SEMI-ISOTOPIC KNOTS

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ABSTRACT. A *knot* is a possibly wild simple closed curve in S^3. A knot J is *semi-isotopic* to a knot K if there is an annulus A in $S^3 \times [0,1]$ such that $A \cap (S^3 \times \{0,1\}) = \partial A = (J \times \{0\}) \cup (K \times \{1\})$ and there is a homeomorphism $e : S^1 \times [0,1) \to A - (K \times \{1\})$ such that $e(S^1 \times \{t\}) \subset S^3 \times \{t\}$ for every $t \in [0,1)$.

Theorem. *Every knot is semi-isotopic to an unknot.*

1. INTRODUCTION

We fix some notation and terminology. Let $I = [0,1]$. A *knot* is the image of an embedding $S^1 \to S^3$. If the composition of this embedding with some homeomorphism of S^3 is a piecewise linear embedding, then the knot is tame. Otherwise, it is wild. An annulus is a space that is homeomorphic to $S^1 \times I$.

Knots J and K are *ambiently isotopic* if there is a level-preserving homeomorphism $h : S^3 \times I \to S^3 \times I$ such that $h(x,0) = (x,0)$ and $h(J \times \{1\}) = K \times \{1\}$. Note that for such an h, $h(J \times \{0\}) = J \times \{0\}$ and $h(J \times \{1\}) = K \times \{1\}$. Of course, classical knot theory is the study of ambient isotopy classes of tame knots in S^3.

Knots J and K are *(non-ambiently) isotopic* if there is a level-preserving embedding $e : J \times I \to S^3 \times I$ such that $e(J \times \{0\}) = J \times \{0\}$ and $e(J \times \{1\}) = K \times \{1\}$.

Observe that every knot that pierces a tame disk is isotopic to an unknot. The sequence of pictures in Figure 1 suggests a proof of this observation.

The *Bing sling* (Figure 2) [2] is a wild knot that pierces no disk. It is not known whether the Bing sling is isotopic to an unknot.

The following conjecture is well-known.

Conjecture. *Every knot is isotopic to an unknot.*

A knot J is *semi-isotopic* to a knot K if there is an annulus A in $S^3 \times I$ such that $\partial A = (J \times \{0\}) \cup (K \times \{1\})$ and there is a homeomorphism $e : S^1 \times [0,1) \to A - (K \times \{1\})$ such that $e(S^1 \times \{t\}) \subset S^3 \times \{t\}$ for every $t \in [0,1)$. Note that e may not extend continuously to homeomorphism from $S^1 \times [0,1]$ onto A.

The main result of this paper is:

Theorem. *Every knot is semi-isotopic to an unknot.*

Thus, the Bing sling is semi-isotopic to an unknot.

Knots J and K are *(topologically) concordant* or *I-equivalent* if there is an annulus A in $S^3 \times I$ such that $A \cap (S^3 \times \{0,1\}) = \partial A = (J \times \{0\}) \cup (K \times \{1\})$.

Note that: *Isotopic* $\Rightarrow$ *semi-isotopic* $\Rightarrow$ *concordant.*

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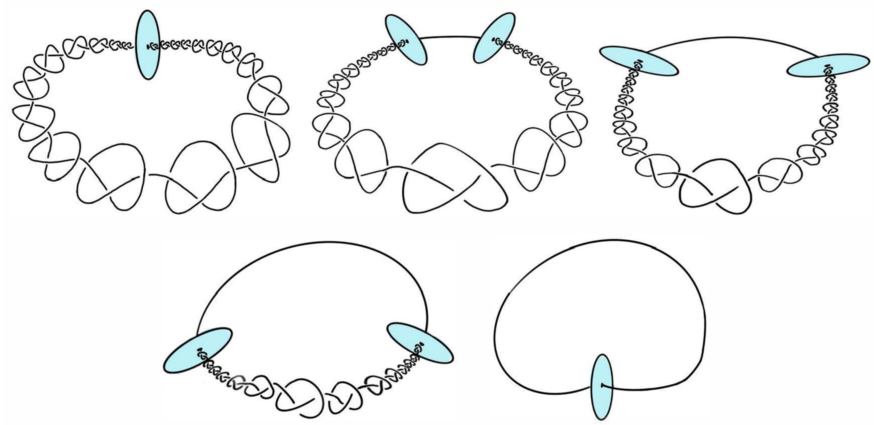

FIGURE 1. An isotopy of a knot to an unknot

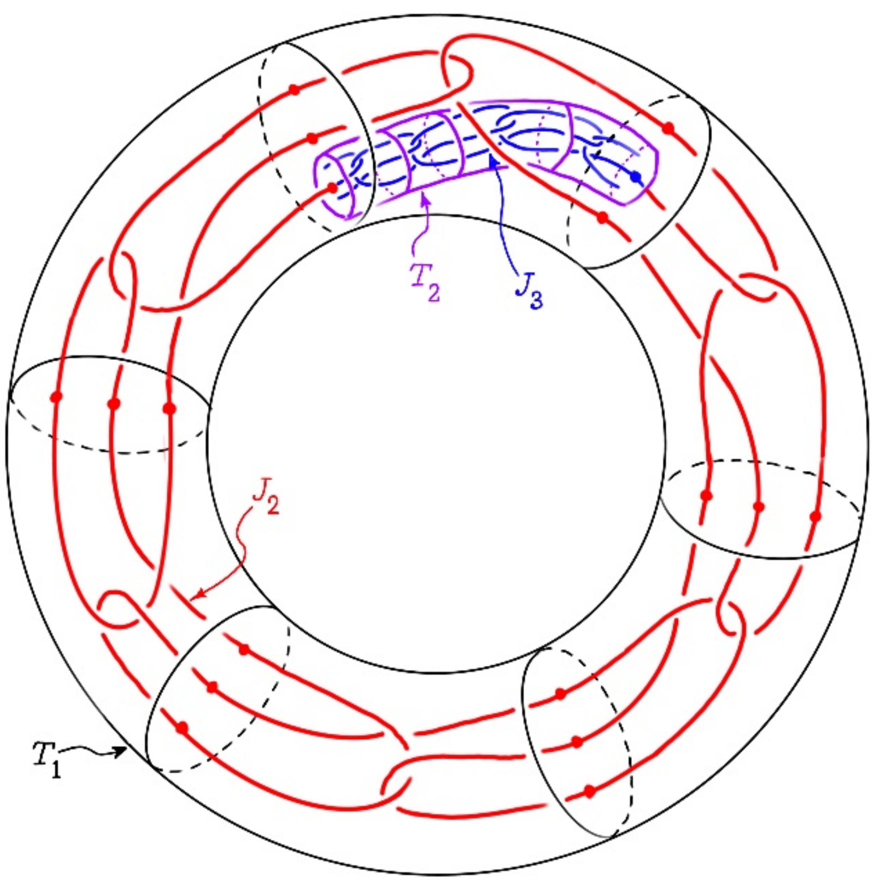

FIGURE 2. The Bing sling

Example. There is a (wild) two-component link in S^3 that is not concordant to any PL link! [7]

Observe that Melikhov's striking example shows that the method of proof of the Theorem won't extend to two-component links, and that the invariants which Melikhov exploits won't work for single component knots.

The Theorem is proved by applying the two main results of [1]. These results are a reinterpretation and formalization of a technique introduced by the topologist C. H. Giffen in the 1960s called *shift-spinning*. A consequence of Giffen's method is that the Mazur 4-manifold [6] has a disk pseudo-spine. (A *pseudo-spine* of a compact manifold M is a compact subset X of $int(M)$ such that $M - X$ is homeomorphic to $\partial M \times [0, 1)$.) After R. D. Edwards' groundbreaking proof that the double suspension of the boundary of the Mazur 4-manifold is homeomorphic to the 5-sphere in 1975, it was observed that had the existence of a disk pseudo-spine in the Mazur 4-manifold been observed before Edwards' proof, then a proof of Edwards' theorem using previously known results could have been given. However, no one made the connection between Giffen's technique and the existence of a disk pseudo-spine until after Edwards announced his proof. Shift-spinning, the existence of a disk pseudo-spine in the Mazur 4-manifold and why this leads to the conclusion that the double suspension of the boundary of the Mazur 4-manifold is homeomorphic to the 5-sphere are clearly explained on pages 102-107 of [3]. In [1] the main results are applied to show that 4-manifolds like the Mazur manifold (constructed by attaching 2-handles to $B^3 \times S^1$ have identifiable pseudo-spines. In particular, all the 4-manifolds constructed by attaching a single 2-handle to $B^3 \times S^1$ along a degree one curve have disk pseudo-spines, and therefore have the property that the double suspensions of their boundaries are homeomorphic to S^5.

Other topologists have explored Giffen shift-spinning in various contexts. (See pages 404-409 of [5] and pages 15-16 of [4].)

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2. Mapping swirls

The central concept of [1] is the *mapping swirl*. We recapitulate its definition and state the two main results of [1] in forms that are convenient for proving the Theorem.

Let X be a compact metric space. Identify the *cone* on X, CX, and the *suspension* of X, ΣX, as the quotient spaces $CX = ([0, \infty] \times X)/(\{\infty\} \times X)$ and $\Sigma X = ([-\infty, \infty] \times X)/\{\{-\infty\}\} \times X, \{\infty\} \times X\}$. Let $(t, x) \mapsto tx$ denote either of the quotient maps $[0, \infty] \times X \to CX$ or $[-\infty, \infty] \times X \to \Sigma X$. For each $t \in [0, \infty]$ or $[-\infty, \infty]$, let tX denote the image of the set $\{t\} \times X$ under the appropriate quotient map. Thus, ∞X denotes the *cone point* of CX, and $(-\infty)X$ and ∞X denote the *suspension points* of ΣX.

Let $f : X \to Y$ be a map between compact metric spaces. Observe that, by exploiting the homeomorphism $x \mapsto (x, f(x))$ from X to the graph of f, the *mapping cylinder* of f, $Cyl(f)$, can be identified with the subset

$$\{(tx, f(x)) \in CX \times Y : (t, x) \in [0, \infty) \times X\} \cup (\infty X \times Y)$$

of $CX \times Y$. Similarly, the *double mapping cylinder* of f, $DblCyl(f)$, which is obtained by identifying two copies of $Cyl(f)$ along their bases, can be identified with the subset

$$\{(tx, f(x)) \in \Sigma X \times Y : (t, x) \in (-\infty, \infty) \times X\} \cup (\{(-\infty)X, \infty X\} \times Y)$$

of $\Sigma X \times Y$. For each $t \in (-\infty, \infty)$, the *t-level* of $DblCyl(f)$ is the set

$$L(f, t) = (tX \times Y) \cap DblCyl(f) = \{(tx, f(x)) : x \in X\}.$$

Furthermore, if $t \in [0, \infty)$, then $L(f, t)$ is also called the *t-level* of $Cyl(f)$. Note that for each $t \in (-\infty, \infty)$, $x \mapsto (tx, f(x)) : X \to L(f, t)$ is a homeomorphism. The ∞-*level* of $Cyl(f)$ is the set $L(f, \infty) = \{\infty X\} \times Y$. The $(-\infty)$-*level* and the ∞-*level* of $DblCyl(f)$ are the sets $L(f, -\infty) = \{(-\infty)X\} \times Y$ and $L(f, \infty) = \{(\infty)X\} \times Y$.

Let X be a compact metric space and let $f : X \to S^1$ be a map. The *mapping swirl* of f is the subset

$$Swl(f) = \{(tx, e^{2\pi it} f(x)) \in CX \times S^1 : (t, x) \in [0, \infty) \times X\} \cup (\{\infty X\} \times S^1)$$

of $CX \times S^1$. The *double mapping swirl* of f is the subset $DblSwl(f) =$

$$\{(tx, e^{2\pi it} f(x)) \in \Sigma X \times S^1 : (t, x) \in (-\infty, \infty) \times X\} \cup (\{(-\infty)X, \infty X\} \times S^1)$$

of $\Sigma X \times S^1$. For each $t \in (-\infty, \infty)$, the *t-level* of $DblSwl(f)$ is the set

$$\mathscr{L}(f, t) = (tX \times S^1) \cap DblSwl(f) = \{(tx, e^{2\pi it} f(x)) : x \in X\}.$$

Furthermore, if $t \in [0, \infty)$, then $\mathscr{L}(f, t)$ is also called the *t-level* of $Swl(f)$. Note that for each $t \in (-\infty, \infty)$, $x \mapsto (tx, e^{2\pi it} f(x)) : X \to \mathscr{L}(f, t)$ is a homeomorphism. The ∞-*level* of $Swl(f)$ is the set $\mathscr{L}(f, \infty) = \{\infty X\} \times S^1$. The $(-\infty)$-*level* and ∞-*level* of $DblSwl(f)$ are the sets $\mathscr{L}(f, -\infty) = \{(-\infty)X\} \times S^1$ and $\mathscr{L}(f, \infty) = \{\infty X\} \times S^1$. For each $x \in X$, the *x-fiber* of $Swl(f)$ is the set

$$\mathscr{F}(f, x) = \{(tx, e^{2\pi it} f(x)) : t \in [0, \infty)\},$$

and the *x-fiber* of $DblSwl(f)$ is the set

$$\mathscr{D}\mathscr{b}\mathscr{l}\mathscr{F}(f, x) = \{(tx, e^{2\pi it} f(x)) : t \in (-\infty, \infty)\}.$$

We now state the two main results of [1].

Theorem 1. *If X is a compact metric space and $f, g : X \to S^1$ are homotopic maps, then there is a homeomorphism $\Omega : Swl(f) \to Swl(g)$ with the following properties.*
1) Ω is fiber-preserving*: for every $x \in X$, $\Omega(\mathscr{F}(f, x)) = \mathscr{F}(g, x)$.*
2) Ω fixes the ∞-level: $\Omega|\mathscr{L}(f, \infty) = id$.

Compare Theorem 1 to the fact that, in general, there is no homeomorphism between mapping cylinders of homotopic maps.

Note that Theorem 1 holds with S^1 replaced by any space homeomorphic to S^1. Also note that the conclusions of the theorem imply that for any subset Y of X, $\Omega(Swl(f|Y)) = Swl(g|Y)$, $\Omega(\mathscr{L}(f|Y, 0) = \mathscr{L}(g|Y, 0)$ and $\Omega(\mathscr{L}(f, \infty)) = \mathscr{L}(g, \infty)$.

Theorem 2. *If Y is a compact metric space, n is a non-zero integer and $f : X \times S^1 \to S^1$ is the map satisfying $f(y, z) = z^n$, then there is a homeomorphism $\Theta : Cyl(f) \to Swl(f)$ with the following properties.*
1) Θ is level-preserving: for every $t \in [0, \infty]$, $\Theta(L(f, t)) = \mathscr{L}(f, t)$.
2) Ω fixes the 0- and ∞-levels: $\Theta|L(f, 0) \cup L(f, \infty) = id$.

Note that Theorem 2 holds for any map $f : X \to J$ for which there is a commutative diagram:

$$\begin{array}{ccc} Y \times S^1 & \xrightarrow{(y,z)\mapsto z^n} & S^1 \\ {\scriptstyle homeomorphism}\downarrow & & \downarrow{\scriptstyle homeomorphism} \\ X & \xrightarrow[f]{} & J \end{array}$$

We will need one other elementary fact:

Proposition 3. *If X is a compact metric space, $f : X \to S^1$ a map and $\lambda : [0, 1) \to [0, \infty)$ is a homeomorphism, then there is a homeomorphism $\Lambda : X \times [0, 1) \to Swl(f) - \mathscr{L}(f, \infty)$ such that for every $t \in [0, 1)$, $\Lambda(X \times \{t\})) = \mathscr{L}(f, \lambda(t))$.*

Proofs of Theorems 1 and 2 are found in [1]. Because we have modified the statements of these theorems for their use in this paper, we will provide outlines of these proofs as well as a proof of Proposition 3 in section 4 below.

3. The proof of the Theorem

Let J be a knot. We will prove that J is semi-isotopic to an unknot.

Step 1: There is an unknotted solid torus T in S^3 such that $J \subset int(T)$ and the inclusion $J \hookrightarrow T$ is a homotopy equivalence.

We can assume $J \subset \mathbb{R}^3 = S^3 - \{\infty\}$. Let p and q be distinct points of J. Let $\mathscr{V}$ be an uncountable family of parallel planes in $\mathbb{R}^3$ that separate p from q. Since J has a countable dense subset, J does not contain an uncountable pairwise disjoint collection of non-empty open sets. Hence, there is a $V \in \mathscr{V}$ such that $J \cap V$ contains no non-empty open subset of J. It follows that $J \cap V$ is a totally disconnected subset of V. Let J_1 and J_2 be arcs such that $J_1 \cup J_2 = J$ and $J_1 \cap J_2 = \partial J_1 = \partial J_2 = \{p, q\}$. Then $J_1 \cap V$ and $J_2 \cap V$ are disjoint compact totally disconnected subsets of V. It follows that there is a disk D in V such that $J_1 \cap V \subset int(D)$ and $(J_2 \cap V) \cap D = \emptyset$. To see this, let D' be a disk in V, let A_1 be a subset of $int(D')$ that is homeomorphic to $J_1 \cap V$, and let A_2 be a subset of $V - D'$ that is homeomorphic to $J_2 \cap V$. Then according to Theorem 13.7 on pages 93-95 of [8], there is a homeomorphism $\phi : V \to V$ such that $\phi(A_i) = J_i \cap V$ for $i = 1, 2$. Simply let $D = \phi(D')$.

We can assume D is a piecewise linear disk in V. Let U be a regular neighborhood of ∂D in S^3 such that $U \cap V$ is a regular neighborhood of ∂D in V and $U \cap J = \emptyset$, and let $T = cl(S^3 - U)$. Then U and, hence, T are unknotted solid tori in S^3, and $J \subset int(T)$. Let $\bar{V} = V \cup \{\infty\}$, and let E_1 and E_2 be the components of $cl(\bar{V} - U)$ such that $E_1 \subset int(D)$ and $\infty \in E_2$. Then E_1 and E_2 are disjoint meridional disks of T such that $T \cap \bar{V} = E_1 \cup E_2$, $J_1 \cap \bar{V} \subset int(E_1)$ and $J_2 \cap \bar{V} \subset int(E_2)$. Thus, $J_1 \cap E_2 = \emptyset = J_2 \cap E_1$.

Clearly, there is a simple closed curve $K \subset int(T)$ such that the inclusion $K \hookrightarrow T$ is a homotopy equivalence, p and $q \in K$, K_1 and K_2 are arcs such that $K_1 \cup K_2 = K$, $K_1 \cap K_2 = \partial K_1 = \partial K_2 = \{p, q\}$, and $K_1 \cap E_2 = \emptyset = K_2 \cap E_1$. At this point we will stretch conventional terminology slightly by saying that for subsets $Z \subset Y$ and $Z \subset Y'$ of a space X, the inclusions $Y \hookrightarrow X$ and $Y' \hookrightarrow X$ are *homotopic in X rel Z* if there is a homotopy $\xi : Y \times I \to X$ such that $\xi_0 = id_Y$, $\xi_1 : Y \to Y'$ is a homeomorphism and $\xi_t|Z = id_Z$ for every $t \in I$. Since $T - E_2$ is contractible, then the inclusions $J_1 \hookrightarrow T - E_2$ and $K_1 \hookrightarrow T - E_2$ are homotopic rel $\{p, q\}$. Similarly, since $T - E_1$ is contractible, the inclusions $J_2 \hookrightarrow T - E_1$ and $K_2 \hookrightarrow T - E_1$ are homotopic rel $\{p, q\}$. Therefore, the inclusions $J \hookrightarrow T$ and $K \hookrightarrow T$ are homotopic. It follows that the inclusion $J \hookrightarrow T$ is a homotopy equivalence.

Step 2: Consider a homeomorphism $\psi : B^2 \times S^1 \to T$, let $o \in int(B^2)$ and let $K = \psi(\{o\} \times S^1)$. Define the homeomorphism $\psi_o : S^1 \to K$ by $\psi_o(z) = \psi(o, z)$. Define the map $\tau : B^2 \times S^1 \to S^1$ by $\tau(y, z) = z$, and define the map $\pi : T \to K$ by $\pi = \psi_o \circ \tau \circ \psi^{-1}$. Then we have a commutative diagram in which the vertical arrows are homeomorphisms:

$$\begin{array}{ccc} B^2 \times S^1 & \xrightarrow{\ \tau\ } & S^1 \\ \psi \downarrow & & \downarrow \psi_o \\ T & \xrightarrow[\ \pi\ ]{} & K \end{array}$$

We can now invoke Theorem 2 to obtain a level-preserving homeomorphism $\Theta : Cyl(\pi) \to Swl(\pi)$ that fixes the 0- and ∞-levels. Also observe that since $\tau : B^2 \times S^1 \to S^1$ is a homotopy equivalence, then so is $\pi : T \to K$.

Step 3: Let $\lambda : I \to [0, \infty]$ be an order-preserving homeomorphism. Clearly, there is an embedding $j : Cyl(\pi) \to S^3 \times I$ with the following properties.
1) j maps $L(\pi, 0)$ "identically" onto $T \times \{0\}$; i.e., $j(0x, \pi(x)) = (x, 0)$ for every $x \in T$.
2) For every $t \in I$, $j(L(\pi, \lambda(t)) \subset S^3 \times \{t\}$, and $j(L(\pi, \lambda(t))$ is a copy of T that is "squeezed" toward K.
3) j maps $L(\pi, \infty)$ "identically" onto $K \times \{1\}$; i.e., for every $y \in K$, $j(\{\infty T\} \times \{y\}) = \{(y, 1)\}$.
Thus, $j \circ \Theta^{-1} : Swl(\pi) \to S^3 \times I$ is an embedding with the following properties.
1) $j \circ \Theta^{-1}$ maps $\mathscr{L}(\pi, 0)$ "identically" onto $T \times \{0\}$; i.e., $j \circ \Theta^{-1}(0x, \pi(x)) = (x, 0)$ for every $x \in T$.
2) For every $t \in I$, $j \circ \Theta^{-1}(\mathscr{L}(\pi, \lambda(t)) \subset S^3 \times \{t\}$.
3) $j \circ \Theta^{-1}$ maps $\mathscr{L}(\pi, \infty)$ "identically" onto $K \times \{1\}$; i.e., for every $y \in K$, $j \circ \Theta^{-1}(\infty T \times \{y\}) = \{(y, 1)\}$.

Step 4: Since $\pi|J \to K$ is the composition of the inclusion $J \hookrightarrow T$ and $\pi : T \to K$, both of which are homotopy equivalences, then $\pi|J \to K$ is a homotopy equivalence. Hence, $\pi|J \to K$ is homotopic to an homeomorphism $\chi : J \to K$. Therefore, Theorem 1 provides a 0- and ∞-level preserving homeomorphism $\omega : Swl(\pi|J) \to Swl(\chi)$. Thus, $j \circ \Theta^{-1} : Swl(\chi) \to S^3 \times I$ is an embedding that maps $\mathscr{L}(\chi, 0)$ into $S^3 \times \{0\}$ and maps $\mathscr{L}(\chi, 1)$ onto $K \times \{1\} \subset S^3 \times \{1\}$.

Step 5: Since $\chi : J \to K$ is a homeomorphism, then Theorem 2 provides a level-preserving homeomorphism from $Cyl(\chi)$ to $Swl(\chi)$. Also, since $\chi : J \to K$ is a homeomorphism, then $Cyl(\chi)$ is an annulus with boundary $L(\chi, 0) \cup L(\chi, 1)$. Therefore, $Swl(\chi)$ is an annulus with boundary $\mathscr{L}(\chi, 0) \cup \mathscr{L}(\chi, 1)$. Since $\omega : Swl(\pi|J) \to Swl(\chi)$ is a 0- and ∞-level preserving homeomorphism, then if follows that $Swl(\pi|J)$ is an annulus with boundary $\mathscr{L}(\pi|J, 0) \cup \mathscr{L}(\pi|J, \infty)$. Let $A = j \circ \Theta^{-1}(Swl(\pi|J))$. Then A is an annulus in $S^3 \times I$ with boundary $j \circ \Theta^{-1}(\mathscr{L}(\pi|J, 0) \cup \mathscr{L}(\pi|J, \infty)) = (J \times \{0\}) \cup (K \times \{1\})$.

Step 6: Recall that for every $t \in I$, $j \circ \Theta^{-1}(\mathscr{L}(\pi|J, \lambda(t)) \subset S^3 \times \{t\}$. Proposition 3 implies that there is a homeomorphism $\Lambda : J \times [0, 1) \to Swl(\pi|J) - \mathscr{L}(\pi|J, \infty)$ such that $\Lambda(J \times \{t\}) = \mathscr{L}(\pi|J, \lambda(t))$ for every $t \in [0, 1)$. Therefore, $j \circ \Theta^{-1} \circ \Lambda : J \times [0, 1) \to A - (K \times \{1\})$ is a homeomorphism such that $j \circ \Theta^{-1} \circ \Lambda(J \times \{t\}) \subset S^3 \times \{t\}$ for every $t \in [0, 1)$. ■

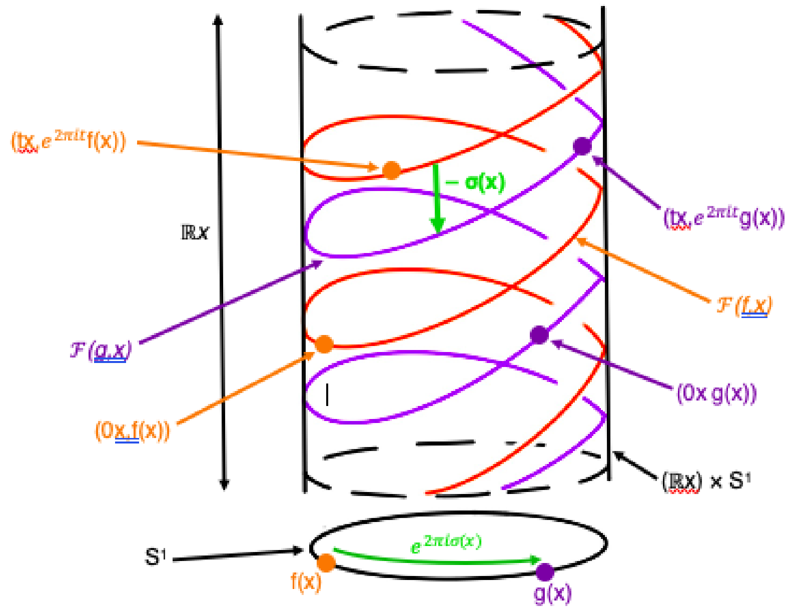

FIGURE 3. Double helix with vertical shift $-\sigma(x)$

4. PROOFS OF THEOREMS 1 AND 2 AND PROPOSITION 3

Proof of Theorem 1. Let X be a compact metric space and let $f, g : X \to S^1$ be homotopic maps.

Step 1: There is a homeomorphism $\Phi : (\Sigma X) \times S^1 \to (\Sigma X) \times S^1$ that carries $DblSwl(f)$ onto $DblSwl(g)$.

For each $x \in X$, observe that $\mathscr{D}b\ell\mathscr{F}(f, x) \cup \mathscr{D}b\ell\mathscr{F}(g, x)$ is a *double helix* in the *cylinder* $(\mathbb{R}x) \times S^1 \subset (\Sigma X) \times S^1$. We will construct a map $\sigma : X \to \mathbb{R}$ such that for each $x \in X$, a first-coordinate shift of $(\mathbb{R}x) \times S^1$ through a distance of $-\sigma(x)$ slides $\mathscr{D}b\ell\mathscr{F}(f, x)$ onto $\mathscr{D}b\ell\mathscr{F}(g, x)$. (See Figure 3.)

Begin the construction of σ by choosing a homotopy $h : X \times I \to S^1$ such that $h_0 = f$ and $h_1 = g$. Using complex division in S^1, define $k : X \times I \to S^1$ by $k(x, t) = h(x, t)/h(x, 0)$. Then for every $x \in X$, $k(x, 0) = 1$ and $k(x, 1)f(x) = g(x)$. Let $e : \mathbb{R} \to S^1$ be the exponential covering map $e(t) = e^{2\pi it}$. The homotopy $k : X \times I \to S^1$ lifts to a homotopy $\widetilde{k} : X \times I \to \mathbb{R}$ such that $e \circ \widetilde{k} = k$ and $\widetilde{k}(x, 0) = 0$ for every $x \in X$. Define $\sigma : X \to \mathbb{R}$ by $\sigma(x) = \widetilde{k}(x, 1)$. Therefore, for every $x \in X$,

$$e^{2\pi i\sigma(x)}f(x) = e \circ \sigma(x)f(x) = e \circ \widetilde{k}(x, 1)f(x) = k(x, 1)f(x) = g(x).$$

Since X is compact, there is a $b > 0$ such that $\sigma(X) \subset (-b, b)$.

Define $\Phi : (\Sigma X) \times S^1 \to (\Sigma X) \times S^1$ by

$$\begin{cases} \Phi(tx, z) = ((t - \sigma(x))x, z) \text{ for } (t, x) \in \mathbb{R} \times X \text{ and } z \in S^1, \text{ and} \\ \Phi = id \text{ on } \{(-\infty)X, \infty X\} \times S^1. \end{cases}$$

Φ is the first-coordinate shift of $(\mathbb{R}x) \times S^1$ through a distance of $-\sigma(x)$ mentioned above. It remains to show that Φ is a homeomorphism of $(\Sigma X) \times S^1$ which slides $\mathscr{D}b\ell\mathscr{F}(f, x)$ onto $\mathscr{D}b\ell\mathscr{F}(g, x)$ for each $x \in X$.

Φ is continuous at points of $\{(-\infty)X, \infty X\} \times S^1$ because for every $x \in X$ and every $z \in S^1$, $\Phi(([t,\infty]x) \times \{z\}) \subset ((t-b,\infty]x) \times \{z\}$ and $\Phi(([-\infty,t]x) \times \{z\}) \subset ([-\infty, t+b)x) \times \{z\}$.

Φ is a homeomorphism because its inverse $\overline{\Phi}$ can be defined explicitly by the equations

$$\begin{cases} \overline{\Phi}(tx,z) = ((t+\sigma(x))x, z) \text{ for } (t,x) \in \mathbb{R} \times X \text{ and } z \in S^1, \text{ and} \\ \overline{\Phi} = id \text{ on } \{(-\infty)X, \infty X\} \times S^1. \end{cases}$$

The verification that $\overline{\Phi} \circ \Phi = id = \Phi \circ \overline{\Phi}$ is straightforward.

For each $x \in X$, a typical point of $\mathscr{Dbl F}(f,x)$ has the form $(tx, e^{2\pi it}f(x))$, and

$$\begin{aligned} &\Phi((tx, e^{2\pi it}f(x)) = ((t-\sigma(x))x, e^{2\pi it}f(x)) = \\ &((t-\sigma(x))x, e^{2\pi i(t-\sigma(x))}e^{2\pi i\sigma(x)}f(x)) = \\ &((t-\sigma(x))x, e^{2\pi i(t-\sigma(x))}g(x)) \in \mathscr{Dbl F}(g,x). \end{aligned}$$

Hence, $\Phi(\mathscr{Dbl F}(f,x)) \subset \mathscr{Dbl F}(g,x)$. A similar calculation shows

$$\overline{\Phi}(\mathscr{Dbl F}(g,x)) \subset \mathscr{Dbl F}(f,x)$$

for every $x \in X$. Thus,

$$\mathscr{Dbl F}(g,x) \subset \Phi(\mathscr{Dbl F}(f,x))$$

for every $x \in X$. We conclude that

$$\Phi(\mathscr{Dbl F}(f,x)) = \mathscr{Dbl F}(g,x)$$

for every $x \in X$.

We have shown that $\Phi : (\Sigma X) \times S^1 \to (\Sigma X) \times S^1$ is a homeomorphism with the property that $\Phi | DblSwl(f) : DblSwl(f) \to DblSwl(g)$ is a fiber-preserving homeomorphism that fixes the $(-\infty)$- and ∞-levels. Unfortunately, we can't conclude that $\Phi(Swl(f))$ and $Swl(g)$ are equal subsets of $DblSwl(g)$. This completes Step 1.

Step 2: There is a homeomorphism $\Psi : (\Sigma X) \times S^1 \to (\Sigma X) \times S^1$ that maps $DblSwl(g)$ onto itself and *twists* $\Phi(Swl(f)$ onto $Swl(g)$.

Observe that for every $x \in X$, $\Phi(\mathscr{F}(f,x))$ and $\mathscr{F}(g,x)$ are contained in the *helix* $\mathscr{Dbl F}(g,x)$ which lies in the *cylinder* $(\mathbb{R}x) \times S^1$. For each $x \in X$, Ψ will map this cylinder to itself with a *screw motion* that preserves $\mathscr{Dbl F}(g,x)$ and *twists* $\Phi(\mathscr{F}(f,x))$ onto $\mathscr{F}(g,x)$.

Recall that $\sigma(X) \subset (-b,b)$. Hence, there is clearly a map $\tau : \mathbb{R} \times X \to \mathbb{R}$ such that for each $x \in X$, $\tau | \mathbb{R} \times \{x\} : \mathbb{R} \times \{x\} \to \mathbb{R}$ is a homeomorphism such that $\tau(-\sigma(x)) = 0$ and $\tau(t,x) = t$ for $t \in (-\infty, -b] \cup [b, \infty)$.

Define $\Psi : (\Sigma X) \times S^1 \to (\Sigma X) \times S^1$ by

$$\begin{cases} \Psi(tx,z) = ((\tau(t,x)x, e^{2\pi i(\tau(t,x)-t)}z) \text{ for } (t,x) \in \mathbb{R} \times X \text{ and } z \in S^1, \text{ and} \\ \Psi = id \text{ on } \{(-\infty)X, \infty X\} \times S^1. \end{cases}$$

Ψ is continuous at points of $\{(-\infty)X, \infty X\} \times S^1$ because $\Psi = id$ on $\{tx : |t| \geq b \text{ and } x \in X\} \times S^1$.

Ψ is a homeomorphism because its inverse $\overline{\Psi}$ can be defined explicitly as follows. First observe that there is a map $\overline{\tau} : \mathbb{R} \times X \to \mathbb{R}$ so that for every $x \in X$, $t \mapsto \overline{\tau}(t,x) : \mathbb{R} \to \mathbb{R}$ is the inverse of the homeomorphism $t \mapsto \tau(t,x) : \mathbb{R} \to \mathbb{R}$. Now define $\overline{\Psi} : (\Sigma X) \times S^1 \to (\Sigma X) \times S^1$ by

$$\begin{cases} \overline{\Psi}(tx,z) = ((\overline{\tau}(t,x)x, e^{2\pi i(\overline{\tau}(t,x)-t)}z) \text{ for } (t,x) \in \mathbb{R} \times X \text{ and } z \in S^1, \text{ and} \\ \overline{\Psi} = id \text{ on } \{(-\infty)X, \infty X\} \times S^1. \end{cases}$$

The verification that $\overline{\Psi} \circ \Psi = id = \Psi \circ \overline{\Psi}$ is straightforward.

Let $x \in X$. To prove that $\Psi(\mathscr{D}\mathscr{b}\mathscr{l}\mathscr{F}(g,x)) = \mathscr{D}\mathscr{b}\mathscr{l}\mathscr{F}(g,x)$, note that a typical point of $\mathscr{D}\mathscr{b}\mathscr{l}\mathscr{F}(g,x)$ has the form $(tx, e^{2\pi it}g(x))$, and

$$\Psi((tx, e^{2\pi it}g(x)) = (\tau(t,x)x, e^{2\pi i(\tau(t,x)-t)}e^{2\pi it}g(x)) = \\ (\tau(t,x)x, e^{2\pi i\tau(t,x)}g(x)) \in \mathscr{D}\mathscr{b}\mathscr{l}\mathscr{F}(g,x).$$

Therefore, $\Psi(\mathscr{D}\mathscr{b}\mathscr{l}\mathscr{F}(g,x)) \subset \mathscr{D}\mathscr{b}\mathscr{l}\mathscr{F}(g,x)$. A similar calculation shows

$$\overline{\Psi}(\mathscr{D}\mathscr{b}\mathscr{l}\mathscr{F}(g,x)) \subset \mathscr{D}\mathscr{b}\mathscr{l}\mathscr{F}(g,x).$$

Hence,

$$\mathscr{D}\mathscr{b}\mathscr{l}\mathscr{F}(g,x) \subset \Psi(\mathscr{D}\mathscr{b}\mathscr{l}\mathscr{F}(g,x)).$$

We conclude that

$$\Psi(\mathscr{D}\mathscr{b}\mathscr{l}\mathscr{F}(g,x)) = \mathscr{D}\mathscr{b}\mathscr{l}\mathscr{F}(g,x).$$

We have shown that $\Psi : (\Sigma X) \times S^1 \to (\Sigma X) \times S^1$ is a homeomorphism with the property that $\Psi|DblSwl(g) : DblSwl(g) \to DblSwl(g)$ is a fiber-preserving homeomorphism that fixes the $(-\infty)$- and ∞-levels.

Again let $x \in X$. To prove that $\Psi(\Phi(\mathscr{F}(f,x))) = \mathscr{F}(g,x)$, note that a typical point of $\Phi(\mathscr{F}(f,x))$ has the form $(tx, e^{2\pi it}g(x)$ where $t \geq -\sigma(x)$, and a typical point of $\mathscr{F}(g,x)$ has the form $(tx, e^{2\pi it}g(x))$ where $t \geq 0$. Also note that τ maps $[-\sigma(x), \infty)$ onto $[0, \infty)$. Since

$$\Psi((tx, e^{2\pi it}g(x)) = (\tau(t,x)x, e^{2\pi i(\tau(t,x)-t)}e^{2\pi it}g(x)) = (\tau(t,x)x, e^{2\pi i\tau(t,x)}g(x)),$$

then clearly $\Psi(\Phi(\mathscr{F}(f,x))) = \mathscr{F}(g,x)$.

It follows that $\Psi \circ \Phi(Swl(f)) = Swl(g)$, completing Step 2.

We conclude that $\Psi \circ \Phi|Swl(f) : Swl(f) \to Swl(g)$ is a fiber-preserving homeomorphism that fixes the ∞-level. ■

Proof of Theorem 2. Let Y be a compact metric space, n a non-zero integer and $f : Y \times S^1 \to S^1$ the map satisfying $f(y,z) = z^n$. We will construct a homeomorphism $\Theta : C(Y \times S^1) \times S^1 \to C(Y \times S^1) \times S^1$ that carries $Cyl(f)$ onto $Swl(f)$.

Define $\zeta : [0,\infty) \to S^1$ by $\zeta(t) = e^{-2\pi it/n}$, and define $\Theta : C(Y \times S^1) \times S^1 \to C(Y \times S^1) \times S^1$ by

$$\begin{cases} \Theta(t(y,z),w) = (t(y,\zeta(t)z),w) \text{ for } t \in [0,\infty), (y,z) \in Y \times S^1 \text{ and } w \in S^1, \text{ and} \\ \Theta = id \text{ on } \{\infty(Y \times S^1)\} \times S^1. \end{cases}$$

Θ is clearly level-preserving and fixes the 0- and ∞-levels.

We argue that Θ is continuous at points of $\{\infty(Y \times S^1)\} \times S^1$. For $a > 0$, let $V_a = \bigcup_{t \in (a,\infty]} t(Y \times S^1)$; and for $w \in S^1$, let $\mathscr{M}_w$ be a basis for the topology on S^1 at w. Then for each $w \in S^1$, $\{V_a \times M : a > 0 \text{ and } M \in \mathscr{M}_w\}$ is a basis for the topology on $C(Y \times S^1) \times S^1$ at $(\infty(Y \times S^1), w)$. Since for each $a > 0$ and each $M \in \mathscr{M}_w$, Θ maps $V_a \times M$ into itself, then Θ is continuous at $(\infty(Y \times S^1), w)$.

Θ is a homeomorphism because its inverse $\overline{\Theta}$ can be defined explicitly as follows. First define the map $\overline{\zeta} : [0,\infty) \to S^1$ by $\overline{\zeta}(t) = e^{2\pi it/n}$. Then define $\overline{\Theta} : C(Y \times$

$S^1) \times S^1 \to C(Y \times S^1) \times S^1$ by

$$\begin{cases} \overline{\Theta}(t(y,z),w) = (t(y,\overline{\zeta}(t)z),w) \text{ for } t \in [0,\infty), (y,z) \in Y \times S^1 \text{ and } w \in S^1, \text{ and} \\ \overline{\Theta} = id \text{ on } \{\infty(Y \times S^1)\} \times S^1 \end{cases}$$

The verification that $\overline{\Theta} \circ \Theta = id = \Theta \circ \overline{\Theta}$ is straightforward.

To prove that $\Theta(Cyl(f)) = Swl(f)$, note that a typical point of $Cyl(f)$ has the form $(t(x,z), f(x,z))$, a typical point of $Swl(f)$ has the form $(t(x,z), e^{2\pi i t} f(x,z))$, and

$$\begin{aligned} &\Theta(t(x,z), f(x,z)) = (t(x,\zeta(t)z), f(x,z)) = \\ &(t(x,\zeta(t)z), z^n) = (t(x,\zeta(t)z), (\zeta(t))^{-n}(\zeta(t)z)^n) = \\ &(t(x,\zeta(t)z), e^{2\pi i t}(\zeta(t)z)^n) = (t(x,\zeta(t)z), e^{2\pi i t} f(x,\zeta(t)z)) \in Swl(f). \end{aligned}$$

Therefore, $\Theta(Cyl(f)) \subset Swl(f)$. A similar calculation shows $\overline{\Theta}(Swl(f)) \subset Cyl(f)$. Hence, $Swl(f) \subset \Theta(Cyl(f))$. We conclude that $\Theta(Cyl(f)) = Swl(f)$.

It follows that $\Theta|Cyl(f) : Cyl(f) \to Swl(f)$ is a level-preserving homeomorphism that fixes the 0- and ∞-levels. ■

Proof of Proposition 3. A compact metric space X, a map $f : X \to S^1$ and a homeomorphism $\lambda : [0,1) \to [0,\infty)$ are given. Define $\Lambda : X \times [0,1) \to Swl(f) - \mathscr{L}(f,\infty)$ by $\Lambda(x,t) = (\lambda(t)x, e^{2\pi i\lambda(t)} f(x))$. We show that Λ is a homeomorphism by exhibiting its inverse. Let $q : X \times [0,\infty] \to CX$ denote the quotient map $q(x,t) = tx$. Then $q|X \times [0,\infty) : X \times [0,\infty) \to CX - \{\infty X\}$ is a homeomorphism; let $r : CX - \{\infty X\} \to X \times [0,\infty)$ denote its inverse. Let $p : (CX - \{\infty X\}) \times S^1 \to CX - \{\infty X\}$ denote projection. Define $\tau : (CX - \{\infty X\}) \times S^1 \to X \times [0,1)$ by $\tau = (id_X \times \lambda^{-1}) \circ r \circ p$. It is easily verified that $\tau \circ \Lambda = id_{X \times [0,1)}$ and $\Lambda \circ (\tau|(Swl(f) - \mathscr{L}(f,\infty))) = id_{Swl(f) - \mathscr{L}(f,\infty)}$. Hence, $\tau|(Swl(f) - \mathscr{L}(f,\infty))$ is the inverse of Λ. ■

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