Summary of Publications

This document aims to be a convenient guide to the papers in this file. Articles are grouped by topic rather than strict chronological order. Not all papers in the file are mentioned in this summary.

Topic 1: Spines of Compact Contractible Manifolds

A subset A of the interior of a compact n-manifold M^n is a *spine* if there is a map $f : \partial M^n \to A$ such that (M^n, A) is homeomorphic to $(Map\,(f)\,, A)$. A still-open question asks whether Mazur's compact contractible 4-manifolds contains disjoint pair of spines. Motivated by this problem, I wrote three papers (two joint with Ric Ancel) about spines of compact contractible manifolds.

[1] *Compact contractible n-manifolds have arc spines ($n \geq 5$)*, Pacific J. Math. **168**(1995), 1-10 (w/ Ancel).

[2] *Some compact contractible manifolds containing disjoint spines*, Topology **34**(1995), 99-108.

[3] *Mapping swirls and pseudo-spines of compact 4-manifolds*, Topology Appl. **71**(1996), 277-293 (w/ Ancel).

Paper [1] shows that, in high dimensions, compact contractible manifolds always have (wild) arc spines. As a by-product, we found a new proof of Kervaire's theorem which guarantees that homology $(n-1)$-spheres bound compact contractible n-manifolds. Paper [2] solved a high-dimensional version of the Mazur manifold problem.

A subset B of $\operatorname{int} M^n$ is a *pseudo-spine* if $M^n - B \approx \partial M^n \times [0, 1)$. Every spine is a pseudo-spine, but not conversely. In [3], Ancel and I generalized a construction by Giffen to show that many compact contractible 4-manifolds contain (wild) disk pseudo-spines. These can be shrunk to arc pseudo-spines if desired. In dimension 4, that is the best one can hope for; the only 4-manifold with an arc spine is B^4.

Topic 2: CAT(0) and CAT(-1) Structures on Manifolds

In [4], we applied the arc spine construction to place $CAT\,(-1)$ metrics on interiors of compact contractible n-manifolds $(n > 4)$, making each a counterexample to a question by Gromov, who asked whether metrically convex geodesic open manifolds are topologically Euclidean spaces. Earlier counterexamples were constructed by Mike Davis and Tadeusz Januszkiewicz.

Item [5] also uses [1], this time to place equivariant $CAT\,(0)$ metrics on many of the exotic coverings from Davis' celebrated Annals paper.

Paper [6] is a broad study of "trees of manifolds". One construction presented there uses a hybrid of [4] and [5] to place $CAT\,(-1)$ metrics on the Davis manifolds (and on topological generalizations of those manifolds). These metrics are not equivariant.

[4] *Interiors of compact contractible n-manifolds are hyperbolic ($n \geq 5$)*, J. Diff. Geom. **45**(1997), 1-32 (w/ Ancel).

[5] *CAT(0) reflection manifolds,* Geometric Topology, W.H. Kazez editor, American Math. Soc./International Press Studies in Advanced Mathematics, Vol. 2, Part 1, 1997, 441-445 (w/ Ancel and Davis).

[6] *On the fundamental groups of trees of manifolds*, Pacific J. Math., **221**, No.1(2005), 49-79 (w/ Fischer).

Topic 3: Ends of Manifolds

A long-time passion is the study of ends of manifolds. The work in this section began with a goal: Develop a theory of ends—in the spirit of Siebenmann's thesis—that applies to manifolds with non-stable fundamental groups at infinity. A manifold V^n with compact boundary is an *open collar* of $V^n \approx \partial V^n \times [0, \infty)$; it is a *homotopy collar* if $\partial V^n \hookrightarrow V^n$ is a homotopy equivalence. A manifold is *collarable* if it contains an open collar neighborhood of infinity; it is *pseudo-collarable* if it contains arbitrarily small homotopy collar neighborhoods of infinity. Collarable manifolds are precisely those which can be compactified by the addition of a manifold boundary; they were characterized by Siebenmann in his dissertation. Davis' exotic covering spaces are examples of pseudo-collarable (but not collarable) open manifolds. The main question: "When is an open n-manifold pseudo-collarable?" was answered in the following series of papers.

[7] *Manifolds with non-stable fundamental groups at infinity*, Geom. Topol. **4**(2000), 537-579.

[8] *Manifolds with non-stable fundamental groups at infinity, II*, Geom. Topol. **7**(2003), 255-286 (w/ Tinsley).

[9] *Manifolds with non-stable fundamental groups at infinity, III*, Geom. Topol., **10**(2006) 541-556 (w/ Tinsley).

In the process of characterizing pseudo-collarable open manifolds, Tinsley and I discovered some interesting examples and surprising properties of *inward tame* open manifolds. This led to a pair of sequels, with a theme: "What can be said about an open manifold if we only assume inward tameness?"

[10] *Spherical alterations of handles: embedding the manifold plus construction*, Algebr. Geom. Topol. 13 (2013) 35-60 (w/ Tinsley).

[11] *Noncompact manifolds that are inward tame*, Pacific J. Math. 288 (2017), no. 1, 87–128. (w/ Tinsley).

One surprising discovery was that, for open manifolds, inward tame implies semistable π_1. By using a variation on that observation, I answered a question by Belegradek regarding products of manifolds with $\mathbb{R}$. That work improved upon a result in Siebenmann's dissertation that applies to products with $\mathbb{R}^2$.

[12] *Products of open manifolds with* $\mathbb{R}$, Fund. Math. 197 (2007), 197-214.

The above papers focused on open manifolds and, more generally, manifolds with compact boundary. Eventually, it was time to take on manifolds with noncompact boundaries, where the classical literature was incomplete. The main open problem was a characterization of "completable" n-manifolds, i.e., those which may be compactified to a compact manifold with boundary by extending the existing boundary. Siebenmann did some work on that topic, and Gary O'Brien completed the 1-ended case. But (unlike the compact case) a manifold with noncompact boundary can be inward tame and still have infinitely many ends. As a consequence, the 1-ended version does not imply a general theorem. In a joint paper with my PhD student Shijie Gu, a full characterization of completable manifolds was obtained.

[13] *Compactifications of manifolds with boundary*, Topol. Anal. 12 (2020), no. 4, 1073–1101. (w/ Gu).

In his dissertation, Gu went on to characterize pseudo-collarable manifolds with noncompact boundaries.

Topic 4: $\mathcal{Z}$-compactifications of Manifolds and Complexes

An alternative point perspective on generalizing Siebenmann's thesis is to look for more flexible compactifications. A *$\mathcal{Z}$-compactification* of X is a compactification $\widehat{X} = X \sqcup Z$ such that Z is a $\mathcal{Z}$-set in $\widehat{X}$. For many reasons, this turns out to be an appropriate generalization of the addition of a boundary to a manifold. Moreover, the idea plays an important role in some attacks on the Borel and Novikov Conjectures. The following papers deal with $\mathcal{Z}$-compactifications of manifolds and complexes.

[14] *$\mathcal{Z}$-compactifications of open manifolds*, Topology **38**(1999), 1265-1280 (w/ Ancel).

[15] *A non-$\mathcal{Z}$-compactifiable polyhedron whose product with the Hilbert cube is $\mathcal{Z}$-compactifiable*, Fund. Math. **168** (2001), 165-197.

Article [14] demonstrates the usefulness of $\mathcal{Z}$-compactifications when applied to open manifolds. For example, if two contractible n-manifolds admit homeomorphic $\mathcal{Z}$-boundaries, then they are homeomorphic and their union along those boundaries is an n-sphere. It follows that, if G is $CAT(0)$ or word hyperbolic (or simply admits a *$\mathcal{Z}$-structure*), then any two aspherical manifolds with fundamental group G (or even quasi-isometric to G) have homeomorphic universal covers. We view this as a weak version of the Borel Conjecture.

A key question becomes: When is a manifold $\mathcal{Z}$-compactifiable? In 1976, Chapman and Siebenmann characterized $\mathcal{Z}$-compactifiable Hilbert cube manifolds, then asked whether their conditions are sufficient for $\mathcal{Z}$-compactifiability of *arbitrary* ANRs. A "yes" answer would, of course, handle finite-dimensional manifolds. Their problem remained open until, in [15], when I constructed a counterexample. The example is a non-manifold 2-dimensional polyhedron, so the Chapman-Siebenmann question remains open for manifolds.

By applying our characterization of completable manifolds and tools from [5]-[7] , Gu and I showed (in [13]) that, if M^n satisfies the Chapman-Siebenmann conditions and $n \geq 5$, then $M^n \times [0,1]$ is $\mathcal{Z}$-compactifiable (in fact, completable). Previously, Ferry had shown that $M^n \times [0,1]^{2k+5}$ is $\mathcal{Z}$-compactifiable. It remains to shave off that final interval!

Topic 5: Topological Properties of CAT(0) Group Boundaries

After the discovery of a CAT(0) group with non-homeomorphic boundaries, Bestvina observed that all such boundaries must be shape equivalent. He then asked whether they satisfy the stronger property of being *cell-like equivalent.* Roughly speaking, Z and Z' are cell-like equivalent if there is a space Y, and cell-like maps $Z \xleftarrow{f_1} Y \xrightarrow{f_2} Z'$. In posing this problem, he asked specifically about the Croke-Kleiner boundaries.

My former PhD student Christopher Mooney and I attacked this problem with good success. In [16], we laid out a program for attacking Bestvina's question, and illustrated its effectiveness in some simple situations. In [17] we proved that, for a larger class of "Croke-Kleiner groups", all boundaries of a given group are *equivariantly* cell-like equivalent. Both Bestvina and the coauthors were enthusiastic about the inclusion of equivariance in the theorem.

[16] *Cell-like equivalences for boundaries of certain CAT(0) groups*, Geom. Dedicata 160 (2012), 119–145 (w/ Mooney).

[17] *Boundaries of Croke–Kleiner-admissible groups and equivariant cell-like equivalence*, J. Topol. (2014), no. 3, 849–868. (w/ Mooney).

Topic 6: $\mathcal{Z}$-boundaries of groups

This set of papers involves *$\mathcal{Z}$-boundaries* of groups—a notion introduced by Bestvina and expanded upon by Dranishnikov. The idea is to provide an axiomatic setting for group boundaries that includes boundaries of hyperbolic and CAT(0) groups as special cases. My involvement is a natural offshoot of my interest in $\mathcal{Z}$-compactifications. It has also given me the opportunity to aid in bringing powerful ideas from the Bing-Borsuk-Anderson school of topology to the field of geometric group theory. It has been a good topic for my PhD students as well. Carrie Tirel, Molly Moran, and Brian Pietsch all wrote dissertations in this area. Several of the following papers involve joint work with these students after they completed their degrees.

[18] *On the dimension of $\mathcal{Z}$-sets*, Topology Appl. 160 (2013) 1849-1852. (w/ Tirel).

[19] *Weak Z-structures for some classes of groups*, Algebr. Geom. Topol. 14 (2014), no. 2, 1123–1152.

[20] *A comparison of large scale dimension of a metric space to the dimension of its boundary*, Topology Appl. 199 (2016), 17–22. (w/ Moran).

[21] *Proper homotopy types and $\mathcal{Z}$-boundaries of spaces admitting geometric group actions*, Expo. Math. 37 (2019), no. 3, 292–313 (w/ Moran).

[22] *$\mathcal{Z}$-structures on Baumslag-Solitar groups*, Algebr. Geom. Topol. 19 (2019), no. 4, 2077–2097 (with Moran and Tirel).

To be a candidate for a $\mathcal{Z}$-structure, G must have a finite $\mathrm{K}(G,1)$—or at least admit a proper cocompact action on an AR (call G *type F* or *type F**, accordingly). A question posed by Bestvina asks whether all type F (or type F*) groups admit $\mathcal{Z}$-structures. Aside from groups with geometric structures (hyperbolic, CAT(0), systolic), there are few general theorems. In [19], I made some progress by looking for "weak $\mathcal{Z}$-structures". For example, I showed that an extension of a type F by a type F always admits a weak $\mathcal{Z}$-structure.

Bestvina, Mess, Dranishnikov, and others have established relationships between topological dimension of a $\mathcal{Z}$-boundary Z and the cohomological dimension of G. Paper [18] made those results less technical by giving an elementary proof that $\dim Z$ is strictly less that of the space it compactifies. [20] contains a similar result, but instead compares $\dim Z$ to the *large-scale* dimension of the space.

In [21], we cleaned up and unified the theory of $\mathcal{Z}$-boundaries. We also generalized some foundational results about large-scale geometry, boundary swapping, and shapes of boundaries, by allowing for groups with torsion and for actions on a more general class of spaces. By contrast, in [22], we got very specific, proving that all generalized Baumslag-Solitar groups admit $\mathcal{Z}$-boundaries

Topic 7: pro-π_1 and the semistability conjecture

In [22], Ross Geoghegan and I published strengthened a famous theorem by David Wright, who had shown that: If a simply connected space X with pro-monomorphic fundamental group at infinity admits a proper $\mathbb{Z}$-action, then pro-π_1 is an inverse sequence of finitely generated free groups. We showed that the inverse sequence is also stable.

[22] *Topological properties of spaces admitting free group actions,* J. Topology 5 (2) (2012), 249-275 (w/ Geoghegan).

In [23], Geoghegan and I teamed with Mike Mihalik to improve upon that work. Using an entirely different approach, we obtained a stronger theorem with weaker hypotheses. The new argument replaces both Wright and [22]. It seems that we have discovered the "real reasons" for those theorems.

[23] *Topological properties of spaces admitting a coaxial homeomorphism*, Algebr. Geom. Topol. 20 (2020), no. 2, 601–642 (w/ Geoghegan and Mihalik).

Next, we pushed those ideas further in an attack on a topic of common interest: *the Semistability Conjecture for groups.* The rough idea of our approach is to find an appropriately chosen $J \leq G$, then break the G-action down into the J-direction, and a direction orthogonal to J. Under the right circumstances (including some previously unsolved cases), we are able to prove semistability of G.:

[24] *Non-cocompact Group Actions and π_1-Semistability at Infinity*, Canad. J. Math. 72 (2020), no. 5, 1275–1303. (w/ Geoghegan and Mihalik).

Topic 8: Some miscellaneous projects

The Absolute Cone Conjecture (de Groot, 1971") posed that: *If X is an n-dimensional compactum and $\forall x \in X$, X can be expressed as a cone with cone-point x, then $X \approx B^n$.* By applying some famous work by Cannon and Edwards, I was able to construct counterexamples in dimensions ≥ 5; then, with a well-timed assist from Perelman, I was able to use powerful techniques from the theory of homology manifolds prove the conjecture when $n \leq 4$.

[25] *A solution to de Groot's absolute cone conjecture*, Topology, **46**(2007), 89-102.

I got hooked on finding an elementary proof of the Topological Radon Theorem. Eventually, I succeeded; this paper was the result.

[26] *An elementary deduction of the Topological Radon Theorem from Borsuk-Ulam*, Discrete & Computational Geometry, 43 (2010), no. 4, 951–954.

In an Annals paper from 1975, D. Edwards and R. Geoghegan characterized compacta with the shape of a finite complex. I found my own self-contained proof, which I later contributed to a conference proceedings in Geoghegan's honor.

[27] *Compacta with shapes of finite complexes: A direct approach to the Edwards-Geoghegan-Wall obstruction*, Topological methods in group theory, 92–110, London Math. Soc. Lecture Note Ser., 451.

After giving a series of talks at the workshop on *Geometrical Methods in High-dimensional Topology* (Ohio State University, 2011), I contributed a paper to the proceedings of that conference—a lengthy survey of the interlocking aspects of my favorite areas of topology and group theory. The article has generated a fair amount of interest; I am considering expanding it into a book.

[28] *Ends, shapes and boundaries in manifold topology and geometric group theory*, Top. and Geom. Group Theory, Springer Proc. Math. Stat., 184, Springer, Cham, 2016.

Topic 9: Very recent work

To close, I will describe some recently accepted and submitted work.

End-sum is an analog of the connected sum operation for manifolds which is appropriate in the proper category. For a pair of sufficiently nice open manifolds, the operation is well-defined, but in [29], Jack Calcut, Patrick Haggerty and I show that it is possible for the end-sum of a pair of 1-ended open manifolds to yield uncountably many distinct results. Our use of the theory of infinitely generated abelian groups seems novel.

[29] *Extreme nonuniqueness of end-sum,* J. Topol. Anal., (online ready; final version to appear), (w/ Calcut and Haggerty).

Items [30]-[32] return to the topic of $\mathcal{Z}$-boundaries of groups. In [30] we define and develop the notion of a *coarse* $\mathcal{Z}$*-boundary*; we show that nearly all theorems about $\mathcal{Z}$-structures remain true; and (most significantly) we show that admitting coarse $\mathcal{Z}$-structure is a quasi-isometry invariant. Papers [31] and [32] (with new collaborators, Kevin Schreve and Burns Healy) identify new groups that admit $\mathcal{Z}$-structures and find $E\mathcal{Z}$-structures (E for *equivariant*) whenever possible. This is important since $E\mathcal{Z}$-structures are useful in attacking the Novikov Conjecture. [31] analyzes a specific type of group—high-dimensional analogs of generalized Baumslag-Solitar groups—while [32] is much broader, with theorems covering all closed 3-manifold groups and all groups of polynomial growth.

[30] *Coarse* $\mathcal{Z}$*-boundaries for groups,* Mich. Math. J. (to appear), (w/ Moran).
[31] *Compressible spaces and EZ-structures,* Fund. Math. (to appear), (w/ Moran and Schreve).
[32] *Group boundaries for semidirect products with* $\mathbb{Z}$, submitted, (w/ Healy and Pietsch).