On the definition of effects in fractional factorial designs

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Abstract

This paper simplifies a previous exposition of Rao's 1947 proof of his inequalities for orthogonal arrays. A key issue in the proof is the way in which one defines effects in a fractional factorial design. Here we replace the definition used in the earlier exposition with a simpler one, based on a more obvious interpretation of what Rao wrote and more in line with common practice, and show that it still leads to the same mathematical results. As in Rao's original paper, all designs are assumed to be unblocked. Two applications are given illustrating alias patterns in certain nonregular fractional designs, the second affording an opportunity to compare this approach with an alternative one due to Box and Wilson.

1 Introduction

How should one define a main effect or interaction in a fractional factorial design?

In a full factorial design, each main effect or interaction is defined as a set of contrasts in cell means [2]. In a fractional factorial design, according to Rao [14, page 135], each linear combination of cell means belonging to a given effect or interaction may be

> obtained from [the corresponding contrast in the full factorial] by retaining only the assemblies [cells] contained in the subset ... .

Thus, for example, we define the ABC interaction in a fraction S simply by restricting the ABC contrasts to S. While this *definition by restriction* is a natural one to use, it may destroy orthogonality, and may in particular destroy the property of being a contrast. That is, two contrasts which are orthogonal in the full factorial design may not be orthogonal when restricted to the fraction, and may even fail to be contrasts altogether. For this reason, the exposition of Rao's paper given in [1] used an altered definition, based on orthogonalization.

However, the altered definition departs from common statistical practice in dealing with regular fractional designs. The purpose of this paper is to redevelop the results of [1] using definition by restriction. The two definitions are compared in Remark 3.6 and in the example below. In order to give a self-contained presentation while staying brief, this note succinctly reviews certain points from [1] and refers to that paper for further information when possible.

Rao called his fractional designs *arrays*, later *orthogonal arrays*. In [14] he deals with so-called *symmetric* arrays, those in which each factor is measured at the same set of levels, and discusses *asymmetric* or *mixed-level* arrays in [17]. These papers are the source for his well-known inequalities concerning arrays of strength t, but these are a by-product of some deeper results which are also of statistical importance, summarized below as Theorem 3.4.

How we decide to define effects in a fractional factorial design is of importance in extending the concept of aliasing to non-regular fractions. In Section 4 we discuss two such applications, the second of which provides an opportunity to compare the present approach to aliasing with one due to Box and Wilson. The oft-stated assumption that certain interactions are absent is taken up in Section 5.

Example. Consider the familiar example of a 2^3 factorial design in which factor levels are coded 0 and 1, as given in Table 1. The main effect of A is described by the contrast

$$\mu(000)+\mu(011)-\mu(101)-\mu(110)+\mu(001)+\mu(010)-\mu(100)-\mu(111)$$

(coefficients in column A), where $\mu(s)$ is the expected response to treatment combination s. If we restrict ourselves to the regular fraction $S = \{000, 011, 101, 110\}$, the main effect of A in the fraction is usually defined by the contrast

$$\mu(000) + \mu(011) - \mu(101) - \mu(110)$$

(given by the first four rows of Table 1) and similarly for the other main effects and interactions. We simply restrict each contrast to S. However, using this approach we see that ABC is given by $[1, 1, 1, 1]'$ and is no longer a contrast when restricted to S. This is a result of defining the ABC interaction by restriction.

The alternative definition posed in [1] is designed to produce only contrasts. (A precise definition is given in Remark 3.6 below.) Following it through in the present example, the contrasts for the main effects and the two-factor interactions would be the same as usual, while the ABC interaction would need to be represented by a contrast that is orthogonal to all the lower-order ones. The only such contrast would be the trivial one given by the coefficients $[0, 0, 0, 0]$. This is certainly at variance with standard statistical practice described above. In fact, the relation $ABC = [1, 1, 1, 1]'$ is the defining relation $ABC = I$ of the fraction. $\square$

This is the motivation for rederiving the results in [1] using definition by restriction, as we do in Section 3. One might also ask

cell	I	A	B	C	AB	AC	BC	ABC
000	1	1	1	1	−1	−1	−1	1
011	1	1	−1	−1	1	1	−1	1
101	1	−1	1	−1	1	−1	1	1
110	1	−1	−1	1	−1	1	1	1
001	1	1	1	−1	−1	1	1	−1
010	1	1	−1	1	1	−1	1	−1
100	1	−1	1	1	1	1	−1	−1
111	1	−1	−1	−1	−1	−1	−1	−1

Table 1: The 2^3 factorial design and its contrasts.

what could be said about the fraction obtained by deleting only two cells of the 2^3 design, and in particular what aliasing patterns would emerge. We answer this in Section 4.1.

Notation and basic definitions. We follow the notation and definitions given in [1]. In particular, the cardinality of a set E is denoted by $|E|$, and the empty set by $\emptyset$. The integers modulo n are denoted by $\mathbb{Z}_n$, the real numbers by $\mathbb{R}$, and the real-valued functions on the set T by $\mathbb{R}^T$. We let 1 denote the function which is constantly 1. Given a finite set T (for us, the set of treatment combinations), $\mathbb{R}^T$ is a Euclidean space with inner product

$$(u, v) = \sum_{s \in T} u(s)v(s) \tag{1}$$

for $u, v \in \mathbb{R}^T$ and norm $\|v\| = \sqrt{(v, v)}$. If we fix an ordering of the elements of T, we may view u and v as ordinary column vectors in the Euclidean space $\mathbb{R}^g$, where $g = |T|$. The function 1 is then a column of g 1's, and formula (1) is the ordinary dot product.

Other notation is introduced as needed.

2 Effects in the full factorial

In this section we give a brief restatement of [1, Section 3], a formulation of the definition of main effects and interactions in a full factorial design according to Bose [2].

Let $T = A_1 \times \cdots \times A_n$ be the set of treatment combinations in an $s_1 \times \cdots \times s_n$ factorial, where A_i indexes the levels of factor i and $s_i = |A_i|$. Contrasts in cell means $\mu(s)$ are expressions of the form

$$\sum_{s \in T} c(s)\mu(s)$$

where $\sum_{s \in T} c(s) = 0$. Which main effect or interaction a contrast belongs to is determined entirely by the coefficients $c(s)$. We may refer to these functions $c \in \mathbb{R}^T$ as *contrast functions* or *contrast vectors*, or (by abuse of language) as *contrasts*.

Any blocking (or partition) $\mathcal{C}$ of T determines a subspace $U_{\mathcal{C}} \subset \mathbb{R}^T$ of dimension $|\mathcal{C}| - 1$ consisting of the contrasts between the blocks of $\mathcal{C}$. These are all the contrast functions which are constant on the blocks of $\mathcal{C}$.

As r ranges over A_i, the sets

$$A_1 \times \cdots \times A_{i-1} \times \{r\} \times A_{i+1} \times \cdots \times A_n$$

form a blocking $\mathcal{A}_i$ of T consisting of s_i blocks of equal size. The contrasts between these blocks define the main effect of factor i. The set of such contrasts is then

$$U_i = U_{\mathcal{A}_i}.$$

We define $\mathcal{A}_i \vee \mathcal{A}_j$ to be the partition of T whose blocks are formed by intersecting a block of $\mathcal{A}_i$ with a block of $\mathcal{A}_j$. For $i < j$ the blocks of $\mathcal{A}_i \vee \mathcal{A}_j$ are thus sets of the form

$$A_1 \times \cdots \times A_{i-1} \times \{r\} \times A_{i+1} \times \cdots \times A_{j-1} \times \{s\} \times A_{j+1} \times \cdots \times A_n$$

where $r \in A_i$ and $s \in A_j$. The contrast functions belonging to the ij-interaction are defined to be the elements of $U_{\mathcal{A}_i \vee \mathcal{A}_j}$ which are orthogonal to both U_i and U_j. They form a subspace which we denote U_{ij}.

In general, for any nonempty subset $I \subset \{1, \ldots, n\}$ the factors $i \in I$ determine the blocking $\vee_{i \in I} \mathcal{A}_i$ of T. Its blocks are formed by taking intersections of blocks, one from each $\mathcal{A}_i$, $i \in I$, and are subsets of T of the form $B_1 \times \cdots \times B_n$ where for fixed elements $r_i \in A_i$ we have

$$\begin{aligned} B_i &= \{r_i\}, \quad i \in I, \\ &= \;A_i, \quad\; i \notin I. \end{aligned}$$

For $\emptyset \neq I \subset \{1, \ldots, n\}$ we define the subspaces U_I inductively as

$$U_I = \{c \in U_{\mathcal{C}} : c \perp U_J \text{ for all } J \subsetneqq I\},$$

where $\mathcal{C} = \vee_{i \in I} \mathcal{A}_i$ and where $U_\emptyset$ is the subspace of constant functions. For nonempty I, the subspace U_I is the set of contrast functions belonging to the interaction between the factors listed in the set I. As noted in [1], this is a modernized version of the definitions given by Bose [2, page 110]. The next result summarizes the basic facts of the analysis of variance.

Theorem 2.1. a. *If $I \neq J$ then $U_I \perp U_J$.*

b. $\mathbb{R}^T = \oplus_I U_I$, *the (orthogonal) sum running over all subsets I of $\{1, \dots, n\}$.*

c. $\dim U_I = \prod_{i \in I} (s_i - 1)$.

The statement in (b) separates all linear functions of cell means into the grand mean and the subsets of contrasts corresponding to main effects and interactions. Item (c) gives the usual formula for the degrees of freedom for each main effect and interaction, where we agree to define the empty product to equal 1.

3 Effects in a fractional factorial

Let $S \subset T$, and let $\hat{U}_I$ be the restriction of all the functions in U_I to the fraction S. As above, U_I represents a main effect or interaction, and using the principle of definition by restriction we agree to call $\hat{U}_I$ the corresponding main effect or interaction in the fraction. Restated in our terms, Rao's primary goal in [14] was to recover as much of Theorem 2.1 as possible in the fraction S, the sets $\hat{U}_I$ naturally replacing U_I.

To make this more precise, let $r : \mathbb{R}^T \to \mathbb{R}^S$ be the map which takes each function $v \in \mathbb{R}^T$ to its restriction to S. Then

$$\hat{U}_I = r(U_I),$$

and we need to investigate the properties of the map r.

For the moment we may let T be an arbitrary finite set and let $S \subset T$. Then r is a linear transformation and is surjective.

Given any partition $\mathcal{C}$ of T, let

$$W_{\mathcal{C}} = \{w \in \mathbb{R}^T : w \text{ is constant on each block of } \mathcal{C}\}.$$

Then $w \in W_{\mathcal{C}}$ iff $w^{-1}(y)$ is a union of blocks of $\mathcal{C}$ for any $y \in \mathbb{R}$.

As in [1], we define π to be the uniform probability measure on T, so that $\pi(A) = |A|/|T|$, and we write $\perp\!\!\!\perp$ to denote independence with respect to π. Thus

$$\begin{aligned} A \perp\!\!\!\perp B \quad &\text{iff} \quad \pi(A \cap B) = \pi(A)\pi(B) \\ &\text{iff} \quad |A \cap B| = |A||B|/|T|. \end{aligned}$$

If $S \subset T$ and $\mathcal{C}$ is a partition of T, we say that

$$\begin{aligned} S \perp\!\!\!\perp \mathcal{C} \quad &\text{iff} \quad S \perp\!\!\!\perp B \quad \text{for every block } B \in \mathcal{C} \\ &\text{iff} \quad S \perp\!\!\!\perp B \quad \text{whenever B is a union of blocks of } \mathcal{C}. \end{aligned}$$

Lemma 3.1. *Let T be finite, $S \subset T$, and $\mathcal{C}$ a partition of T such that $S \perp\!\!\!\perp \mathcal{C}$. If $w \in W_{\mathcal{C}}$, then $\sum_{s\in S} w(s) = \pi(S) \sum_{s\in T} w(s)$.*

Proof. Let the range of w be $\{y_1, \dots y_k\}$, and let $A_j = w^{-1}(y_j)$. Then A_j is a union of blocks of $\mathcal{C}$, and so

$$\begin{aligned} \sum_{s\in S} w(s) &= \sum_{j=1}^{k} \sum_{s \in S\cap A_j} w(s) = \sum_{j=1}^{k} y_j |S \cap A_j| \\ &= \sum_{j=1}^{k} y_j \frac{|S||A_j|}{|T|} = \pi(S) \sum_{j=1}^{k} y_j |A_j| = \pi(S) \sum_{s\in T} w(s). \end{aligned}$$

$\square$

Proposition 3.2. *Let T be finite, $S \subset T$, and $\mathcal{C}$ a partition of T such that $S \perp\!\!\!\perp \mathcal{C}$. Then the restriction map r*

a. *preserves orthogonality on $W_{\mathcal{C}}$. More precisely, if $u, v \in W_{\mathcal{C}}$ then $r(u) \perp r(v)$ (in $\mathbb{R}^S$) iff $u \perp v$ (in $\mathbb{R}^T$).*

b. *maps $W_{\mathcal{C}}$ one-to-one into $\mathbb{R}^S$.*

Proof. Applying the lemma to the pointwise product $w = uv$, we see that

$$(r(u), r(v)) = \pi(S)(u, v) \tag{2}$$

for all $u, v \in W_{\mathcal{C}}$ (inner products in $\mathbb{R}^S$ and $\mathbb{R}^T$, respectively). Thus $r(u) \perp r(v)$ iff $u \perp v$, as claimed. Moreover, from (2) we have in particular that $\|u\|^2 = \|r(u)\|^2/\pi(S)$ (norms respectively in $\mathbb{R}^T$ and $\mathbb{R}^S$). But then $u = 0$ if (and only if) $r(u) = 0$, and so r, viewed as a linear transformation on the vector space $W_{\mathcal{C}}$, has trivial nullspace. In other words, r is one-to-one on $W_{\mathcal{C}}$, $\square$

Assume now that $T = A_1 \times \cdots \times A_n$, and let $S \subset T$.

Definition 3.3. *S has* strength t *if, for every $I \subset \{1, \ldots, n\}$ of size t,*

$$S \perp\!\!\!\perp \vee_{i \in I} \mathcal{A}_i.$$

It can be shown [1, Corollary 5.2] that this is equivalent to the usual definition of strength in a mixed-level orthogonal array. The definition given above is more convenient for the present application.

We are now in a position to recapture as much as possible of Theorem 2.1 in the fraction S. The following results correspond to Theorem 6.1 and Corollaries 6.2 and 6.3 of [1], with $\hat{U}_I$ replacing U_I^S. As before, we use $[x]$ to denote the greatest integer not exceeding x, and $\oplus$ to denote orthogonal sums.

Theorem 3.4. *Let $T = A_1 \times \cdots \times A_n$, and let $S \subset T$ be a fraction of strength t.*

a. *Let $I, J \subset \{1, \ldots, n\}$ with $|I \cup J| \leq t$. If $I \neq J$, then $\hat{U}_I \perp \hat{U}_J$.*

b. *If $e = [t/2]$ and $\mathcal{E}$ is the family of all $I \subset \{1, \ldots, n\}$ of size at most e, then*

$$\mathbb{R}^S \supset \oplus_{I \in \mathcal{E}} \hat{U}_I.$$

If in addition t is odd then

$$\mathbb{R}^S \supset \oplus_{I \in \mathcal{E} \cup \mathcal{F}} \hat{U}_I,$$

where $\mathcal{F}$ is an intersecting family of subsets of size $e+1$ of $\{1, \ldots, n\}$.

c. $\dim \hat{U}_I = \prod_{i \in I}(s_i - 1)$ *for $|I| \leq t$.*

($\mathcal{F}$ is an *intersecting family* if $I \cap J \neq \emptyset$ for all $I, J \in \mathcal{F}$.)

Proof. (a) This follows from Theorem 2.1(a) and Proposition 3.2(a) by taking $\mathcal{C} = \vee_{i \in I \cup J} \mathcal{A}_i$.

(b) These statements follow from part (a) as in [1].

(c) Note that for any I, the linear transformation r maps U_I onto $\hat{U}_I$ and that $U_I \subset W_{\mathcal{C}}$ where $\mathcal{C} = \vee_{i \in I} \mathcal{A}_i$. But if $|I| \leq t$, then Proposition 3.2(b) shows that r is an isomorphism of U_I, so that $\dim \hat{U}_I = \dim U_I$, and the conclusion follows from Theorem 2.1(c). $\square$

Counting dimensions in Theorem 3.4(b) is what gives us Rao's inequalities [1, Corollaries 6.4 and 6.5].

We say that U_I and U_J are *completely aliased* in the fraction S if $\hat{U}_I = \hat{U}_J$, *unaliased* in S if $\hat{U}_I \perp \hat{U}_J$, and *partially aliased* in S otherwise. For example, Theorem 3.4(a) says that U_I and U_J are unaliased in S as long as the number of factors listed in $I \cup J$ does not exceed t, the strength of S. We say that S has *resolution* R if, for each p, every p-factor effect is unaliased with every effect having fewer than $R - p$ factors [3].

Corollary 3.5. *If S has strength t then it has resolution $t + 1$.*

This result is a direct consequence of Theorem 3.4(a). It is stated as Corollary 8.1 in [1], and its proof is given there along with further discussion.

Remark 3.6. The subspaces U_I^S of [1] are defined as follows. Every partition $\mathcal{C}$ of T induces a partition $\mathcal{C}S$ of S by

$$\mathcal{C}S = \{C \cap S : C \in \mathcal{C}\}.$$

The subspaces $U_I^S \subset \mathbb{R}^S$ are now defined exactly as the subspaces $U_I \subset \mathbb{R}^T$ were in Section 2, with $\mathcal{A}_i S$ replacing $\mathcal{A}_i$ everywhere. This is spelled out in more detail in [1, Section 4]. The main point is that the set U_I^S will always consist of contrasts, for every $I \subset \{1, \dots, n\}$, although it may contain only the trivial contrast 0.

We seem to have two ways of defining effects in a fractional factorial experiment that lead to the same theorems. This may be explained by the following fact, whose proof is omitted:

Proposition 3.7. *If the fraction S has strength $t \geq 1$, then $U_I^S = \hat{U}_I$ whenever $|I| \leq t$. In particular, $\hat{U}_I$ consists of contrasts whenever $|I| \leq t$.*

The second statement of the proposition also follows directly from Proposition 3.2(a).

4 Some applications

In this section we give brief applications of the approach we have followed in defining the effects in a fractional factorial design.

4.1 Aliasing in deletion designs

Consider the regular fractions of a 2^3 factorial experiment with defining relation $I = A = BC$. There are four such fractions – for example, that given by $\{(000), (011)\}$ (the first two cells in Table 1). If we restrict the columns of Table 1 to one of these fractions (say the first two lines), we see that the columns A, BC, and ABC will be equal to I except possibly for a scale factor of -1 and consequently will not be contrasts, while the other columns will be orthogonal to these (and therefore contrasts) and will be equal to each other except again possibly for a scalar factor of -1. The usual group theory keeps track of this for us: the full defining relation is $I = A = BC = ABC$, and by multiplying this by B we get the aliases $B = AB = C = AC$. We recognize $H = \{I, A, BC, ABC\}$ as a subgroup of the effects group, with coset $BH = \{B, AB, C, AC\}$.

Now consider in particular the fraction obtained by *deleting* one of these regular fractions, say the cells (000) and (011). Covering up the first two rows of Table 1, we see that each of the columns I, A, BC, and ABC is still orthogonal to each of the columns B, C, AB, and AC, and that the latter four vectors are still contrasts. However, the vectors in each set are not scalar multiples of each other. Rather, each set has rank 3, so that the vectors of each set are connected by a linear relation. That is, the corresponding effects are partially aliased in the given fraction.

We should have expected these results, since *the remaining six cells are the union of three other fractions sharing the same defining relation.* Any two effects which are orthogonal (unaliased) in one of these fractions must be orthogonal in the other three as well, and when these vectors are stacked up on each other, this orthogonality is maintained. On the other hand, stacking three of the four fractions will typically destroy the property of being scalar multiples (complete aliasing). At the same time, eight vectors in $\mathbb{R}^6$ must exhibit some linear dependence, and in this case each set of four vectors has rank 3.

This is a particular case of a fraction S of size $2^n - 2$ (a "delete 2" design) formed by deleting two points from a 2^n factorial design. (Deletion designs are studied in [7], [8], and [12], and are actually discussed briefly by Rao himself in an early paper [16, p. 84].) It is not hard to show that *any* two points in a 2^n design form a regular fraction – that is, one determined by $n-1$ independent defining words, and constituting the solution set of $n-1$ independent linear equations in n variables over $\mathbb{Z}_2$. Thus, fixing the two deleted points, the fraction S itself is the union of the other regular fractions sharing the same defining relation. Again, the main effects and interactions, when restricted to S, will separate into two mutually orthogonal sets, a subgroup H of the effects group and a single coset – the *same* subgroup and coset which define the alias relations of the deleted fraction.

Deleting a regular fraction of size 2^k will result similarly in 2^k mutually orthogonal sets, namely a subgroup H and its cosets. Computer experiments [6] on 2^n-2, 2^n-4 and 2^n-8 designs with $n \leq 6$ and where the deleted fraction is regular indicate that the rank of each alias set will always be $|H|-1$.

We can similarly determine the alias structure of fractions of s^n experiments which are formed by deleting one or more regular fractions sharing the same defining relation. Here all factors are measured at s levels, where s is a prime or prime power. In the usual way, a defining relation with k independent words will determine a fraction of size s^{n-k} – in fact, s^k disjoint fractions, all having the same alias structure [9, Section 21.1]. (Each is generated by k independent words, and is the solution set of a system of k independent linear equations over the finite field $GF(s)$. Together the fractions form the blocks of a confounded design.)

A fractional design S formed by deleting one or more such fractions is the union of the remaining fractions, and the effects in S are defined by "stacking" the corresponding effects from the component fractions. Thus, from the set of s^k fractions described above pick any m to be retained, say $S_1, \dots, S_m$, and consider the fraction S of the form

$$S = S_1 \cup \cdots \cup S_m.$$

Then two effects F and G (main effects or interactions) which are unaliased (orthogonal) in any one of the fractions S_i are unaliased in all of them, and since

$$(u,v)_S = \sum_{s \in S} u(s)v(s) = \sum_{i=1}^{m} \sum_{s \in S_i} u(s)v(s) = \sum_{i=1}^{m} (u,v)_{S_i},$$

they are also unaliased in S. Thus *a nonregular fraction S which is the union of regular fractions S_i having the same defining relation will have the same alias sets as the components S_i.* However, for the fraction S the effects in a given alias set will *not* be completely aliased in general.

4.2 Aliasing in Plackett-Burman designs

As we see in the previous example, Rao's definition of effects in a fractional factorial design leads naturally to a particular approach to the study of aliasing. A rather different approach is that introduced by Box and Wilson [4], who identify aliasing with bias due to lack of fit. Lin and Draper [11] apply the Box-Wilson approach to compute the alias relations between main effects and two-factor interactions in a 12-run, 11-factor Plackett-Burman design. Their design is obtained by cyclically permuting the 11×1 column vector $[1, -1, 1, -1, -1, -1, 1, 1, 1, -1, 1]'$ and then adding a row of -1's, resulting in the design given in columns A through K of Table 2. (This starting vector is taken from their equation (2.1) rather than from the design they list in their Table 1, which is actually not the one they analyze.) The interaction columns AB through JK may be computed by componentwise multiplication of the corresponding main effects contrasts, since this is true in the full (2^{11}) factorial and since restriction respects pointwise multiplication.

Let $\mathbf{Y}$ denote the vector of responses to the 12 runs in this design. The Box-Wilson approach posits a model of the form

$$\mathrm{E}(\mathbf{Y}) = \mathbf{X}\boldsymbol{\beta} + \mathbf{X}_1\boldsymbol{\beta}_1 \tag{3}$$

for some matrices $\mathbf{X}$ and $\mathbf{X}_1$. Let $\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$. Under the assumption that $\boldsymbol{\beta}_1 = \mathbf{0}$ in the model (3), $\hat{\boldsymbol{\beta}}$ is the usual least squares estimator of $\boldsymbol{\beta}$ and is unbiased. However, without this assumption we have

$$\mathrm{E}(\hat{\boldsymbol{\beta}}) = \boldsymbol{\beta} + \mathbf{A}\boldsymbol{\beta}_1, \tag{4}$$

where $\mathbf{A} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}_1$ is called the *alias matrix* or, for obvious reasons, the *bias matrix* of the design. What $\mathbf{A}$ tells us about aliasing is less obvious.

run	*1*	A	B	C	D	E	F	G	H	I	J	K	AB	AC	$\dots$	JK
1	1	1	−1	1	−1	−1	−1	1	1	1	−1	1	−1	1	...	−1
2	1	−1	1	−1	−1	−1	1	1	1	−1	1	1	−1	1	...	1
3	1	1	−1	−1	−1	1	1	1	−1	1	1	−1	−1	−1	...	−1
4	1	−1	−1	−1	1	1	1	−1	1	1	−1	1	1	1	...	−1
5	1	−1	−1	1	1	1	−1	1	1	−1	1	−1	1	−1	...	−1
6	1	−1	1	1	1	−1	1	1	−1	1	−1	−1	−1	−1	...	1
7	1	1	1	1	−1	1	1	−1	1	−1	−1	−1	1	1	...	1
8	1	1	1	−1	1	1	−1	1	−1	−1	−1	1	1	−1	...	−1
9	1	1	−1	1	1	−1	1	−1	−1	−1	1	1	−1	1	...	1
10	1	−1	1	1	−1	1	−1	−1	−1	1	1	1	−1	−1	...	1
11	1	1	1	−1	1	−1	−1	−1	1	1	1	−1	1	−1	...	−1
12	1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	1	1	...	1

Table 2: The 12-run Plackett-Burman design from Lin and Draper [11, (2.1)]. Columns A through K give the contrasts for main effects in this fraction, and also encode its 12 cells or treatment combinations. If we code the levels by 1 and 0 instead of 1 and -1, then the first run is the cell (10100011101).

Lin and Draper take $\mathbf{X}$ to be composed of the 12 columns 1 through K of Table 2, and $\mathbf{X}_1$, the 55 columns AB through JK. They assume the model (3) and, in addition, that all interactions of three or more factors are absent [11, page 149]. The resulting 55×12 alias matrix A is given in their Table 3, with the column corresponding to the constant term omitted (it consists of zeros).

Let us approach the aliasing problem from the point of view we have discussed previously. First, since the design has strength 2, Proposition 3.7 and Theorem 3.4(a) guarantee that the two-factor interactions are still represented by contrasts in this fraction and that the interaction of factors i and j will be orthogonal to (unaliased with) the main effects of factors i and j. What more can be said?

The interaction vectors belong to $\mathbb{R}^{12}$. But the vectors 1 through K are already a basis for $\mathbb{R}^{12}$ – in fact, an orthogonal basis – and either by least squares or by orthogonality we see that a vector $\mathbf{v} \in \mathbb{R}^{12}$ has the coefficients $(1/12)\mathbf{X}'\mathbf{v}$ with respect to this basis. These coefficients form the rows of the matrix $(1/12)\mathbf{X}'\mathbf{X}_1$ – precisely the alias matrix $\mathbf{A}$ given above. Thus, at least in this example, the alias matrix lists the coefficients of linear combinations that represent two-factor contrasts in terms of main-effects contrasts. In particular, Table 3 of [11] shows that the AB-interaction is partially aliased with all the main effects except A and B.

The fact that both approaches give rise to the same answer is in some sense a coincidence, since they arise from very different assumptions. As expressed in [13, page 95], the Box-Wilson definition of aliasing is model-dependent, while the present approach (arising from Rao) assumes only a cell-means model and, moreover, has nothing to do with bias. A clear analysis of the relation of the two conceptions of aliasing presented here would certainly be a valuable addition to the literature.

Definition by restriction happens to have a convenient property hinted at in the above example. We know [10] that the pointwise (or componentwise) product cd of $c \in U_I$ and $d \in U_J$ belongs to $U_{I \cup J}$, and we may use this to generate a full set of contrasts for all interactions from the main-effects contrasts. But since $r(cd) = r(c)r(d)$ (restriction respects pointwise products), we may generate interactions in a fraction the same way, rather than having to construct the contrasts in the full factorial first, which of course have many more components. (In the language of Section 3, the map r is not just a linear transformation but an algebra homomorphism.) That is how the interaction columns of Table 2 were computed.

5 The assumption of absent effects

The assumption that all interactions of sufficiently high order are absent is necessary in the Box-Wilson approach to aliasing. It determines the number of parameters in a given fractional factorial design, while the design matrix $\mathbf{X}$ determines which parameters represent main effects or interactions of particular factors.

Using a different approach, Dey and Mukerjee [5, Section 2.3] choose to retain all the cell means of a full factorial as parameters in a fractional design, so that all effects can continue to be defined as they are in the full model. However, their choice obviously yields an overparametrized model, and as a consequence, certain parametric functions of the cell means become inestimable. They are thus forced to impose the assumption of absent effects to restore estimability.

The present approach retains only those means $\mu(s)$ belonging to cells s of the observed fraction. However, effects in the fraction are not defined by the design matrix but by restricting contrasts to the fraction.

There is one type of contrast $\sum_{s\in T} c(s)\mu(s)$ which does not need any special definition when restricted to a fraction $S \subset T$, namely one in which $c(s) = 0$ for all cells $s \notin S$. In this case the contrast is computed entirely from cells in the fraction. Dey and Mukerjee [5, Section 8.2] use this in a novel way to justify assuming interactions are absent, at least in the context of regular fractions. Their idea is as follows.

In a symmetric factorial design, fix a regular fraction, and let $\mathbf{b}$ be a word (or *pencil*) that is not a defining word of the fraction. Dey and Mukerjee construct a contrast function c that vanishes outside the fraction and that will belong to the $\mathbf{b}$ effect if all the aliases of $\mathbf{b}$ are absent. More precisely, from a contrast function d belonging to $\mathbf{b}$ (in the sense of their equation (8.2.3)) they construct a contrast function c (equation (8.2.13)) which (i) vanishes outside the fraction and (ii) is such that $\sum_{s\in T} c(s)\mu(s) = \sum_{s\in T} d(s)\mu(s)$ for all μ whenever the effects aliased with $\mathbf{b}$ are absent. Thus, *if these aliases are assumed absent*, then the contrasts belonging to $\mathbf{b}$ are *automatically* computable from only the cells of the fraction. In other words, the assumption that the aliased effects are absent means that the effect of $\mathbf{b}$ may be defined naturally in the fraction. Definition by restriction is then unnecessary, but at the price of introducing an assumption on the model.

The assumption that all interactions of sufficiently high order are absent does occur in Rao's original paper [14, pp.134-136], but it is not needed to derive his inequalities or Theorem 3.4. Indeed, in a subsequent exposition [15] Rao omits it altogether. In practice one sometimes makes the assumption in order to interpret the analysis of variance unambiguously, or one may simply report whatever aliasing occurs. That is the only role it plays in this approach.

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