 1.1 [Kostant and Wallach 2006a, Theorems 3.23 and 3.28]. *The elements of $\mathfrak{gl}(n)_\Omega$ are strongly regular. If $c \in \Omega_n$, then $\mathfrak{gl}(n)_c = \mathfrak{gl}(n)_c^{\mathrm{sreg}}$ is precisely one orbit under the action of the group A. Moreover, $\mathfrak{gl}(n)_c$ is a homogeneous space for a free, algebraic action of the torus $(\mathbb{C}^\times)^{n(n-1)/2}$.*

In Section 4, we give a construction that describes an A-orbit in an arbitrary fiber $\mathfrak{gl}(n)_c^{\text{sreg}}$ as the image of a certain morphism of a commutative, connected algebraic group into $\mathfrak{gl}(n)_c^{\text{sreg}}$. The construction gives a bijection between A-orbits in $\mathfrak{gl}(n)_c^{\text{sreg}}$ and orbits of a product of connected, commutative algebraic groups acting freely on a concrete, well-understood variety, but it does not enumerate the A-orbits in $\mathfrak{gl}(n)_c^{\text{sreg}}$. In Section 5, we use the construction and combinatorial data of the fiber $\mathfrak{gl}(n)_c^{\text{sreg}}$ to give explicit descriptions of the A-orbits in $\mathfrak{gl}(n)_c^{\text{sreg}}$. The main result is Theorem 5.11, which contrasts substantially with the generic case described in Theorem 1.1.

Theorem 1.2. *Let $c = (c_1, c_2, \ldots, c_n) \in \mathbb{C}^1 \times \mathbb{C}^2 \times \cdots \times \mathbb{C}^n = \mathbb{C}^{n(n+1)/2}$ be such that there are $0 \le j_i \le i$ roots in common between the monic polynomials $p_{c_i}(t)$ and $p_{c_{i+1}}(t)$. Then the number of A-orbits in $\mathfrak{gl}(n)_c^{\text{sreg}}$ is exactly 2^j, where $j = \sum_{i=1}^{n-1} j_i$. For $x \in \mathfrak{gl}(n)_c^{\text{sreg}}$, let Z_i denote the centralizer of the Jordan form of x_i in $\mathfrak{gl}(i)$. The orbits of A on $\mathfrak{gl}(n)_c^{\text{sreg}}$ are the orbits of a free algebraic action of the complex, commutative, connected algebraic group $Z = Z_1 \times \cdots \times Z_{n-1}$ on $\mathfrak{gl}(n)_c^{\text{sreg}}$.*

Remark 1.3. After the results of this paper were established, a very interesting paper by Roger Bielawski and Victor Pidstrygach [2008] appeared proving similar results. They show there are 2^j orbits, where $j = \sum_{i=1}^{n-1} j_i$, using symplectic reduction and rational maps of fixed degree from the Riemann sphere into the flag manifold for $\mathrm{GL}(n+1)$. Their arguments are independent and completely different from ours. Our work is more precise in that we provide explicit representatives for the A-orbits, and show that the A-orbits have a simply transitive algebraic action of $Z_1 \times \cdots \times Z_{n-1}$. These ideas were useful in the writing of [Colarusso and Evens 2010].

The nilfiber $\mathfrak{gl}(n)_0 = \Phi^{-1}(0)$ contains some of the most interesting structure in regard to the action of A. The fiber $\mathfrak{gl}(n)_0$ has been studied extensively by Lie theorists and numerical linear algebraists. Parlett and Strang [2008] studied matrices in $\mathfrak{gl}(n)_0$ and obtained interesting results. Ovsienko [2003] also studied $\mathfrak{gl}(n)_0$ and showed that it is a complete intersection. It turns out that the A-orbits in $\mathfrak{gl}(n)_0^{\text{sreg}}$ correspond to 2^{n-1} Borel subalgebras of $\mathfrak{gl}(n)$. The main results are contained in Theorems 5.2 and 5.5. We combine them into a single statement here.

Theorem 1.4. *The nilfiber $\mathfrak{gl}(n)_0^{\text{sreg}}$ contains 2^{n-1} A-orbits. For $x \in \mathfrak{gl}(n)_0^{\text{sreg}}$, let $\overline{A \cdot x}$ denote the Zariski closure of $A \cdot x$ (which is the same as its Hausdorff closure). Then $\overline{A \cdot x}$ is a nilradical of a Borel subalgebra in $\mathfrak{gl}(n)$ that contains the standard Cartan subalgebra of diagonal matrices.*

The nilradicals obtained as closures of A-orbits in $\mathfrak{gl}(n)_0^{\text{sreg}}$ are described explicitly in Theorem 5.5. In Theorem 5.7, we also describe the permutations that conjugate the strictly lower triangular matrices into each of these 2^{n-1} nilradicals.

Theorem 1.2 lets us identify exactly where the action of the group A is transitive on $\mathfrak{gl}(n)_c^{\text{sreg}}$. (See Corollary 5.13 and Remark 5.14.)

Corollary 1.5. *The action of A is transitive on $\mathfrak{gl}(n)_c^{\text{sreg}}$ if and only if $p_{c_i}(t)$ and $p_{c_{i+1}}(t)$ are relatively prime for each i in $1 \le i \le n-1$. Moreover, for such $c \in \mathbb{C}^{n(n+1)/2}$, we have $\mathfrak{gl}(n)_c = \mathfrak{gl}(n)_c^{\text{sreg}}$.*

This corollary allows us to identify the maximal subset of $\mathfrak{gl}(n)$ on which the action of A is transitive on the fibers of the map Φ in (1-1) over this set. The set $\mathfrak{gl}(n)_\Omega$ is a proper open subset of this maximal set. This is discussed in detail in Section 5c.

2. The group A

We briefly discuss the construction of an analytic action of a group $A \cong \mathbb{C}^{n(n-1)/2}$ on $\mathfrak{gl}(n)$ that appears in [Kostant and Wallach 2006a]; see also [Colarusso 2009].

We regard $\mathfrak{gl}(n)^*$ as a Poisson manifold with the Lie–Poisson structure; see [Vaisman 1994; Chriss and Ginzburg 1997]. The Lie–Poisson structure is the unique Poisson structure on the symmetric algebra $S(\mathfrak{gl}(n)) = \mathbb{C}[\mathfrak{gl}(n)^*]$ such that for $x, y \in S^1(\mathfrak{gl}(n))$, the Poisson bracket $\{x, y\}$ is the Lie bracket $[x, y]$. We use the trace form to transfer the Poisson structure from $\mathfrak{gl}(n)^*$ to $\mathfrak{gl}(n)$. For $i \le n$, we can view $\mathfrak{gl}(i) \hookrightarrow \mathfrak{gl}(n)$ as a subalgebra simply by embedding an $i \times i$ matrix in the upper left corner of an $n \times n$ matrix, that is, via

$$Y \hookrightarrow \begin{bmatrix} Y & 0 \\ 0 & 0 \end{bmatrix}. \tag{2-1}$$

We have a corresponding embedding of the adjoint groups $\mathrm{GL}(i) \hookrightarrow \mathrm{GL}(n)$ via

$$g \hookrightarrow \begin{bmatrix} g & 0 \\ 0 & \mathrm{Id}_{n-i} \end{bmatrix}.$$

In this paper, we always think of $\mathfrak{gl}(i) \hookrightarrow \mathfrak{gl}(n)$ and $\mathrm{GL}(i) \hookrightarrow \mathrm{GL}(n)$ via these embeddings, unless otherwise stated.

We can use the embedding (2-1) to realize $\mathfrak{gl}(i)$ as a summand of $\mathfrak{gl}(n)$. Indeed, we have

$$\mathfrak{gl}(n) = \mathfrak{gl}(i) \oplus \mathfrak{gl}(i)^\perp, \tag{2-2}$$

where $\mathfrak{gl}(i)^\perp$ denotes the orthogonal complement of $\mathfrak{gl}(i)$ in $\mathfrak{gl}(n)$ with respect to the trace form. It is convenient for us to have a coordinate description of this decomposition.

Definition 2.1. For $x \in \mathfrak{gl}(n)$, we let $x_i \in \mathfrak{gl}(i)$ be the upper left corner of x, that is, $(x_i)_{k,l} = x_{k,l}$ for $1 \le k, l \le i$. We refer to x_i as the $i \times i$ cutoff of x.

The decomposition of $y \in \mathfrak{gl}(n)$ in (2-2) is written $y = y_i \oplus y_i^{\perp}$, where $y_i^{\perp}$ denotes the entries $y_{k,l}$ where k and l are not both in the set $\{1, \dots, i\}$. Using this decomposition, we can think of the polynomials on $\mathfrak{gl}(i)$, which we denote by $P(\mathfrak{gl}(i))$, as a Poisson subalgebra of $P(\mathfrak{gl}(n))$, the polynomials on $\mathfrak{gl}(n)$. Explicitly, if $f \in P(\mathfrak{gl}(i))$, then (2-2) gives $f(x) = f(x_i)$ for $x \in \mathfrak{gl}(n)$. The Poisson structure on $P(\mathfrak{gl}(i))$ inherited from $P(\mathfrak{gl}(n))$ agrees with the Lie–Poisson structure on $P(\mathfrak{gl}(i))$; see [Kostant and Wallach 2006a, page 330].

Since $\mathfrak{gl}(n)$ is a Poisson manifold, we have the notion of a Hamiltonian vector field ξ_f for any holomorphic function $f \in \mathbb{O}(\mathfrak{gl}(n))$. If $g \in \mathbb{O}(\mathfrak{gl}(n))$, then $\xi_f(g) = \{f, g\}$. The group A is defined as the simply connected, complex Lie group that corresponds to a certain Lie algebra of Hamiltonian vector fields on $\mathfrak{gl}(n)$. To define this Lie algebra of vector fields, we consider the subalgebra of $P(\mathfrak{gl}(n))$ generated by the adjoint invariant polynomials for each of the n subalgebras $\mathfrak{gl}(i)$. We denote this subalgebra by $J(\mathfrak{gl}(n))$. We will soon see that

$$J(\mathfrak{gl}(n)) \cong P(\mathfrak{gl}(1))^{\mathrm{GL}(1)} \otimes \cdots \otimes P(\mathfrak{gl}(n))^{\mathrm{GL}(n)}. \tag{2-3}$$

This algebra may be viewed as a classical analogue of the Gelfand–Zeitlin subalgebra of the universal enveloping algebra $U(\mathfrak{gl}(n))$; see [Drozd et al. 1994]. Since $P(\mathfrak{gl}(i))^{\mathrm{GL}(i)}$ is the Poisson center of $P(\mathfrak{gl}(i))$, it is easy to see that $J(\mathfrak{gl}(n))$ is Poisson commutative; see [Kostant and Wallach 2006a, Proposition 2.1]. Let $f_{i,1}, \dots, f_{i,i}$ generate the ring $P(\mathfrak{gl}(i))^{\mathrm{GL}(i)}$. Then the set $f_{i,i} \mid 1 \le i \le n\}$ generates $J(\mathfrak{gl}(n))$. Note that the sum $\sum_{i=1}^{n-1} i = \frac{1}{2}n(n-1) = n(n-1)/2$ is half the dimension of a regular adjoint orbit in $\mathfrak{gl}(n)$. We will see shortly that the functions $\{f_{i,1}, \dots, f_{i,i} \mid 1 \le i \le n-1\}$ form a completely integrable system on a regular adjoint orbit.

The surprising fact about this integrable system proved by Kostant and Wallach is that the corresponding Hamiltonian vector fields $\xi_{f_{i,j}}$ for $(i, j) \in \Delta_{i,j}^{n-1}$ are complete; [Kostant and Wallach 2006a, Theorem 3.5]. Let $f_{i,j} = \operatorname{tr}(x_i^j)$ and $\mathfrak{a} = \operatorname{span}\{\xi_{f_{i,j}} \mid (i, j) \in \Delta_{i,j}^{n-1}\}$. We define A as the simply connected, complex Lie group corresponding to the Lie algebra $\mathfrak{a}$. Since the vector fields $\xi_{f_{i,j}}$ commute for all i and j, the corresponding (global) flows define a global action of $\mathbb{C}^{n(n-1)/2}$ on $\mathfrak{gl}(n)$. The group $A \cong \mathbb{C}^{n(n-1)/2}$, and it acts on $\mathfrak{gl}(n)$ by composing these flows in any order. The action of A also preserves adjoint orbits [Kostant and Wallach 2006a, Theorems 3.3 and 3.4].

The action of $A \cong \mathbb{C}^{n(n-1)/2}$ may seem at first glance to be noncanonical. However, one can show that the orbit structure of $\mathbb{C}^{n(n-1)/2}$ given by integrating the complete vector fields $\xi_{f_{i,j}}$ is independent of the choice of generators $f_{i,j}$ for $P(\mathfrak{gl}(i))^{\mathrm{GL}(i)}$. See [Kostant and Wallach 2006a, Theorem 3.5]. Since we are interested in studying the geometry of these orbits, we lose no information by fixing a choice of generators.

Remark 2.2. Using the Gelfand–Zeitlin algebra for complex orthogonal Lie algebras $\mathfrak{so}(n)$, we can define an analogous group $\mathbb{C}^d$, where d is half the dimension of a regular adjoint orbit in $\mathfrak{so}(n)$. The construction of the group and the study of its orbit structure on certain regular semisimple elements of $\mathfrak{so}(n)$ are discussed in detail in [Colarusso 2009].

For our choice of generators, we can write down the Hamiltonian vector fields $\xi_{f_{i,j}}$ in coordinates and their corresponding global flows. We use the following notation. Given $x, z \in \mathfrak{gl}(n)$, we denote the directional derivative in the direction of z evaluated at x by ∂_x^z. Its action on function on a holomorphic function f is

$$\partial_x^z f = \frac{d}{dt}\Big|_{t=0} f(x+tz). \tag{2-4}$$

By [Kostant and Wallach 2006a, Theorem 2.12],

$$(\xi_{f_{i,j}})_x = \partial_x^{[-jx_i^{j-1},x]}. \tag{2-5}$$

We see that $\xi_{f_{i,j}}$ integrates to an action of $\mathbb{C}$ on $\mathfrak{gl}(n)$ given by

$$\operatorname{Ad}(\exp(tjx_i^{j-1}))\cdot x \quad \text{for } t \in \mathbb{C}, \tag{2-6}$$

where $x_i^0 = \mathrm{Id}_i \in \mathfrak{gl}(i)$.

Remark 2.3. The orbits of A are the composition of the (commuting) flows in (2-6) for $(i, j) \in \Delta_{i,j}^{n-1}$, in any order acting on $x \in \mathfrak{gl}(n)$. Clearly, the action of A stabilizes adjoint orbits.

Equation (2-5) gives us a convenient description of the tangent space to the action of A on $\mathfrak{gl}(n)$. We first need some notation. For $x \in \mathfrak{gl}(n)$, let Z_{x_i} be the associative subalgebra of $\mathfrak{gl}(i)$ generated by the elements $\mathrm{Id}_i, x_i, x_i^2, \dots, x_i^{i-1}$. We then let $Z_x = \sum_{i=1}^n Z_{x_i}$. Let $x \in \mathfrak{gl}(n)$ and let $A\cdot x$ denote its A-orbit. Then (2-5) gives us

$$T_x(A\cdot x) = \operatorname{span}\{(\xi_{f_{i,j}})_x \mid (i, j) \in \Delta_{i,j}^{n-1}\} = \operatorname{span}\{\partial_x^{[z,x]} \mid z \in Z_x\}.$$

Following the notation in [Kostant and Wallach 2006a], we let

$$V_x := \operatorname{span}\{\partial_x^{[z,x]} \mid z \in Z_x\} = T_x(A\cdot x) \subset T_x(\mathfrak{gl}(n)). \tag{2-7}$$

Our work focuses on orbits of A of maximal dimension $n(n-1)/2$, since such orbits form Lagrangian submanifolds of regular adjoint orbits. (If such orbits exist, they are the leaves of maximal dimension of the Gelfand–Zeitlin integrable system.) Accordingly, we make the following theorem/definition. See [Kostant and Wallach 2006a, Theorem 2.7 and Remark 2.8].

Theorem/definition 2.4. *An element $x \in \mathfrak{gl}(n)$ is called strongly regular if and only if the differentials $\{(df_{i,j})_x \mid (i,j) \in \Delta^n_{i,j}\}$ are linearly independent at x. Equivalently, x is strongly regular if the A-orbit $A \cdot x$ of x has* $\dim(A \cdot x) = n(n-1)/2$. *We denote the set of strongly regular elements of $\mathfrak{gl}(n)$ by $\mathfrak{gl}(n)^{\mathrm{sreg}}$.*

Our goal is to determine the A-orbit structure of $\mathfrak{gl}(n)^{\mathrm{sreg}}$. In [Kostant and Wallach 2006a], Kostant and Wallach produce strongly regular elements using the map

$$\Phi : \mathfrak{gl}(n) \to \mathbb{C}^{n(n+1)/2}, \quad x \mapsto (p_{1,1}(x_1), p_{2,1}(x_2), \ldots, p_{n,n}(x)), \tag{2-8}$$

where $p_{i,j}(x_i)$ is the coefficient of t^{j-1} in the characteristic polynomial of x_i.

Theorem 2.5 [Kostant and Wallach 2006a, Theorem 2.3]. *Let $\mathfrak{b} \subset \mathfrak{gl}(n)$ denote the standard Borel subalgebra of upper triangular matrices in $\mathfrak{gl}(n)$. Let f be the sum of the negative simple root vectors. Then the restriction of Φ to the affine variety $f + \mathfrak{b}$ is an algebraic isomorphism.*

We will refer to the elements of $f+\mathfrak{b}$ as Hessenberg matrices. They are matrices of the form

$$f + \mathfrak{b} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n-1} & a_{1n} \\ 1 & a_{22} & \cdots & a_{2n-1} & a_{2n} \\ 0 & 1 & \cdots & a_{3n-1} & a_{3n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a_{nn} \end{bmatrix}.$$

Theorem 2.5 implies that if $x \in f + \mathfrak{b}$, the differentials $(dp_{i,j})_x$ for $(i,j) \in \Delta^n_{i,j}$ are linearly independent. For the same range of i and j, the sets of functions $\{f_{i,j}\}$ and $\{p_{i,j}\}$ both generate the classical analogue of the Gelfand–Zeitlin algebra $J(\mathfrak{gl}(n))$. It follows that

$$\mathrm{span}\{(df_{i,j})_x \mid (i,j) \in \Delta^n_{i,j}\} = \mathrm{span}\{(dp_{i,j})_x \mid (i,j) \in \Delta^n_{i,j}\} \quad \text{for any } x \in \mathfrak{gl}(n)$$

by the Leibniz rule. Theorem 2.5 then implies $f + \mathfrak{b} \subset \mathfrak{gl}(n)^{\mathrm{sreg}}$ and therefore $\mathfrak{gl}(n)^{\mathrm{sreg}}$ is a nonempty Zariski open subset of $\mathfrak{gl}(n)$. Thus the functions $\{f_{i,j} \mid (i,j) \in \Delta^n_{i,j}\}$ are algebraically independent, and we obtain (2-3).

For $c = (c_1, c_2, \ldots, c_n) \in \mathbb{C} \times \mathbb{C}^2 \times \cdots \times \mathbb{C}^n = \mathbb{C}^{n(n+1)/2}$, we denote the fiber $\Phi^{-1}(c)$ by $\mathfrak{gl}(n)_c$, with Φ as in (2-8). For $c_i \in \mathbb{C}^i$, we define a monic polynomial $p_{c_i}(t)$ with coefficients given by c_i as in (1-2). Then $x \in \mathfrak{gl}(n)_c$ if and only if x_i has characteristic polynomial $p_{c_i}(t)$ for all i. Then for any $c \in \mathbb{C}^{n(n+1)/2}$, Theorem 2.5 says that $\mathfrak{gl}(n)_c$ is nonempty and contains a unique Hessenberg matrix. We denote the strongly regular elements of the fiber $\mathfrak{gl}(n)_c$ by $\mathfrak{gl}(n)_c^{\mathrm{sreg}}$, that is,

$$\mathfrak{gl}(n)_c^{\mathrm{sreg}} = \mathfrak{gl}(n)_c \cap \mathfrak{gl}(n)^{\mathrm{sreg}}.$$

Since Hessenberg matrices are strongly regular, $\mathfrak{gl}(n)_c^{\mathrm{sreg}}$ is a nonempty Zariski open subset of $\mathfrak{gl}(n)_c$ for any $c \in \mathbb{C}^{n(n+1)/2}$.

Theorem 2.5 implies that every regular adjoint orbit contains strongly regular elements. This follows from the fact that a regular adjoint orbit contains a companion matrix, which is Hessenberg. We can then use A-orbits of dimension $n(n-1)/2$ to construct polarizations of dense, open submanifolds of regular adjoint orbits. Hence, the Gelfand–Zeitlin system is completely integrable on each regular adjoint orbit [Kostant and Wallach 2006a, Theorem 3.36].

Our goal is to give a complete description of the A-orbit structure of $\mathfrak{gl}(n)^{\mathrm{sreg}}$. It follows from the Poisson commutativity of the algebra $J(\mathfrak{gl}(n))$ in (2-3) that the fibers $\mathfrak{gl}(n)_c$ are A-stable. Whence, the fibers $\mathfrak{gl}(n)_c^{\mathrm{sreg}}$ are A-stable. Moreover, we have by [Kostant and Wallach 2006a, Theorem 3.12] that the A-orbits in $\mathfrak{gl}(n)^{\mathrm{sreg}}$ are the irreducible components of the fibers $\mathfrak{gl}(n)_c^{\mathrm{sreg}}$. From this it follows that

$$\text{there are only finitely many } A\text{-orbits in the fiber } \mathfrak{gl}(n)_c^{\mathrm{sreg}}\ .$$

In this paper, we describe the A-orbit structure of an arbitrary fiber $\mathfrak{gl}(n)_c^{\mathrm{sreg}}$ and count the exact number of A-orbits in the fiber. This gives a complete description of the A-orbit structure of $\mathfrak{gl}(n)^{\mathrm{sreg}}$.

Remark 2.6. The set of fibers of the map Φ is the same as the set of fibers of the moment map for the A-action $x \to (f_{1,1}(x_1), f_{2,1}(x_2), \dots, f_{n,n}(x))$. Thus, studying the action of A on the fibers of Φ is essentially studying the action of A on the fibers of the corresponding moment map. We use the map Φ instead of the moment map, since it is easier to describe the fibers of Φ.

For our purposes, it is convenient to have a more concrete characterization of strongly regular elements.

Proposition 2.7 [Kostant and Wallach 2006a, Theorem 2.14]. *Let $x \in \mathfrak{gl}(n)$ and let $\mathfrak{z}_{\mathfrak{gl}(i)}(x_i)$ denote the centralizer in $\mathfrak{gl}(i)$ of x_i. Then x is strongly regular if and only if the following two conditions hold.*

(a) *$x_i \in \mathfrak{gl}(i)$ is regular for all $1 \le i \le n$.*

(b) *$\mathfrak{z}_{\mathfrak{gl}(i-1)}(x_{i-1}) \cap \mathfrak{z}_{\mathfrak{gl}(i)}(x_i) = 0$ for all $2 \le i \le n$.*

3. The action of A on generic matrices

For $x \in \mathfrak{gl}(i)$, let $\sigma(x)$ denote the spectrum of x, where x is viewed as an $i \times i$ matrix. We consider the following Zariski open subset of regular semisimple elements of $\mathfrak{gl}(n)$

$$\begin{aligned}(3\text{-}1)\quad & \mathfrak{gl}(n)_\Omega \\ &= \{x \in \mathfrak{gl}(n) \mid x_i \text{ is regular semisimple}, \ \sigma(x_{i-1}) \cap \sigma(x_i) = \varnothing,\ 2 \le i \le n\}.\end{aligned}$$

Kostant and Wallach give a complete description of the action of A on $\mathfrak{gl}(n)_\Omega$. We give an example of a matrix in $\mathfrak{gl}(3)_\Omega$.

Example 3.1. Consider the matrix in $\mathfrak{gl}(3)$

$$X = \begin{bmatrix} 1 & 2 & 16 \\ 1 & 0 & 4 \\ 0 & 1 & -3 \end{bmatrix}.$$

Because X has eigenvalues $\sigma(X) = \{-3, 3, -2\}$, it is regular semisimple and $\sigma(X_2) = \{2, -1\}$. Clearly, $\sigma(X_1) = \{1\}$. Thus $X \in \mathfrak{gl}(3)_\Omega$.

We recall the notation introduced in (1-2). (If $c_i = (z_1, z_2, \dots, z_i) \in \mathbb{C}^i$, then $p_{c_i}(t) = z_1 + z_2 t + \cdots + z_i t^{i-1} + t^i$.) Let $\Omega_n \subset \mathbb{C}^{n(n+1)/2}$ be the Zariski open subset consisting of $c \in \mathbb{C}^{n(n+1)/2}$ with $c = (c_1, \dots, c_i, \dots, c_n)$ such that $p_{c_i}(t)$ has distinct roots and $p_{c_i}(t)$ and $p_{c_{i+1}}(t)$ have no roots in common [Kostant and Wallach 2006a, Remark 2.16]. It is easy to see that $\mathfrak{gl}(n)_\Omega = \bigcup_{c \in \Omega_n} \mathfrak{gl}(n)_c$.

Kostant and Wallach described the A-orbit structure on $\mathfrak{gl}(n)_\Omega$, as summarized in Theorem 1.1. We sketch the ideas behind a possible proof in the case of $\mathfrak{gl}(3)$. See [Kostant and Wallach 2006a] or [Colarusso 2009] for complete proofs and a more thorough explanation.

The A-orbit of $x \in \mathfrak{gl}(3)$ is

$$\mathrm{Ad}\left(\begin{bmatrix} z_1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} z_2 & & \\ & z_2 & \\ & & 1 \end{bmatrix} \begin{bmatrix} \exp(t x_2) & \\ & 1 \end{bmatrix} \right) \cdot x, \tag{3-2}$$

where $z_1, z_2 \in \mathbb{C}^\times$ and $t \in \mathbb{C}$; see Equation (2-6).

If we let $Z_i \subset \mathrm{GL}(i)$ be the centralizer of x_i in $\mathrm{GL}(i)$, we notice from (3-2) that the action of A appears to push down to an action of $Z_1 \times Z_2$. For $x \in \mathfrak{gl}(3)_\Omega$, we should then expect to see an action of $(\mathbb{C}^\times)^3$ as realizing the action of A.

Working directly from the definition of the action of A in (3-2) is cumbersome. The action of Z_2 on x_2 would be much easier to write down if x_2 were diagonal. However, x_2 is not diagonal for $x \in \mathfrak{gl}(3)_\Omega$, but it is diagonalizable. So, we first diagonalize x_2 and then conjugate by the centralizer $Z_2 = (\mathbb{C}^\times)^2$. If $\gamma(x) \in \mathrm{GL}(2)$ is such $(\mathrm{Ad}(\gamma(x)) \cdot x)_2$ is diagonal, then we can define an action of $(\mathbb{C}^\times)^3$ on $\mathfrak{gl}(3)_c$ for $c \in \Omega_3$ by

$$(z_1', z_2', z_3') \cdot x = \mathrm{Ad}\left(\begin{bmatrix} z_1' & & \\ & 1 & \\ & & 1 \end{bmatrix} \gamma(x)^{-1} \begin{bmatrix} z_2' & & \\ & z_3' & \\ & & 1 \end{bmatrix} \gamma(x) \right) \cdot x, \tag{3-3}$$

with $z_i' \in \mathbb{C}^\times$.

We can show (3-3) is a simply transitive algebraic group action on $\mathfrak{gl}(3)_c$ by explicit computation. Comparing (3-3) and (3-2), it is not hard to believe that the action of $(\mathbb{C}^\times)^3$ in (3-3) has the same orbits as the action of A on $\mathfrak{gl}(3)_c$. To prove this precisely, one needs to see that $\mathfrak{gl}(3)_c^{\mathrm{sreg}} = \mathfrak{gl}(3)_c$. This can be proved by computing the tangent space to the action of $(\mathbb{C}^\times)^3$ in (3-3) and showing that it is same as the subspace V_x in (2-7), or by appealing to [Kostant and Wallach 2006a, Theorem 2.17]. The fact that $\mathfrak{gl}(3)_c$ is one A-orbit then follows by applying [Kostant and Wallach 2006a, Theorem 3.12].

This line of argument is not the one used in [Kostant and Wallach 2006a] to prove Theorem 1.1. The ideas here go back to a preliminary approach by Kostant and Wallach. However, it is this method that generalizes to describe all orbits of A in $\mathfrak{gl}(n)^{\text{sreg}}$. We describe the general construction in the next section.

4. Constructing nongeneric A-orbits

4a. ***Overview.*** In the next three sections, we classify A-orbits in $\mathfrak{gl}(n)^{\text{sreg}}$ by determining the A-orbit structure of an arbitrary fiber $\mathfrak{gl}(n)_c^{\text{sreg}}$. Let $c_i \in \mathbb{C}^i$ and $p_{c_i}(t) = (t-\lambda_1)^{n_1} \cdots (t-\lambda_r)^{n_r}$ with $\lambda_j \neq \lambda_k$ for $j \neq k$; see (1-2). To study the action of A on $\mathfrak{gl}(n)_c$ with

$$c = (c_1, \dots, c_i, c_{i+1}, \dots, c_n) \in \mathbb{C}^1 \times \cdots \times \mathbb{C}^i \times \mathbb{C}^{i+1} \times \cdots \times \mathbb{C}^n = \mathbb{C}^{n(n+1)/2},$$

we consider elements of $\mathfrak{gl}(i+1)$ of the form

$$\left[\begin{array}{ccccc}
\begin{bmatrix} \lambda_1 & 1 & \cdots & 0 \\ 0 & \lambda_1 & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & \lambda_1 \end{bmatrix} & & 0 & & \begin{matrix} y_{1,1} \\ \vdots \\ \vdots \\ y_{1,n_1} \end{matrix} \\
 & \ddots & & & \vdots \\
0 & & \begin{bmatrix} \lambda_r & 1 & \cdots & 0 \\ 0 & \lambda_r & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & \lambda_r \end{bmatrix} & & \begin{matrix} y_{r,1} \\ \vdots \\ \vdots \\ y_{r,n_r} \end{matrix} \\
z_{1,1} \ \cdots \ \cdots \ z_{1,n_1} & \cdots & z_{r,1} \ \cdots \ \cdots \ z_{r,n_r} & & w
\end{array}\right] \tag{4-1}$$

with characteristic polynomial $p_{c_{i+1}}(t)$.

To avoid ambiguity, it is necessary to order the Jordan blocks of the $i \times i$ cutoff of the matrix in (4-1). To do this, we introduce a lexicographical ordering on $\mathbb{C}$ defined as follows. Let $z_1, z_2 \in \mathbb{C}$. We say that $z_1 > z_2$ if and only if $\operatorname{Re} z_1 > \operatorname{Re} z_2$, or $\operatorname{Re} z_1 = \operatorname{Re} z_2$ and $\operatorname{Im} z_1 > \operatorname{Im} z_2$.

Definition 4.1. Let $c_i \in \mathbb{C}^i$ be such that $p_{c_i}(t) = (t-\lambda_1)^{n_1} \cdots (t-\lambda_r)^{n_r}$ with $\lambda_j \neq \lambda_k$, as in (1-2), and let $\lambda_1 > \lambda_2 > \cdots > \lambda_r$ in the lexicographical ordering on $\mathbb{C}$. For $c_{i+1} \in \mathbb{C}^{i+1}$, we define $\Xi^i_{c_i, c_{i+1}}$ as the set of elements $x \in \mathfrak{gl}(i+1)$ of the form (4-1) whose characteristic polynomial is $p_{c_{i+1}}(t)$. We refer to $\Xi^i_{c_i,c_{i+1}}$ as the solution variety at level i.

We know from Theorem 2.5 that $\Xi^i_{c_i,c_{i+1}}$ is nonempty for any $c_i \in \mathbb{C}^i$ and any $c_{i+1} \in \mathbb{C}^{i+1}$. Let us denote the regular Jordan form that is the $i \times i$ cutoff of the matrix in (4-1) by J. Let Z_i denote the centralizer of J in $\mathrm{GL}(i)$. Since J is regular, Z_i is a connected, abelian algebraic group [Kostant 1963, Proposition 14]. The group Z_i acts algebraically on the solution variety $\Xi^i_{c_i,c_{i+1}}$ by conjugation. In the remainder of Section 4, we give a bijection between A-orbits in $\mathfrak{gl}(n)^{\mathrm{sreg}}_c$ and free $Z_1 \times \cdots \times Z_{n-1}$ orbits on $\Xi^1_{c_1,c_2} \times \cdots \times \Xi^{n-1}_{c_{n-1},c_n}$. In Section 5, we will classify the Z_i-orbits on $\Xi^i_{c_i,c_{i+1}}$ using combinatorial data of the tuple $c \in \mathbb{C}^{n(n+1)/2}$. We will then have a complete picture of the A-action on $\mathfrak{gl}(n)^{\mathrm{sreg}}_c$.

We now briefly outline the construction, which gives the bijection between A-orbits in $\mathfrak{gl}(n)^{\mathrm{sreg}}_c$ and $Z_1 \times \cdots \times Z_{n-1}$ orbits in $\Xi^1_{c_1,c_2} \times \cdots \times \Xi^{n-1}_{c_{n-1},c_n}$. This construction not only describes A-orbits in $\mathfrak{gl}(n)^{\mathrm{sreg}}_c$, but all A-orbits in the larger set $\mathfrak{gl}(n)_c \cap S$, where S is the Zariski open subset of $\mathfrak{gl}(n)$ consisting of elements x whose cutoffs x_i for $1 \le i \le n-1$ are regular. We know by Proposition 2.7(a) that $\mathfrak{gl}(n)^{\mathrm{sreg}}_c \subset \mathfrak{gl}(n)_c \cap S$, and it is in general a proper subset; see Example 5.4.

For $1 \le i \le n-2$, choose a Z_i-orbit $\mathbb{O}^i_{a_i} \in \Xi^i_{c_i,c_{i+1}}$ consisting of regular elements of $\mathfrak{gl}(i+1)$. For $i = n-1$, choose any orbit $\mathbb{O}^{n-1}_{a_{n-1}}$ of Z_{n-1} in $\Xi^{n-1}_{c_{n-1},c_n}$. Then define a morphism

$$\text{(4-2)}\quad \begin{aligned} \Gamma_n^{a_1,a_2,\dots,a_{n-1}} &: \mathbb{O}^1_{a_1} \times \cdots \times \mathbb{O}^{n-1}_{a_{n-1}} \to \mathfrak{gl}(n)_c \cap S, \\ &(x_1,\dots,x_{n-1}) \mapsto \mathrm{Ad}(g_{1,2}(x_1)^{-1} g_{2,3}(x_2)^{-1} \cdots g_{n-2,n-1}(x_{n-2})^{-1})x_{n-1}. \end{aligned}$$

where $g_{i,i+1}(x_i)$ conjugates x_i into Jordan canonical form (with eigenvalues in decreasing lexicographical order). For brevity, we define

$$\mathbf{\Gamma}_n := \Gamma_n^{a_1,a_2,\dots,a_{n-1}}.$$

We denote the image of this morphism by $\operatorname{im}\mathbf{\Gamma}_n$.

Theorem 4.2. *Every A-orbit in $\mathfrak{gl}(n)_c \cap S$ is of the form $\operatorname{im}\mathbf{\Gamma}_n$ for some choice of orbits $\mathbb{O}^i_{a_i} \subset \Xi^i_{c_i,c_{i+1}}$, with $\mathbb{O}^i_{a_i}$ consisting of regular elements of $\mathfrak{gl}(i+1)$ for $1 \le i \le n-2$.*

In Section 4c, we prove Theorem 4.2 for A-orbits in $\mathfrak{gl}(n)^{\mathrm{sreg}}_c$ (see Theorem 4.9). In Section 4d, we establish the results needed to prove Theorem 4.2 for $\mathfrak{gl}(n)_c \cap S$.

4b. *Definition and properties of the $\mathbf{\Gamma}_n$ maps.* We first define the map $\mathbf{\Gamma}_n$ only for Z_i-orbits $\mathbb{O}^i_{a_i} \subset \Xi^i_{c_i,c_{i+1}}$ on which Z_i acts freely. To define $\mathbf{\Gamma}_n$, we must define a morphism $\mathbb{O}^i_{a_i} \to \mathrm{GL}(i+1)$ that sends $y \mapsto g_{i,i+1}(y)$, where $g_{i,i+1}(y)$ conjugates y into Jordan form with eigenvalues in decreasing lexicographical order. Since Z_i acts freely on $\mathbb{O}^i_{a_i}$, we can identify $\mathbb{O}^i_{a_i} \cong Z_i$ as algebraic varieties. Let x_{a_i} be an arbitrary choice of base point for the orbit $\mathbb{O}^i_{a_i}$, that is, $\mathbb{O}^i_{a_i} = \mathrm{Ad}(Z_i)\cdot x_{a_i}$. We choose an element $g_{i,i+1}(x_{a_i}) \in \mathrm{GL}(i+1)$ that conjugates the base point x_{a_i} into Jordan form (with eigenvalues in decreasing lexicographical order). For $y = \mathrm{Ad}(k_i)\cdot x_{a_i}$, with $k_i \in Z_i$, we define

$$g_{i,i+1}(y) = g_{i,i+1}(x_{a_i})k_i^{-1}. \tag{4-3}$$

For each choice of orbit $\mathbb{O}^i_{a_i} \subset \Xi^i_{c_i,c_{i+1}}$ for $1 \le i \le n-1$, we define a morphism $\mathbf{\Gamma}_n : Z_1 \times \cdots \times Z_{n-1} \to \mathfrak{gl}(n)$ by

$$\begin{aligned}\mathbf{\Gamma}_n(k_1,\dots,k_{n-1}) \\ = \mathrm{Ad}(k_1 g_{1,2}(x_{a_1})^{-1} k_2 g_{2,3}(x_{a_2})^{-1} \cdots k_{n-2} g_{n-2,n-1}(x_{a_{n-2}})^{-1} k_{n-1}) x_{a_{n-1}}.\end{aligned} \tag{4-4}$$

We want to give a more intrinsic characterization of $\mathrm{im}\,\mathbf{\Gamma}_n$.

Proposition 4.3. *We have*

$$\mathrm{im}\,\mathbf{\Gamma}_n \subset \mathfrak{gl}(n)_c \cap S,$$

$$\mathrm{im}\,\mathbf{\Gamma}_n = \{x \in \mathfrak{gl}(n) \mid x_{i+1} \in \mathrm{Ad}(\mathrm{GL}(i))\cdot x_{a_i} \text{ for all } 1 \le i \le n-1\}. \tag{4-5}$$

Thus, $\mathrm{im}\,\mathbf{\Gamma}_n$ *is a quasiaffine subvariety of* $\mathfrak{gl}(n)$.

The following simple observation is useful in proving Proposition 4.3.

Remark 4.4. Let $x \in \mathfrak{gl}(n)_c \cap S$, and suppose that $g \in \mathrm{GL}(i)$ is such that $[\mathrm{Ad}(g)\cdot x]_i = \mathrm{Ad}(g)\cdot x_i$ is in Jordan canonical form with eigenvalues in decreasing lexicographical order for $1 \le i \le n-1$. Then $[\mathrm{Ad}(g)\cdot x]_{i+1} = \mathrm{Ad}(g)\cdot x_{i+1} \in \Xi^i_{c_i,c_{i+1}}$.

Proof of Proposition 4.3. Denote the set on the right side of (4-5) by T. We note $T \subset \mathfrak{gl}(n)_c \cap S$. Indeed, let $Y \in T$. Then $Y_{i+1} \in \mathrm{Ad}(\mathrm{GL}(i))\cdot x_{a_i}$ for $1 \le i \le n-1$. Since $x_{a_i} \in \Xi^i_{c_i,c_{i+1}}$, the characteristic polynomial of Y_{i+1} is $p_{c_{i+1}}(t)$. For $1 \le i \le n-2$, note that x_{a_i} is regular and hence so is Y_{i+1}. Lastly, using the fact that $k_1 \in \mathrm{GL}(1) = Z_1$ centralizes the $(1,1)$ entry of $x_{a_1} \in \Xi^1_{c_1,c_2}$, it follows that the $(1,1)$ entry of Y is given by $c_1 \in \mathbb{C}$.

The inclusion $\mathrm{im}\,\mathbf{\Gamma}_n \subset T$ is clear from the definition of $\mathbf{\Gamma}_n$ in (4-4). To see the opposite inclusion we use induction. Let $y \in T$. Then y_2 is in $\mathrm{Ad}(\mathrm{GL}(1))\cdot x_{a_1} = \mathbb{O}^1_{a_1}$ since $Z_1 = \mathrm{GL}(1)$. Thus, there exists a $k_1 \in Z_1$ such that $y_2 = \mathrm{Ad}(k_1)\cdot x_{a_1}$. It follows that

$$z_2 = [\mathrm{Ad}(g_{1,2}(x_{a_1}))\,\mathrm{Ad}(k_1^{-1})\cdot y]_3 = [\mathrm{Ad}(g_{1,2}(x_{a_1}))\,\mathrm{Ad}(k_1^{-1})\cdot y_3] \in \Xi^2_{c_2,c_3}.$$

But $y_3 \in \mathrm{Ad}(\mathrm{GL}(2))\cdot x_{a_2}$, so that $z_2 \in \Xi^2_{c_2,c_3} \cap \mathrm{Ad}(\mathrm{GL}(2))\cdot x_{a_2}$, from which it follows easily that $z_2 \in \mathbb{O}^2_{a_2}$. Thus, there exists a $k_2 \in Z_2$ such that

$$[\mathrm{Ad}(g_{2,3}(x_{a_2}))\,\mathrm{Ad}(k_2^{-1})\,\mathrm{Ad}(g_{1,2}(x_{a_1}))\,\mathrm{Ad}(k_1^{-1})\cdot y]_4 \in \Xi^3_{c_3,c_4}.$$

This completes the first two steps of the induction. We now assume that there exist $k_1, \dots, k_{j-1} \in Z_1, \dots, Z_{j-1}$ respectively such that

$$z_j = [\mathrm{Ad}(g_{j-1,j}(x_{a_{j-1}}))\,\mathrm{Ad}(k_{j-1}^{-1}) \cdots \mathrm{Ad}(g_{1,2}(x_{a_1}))\,\mathrm{Ad}(k_1^{-1}) \cdot y]_{j+1} \in \Xi^j_{c_j, c_{j+1}}. \tag{4-6}$$

Since $y_{j+1} \in \mathrm{Ad}(\mathrm{GL}(j)) \cdot x_{a_j}$, it follows that $z_j \in \Xi^j_{c_j,c_{j+1}} \cap \mathrm{Ad}(\mathrm{GL}(j)) \cdot x_{a_j}$. As above, it follows that $z_j \in \mathbb{O}^j_{a_j}$, so that there exists an element $k_j \in K_j$ such that

$$[\mathrm{Ad}(g_{j,j+1}(x_{a_j}))\,\mathrm{Ad}(k_j^{-1})\,\mathrm{Ad}(g_{j-1,j}(x_{a_{j-1}}))\,\mathrm{Ad}(k_{j-1}^{-1}) \cdots \mathrm{Ad}(g_{1,2}(x_{a_1}))\,\mathrm{Ad}(k_1^{-1}) \cdot y]_{j+2} \in \Xi^{j+1}_{c_{j+1},c_{j+2}}.$$

We have made use of Remark 4.4 throughout. By induction, we conclude that there exist $k_1, \dots, k_{n-1} \in Z_1, \dots, Z_{n-1}$ respectively such that

$$x_{a_{n-1}} = \mathrm{Ad}(k_{n-1}^{-1})\,\mathrm{Ad}(g_{n-2,n-1}(x_{a_{n-1}}))\,\mathrm{Ad}(k_{n-2}^{-1}) \cdots \mathrm{Ad}(g_{1,2}(x_{a_1}))\,\mathrm{Ad}(k_1^{-1}) \cdot y,$$

from which it follows that $y = \mathbf{\Gamma}_n(k_1, \dots, k_{n-1})$.

To see the final statement, we observe T is a Zariski locally closed subset of $\mathfrak{gl}(n)$. Indeed, the set $U_i = \{x \mid x_{i+1} \in \mathrm{Ad}(\mathrm{GL}(i)) \cdot x_{a_i}\}$ is locally closed, since it is the preimage of the orbit $\mathrm{Ad}(\mathrm{GL}(i)) \cdot x_{a_i} \subset \mathfrak{gl}(i+1)$ under the projection morphism $\pi_{i+1}(x) = x_{i+1}$. The set $T = U_1 \cap \cdots \cap U_{n-1}$ is locally closed. □

Remark 4.5. From Proposition 4.3 it follows that the set $\mathrm{im}\,\mathbf{\Gamma}_n$ depends only on the orbits $\mathbb{O}^i_{a_i}$ for $1 \le i \le n-1$, and is thus independent of the choices involved in defining the map $\mathbf{\Gamma}_n$ in (4-4).

4c. $\mathbf{\Gamma}_n$ *and A-orbits in* $\mathfrak{gl}(n)_c^{\mathrm{sreg}}$. In this section, we show that the image of the morphism $\mathbf{\Gamma}_n$ is an A-orbit in $\mathfrak{gl}(n)_c^{\mathrm{sreg}}$. The first step is to see $\mathrm{im}\,\mathbf{\Gamma}_n$ is smooth variety.

Theorem 4.6. *The morphism*

$$\mathbf{\Gamma}_n : Z_1 \times \cdots \times Z_{n-1} \to \mathfrak{gl}(n)_c \cap S$$

is an isomorphism onto its image. Hence, $\mathrm{im}\,\mathbf{\Gamma}_n$ *is a smooth*, *irreducible subvariety of* $\mathfrak{gl}(n)$ *of dimension* $n(n-1)/2$.

Proof. We explicitly construct an inverse $\Psi : \mathrm{im}\,\mathbf{\Gamma}_n \to Z_1 \times \cdots \times Z_{n-1}$ of $\mathbf{\Gamma}_n$ and show that it is a morphism. Specifically, we show that there exist morphisms $\psi_i : \mathrm{im}\,\mathbf{\Gamma}_n \to Z_i$ for $1 \le i \le n-1$ such that the morphism

$$\Psi = (\psi_1, \dots, \psi_{n-1}) : \mathrm{im}\,\mathbf{\Gamma}_n \to Z_1 \times \cdots \times Z_{n-1} \tag{4-7}$$

is an inverse of $\mathbf{\Gamma}_n$. The morphisms ψ_i are constructed inductively.

Given $y \in \operatorname{im} \boldsymbol{\Gamma}_n$, we have $y_2 \in \mathbb{O}^1_{a_1} \subset \Xi^1_{c_1,c_2}$ by Proposition 4.3. Thus, $y_2 = \operatorname{Ad}(k_1) \cdot x_{a_1}$ for a unique k_1 in Z_1. The map $\mathbb{O}^1_{a_1} \to Z_1$, $\operatorname{Ad}(k_1) \cdot x_{a_1} \mapsto k_1$ is an isomorphism of smooth affine varieties. Hence, the map $\psi_1(y) = k_1$ is a morphism.

Arguing as in the proof of Proposition 4.3, suppose we have defined morphisms $\psi_1, \ldots, \psi_{j-1}$, with $\psi_i : \operatorname{im} \boldsymbol{\Gamma}_n \to Z_i$ for $1 \le i \le j-1$. Then the function $\operatorname{im} \boldsymbol{\Gamma}_n \to \mathbb{O}^j_{a_j}$ given by (4-6), that is,

$$y \mapsto [\operatorname{Ad}(g_{j-1,j}(x_{a_{j-1}})) \operatorname{Ad}(\psi_{j-1}(y)^{-1}) \cdots \operatorname{Ad}(g_{1,2}(x_{a_1})) \operatorname{Ad}(\psi_1(y)^{-1}) \cdot y]_{j+1},$$

is a morphism. We can then define a morphism $\psi_j : \operatorname{im} \boldsymbol{\Gamma}_n \to Z_j$ by $y \mapsto k_j$, where k_j is the unique element of Z_j such that

$$\begin{aligned} \operatorname{Ad}(k_j) \cdot x_{a_j} = [&\operatorname{Ad}(g_{j-1,j}(x_{a_{j-1}})) \operatorname{Ad}(\psi_{j-1}(y)^{-1}) \\ &\cdots \operatorname{Ad}(g_{1,2}(x_{a_1})) \operatorname{Ad}(\psi_1(y)^{-1}) \cdot y]_{j+1}. \end{aligned} \tag{4-8}$$

This completes the induction.

We now show that Ψ is an inverse of $\boldsymbol{\Gamma}_n$. That $\boldsymbol{\Gamma}_n(\psi_1(y), \ldots, \psi_{n-1}(y)) = y$ follows exactly as in the proof of the inclusion $T \subset \operatorname{im} \boldsymbol{\Gamma}_n$ in Proposition 4.3.

Finally, we show that $\Psi(\boldsymbol{\Gamma}_n(k_1, \ldots, k_{n-1})) = (k_1, \ldots, k_{n-1})$. Consider the element

$$\operatorname{Ad}(k_j g_{j,j+1}(x_{a_j})^{-1} \cdots g_{n-2,n-1}(x_{a_{n-2}})^{-1} k_{n-1}) \cdot x_{a_{n-1}}.$$

The $(j+1) \times (j+1)$ cutoff of this element is equal to $k_j \cdot x_{a_j}$. This fact with $j = 1$ gives $\psi_1(y) = k_1$. Assume that we have $\psi_2(y) = k_2, \ldots, \psi_l(y) = k_l$ for $2 \le l \le j-1$. Using the definition of ψ_j in (4-8), we obtain

$$\begin{aligned} \operatorname{Ad}(\psi_j(y)) \cdot x_{a_j} &= [\operatorname{Ad}(k_j) \operatorname{Ad}(g_{j,j+1}(x_{a_j})^{-1} \cdots g_{n-2,n-1}(x_{a_{n-2}})^{-1} k_{n-1}) x_{a_{n-1}}]_{j+1} \\ &= \operatorname{Ad}(k_j) \cdot x_{a_j}. \end{aligned}$$

Thus by induction, $\Psi \circ \boldsymbol{\Gamma}_n$ is the identity. Hence, Ψ is a regular inverse of the map $\boldsymbol{\Gamma}_n$ and Ψ is an isomorphism of varieties. □

The image of $\boldsymbol{\Gamma}_n$ is a smooth irreducible quasiaffine subvariety of $\mathfrak{gl}(n)$. Thus $\operatorname{im} \boldsymbol{\Gamma}_n$ has the structure of a connected analytic submanifold of $\mathfrak{gl}(n)$, and $\boldsymbol{\Gamma}_n$ is an analytic isomorphism.

Proposition 4.7. *The action of the analytic group A on $\mathfrak{gl}(n)$ preserves the submanifolds* $\operatorname{im} \boldsymbol{\Gamma}_n$.

Proof. The action of A on $\mathfrak{gl}(n)$ is given by the composition of the flows in (2-6) in any order; see Remark 2.3. Thus, to see that the action of A preserves $\operatorname{im} \boldsymbol{\Gamma}_n$, it suffices to see that the action of $\mathbb{C}$ in (2-6) preserves $\operatorname{im} \boldsymbol{\Gamma}_n$ for any $(i, j) \in \Delta^{n-1}_{i,j}$. Suppose that $x \in \operatorname{im} \boldsymbol{\Gamma}_n$. Then by Proposition 4.3, $x_{k+1} \in \operatorname{Ad}(\mathrm{GL}(k)) \cdot x_{a_k}$ for any $1 \le k \le n-1$. Define an element $h = \exp(tjx_i^{j-1}) \in \mathrm{GL}(i)$ with $t \in \mathbb{C}$ fixed and consider $\operatorname{Ad}(h) \cdot x$ as in (2-6). We claim that $(\operatorname{Ad}(h) \cdot x)_{k+1} \in \operatorname{Ad}(\mathrm{GL}(k)) \cdot x_{a_k}$ for

$1 \le k \le n-1$. We consider two cases. Suppose $k \ge i$ and consider $(\mathrm{Ad}(h) \cdot x)_{k+1}$. We have $(\mathrm{Ad}(h) \cdot x)_{k+1} = \mathrm{Ad}(h) \cdot x_{k+1}$. But $x_{k+1} \in \mathrm{Ad}(\mathrm{GL}(k)) \cdot x_{a_k}$, so that $\mathrm{Ad}(h) \cdot x_{k+1} \in \mathrm{Ad}(\mathrm{GL}(k)) \cdot x_{a_k}$, since $\mathrm{GL}(i) \subset \mathrm{GL}(k)$. Next, we suppose that $k < i$, so that $k+1 \le i$. Since $h \in \mathrm{GL}(i)$ centralizes x_i,

$$(\mathrm{Ad}(h)x)_{k+1} = (\mathrm{Ad}(h)(x_i))_{k+1} = (x_i)_{k+1} = x_{k+1} \in \mathrm{Ad}(\mathrm{GL}(k)) \cdot x_{a_k}.$$

By Proposition 4.3, $\mathrm{Ad}(h) \cdot x \in \operatorname{im} \mathbf{\Gamma}_n$. □

The main theorem of this section depends on a technical result about the action of Z_i on the solution varieties $\Xi^i_{c_i, c_{i+1}}$; this result will be proved independently in Section 4d.

Lemma 4.8. *For $x \in \Xi^i_{c_i, c_{i+1}}$, the isotropy group* $\mathrm{Stab}(x)$ *of x under the action of Z_i is a connected algebraic group.*

Thus, given an orbit $\mathbb{O} \subset \Xi^i_{c_i, c_{i+1}}$ of Z_i,

$$\dim(\mathbb{O}) = i \quad \text{if and only if } Z_i \text{ acts freely on } \mathbb{O}. \tag{4-9}$$

Theorem 4.9. *The submanifold* $\operatorname{im} \mathbf{\Gamma}_n \subset \mathfrak{gl}(n)_c \cap S$ *is a single A-orbit in $\mathfrak{gl}(n)_c^{\mathrm{sreg}}$. Every A-orbit in $\mathfrak{gl}(n)_c^{\mathrm{sreg}}$ is of the form* $\operatorname{im} \mathbf{\Gamma}_n$ *with $\mathbf{\Gamma}_n = \Gamma_n^{a_1, a_2, \dots, a_{n-1}}$, where $\mathbb{O}^i_{a_i} \subset \Xi^i_{c_i, c_{i+1}}$ are free Z_i-orbits consisting of regular elements of $\mathfrak{gl}(i+1)$ for $1 \le i \le n-1$.*

Proof. First, we show that $\operatorname{im} \mathbf{\Gamma}_n$ is an A-orbit. For this, we need to describe the tangent space $T_y(\operatorname{im} \mathbf{\Gamma}_n) = (d\mathbf{\Gamma}_n)_{\underline{k}}$, where $\underline{k} = (k_1, \dots, k_{n-1}) \in Z_1 \times \cdots \times Z_{n-1}$ and $y = \mathbf{\Gamma}_n(\underline{k})$. Let $\{\alpha_{i1}, \dots, \alpha_{ii}\}$ be a basis for $\mathrm{Lie}(Z_i) = \mathfrak{z}_i$. Working analytically, we compute

$$(d\mathbf{\Gamma}_n)_{\underline{k}}(0, \dots, \alpha_{ij}, \dots, 0) = \tfrac{d}{dt}\Big|_{t=0} \mathbf{\Gamma}_n(k_1, \dots, k_i \exp(t\alpha_{ij}), \dots, k_{n-1})$$

for $1 \le j \le i$. Using the definition of the morphism $\mathbf{\Gamma}_n$, the right side of this becomes

$$\frac{d}{dt}\Big|_{t=0} \mathrm{Ad}(k_1 g_{1,2}(x_{a_1})^{-1} \cdots k_i \exp(t\alpha_{ij}) g_{i,i+1}(x_{a_i})^{-1} \cdots k_{n-2} g_{n-2,n-1}(x_{a_{n-2}})^{-1} k_{n-1}) x_{a_{n-1}},$$

which, after defining

$$l_i = k_1 g_{1,2}(x_{a_1})^{-1} \cdots k_i \quad \text{and} \quad h_i = g_{i,i+1}(x_{a_i})^{-1} \cdots k_{n-2} g_{n-2,n-1}(x_{a_{n-2}})^{-1} k_{n-1},$$

becomes

$$\tfrac{d}{dt}\Big|_{t=0} \mathrm{Ad}(l_i \exp(t\alpha_{ij}) h_i) \cdot x_{a_{n-1}},$$

which in turn has differential

$$\mathrm{ad}(\mathrm{Ad}(l_i) \cdot \alpha_{ij}) \cdot (\mathrm{Ad}(l_i h_i) \cdot x_{a_{n-1}}). \tag{4-10}$$

By definition of the element $l_i \in \mathrm{GL}(i)$, the $i \times i$ cutoff of $\mathrm{Ad}(l_i^{-1}) \cdot y = \mathrm{Ad}(l_i^{-1}) \cdot y_i$ is in Jordan form (with eigenvalues in decreasing lexicographical order). Hence elements of the form $\mathrm{Ad}(l_i) \cdot \alpha_{ij} = \gamma_{ij}$ for $1 \le j \le i$ form a basis for $\mathfrak{z}_{\mathfrak{gl}(i)}(y_i)$. Since $\mathrm{Ad}(l_i h_i) \cdot x_{a_{n-1}} = y$, (4-10) implies that the image of $(d\mathbf{\Gamma}_n)_{\underline{k}}$ is

$$\mathrm{im}((d\mathbf{\Gamma}_n)_{\underline{k}}) = \mathrm{span}\{\partial_y^{[\gamma_{i,j},y]} \mid (i,j) \in \Delta_{i,j}^{n-1}\} = T_y(\mathrm{im}\,\mathbf{\Gamma}_n). \tag{4-11}$$

Equation (2-7), with $y \in \mathrm{im}\,\mathbf{\Gamma}_n$ instead of x, reads

$$T_y(A \cdot y) = \mathrm{span}\{\partial_y^{[z,y]} \mid z \in Z_y\} := V_y.$$

Now, y has the property that y_i is regular for all $i \le n-1$, so that $\mathfrak{z}_{\mathfrak{gl}(i)}(y_i)$ has basis $\{\mathrm{Id}_i, y_i, \ldots, y_i^{i-1}\}$; see [Kostant 1963, page 382]. Thus,

$$T_y(\mathrm{im}\,\mathbf{\Gamma}_n) = \mathrm{span}\{\partial_y^{[z,y]} \mid z \in Z_y\} = V_y.$$

This gives

$$\dim V_y = \dim(A \cdot y) = n(n-1)/2, \tag{4-12}$$

which implies $\mathrm{im}\,\mathbf{\Gamma}_n \subset \mathfrak{gl}(n)_c^{\mathrm{sreg}}$. By Proposition 4.7, A acts on $\mathrm{im}\,\mathbf{\Gamma}_n$. We claim that the action of A is transitive on $\mathrm{im}\,\mathbf{\Gamma}_n$. Indeed, an A-orbit $A \cdot y$ with $y \in \mathrm{im}\,\mathbf{\Gamma}_n$ is a submanifold of $\mathrm{im}\,\mathbf{\Gamma}_n$ of the same dimension as $\mathrm{im}\,\mathbf{\Gamma}_n$ by (4-12), and thus must be open. The action of A is then clearly transitive on $\mathrm{im}\,\mathbf{\Gamma}_n$ since $\mathrm{im}\,\mathbf{\Gamma}_n$ is connected.

We now show that every A-orbit in $\mathfrak{gl}(n)_c^{\mathrm{sreg}}$ is obtained in this manner. For $x \in \mathfrak{gl}(n)_c^{\mathrm{sreg}}$, by Proposition 2.7(a) and Remark 4.4 there exists a matrix $g_i \in \mathrm{GL}(i)$ such that $z_i = \mathrm{Ad}(g_i) \cdot x_{i+1} \in \Xi_{c_i,c_{i+1}}^i$ and z_i is regular for each $1 \le i \le n-1$. Thus $z_i \in \mathbb{O}_{a_i}^i$, with $\mathbb{O}_{a_i}^i$ an orbit of Z_i in $\Xi_{c_i,c_{i+1}}^i$ consisting of regular elements of $\mathfrak{gl}(i+1)$. We claim that Z_i must act freely on $\mathbb{O}_{a_i}^i$. Suppose to the contrary that $\mathrm{Stab}(x_{a_i})$ is nontrivial. Lemma 4.8 gives that $\dim(\mathrm{Stab}(x_{a_i})) \ge 1$. But, this implies $\dim(Z_{\mathrm{GL}(i)}(x_i) \cap Z_{\mathrm{GL}(i+1)}(x_{i+1})) \ge 1$, contradicting Proposition 2.7(b). By Proposition 4.3, x is in $\mathrm{im}\,\mathbf{\Gamma}_n$, with $\mathbf{\Gamma}_n = \Gamma_n^{a_1,a_2,\ldots,a_{n-1}}$ for some choice of free Z_i-orbits $\mathbb{O}_{a_i}^i \subset \Xi_{c_i,c_{i+1}}^i$. $\square$

Remark 4.10. Let $\mathbf{\Gamma}_n$ be defined using Z_i-orbits $\mathbb{O}_{a_i}^i$, and let $\tilde{\mathbf{\Gamma}}_n := \Gamma_{n-1}^{\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_{n-1}}$ be defined using Z_i-orbits $\mathbb{O}_{\tilde{a}_i}^i = \mathrm{Ad}(Z_i) \cdot x_{\tilde{a}_i}$, where $\mathbb{O}_{a_i}^i \cap \mathbb{O}_{\tilde{a}_i}^i = \varnothing$ for some i in $1 \le i \le n-1$. Then the A-orbits $\mathrm{im}\,\mathbf{\Gamma}_n$ and $\mathrm{im}\,\tilde{\mathbf{\Gamma}}_n$ are distinct: Suppose to the contrary that $y \in \mathrm{im}\,\mathbf{\Gamma}_n \cap \mathrm{im}\,\tilde{\mathbf{\Gamma}}_n$. By Proposition 4.3, $y_{i+1} \in \mathrm{Ad}(\mathrm{GL}(i)) \cdot x_{a_i} \cap \mathrm{Ad}(\mathrm{GL}(i)) \cdot x_{\tilde{a}_i}$. This implies that there exists $h \in \mathrm{GL}(i)$ such that $\mathrm{Ad}(h) \cdot x_{a_i} = x_{\tilde{a}_i}$. Since $x_{a_i}, x_{\tilde{a}_i} \in \Xi_{c_i,c_{i+1}}^i$, the previous equation forces $h \in Z_i$, which implies $\mathbb{O}_{a_i}^i = \mathbb{O}_{\tilde{a}_i}^i$, a contradiction. We have thus established a bijection between free $Z_1 \times \cdots \times Z_{n-1}$ orbits on the product of solution varieties $\Xi_{c_1,c_2}^1 \times \cdots\cdots \times \Xi_{c_{n-1},c_n}^{n-1}$ and A-orbits in $\mathfrak{gl}(n)_c^{\mathrm{sreg}}$.

On the subvariety $\operatorname{im} \boldsymbol{\Gamma}_n$, we have a free and transitive algebraic action of the algebraic group $Z = Z_1 \times \cdots \times Z_{n-1}$. This action is defined as follows:

$$\begin{aligned} &\text{if} && {\boldsymbol{\Gamma}_n}^{-1}(y) = (k_1, \ldots, k_{n-1}), \\ &\text{then} && (k'_1, \ldots, k'_{n-1}) \cdot y = \boldsymbol{\Gamma}_n(k'_1 k_1, \ldots, k'_{n-1} k_{n-1}). \end{aligned} \tag{4-13}$$

Remark 4.11. The action in (4-13) generalizes the action of $(\mathbb{C}^\times)^3$ in (3-3) to the nongeneric case.

Thus, the A-orbit $\operatorname{im} \boldsymbol{\Gamma}_n$ is the orbit of an algebraic group acting on a quasiaffine variety. We now show that $Z = Z_1 \times \cdots \times Z_{n-1}$ acts algebraically on the fiber $\mathfrak{gl}(n)_c^{\text{sreg}}$. By [Kostant and Wallach 2006a, Theorem 3.12], the A-orbits in $\mathfrak{gl}(n)_c^{\text{sreg}}$ are the irreducible components of $\mathfrak{gl}(n)_c^{\text{sreg}}$. Since they are disjoint, these components are both open and closed in $\mathfrak{gl}(n)_c^{\text{sreg}}$ (in the Zariski topology on $\mathfrak{gl}(n)_c^{\text{sreg}}$). Following [Kostant and Wallach 2006a], we index these components by $\mathfrak{gl}_{c,i}^{\text{sreg}}(n) = A \cdot x(i)$, with $x(i) \in \mathfrak{gl}(n)_c^{\text{sreg}}$. We have morphisms $\phi_i : Z \times \mathfrak{gl}_{c,i}^{\text{sreg}}(n) \to \mathfrak{gl}_c^{\text{sreg}}(n)$ given by the action of Z on $\operatorname{im} \boldsymbol{\Gamma}_n$. The sets $Z \times \mathfrak{gl}_{c,i}^{\text{sreg}}(n)$ are (Zariski) open in the product $Z \times \mathfrak{gl}(n)_c^{\text{sreg}}$ and are disjoint. Thus, the morphisms ϕ_i glue to a unique morphism

$$\Phi : Z \times \mathfrak{gl}(n)_c^{\text{sreg}} \to \mathfrak{gl}(n)_c^{\text{sreg}} \quad \text{such that } \Phi\big|_{Z \times \mathfrak{gl}_{c,i}^{\text{sreg}}(n)} = \phi_i.$$

The morphism Φ defines an algebraic action of the group Z on $\mathfrak{gl}(n)_c^{\text{sreg}}$ whose orbits are the orbits of A in $\mathfrak{gl}(n)_c^{\text{sreg}}$. We have thus proved the following theorem.

Theorem 4.12. *Let $x \in \mathfrak{gl}(n)_c^{\text{sreg}}$ be arbitrary and let Z_i be the centralizer in* $\mathrm{GL}(i)$ *of the Jordan form of x_i (with eigenvalues in decreasing lexicographical order). On $\mathfrak{gl}(n)_c^{\text{sreg}}$ the orbits of the group A are orbits of a free algebraic action of the connected abelian algebraic group $Z = Z_1 \times \cdots \times Z_{n-1}$.*

We end this section with a result that will be of great use in Section 5 where we count the number of A-orbits in the fiber $\mathfrak{gl}(n)_c^{\text{sreg}}$.

It turns out that the condition in Theorem 4.9 that $\mathbb{O}^i_{a_i} \subset \mathfrak{gl}(i+1)^{\text{reg}}$ is superfluous.

Theorem 4.13. *If $\mathbb{O}^i_{a_i} \subset \Xi^i_{c_i, c_{i+1}}$ is a free Z_i-orbit, then $\mathbb{O}^i_{a_i} \subset \mathfrak{gl}(i+1)^{\text{reg}}$.*

Proof. Let $c = (c_1, c_2, \ldots, c_j, c_{j+1}, \ldots, c_n) \in \mathbb{C}^{n(n+1)/2}$, with $c_j \in \mathbb{C}^j$, be given. By Theorem 2.5, there is a unique upper Hessenberg matrix $h \in \mathfrak{gl}(n)_c^{\text{sreg}}$. This implies by Remark 4.4 that for any j in $1 \le j \le n-1$, there exists a $g_j \in \mathrm{GL}(j)$ such that $(\mathrm{Ad}(g_j) \cdot h)_{j+1} \in \Xi^j_{c_j, c_{j+1}}$. Thus, $\mathrm{Ad}(g_j) \cdot h_{j+1} \in Z_j \cdot x_{a_j} = \mathbb{O}^j_{a_j}$ for some $x_{a_j} \in \Xi^j_{c_j, c_{j+1}}$. But $h \in \mathfrak{gl}(n)^{\text{sreg}}$ and therefore h_{j+1} is regular by Proposition 2.7(a), which implies that $\mathbb{O}^j_{a_j} \subset \mathfrak{gl}(j+1)^{\text{reg}}$. Also, by Proposition 2.7(b), Z_j acts freely on $\mathbb{O}^j_{a_j}$, as in the proof of the last statement of Theorem 4.9. Thus, for any j in $1 \le j \le n-1$, there exists a free Z_j-orbit in $\Xi^j_{c_j, c_{j+1}}$ consisting of regular elements of $\mathfrak{gl}(j+1)$.

Now, let $\mathbb{O}^i_{a_i} \subset \Xi^i_{c_i,c_{i+1}}$ be any free Z_i-orbit. Now, we use the free Z_j-orbit $\mathbb{O}^j_{a_j} \subset \mathfrak{gl}(j+1)^{\text{reg}}$ as above for $1 \leq j \leq i-1$ and we use $\mathbb{O}^i_{a_i}$ to construct a morphism

$$\mathbf{\Gamma}_{i+1} := \Gamma^{a_1,a_2,\ldots,a_i}_{i+1} : Z_1 \times \cdots \times Z_i \to \mathfrak{gl}(n)_c \cap S.$$

By Theorem 4.9, $\operatorname{im} \mathbf{\Gamma}_{i+1} \subset \mathfrak{gl}(i+1)^{\text{sreg}}$. Proposition 2.7(a) then implies

$$\operatorname{im} \mathbf{\Gamma}_{i+1} \subset \mathfrak{gl}(i+1)^{\text{reg}}.$$

Then $\mathbb{O}^i_{a_i} \subset \mathfrak{gl}(i+1)^{\text{reg}}$ since elements of $\mathbb{O}^i_{a_i}$ are conjugate to those of $\operatorname{im} \mathbf{\Gamma}_{i+1}$. □

4d. ***A-orbits in*** $\mathfrak{gl}(n)_c \cap S$**.** We now discuss how the construction in Sections 4b and 4c can be generalized to describe A-orbits of dimension strictly less than $n(n-1)/2$ in the Zariski open subset of the fiber $\mathfrak{gl}(n)_c \cap S$. In this case, it is more difficult to define the morphism $\mathbf{\Gamma}_n$ of (4-2). The problem is that it is not clear how to define a morphism $\mathbb{O}^i_{a_i} \to \mathrm{GL}(i+1)$ that sends $x \to g_{i,i+1}(x)$, where $\mathrm{Ad}(g_{i,i+1}(x)) \cdot x$ is in Jordan form (with eigenvalues in decreasing lexicographical order). This is not difficult in the strongly regular case, since we are dealing with free Z_i-orbits $\mathbb{O}^i_{a_i} \cong Z_i$ so that $g_{i,i+1}(x)$ can be defined as in (4-3). The fortunate fact is that even for an orbit $\mathbb{O}^i_{a_i} \subset \Xi^i_{c_i,c_{i+1}}$ of dimension strictly less than i, there exists a connected, Zariski closed subgroup $K_i \subset Z_i$ with K_i acting freely on $\mathbb{O}^i_{a_i} \cong K_i$. Therefore, we can mimic what we did in (4-3).

To prove this, we need to understand better the action of Z_i on $\Xi^i_{c_i,c_{i+1}}$. As in Section 4a, let $J = J_1 \oplus \cdots \oplus J_r$ be the $i \times i$ cutoff of the matrix in (4-1), where $J_j \in \mathfrak{gl}(n_j)$ is the Jordan block corresponding to eigenvalue λ_j. We note since J is regular, Z_i is an abelian connected algebraic group, which is the product $\prod_{j=1}^r Z_{J_j}$ of groups, where Z_{J_j} denotes the centralizer of J_j. It is then easy to see that the action of Z_i is the diagonal action of the product $\prod_{j=1}^r Z_{J_j}$ on the last column of $x \in \Xi^i_{c_i,c_{i+1}}$ and the dual action on the last row of x; see (4-1). In other words, Z_{J_j} acts only on the columns and rows of x that contain the Jordan block J_j; see (4-1). This leads us to define an action of Z_{J_j} on $\mathbb{C}^{2n_j}$ by

$$z \cdot ([t_1,\ldots,t_{n_j}],[s_1,\ldots,s_{n_j}]^T) = ([t_1,\ldots,t_{n_j}] \cdot z^{-1}, z \cdot [s_1,\cdots,s_{n_j}]^T). \tag{4-14}$$

Let $\mathbb{O}$ be the Z_i-orbit of some $x \in \Xi^i_{c_i,c_{i+1}}$, and let $\mathbb{O}_j \subset \mathbb{C}^{2n_j}$ be the Z_{J_j}-orbit of

$$x[j] = ([z_{j,1},\ldots,z_{j,n_j}],[y_{j,1},\ldots,y_{j,n_j}]),$$

where the coordinates for x are as in (4-1). It follows directly from our remarks above that

$$\mathbb{O} \cong \mathbb{O}_1 \times \cdots \times \mathbb{O}_r, \tag{4-15}$$

where the isomorphism is Z_i-equivariant. It is easy to describe the structure of the isotropy groups for the Z_i-action using this description of a Z_i-orbit $\mathbb{O} \subset \Xi^i_{c_i,c_{i+1}}$.

Lemma 4.14. *Let $x \in \Xi^i_{c_i,c_{i+1}}$ and let* $\mathrm{Stab}(x) \subset Z_i$ *be its isotropy group under the action of Z_i on $\Xi^i_{c_i,c_{i+1}}$. Then, up to reordering,*

$$\mathrm{Stab}(x) = \prod_{j=1}^{q} Z_{J_j} \times \prod_{j=q+1}^{r} U_j, \tag{4-16}$$

where $U_j \subset Z_{J_j}$ is a unipotent Zariski closed subgroup (possibly trivial) for some q in $0 \le q \le r$.

Proof. Suppose that $x \in \Xi^i_{c_i,c_{i+1}}$ is given by (4-1). By (4-15), to compute the stabilizer of x we need only compute the stabilizers for each of the Z_{J_k} orbits $\mathbb{O}_k = Z_{J_k} \cdot x[k]$, where $1 \le k \le r$. To compute the stabilizer of $x[k]$, suppose that there exists an i with $1 \le i \le n_k$ such that $y_{k,i} \ne 0$ and $y_{k,l} = 0$ for $i < l \le n_k$. We consider the matrix equation

$$A_k \cdot \underline{y}_k = \underline{y}_k, \tag{4-17}$$

where $A_k \in Z_{J_k}$ is an invertible upper triangular Toeplitz matrix and $\underline{y}_k \in \mathbb{C}^{n_k}$ is the column vector $\underline{y}_k = (y_{k,1}, \dots, y_{k,i}, 0, \dots, 0)^T$. Since A_k is an upper triangular Toeplitz matrix, we see by considering the i-th row in (4-17) that A_k is forced to be unipotent. If on the other hand, all $y_{k,j} = 0$ for $1 \le j \le n_k$, we can argue similarly using the $z_{k,j}$ and the dual action.

If $y_{k,l} = 0$ for all l and $z_{k,l} = 0$ for all l, then clearly the stabilizer of $x[k]$ is Z_{J_k} itself. Repeating this analysis for each k in $1 \le k \le r$ and after possibly reordering the Jordan blocks of x_i, we get the desired result. □

Proof of Lemma 4.8. Upon reordering the eigenvalues, we can always assume that $\mathrm{Stab}(x)$ has the form given in (4-16) in Lemma 4.14. This proves the result since unipotent algebraic groups are always connected and the groups Z_{J_j} are connected since they are centralizers of regular elements in $\mathfrak{gl}(n_j)$. □

We can now prove the structural theorem about the group Z_i that lets us construct the morphism $\mathbf{\Gamma}_n$ in the general case.

Theorem 4.15. *Let $x \in \Xi^i_{c_i,c_{i+1}}$ and let* $\mathrm{Stab}(x) \subset Z_i$ *denote the isotropy group of x under the action of Z_i on $\Xi^i_{c_i,c_{i+1}}$. Then as an algebraic group,*

$$Z_i = \mathrm{Stab}(x) \times K,$$

where K is a connected, Zariski closed algebraic subgroup of Z_i.

Proof. For the purposes of this proof we denote by H the group $\mathrm{Stab}(x)$. Without loss of generality, we assume H is as given in (4-16). Let $\mathfrak{z}_i = \mathrm{Lie}(Z_i)$ and let

$\mathfrak{h} = \mathrm{Lie}(H)$. Now, by Lemma 4.14,

$$\mathfrak{h} = \bigoplus_{j=1}^{q} \mathfrak{z}_{J_j} \oplus \bigoplus_{j=q+1}^{r} \mathfrak{n}_j, \tag{4-18}$$

where $\mathfrak{z}_{J_j}$ is the Lie algebra of the abelian algebraic group Z_{J_j} and $\mathfrak{n}_j = \mathrm{Lie}(U_j)$ is a Lie subalgebra of $\mathfrak{n}^+(n_j)$, the strictly upper triangular matrices in $\mathfrak{gl}(n_j)$.

The proof takes two steps. We first find an algebraic Lie subalgebra $\mathfrak{k} \subset \mathfrak{z}_i$ such that $\mathfrak{z}_i = \mathfrak{h} \oplus \mathfrak{k}$ as Lie algebras. We then show that if $K \subset Z_i$ is the corresponding Zariski closed subgroup, then $Z_i = HK$ and $H \cap K = \{e\}$. To find $\mathfrak{k}$, consider the abelian Lie algebra $\mathfrak{z}_{J_j}$ for $q+1 \leq j \leq r$. Since $\mathfrak{z}_{J_j}$ is abelian, it has a Jordan decomposition as a direct sum of Lie algebras $\mathfrak{z}_{J_j} = \mathfrak{z}_{J_j}^{ss} \oplus \mathfrak{z}_{J_j}^{n}$, where $\mathfrak{z}_{J_j}^{ss}$ are the semisimple elements of $\mathfrak{z}_{J_j}$ and $\mathfrak{z}_{J_j}^{n}$ are the nilpotent elements. Now the Lie algebra $\mathfrak{n}_j$ in (4-18) is a subalgebra of $\mathfrak{z}_{J_j}^{n}$. Take $\tilde{\mathfrak{n}}_j$ such that $\mathfrak{z}_{J_j}^{n} = \mathfrak{n}_j \oplus \tilde{\mathfrak{n}}_j$. Let

$$\mathfrak{m}_j = \mathfrak{z}_{J_j}^{ss} \oplus \tilde{\mathfrak{n}}_j.$$

Note that $\mathfrak{m}_j \oplus \mathfrak{n}_j = \mathfrak{z}_{J_j}$. We claim that $\mathfrak{m}_j$ is an algebraic subalgebra of $\mathfrak{z}_{J_j}$. Indeed, $\tilde{\mathfrak{n}}_j$ is algebraic since it is a nilpotent Lie algebra; see [Tauvel and Yu 2005, page 383]. Let $\tilde{N}_j$ be the corresponding algebraic subgroup. Then $M_j = \mathbb{C}^\times \times \tilde{N}_j$ has $\mathrm{Lie}(M_j) = \mathfrak{m}_j$ since $\mathbb{C}^\times$ is the semisimple part of group Z_{J_j}; see (4-1). We then take $\mathfrak{k} = \bigoplus_{j=q+1}^{r} \mathfrak{m}_j$. This finishes the first step.

Let $K = \prod_{j=q+1}^{r} M_j$ be the Zariski closed, connected algebraic subgroup of $\prod_{j=q+1}^{r} Z_{J_j}$ that corresponds to the algebraic Lie algebra $\mathfrak{k}$. We now show that $Z_i = H \times K$. $H \cap K$ is finite by our choice of K. But also $H \cap K \subset \prod_{j=q+1}^{r} U_j$ and it is thus unipotent; see (4-16). Since any unipotent group must be connected, we have $H \cap K = \{e\}$. Now, it is clear that $Z_i = HK$, since HK is a closed, connected subgroup of Z_i of dimension $\dim Z_i$. □

Proof of Theorem 4.2. With Theorem 4.15 in hand, we can now define the general $\mathbf{\Gamma}_n$ morphism of (4-2) as we did in the strongly regular case. Now, suppose we are given Z_i-orbits $\mathbb{O}_{a_i}^{i}$ in Ξ_{c_i, c_i+1}^{i}, with $\mathbb{O}_{a_i}^{i} = K_{a_i} \cdot x_{a_i} \cong K_{a_i}$ with K_{a_i} as in Theorem 4.15 for $1 \leq i \leq n-1$, and with $\mathbb{O}_{a_i}^{i}$ consisting of regular elements of $\mathfrak{gl}(i+1)$ for $1 \leq i \leq n-2$. As in (4-4), we define a morphism

$$\mathbf{\Gamma}_n := \Gamma_n^{a_1, \dots, a_{n-1}} : K_{a_1} \times \cdots \times K_{a_{n-1}} \to \mathfrak{gl}(n)_c \cap S.$$

Propositions 4.3 and 4.7, Theorem 4.6, and Remark 4.10 from the strongly regular case remain valid in this case by simply replacing the groups Z_i by the groups K_{a_i}. We recall that the main ingredient in proving Theorem 4.6 is the fact that the group Z_i acts freely on $\mathbb{O}_{a_i}^{i}$. The analogue of Theorem 4.9 remains valid in this case, since it is easy to show that $T_y(\mathrm{im}\, \mathbf{\Gamma}_n) = V_y$ for V_y as in (2-7). □

The following corollary of Theorem 4.2 generalizes [Kostant and Wallach 2006a, Theorem 3.14] to include elements that are not necessarily strongly regular.

Corollary 4.16. *Let $x \in \mathfrak{gl}(n)_c \cap S$. The A-orbit $A \cdot x$ of x is a smooth, irreducible subvariety of $\mathfrak{gl}(n)$ that is isomorphic as an algebraic variety to a closed subgroup $K_{a_1} \times \cdots \times K_{a_{n-1}}$ of the connected algebraic group $Z_1 \times \cdots \times Z_{n-1}$.*

5. Counting A-orbits in $\mathfrak{gl}(n)_c^{\mathrm{sreg}}$

Using Theorem 4.9, we can count the number of A-orbits in $\mathfrak{gl}(n)_c^{\mathrm{sreg}}$ for any $c \in \mathbb{C}^{n(n+1)/2}$ and explicitly describe the orbits. We know from Theorem 4.9 and Remark 4.10 that counting the number of A-orbits in $\mathfrak{gl}(n)_c^{\mathrm{sreg}}$ is equivalent to counting the number of Z_i-orbits in $\Xi^i_{c_i,c_{i+1}}$ on which Z_i acts freely. We show in this section that the number of such orbits is directly related to the number of degeneracies in the roots of the monic polynomials $p_{c_i}(t)$ and $p_{c_{i+1}}(t)$; see (1-2). The study of this problem can be reduced to studying the structure of nilpotent solution varieties $\Xi^i_{0,0}$. Thus, we begin our discussion by describing the A-orbit structure of the nilfiber $\mathfrak{gl}(n)_0^{\mathrm{sreg}}$.

5a. *Nilpotent solution varieties and A-orbits in the nilfiber.* In this section, we study strongly regular matrices in the fiber $\mathfrak{gl}(n)_0$. By definition, $x \in \mathfrak{gl}(n)_0$ if and only if $x_i \in \mathfrak{gl}(i)$ is nilpotent for all i. Such matrices have been studied by Ovsienko [2003] and Parlett and Strang [2008].

We restate Definition 4.1 of the solution variety $\Xi^i_{c_i,c_{i+1}}$ in this case. Elements of $\mathfrak{gl}(i+1)$ of the form

$$X = \begin{bmatrix} 0 & 1 & \cdots & 0 & y_1 \\ 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & & \ddots & 1 & \vdots \\ 0 & \cdots & \cdots & 0 & y_i \\ z_1 & \cdots & \cdots & z_i & w \end{bmatrix} \tag{5-1}$$

that are nilpotent define the nilpotent solution variety at level i, which we denote by $\Xi^i_{0,0}$. In this case, it is easy to write down elements in $\Xi^i_{0,0}$. For example, we can take all of the z_j, y_j, and w to be 0. However, such an element is not regular, and so cannot be used to construct a $\mathbf{\Gamma}_n$ mapping that gives rise to a strongly regular orbit in $\mathfrak{gl}(n)_0^{\mathrm{sreg}}$. To describe A-orbits in $\mathfrak{gl}(n)_0^{\mathrm{sreg}}$, we focus our attention on free Z_i-orbits in $\Xi^i_{0,0}$; see Theorem 4.9. To find such orbits, we need to compute the characteristic polynomial of X.

Proposition 5.1. *The characteristic polynomial of the matrix in* (5-1) *is*

$$\det(X - t) = (-1)^i \big[-t^{i+1} + wt^i + \textstyle\sum_{l=0}^{i-1} \sum_{j=1}^{i-l} z_j y_{j+l} t^{i-1-l}\big]. \tag{5-2}$$

Proof. We compute the characteristic polynomial of the matrix in (5-1) using the Schur complement formula for the determinant; see [Horn and Johnson 1985, pages 21 and 22]. In the notation of that reference, $\alpha = \{1, \dots, n-1\}$ and $\alpha' = \{n\}$. Let $J = X_i$ denote the principal nilpotent Jordan block. Then the formula gives

$$\det(X - t) = \det(J - t)\,(w - t) - \underline{z}\,\mathrm{adj}(J - t)\,\underline{y}, \tag{5-3}$$

where $\mathrm{adj}(J - t) \in \mathfrak{gl}(i)$ denotes the classical adjoint matrix, $\underline{z} = [z_1, \dots, z_i]$ is a row vector, and $\underline{y} = [y_1, \dots, y_i]^T$ is a column vector. We easily compute that $\det(J - t) = (-1)^i t^i$. It is not difficult to see that

$$\mathrm{adj}(J-t) = (-1)^{i-1} \begin{bmatrix} t^{i-1} & t^{i-2} & \cdots & \cdots & t & 1 \\ 0 & t^{i-1} & t^{i-2} & \cdots & \cdots & t \\ \vdots & 0 & t^{i-1} & \ddots & & \vdots \\ & & 0 & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & t^{i-2} \\ 0 & \cdots & & \cdots & 0 & t^{i-1} \end{bmatrix}.$$

Now, we compute that the coefficient of t^{i-1-l} for $0 \le l \le i-1$ in the product $\underline{z}\,\mathrm{adj}(J - t)\,\underline{y}^T$ is $(-1)^{i-1} \sum_{j=1}^{i-l} z_j y_{j+l}$. Summing up these terms for $0 \le l \le i-1$ and using (5-3), we obtain the polynomial in (5-2). □

For the matrix in (5-1) to be nilpotent, we require that all of the coefficients of the polynomial in (5-2) (excluding the leading coefficient) vanish, that is

$$\begin{aligned} z_1 y_i &= 0, \\ z_1 y_{i-1} + z_2 y_i &= 0, \\ &\vdots \\ z_1 y_1 + \cdots + z_i y_i &= 0. \end{aligned} \tag{5-4}$$

We claim that $\Xi^i_{0,0}$ has exactly two free Z_i-orbits. These correspond to choosing either $z_1 \in \mathbb{C}^\times$ and $y_i = 0$, or $y_i \in \mathbb{C}^\times$ and $z_1 = 0$ in the first equation of (5-4). We claim that any point in $\Xi^i_{0,0}$ with $z_1 \neq 0$ is in

$$\mathbb{O}^i_L = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & & \ddots & 1 & \vdots \\ 0 & \cdots & \cdots & 0 & 0 \\ z_1 & \cdots & \cdots & z_i & 0 \end{bmatrix}, \tag{5-5}$$

with $z_j \in \mathbb{C}$ for $2 \le j \le i$. Any point in $\Xi^i_{0,0}$ with $y_i \in \mathbb{C}^\times$ is in

$$\mathbb{O}^i_U = \begin{bmatrix} 0 & 1 & \cdots & 0 & y_1 \\ 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & & \ddots & 1 & \vdots \\ 0 & \cdots & \cdots & 0 & y_i \\ 0 & \cdots & \cdots & 0 & 0 \end{bmatrix}, \tag{5-6}$$

with $y_j \in \mathbb{C}$ for $1 \le j \le i-1$. To verify this claim, note that if $z_1 \ne 0$ and $y_i = 0$, then $y_1 = 0$ and $y_2 = 0, \dots, y_{i-1} = 0$ by successive use of equations (5-4). The case $y_i \ne 0$ and $z_1 = 0$ is similar. An easy computation in linear algebra, as in the proof of Lemma 4.14 gives that Z_i acts freely on $\mathbb{O}^i_U$ and $\mathbb{O}^i_L$. We think of $\mathbb{O}^i_U$ as the "upper orbit" in $\Xi^i_{0,0}$ and $\mathbb{O}^i_L$ as the "lower orbit". Both orbits consist of regular elements of $\mathfrak{gl}(i+1)$ by Theorem 4.13.

Now, suppose that both $z_1 = 0 = y_i$ in (5-4). It is easy to see that such an element has a nontrivial isotropy group in Z_i containing the one-dimensional subgroup of matrices consisting of identity matrices with an element $c \in \mathbb{C}^\times$ inserted in the upper right corner. It does not belong to a Z_i-orbit of dimension i.

Thus, to analyze $\mathfrak{gl}(n)^{\mathrm{sreg}}_0$, we consider only the Z_i-orbits $\mathbb{O}^i_U, \mathbb{O}^i_L$. We can construct 2^{n-1} morphisms $\mathbf{\Gamma}_n = \Gamma_n^{a_1,a_2,\dots,a_{n-1}}$, where $\mathbb{O}^i_{a_i} = \mathbb{O}^i_U, \mathbb{O}^i_L$ for $1 \le i \le n-1$.

The following result follows immediately from Theorems 4.9 and 4.12 and Remark 4.10.

Theorem 5.2. *The nilfiber* $\mathfrak{gl}(n)^{\mathrm{sreg}}_0$ *contains* 2^{n-1} *A-orbits. On* $\mathfrak{gl}(n)^{\mathrm{sreg}}_0$, *the orbits of A are orbits of a free action of the algebraic group* $(\mathbb{C}^\times)^{n-1} \times \mathbb{C}^{n(n-1)/2-n+1}$.

The nilfiber has much more structure than Theorem 5.2 indicates, which we can see by considering an example of an A-orbit given as the image of a morphism $\mathbf{\Gamma}_n$ with $\mathbb{O}^i_{a_i} = \mathbb{O}^i_U$, $\mathbb{O}^i_L$ and its closure. Closure here means either closure in the Zariski topology in $\mathfrak{gl}(n)$ or in the Euclidean topology, since A-orbits are constructible sets these two different types of closure agree; see [Kostant and Wallach 2006a, Theorem 3.7]. We will abbreviate from now on

$$\mathbb{O}^i_{a_i} = a_i, \quad \mathbb{O}^i_L = L, \quad \mathbb{O}^i_U = U.$$

Example 5.3. Let us take our A-orbit in $\mathfrak{gl}(4)^{\mathrm{sreg}}_0$ to be the image of $\Gamma_4^{a_1,a_2,a_3}$ with $a_1 = L$, $a_2 = L$ and $a_3 = U$. For coordinates, let us take

$$\begin{array}{lll} z_1 \in \mathbb{C}^\times & & \text{for } \mathbb{O}^1_L, \\ z_2 \in \mathbb{C}^\times, & z_3 \in \mathbb{C} & \text{for } \mathbb{O}^2_L, \\ y_1, y_2 \in \mathbb{C}, & y_3 \in \mathbb{C}^\times & \text{for } \mathbb{O}^3_U. \end{array}$$

In these coordinates, we compute that $\operatorname{im}\Gamma_4^{L,L,U}$ is

$$\operatorname{im}\Gamma_4^{L,L,U} = \begin{bmatrix} 0 & 0 & 0 & y_3/(z_1z_2) \\ z_1 & 0 & 0 & y_2/z_2 - y_3z_3/z_2^2 \\ z_1z_3 & z_2 & 0 & y_1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{5-7}$$

We compute the closure as

$$\overline{\operatorname{im}\Gamma_4^{L,L,U}} = \begin{bmatrix} 0 & 0 & 0 & a_1 \\ a_2 & 0 & 0 & a_3 \\ a_4 & a_5 & 0 & a_6 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{5-8}$$

with $a_i \in \mathbb{C}$ for $1 \le i \le 6$. It is a nilradical of a Borel subalgebra that contains the standard Cartan subalgebra of diagonal matrices in $\mathfrak{gl}(4)$. The easiest way to see this is to note that the strictly lower triangular matrices in $\mathfrak{gl}(4)$ are conjugate to it by the permutation $\tau = (1432)$.

This example illustrates that the A-orbits in $\mathfrak{gl}(n)_0^{\text{sreg}}$ are essentially parametrized by prescribing whether or not the $i \times i$ cutoff of an element $x \in \mathfrak{gl}(n)_0$ has zeroes in its i-th column or zeroes in its i-th row. This is because for an $x \in \mathfrak{gl}(n)_0$ to be in the image of a morphism $\mathbf{\Gamma}_n = \Gamma_n^{a_1,a_2,\dots,a_{n-1}}$ with $a_i = L, U$, the i-th row or the i-th column of x_i must entirely consist of zeroes for each i by Proposition 4.3.

Contrast this with the following example of a matrix $x \in \mathfrak{gl}(n)_0$ each of whose cutoffs is regular, but that is not itself strongly regular.

Example 5.4. Consider $x \in \mathfrak{gl}(4)_0$ defined by

$$x = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & x_2 \\ 0 & 1 & 0 & x_3 \\ y_1 & 0 & 0 & 0 \end{bmatrix}, \tag{5-9}$$

where $x_2 \in \mathbb{C}^\times$, $y_1 \in \mathbb{C}^\times$ and $x_3 \in \mathbb{C}$. Both the 4-th column and row of this matrix have nonzero entries. Thus, this matrix cannot be in the image of a morphism $\mathbf{\Gamma}_n$ with $a_i = L, U$ and is not strongly regular. However, one can easily check that each cutoff of this matrix is regular, so that $x \in \mathfrak{gl}(4)_0 \cap S$. Thus, $\mathfrak{gl}(4)_0^{\text{sreg}}$ is a proper subset of $\mathfrak{gl}(4)_0 \cap S$. (One can also see that this matrix is not strongly regular directly by observing that $\mathfrak{z}_{\mathfrak{gl}(3)}(x_3) \cap \mathfrak{z}_{\mathfrak{gl}(4)}(x) \neq 0$.)

Example 5.3 demonstrates that although the A-orbits $\operatorname{im}\mathbf{\Gamma}_n$ may be complicated, their closures are relatively simple. In this example, the closure is a nilradical of a Borel subalgebra that contains the standard Cartan subalgebra of diagonal matrices in $\mathfrak{gl}(n)$. This is in fact the case in general.

Theorem 5.5. *Let $x \in \mathfrak{gl}(n)_0^{\mathrm{sreg}}$ and let $A \cdot x$ denote the A-orbit of x. Then $\overline{A \cdot x}$ is a nilradical of a Borel subalgebra in $\mathfrak{gl}(n)$ that contains the standard Cartan subalgebra of diagonal matrices. More explicitly, if the A-orbit is given by $\mathbf{\Gamma}_n = \Gamma_n^{a_1,a_2,\dots,a_{n-1}}$, where $a_i = U$ or L for $1 \le i \le n-1$, then $\overline{A \cdot x}$ is the set of matrices of the form*

$$\mathfrak{n}_{a_1,\dots,a_{n-1}} := \left\{ x : x_{i+1} = \begin{bmatrix} & b_1 \\ x_i & \vdots \\ & b_i \\ 0 & 0 \end{bmatrix} \right\} \qquad \textit{if } a_i = U, \textit{ or}$$

$$\mathfrak{n}_{a_1,\dots,a_{n-1}} := \left\{ x : x_{i+1} = \begin{bmatrix} & x_i & & 0 \\ b_1 & \cdots & b_i & 0 \end{bmatrix} \right\} \quad \textit{if } a_i = L$$

with $b_j \in \mathbb{C}$.

Proof. Let $x \in \mathfrak{gl}(n)_0^{\mathrm{sreg}}$. By Gerstenhaber's theorem [1958], it suffices to show the second statement of the theorem. Then $\overline{A \cdot x}$ is a linear space consisting of nilpotent matrices of dimension $n(n-1)/2$ and is clearly normalized by the diagonal matrices in $\mathfrak{gl}(n)$.

Suppose that $A \cdot x = \operatorname{im} \mathbf{\Gamma}_n$ with $a_i = U,\ L$. Since $A \cdot x$ is an irreducible variety of dimension $n(n-1)/2$, $\overline{A \cdot x} \subset \mathfrak{n}_{a_1,\dots,a_{n-1}}$ is an irreducible, closed subvariety of dimension $n(n-1)/2 = \dim \mathfrak{n}_{a_1,\dots,a_{n-1}}$, and therefore $\overline{A \cdot x} = \mathfrak{n}_{a_1,\dots,a_{n-1}}$. □

Remark 5.6. The set of strictly lower triangular matrices $\mathfrak{n}^-$ is the closure of the A-orbit $\Gamma_n^{L,\dots,L}$, and the set of strictly upper triangular matrices $\mathfrak{n}^+$ is the closure of the A-orbit $\Gamma_n^{U,\dots,U}$.

By Theorem 5.5, the A-orbits in $\mathfrak{gl}(n)_0^{\mathrm{sreg}}$ give rise to 2^{n-1} Borel subalgebras of $\mathfrak{gl}(n)$ that contain the diagonal matrices. Moreover, each of the nilradicals $\mathfrak{n}_{a_1,\dots,a_{n-1}}$ is conjugate to the strictly lower triangular matrices by a unique permutation in $\mathcal{S}_n$, the symmetric group on n letters. The A-orbits in $\mathfrak{gl}(n)_0^{\mathrm{sreg}}$ thus determine 2^{n-1} permutations. We now describe these permutations.

Theorem 5.7. *Let $\mathfrak{n}^-$ denote the strictly lower triangular matrices in $\mathfrak{gl}(n)$ and let $\mathfrak{n}_{a_1,\dots,a_{n-1}}$ be as in Theorem 5.5. Then $\mathfrak{n}_{a_1,\dots,a_{n-1}}$ is obtained from $\mathfrak{n}^-$ by conjugating by a permutation $\sigma = \tau_1\tau_2 \cdots \tau_{n-1}$, where $\tau_i \in \mathcal{S}_{i+1}$ is either the long element $w_{i,0}$ of $\mathcal{S}_{i+1}$ or the identity permutation, id_i. The τ_i are determined by the values of a_i as follows. Let $a_n = L$. Starting with $i = n-1$, we compare a_i and a_{i+1}. If $a_i = a_{i+1}$, then $\tau_i = id_i$, but if $a_i \neq a_{i+1}$, then $\tau_i = w_{0,i}$.*

The same procedure beginning with $a_n = U$ produces a permutation that conjugates the strictly upper triangular matrices $\mathfrak{n}^+$ into $\mathfrak{n}_{a_1,\dots,a_{n-1}}$.

Before proving Theorem 5.7, let us see it in action in Example 5.3. In that case the nilradical in (5-8) is $\mathfrak{n}_{L,L,U}$. Thus, according to Theorem 5.7, $\sigma = (13)(14)(23)$, the product of the long elements for $\mathcal{S}_3$ and $\mathcal{S}_4$. Notice that $\sigma = (1432)$, which is precisely the permutation that we observed conjugates the strictly lower triangular matrices in $\mathfrak{gl}(4)$ into $\mathfrak{n}_{L,L,U}$ in Example 5.3.

Proof of Theorem 5.7. Let $\pi_i : \mathfrak{gl}(n) \to \mathfrak{gl}(i)$ be the projection $\pi_i(x) = x_i$. For any subset $S \subset \mathfrak{gl}(n)$, we will denote by S_i the image $\pi_i(S)$.

Suppose that $L = a_n = a_{n-1} = \cdots = a_{i+1}$, but $a_i = U$. Conjugating $\mathfrak{n}^-$ by $\tau_i = w_{0,i}$ produces the nilradical $\mathrm{Ad}(\tau_i) \cdot \mathfrak{n}^-$ with $(\mathrm{Ad}(\tau_i) \cdot \mathfrak{n}^-)_{i+1} = \mathfrak{n}^+_{i+1}$. Thus, $(\mathfrak{n}_{a_1,\dots,a_{n-1}})_{i+1}$ and $(\mathrm{Ad}(\tau_i)\cdot\mathfrak{n}^-)_{i+1}$ now have the same $(i+1)$-st columns. We also note that the components of $\mathrm{Ad}(\tau_i)\cdot\mathfrak{n}^-$ and $\mathfrak{n}_{a_1,\dots,a_{n-1}}$ in $\mathfrak{gl}(i+1)^\perp$ also agree, since τ_i permutes the strictly lower triangular entries of the rows below the $(i+1)$-st row of $\mathfrak{n}^-$ amongst themselves. Now, we start the procedure again with $(\mathrm{Ad}(\tau_i)\cdot\mathfrak{n}^-)_{i+1}$ and $a_i = U$ and use induction. We note that conjugating $\mathrm{Ad}(\tau_i)\cdot\mathfrak{n}^-$ by a permutation in $\mathcal{S}_k$ with $k \le i+1$ leaves the component of $\mathrm{Ad}(\tau_i)\cdot\mathfrak{n}^-$ in $\mathfrak{gl}(i+1)^\perp$ unchanged. This proves the theorem. □

Remark 5.8. A related result is [Parlett and Strang 2008, Lemma 1, page 1736].

5b. ***General solution varieties $\Xi^i_{c_i,c_{i+1}}$ and counting A-orbits in $\mathfrak{gl}(n)^{\mathrm{sreg}}_c$.*** Now, we use our understanding of the nilpotent case to count A-orbits in the general case. Recall the definition of the solution variety $\Xi^i_{c_i,c_{i+1}}$ in Section 4a. We also recall some notation. Given $c \in \mathbb{C}^{n(n+1)/2}$, we write $c = (c_1, \dots, c_i, \dots, c_n)$ with $c_i = (z_1, \dots, z_i) \in \mathbb{C}^i$ and define a corresponding monic polynomial $p_{c_i}(t)$ with coefficients given by c_i; see (1-2). Recall that $J = J_1 \oplus \cdots \oplus J_r$, where $J_k \in \mathfrak{gl}(n_k)$, denotes the regular Jordan form that is the $i \times i$ cutoff of the matrix in (4-1). We now describe the Z_i-orbit structure of the variety $\Xi^i_{c_i,c_{i+1}}$ for any $c_i \in \mathbb{C}^i$ and $c_{i+1} \in \mathbb{C}^{i+1}$.

As in the nilpotent case, to understand $\Xi^i_{c_i,c_{i+1}}$ we must compute the characteristic polynomial of the matrix in (4-1).

Proposition 5.9. *The characteristic polynomial of the matrix in* (4-1) *is*

$$\begin{aligned}(w-t)\prod_{k=1}^{r}(\lambda_k - t)^{n_k} \\ + \sum_{j=1}^{r}\left((-1)^{n_j}\prod_{k=1,k\neq j}^{r}(\lambda_k - t)^{n_k}\sum_{l=0}^{n_j-1}\sum_{j'=1}^{n_j-l} z_{j,j'}y_{j,j'+l}(t-\lambda_j)^{n_j-1-l}\right).\end{aligned} \tag{5-10}$$

The proof of this proposition reduces to the case where J is a single Jordan block of eigenvalue λ. The case of a single Jordan block follows easily from the nilpotent case in Proposition 5.1 by a simple change of variables.

We need to understand the conditions that w, $z_{i,j}$, and $y_{i,j}$ must satisfy so that polynomial in (5-10) is equal to the monic polynomial $p_{c_{i+1}}(t)$. The first is easily determined by considering the trace of the matrix in (4-1). The values of the $z_{i,j}$ and the $y_{i,j}$ are directly related to the number of roots in common between the polynomials $p_{c_i}(t)$ and $p_{c_{i+1}}(t)$. Suppose that the polynomials $p_{c_i}(t)$ and $p_{c_{i+1}}(t)$ have j roots in common, where $1 \leq j \leq r$. Then we claim that $\Xi^i_{c_i,c_{i+1}}$ has precisely 2^j free Z_i-orbits. Consider the Jordan block corresponding to the eigenvalue λ_k. First, suppose that λ_k is a root of $p_{c_{i+1}}(t)$. Then Proposition 5.9 implies

$$z_{k,1} y_{k,n_k} = 0. \tag{5-11}$$

However, if λ_k is not a root of $p_{c_{i+1}}(t)$, then Proposition 5.9 gives

$$z_{k,1} y_{k,n_k} \in \mathbb{C}^\times. \tag{5-12}$$

As in the nilpotent case, (5-11) gives rise to two separate cases.

$$z_{k,1} \in \mathbb{C}^\times \quad \text{and} \quad y_{k,n_k} = 0 \tag{5-13}$$

and

$$y_{k,n_k} \in \mathbb{C}^\times \quad \text{and} \quad z_{k,1} = 0. \tag{5-14}$$

In case (5-13), we can argue using (5-10) that the coordinates $y_{k,i}$ for $1 \leq i \leq n_k$ can be solved uniquely as regular functions of $z_{k,1} \in \mathbb{C}^\times$ and $z_{k,2}, \dots, z_{k,n_k} \in \mathbb{C}$. In case (5-14), we can solve for $z_{k,i}$ as regular functions of $y_{k,n_k} \in \mathbb{C}^\times$ and $y_{k,i} \in \mathbb{C}$ for $1 \leq i \leq n_k - 1$. In the case of (5-12), we can take either the $z_{k,i}$ as coordinates that determine the $y_{k,i}$ or vice versa. For concreteness, we take $y_{k,i} = p_i(z_{k,1}, \dots, z_{k,n_k})$ to be regular functions of $z_{k,1} \in \mathbb{C}^\times$ and $z_{k,2}, \dots, z_{k,n_k} \in \mathbb{C}$.

Remark 5.10. The solutions in the cases of (5-11) and (5-12) are obtained by setting the derivatives of the polynomial in (5-10) up to order $n_p - 1$ evaluated at λ_p equal to the corresponding derivatives of the polynomial $p_{c_{i+1}}(t)$ evaluated at λ_p for $1 \leq p \leq r$. This produces r systems of linear equations. Each system involves only the coordinates $z_{p,k}$ and $y_{p,k}$ from the p-th Jordan block. This follows directly from the fact that the eigenvalues λ_s are all distinct. Each system can then be solved inductively using the fact that the coefficient of $(-1)^{n_p}(t-\lambda_p)^q \prod_{k=1,\, k\neq p}^r (\lambda_k - t)^{n_r}$ is given by the $(n-q)$-th row of the matrix product

$$\begin{bmatrix} z_{p,1} & z_{p,2} & \cdots & z_{p,n_p} \\ 0 & z_{p,1} & \ddots & \vdots \\ \vdots & & \ddots & z_{p,2} \\ 0 & \cdots & 0 & z_{p,1} \end{bmatrix} \cdot \begin{bmatrix} y_{p,1} \\ \vdots \\ \vdots \\ y_{p,n_p} \end{bmatrix}. \tag{5-15}$$

Recall that Z_i is the direct product $Z_i = Z_{J_1} \times \cdots \times Z_{J_r}$, with Z_{J_s} the centralizer of J_s. The adjoint action of Z_i on $\Xi^i_{c_i,c_{i+1}}$ is a diagonal action in which Z_{J_s} acts only on the columns and rows of an $x \in \Xi^i_{c_i,c_{i+1}}$ containing J_s. This observation allowed us to decompose a Z_i-orbit $\mathbb{O}$ into the product $\mathbb{O}_k \subset \mathbb{C}^{2n_k}$ of Z_{J_k}-orbits as in (4-15). If λ_k is a root of $p_{c_{i+1}}(t)$, then (5-11) gives rise to two free Z_{J_k}-orbits, an "upper" orbit $\mathbb{O}_{k,U}$ in the case of (5-14) and a "lower" orbit $\mathbb{O}_{k,L}$ in the case of (5-13). This is proved similarly to the nilpotent case. If on the other hand, λ_k is not a root of $p_{c_{i+1}}(t)$, and we have (5-12), then the vector

$$([z_{k,1}, \dots, z_{k,n_k}], [p_1(z_{k,1}, \dots, z_{k,n_k}), \dots, p_k(z_{k,1}, \dots, z_{k,n_k})]^T) \in \mathbb{C}^{2n_k} \tag{5-16}$$

is a free Z_{J_k}-orbit under the action of Z_{J_k} defined in (4-14). Thus, using the orbits $\mathbb{O}_{k,U}$ and $\mathbb{O}_{k,L}$ for $1 \le k \le j$, we can construct 2^j free Z_i-orbits in $\Xi^i_{c_i,c_{i+1}}$ by (4-15).

Now, using Theorem 4.13, we can construct $2^{\sum_{i=1}^{n-1} j_i}$ $\Gamma_n^{a_1,a_2,\dots,a_{n-1}}$ morphisms into $\mathfrak{gl}(n)_c^{\text{sreg}}$, where j_i is the number of roots in common to the monic polynomials $p_{c_i}(t)$ and $p_{c_{i+1}}(t)$. The following result follows immediately from Theorem 4.9 and Theorem 4.12 and Remark 4.10.

Theorem 5.11. *Let $c = (c_1, c_2, \dots, c_i, c_{i+1}, \dots, c_n) \in \mathbb{C}^{n(n+1)/2}$. Suppose there are $0 \le j_i \le i$ roots in common between the monic polynomials $p_{c_i}(t)$ and $p_{c_{i+1}}(t)$. Then the number of A-orbits in $\mathfrak{gl}(n)_c^{\text{sreg}}$ is exactly $2^{\sum_{i=1}^{n-1} j_i}$. Further, on $\mathfrak{gl}(n)_c^{\text{sreg}}$ the orbits of A are the orbits of a free algebraic action of the commutative, connected algebraic group $Z = Z_1 \times \cdots \times Z_{n-1}$ on $\mathfrak{gl}(n)_c^{\text{sreg}}$.*

Remark 5.12. A similar result is obtained in [Bielawski and Pidstrygach 2008]. See Remark 1.3 in the introduction.

Theorem 5.11 lets us identify exactly where the action of the group A is transitive on $\mathfrak{gl}(n)_c^{\text{sreg}}$. Let Θ_n be the set of $c \in \mathbb{C}^{n(n+1)/2}$ such that the monic polynomials $p_{c_i}(t)$ and $p_{c_{i+1}}(t)$ have no roots in common. From [Kostant and Wallach 2006a, Remark 2.16], it follows that $\Theta_n \subset \mathbb{C}^{n(n+1)/2}$ is Zariski principal open.

Corollary 5.13. *The action of A is transitive on $\mathfrak{gl}(n)_c^{\text{sreg}}$ if and only if $c \in \Theta_n$.*

Remark 5.14. We will see in the next section that $\mathfrak{gl}(n)_c^{\text{sreg}} = \mathfrak{gl}(n)_c$ for $c \in \Theta_n$. Thus, the fiber $\mathfrak{gl}(n)_c$ consists entirely of strongly regular elements.

Corollary 5.13 allows us to enlarge the set of generic matrices $\mathfrak{gl}(n)_\Omega$ studied by Kostant and Wallach.

5c. ***The new set of generic matrices $\mathfrak{gl}(n)_\Theta$.*** We can expand the set of matrices $\mathfrak{gl}(n)_\Omega$ studied by Kostant and Wallach by relaxing the condition that each cutoff is regular semisimple. More precisely, let $\sigma(x_i)$ denote the spectrum of $x_i \in \mathfrak{gl}(i)$, where x_i is viewed as an $i \times i$ matrix. We define a Zariski open subset of elements of $\mathfrak{gl}(n)$ by $\mathfrak{gl}(n)_\Theta = \{x \in \mathfrak{gl}(n) \mid \sigma(x_{i-1}) \cap \sigma(x_i) = \varnothing, 2 \le i \le n\}$. Clearly, $\mathfrak{gl}(n)_\Theta = \bigcup_{c \in \Theta_n} \mathfrak{gl}(n)_c$.

Theorem 5.15. *The elements of $\mathfrak{gl}(n)_\Theta$ are strongly regular and hence $\mathfrak{gl}(n)_c^{\text{sreg}} = \mathfrak{gl}(n)_c$ for $c \in \Theta_n$. Moreover, $\mathfrak{gl}(n)_\Theta$ is the maximal subset of $\mathfrak{gl}(n)$ for which the action of A is transitive on the fibers of Φ.*

Proof. If $p_{c_i}(t)$ and $p_{c_{i+1}}(t)$ are relatively prime polynomials, then we claim $\Xi^i_{c_i,c_{i+1}}$ is exactly one free Z_i-orbit. Indeed, in this case we only have the conditions (5-12) for $1 \le k \le r$. Thus, we can apply our observation in (5-16) to see that $\Xi^i_{c_i,c_{i+1}}$ is one free Z_i-orbit and hence consists of regular elements of $\mathfrak{gl}(i+1)$ by Theorem 4.13. Given $x \in \mathfrak{gl}(n)_c$ with $c \in \Theta_n$, we claim that $x \in \operatorname{im} \Gamma_n^{a_1,a_2,\dots,a_{n-1}}$ with $a_i = \Xi^i_{c_i,c_{i+1}}$ for $1 \le i \le n-1$. Indeed, $x_2 \in \Xi^1_{c_1,c_2}$ and is therefore regular. Thus, by Remark 4.4, there exists a $g_2 \in GL(2)$ such that $(\mathrm{Ad}(g_2) \cdot x)_3 = (\mathrm{Ad}(g_2) \cdot x_3) \in \Xi^2_{c_2,c_3}$. Now, suppose $x_{i+1} \in \mathrm{Ad}(GL(i)) \cdot \Xi^i_{c_i,c_{i+1}}$. Thus, $x_{i+1} \in \mathfrak{gl}(i+1)$ is regular and Remark 4.4 provides a $g_{i+1} \in GL(i+1)$ such that $(\mathrm{Ad}(g_{i+1}) \cdot x)_{i+2} = \mathrm{Ad}(g_{i+1}) \cdot x_{i+2} \in \Xi^{i+1}_{c_{i+1},c_{i+2}}$. By induction, $x_{j+1} \in \mathrm{Ad}(GL(j)) \cdot \Xi^j_{c_j,c_{j+1}}$ for any j in $1 \le j \le n-1$. Proposition 4.3 implies that $x \in \operatorname{im} \boldsymbol{\Gamma}_n$. Thus, $\mathfrak{gl}(n)_\Theta \subset \mathfrak{gl}(n)^{\mathrm{sreg}}$ by Theorem 4.9. The rest of the theorem follows from Corollary 5.13. □

Remark 5.16. The strictly upper triangular part of a matrix $x \in \mathfrak{gl}(n)_c$ where $c \in \Theta_n$ is determined by its strictly lower triangular part. This follows from the definition of the morphisms $\boldsymbol{\Gamma}_n$ and the fact that all of the $y_{k,i}$ can be solved uniquely as regular functions of the $z_{k,i}$ for $1 \le i \le n_k$ and $1 \le k \le r$.

Because elements of $\mathfrak{gl}(n)_\Theta$ are strongly regular, we have the following:

Corollary 5.17. *Let $x \in \mathfrak{gl}(n)_\Theta$. Then $x_i \in \mathfrak{gl}(i)$ is regular for all i.*

Using Corollary 5.13 and Theorem 5.11, we can directly generalize [Kostant and Wallach 2006a, Theorem 3.23] for the case of Θ_n.

Corollary 5.18. *For $c \in \Theta_n \subset \mathbb{C}^{n(n+1)/2}$, we have $\mathfrak{gl}(n)_c \cong Z_1 \times \cdots \times Z_{n-1}$ as algebraic varieties.*

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