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On the dimension of $\mathcal{Z}$-sets [☆]

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ABSTRACT

We offer a short and elementary proof that, for a $\mathcal{Z}$-set A in a finite-dimensional ANR Y, $\dim A < \dim Y$. This result is relevant to the study of group boundaries. The original proof by Bestvina and Mess relied on cohomological dimension theory.

1. Introduction

The notion of a $\mathcal{Z}$-set, developed by R.D. Anderson [1], played an important role in the infinite-dimensional topology boom of the 1970's (see [4]). More recently, work by Bestvina, Mess, Geoghegan and others (see [3,2,6]) has revived interest in $\mathcal{Z}$-sets. In this more recent work, the point of view is the following: for an infinite group G acting nicely on an AR X, there is frequently a compactification $\overline{X} = X \sqcup Z$, where Z is a "boundary" for G and a $\mathcal{Z}$-set in $\overline{X}$. From there, well-known properties of $\mathcal{Z}$-sets reveal connections between the group and its boundary. In contrast with earlier applications of $\mathcal{Z}$-sets, the spaces involved in this new setting are usually finite-dimensional and the precise dimensions are of utmost importance. For example, [3] shows that, for torsion free hyperbolic G, the topological dimension of the Gromov boundary ∂G is always one less than the (algebraically defined) cohomological dimension of G; the latter is bounded above by the topological dimension of X. That result was expanded upon in [2,7,5].

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A useful lemma from [3] asserts that, in the above setting, the dimension of Z is strictly less than the dimension of X. The result was obtained using Čech cohomology and the fact that, for finite-dimensional spaces, the "cohomological dimension" of a space agrees with its topological dimension. The aim of this note is an elementary proof of the same fact. No knowledge of cohomological dimension theory is needed and no algebraic topology will be used. By "topological dimension", we mean "Lebesgue covering dimension", a concept to be defined shortly. Our aim is the following:

Theorem 1.1. *([3, Prop. 2.6]) If A is a $\mathcal{Z}$-set in a compact finite-dimensional metric ANR Y, then* $\dim Y = \dim(Y - A)$ *and* $\dim A < \dim Y$.

Remark 1. More general versions of this theorem are possible. For example, we make no use of the ANR properties of Y or $Y - A$; if A is a subset of any compact finite-dimensional metric space and it is possible to instantly homotope Y off A (see the first bullet point in Section 2), then Theorem 1.1 and its proof are still valid. One may also relax the compactness condition on Y to local compactness without much additional effort. We have chosen to focus on the case of primary interest with the fewest technicalities.

2. Definitions and background

In this paper, all spaces are assumed to be separable metric. A locally compact space Y is an ANR (absolute neighborhood retract) if it can be embedded as a closed subset of $\mathbb{R}^n$ or $\mathbb{R}^\infty$ so that it is a retract of one of its neighborhoods. A closed subset A of an ANR Y is a *$\mathcal{Z}$-set* if either of the following equivalent conditions is satisfied:

- There exists a homotopy $H : Y \times [0, 1] \to Y$ such that $H_0 = \mathrm{id}_Y$ and $H_t(X) \subseteq Y - A$ for all $t > 0$. (We say that H *instantly homotopes* Y off from A.)
- For every open set U in Y, $U - A \hookrightarrow U$ is a homotopy equivalence.

A *$\mathcal{Z}$-compactification* of a space X is a compactification $\overline{X} = X \sqcup Z$ with the property that Z is a $\mathcal{Z}$-set in $\overline{X}$. In this case, Z is called a *$\mathcal{Z}$-boundary* for X. Implicit in this definition is the requirement that $\overline{X}$ be an ANR. Since an open subset of an ANR is an ANR, X itself must be an ANR to be a candidate for $\mathcal{Z}$-compactification. Hanner's Theorem [9] ensures that every compactification $\overline{X}$ of an ANR X, for which $\overline{X} - X$ satisfies either of the above bullet points is necessarily an ANR; hence, it is a $\mathcal{Z}$-compactification.

A collection $\mathcal{A}$ of subsets of a space X has *order k* (k an integer) if some $x \in X$ belongs to $k+1$ elements of $\mathcal{A}$ but none belongs to more than $k + 1$ elements of $\mathcal{A}$; it has *mesh ϵ* ($\epsilon \geqslant 0$) if the diameters of its elements are bounded above by ε. The collection $\mathcal{A}$ *refines* a second collection $\mathcal{B}$ of subsets of X if, for each $A \in \mathcal{A}$ there exists $B \in \mathcal{B}$ such that $A \subseteq B$.

A space X *has dimension* $\leqslant k$ if, for every open cover $\mathcal{U}$ of X, there exists an open cover $\mathcal{V}$ of X that refines $\mathcal{U}$ and has order $\leqslant k$. The smallest such integer (when it exists) is called the *dimension* of X and is denoted $\dim X$. If no such integer exists, X is called *infinite-dimensional*.

Remark 2. There are several equivalent definitions of dimension. We have chosen the most elementary; it is frequently referred to as *Lebesgue covering dimension*.

The following is immediate from the existence of Lebesgue numbers.

Lemma 2.1. *A compact metric space X has dimension $\leqslant k$ if and only if it admits open covers of order k with arbitrarily small mesh.*

We call a map $f : X \to Y$ between metric spaces a δ-ϵ-map if, for each $A \subseteq Y$ of diameter $\leqslant \delta$, $f^{-1}(A)$ has diameter $\leqslant \epsilon$.

Lemma 2.2. *A compact metric space X has dimension $\leqslant k$ if and only if, for every $\epsilon > 0$, there exists $\delta > 0$ and a δ-ϵ-map from X to a space Y of dimension $\leqslant k$.*

Proof. The identity map satisfies the forward implication, so we proceed to the converse. Since every subspace of Y also has dimension $\leqslant k$, we may assume that f is onto and Y is compact. Choose an open cover $\mathcal{V}$ of Y of order $\leqslant k$ and mesh $\leqslant \delta$. Then $\mathcal{U} = \{f^{-1}(V) \mid V \in \mathcal{V}\}$ has order $\leqslant k$ has mesh $\leqslant \epsilon$. $\square$

The final lemma of this section is less trivial than the previous two, but it goes back to the beginnings of the subject. We refer the reader to the original source [8] for a proof.

Lemma 2.3. *For any nonempty locally metric compact space X, $\dim(X \times [0,1]) = \dim X + 1$.*

Remark 3. Although elementary, Lemma 2.3 is not immediate. There exist compact spaces X and Y with $\dim(X \times Y) < \dim X + \dim Y$. The point of the Hurewicz paper is that this does not happen when Y is 1-dimensional. As with our main theorem, more general results are possible.

3. Proof of Theorem 1.1

Proof. Suppose A is a $\mathcal{Z}$-set in a compact finite-dimensional metric ANR Y. Let $\epsilon > 0$, and set $\delta := \frac{\epsilon}{3}$. By definition there is a homotopy $J : Y \times [0,1] \to Y$ satisfying $J_0 = \mathrm{id}_Y$ and $J_t(Y) \subseteq Y - A$ for all $t > 0$.

By compactness of Y, we may choose $T > 0$ so that $d(y, J(y,t)) < \frac{\epsilon}{3}$ for all $y \in Y$ and $t \in [0,T]$. Define $H : Y \times [0,1] \to Y$ by $H(y,t) = J(y, t \cdot T)$. Now $d(y, H(y,t)) < \frac{\epsilon}{3}$ for all $(y,t) \in Y \times [0,1]$, so H_1 is a δ-ϵ-map from Y to $Y - A$, and by Lemma 2.2 $\dim Y = \dim(Y - A)$.

To show that $\dim A < \dim Y$, we will define $\delta' > 0$ and use the homotopy H to construct a δ'-ϵ-map from $A \times [0,1]$ into Y, where $A \times [0,1]$ is endowed with the ℓ_∞-metric $d = \max\{d_1, d_2\}$. From there, an application of Lemma 2.3 completes the proof.

Pick $k \in \mathbb{N}$ so that $\frac{1}{k} < \frac{\epsilon}{3}$. Then choose open subsets $U_0, U_1, \dots, U_{k+1} \subseteq Y$, and $t_1, t_2, t_3, \dots, t_{k+1} \in [0,1]$ in the following way:

- $U_0 = \emptyset$ and $t_1 = 1$;
- $U_1 \supseteq H(Y \times \{1\})$ and $\overline{U}_1 \cap A = \emptyset$;
- for $i = 2, 3, \dots, k$, choose t_i so that $H(A \times [0, t_i]) \cap \overline{U}_{i-1} = \emptyset$ and choose U_i containing $H(Y \times [t_i, 1]) \cup \overline{U}_{i-1}$ with $\overline{U}_i \cap A = \emptyset$; and
- let $t_{k+1} = 0$, and $U_{k+1} = Y$.

Then $0 = t_{k+1} < t_k < \cdots < t_2 < t_1 = 1$ and $\emptyset = U_0 \subseteq U_1 \subseteq U_2 \subseteq \cdots \subseteq U_k \subseteq U_{k+1} = Y$.

Restrict and reparametrize H via the piecewise linear homeomorphism $\lambda : [0,1] \to [0,1]$ satisfying $\lambda(0) = 0$, $\lambda(1) = 1$, and $\lambda(\frac{i}{k}) = t_{k-i+1}$, to get a new homotopy $F : A \times [0,1] \to Y$ defined by $F(z,s) = H(z, \lambda(s))$.

For each $i = 1, 2, \dots, k$, choose $\delta_i > 0$ so that $B(y, \delta_i) \subseteq U_{i+1}$ for all $y \in U_i$, and let $\delta' := \min\{\frac{\epsilon}{3}, \delta_i \mid i = 1, 2, \dots, k\}$.

Claim. *F is a δ'-ϵ-map.*

Suppose $(z,s), (z',s') \in F^{-1}(V)$, where $\operatorname{diam} V < \delta'$. Let $y = F(z,s)$, $y' = F(z',s')$, $t = \lambda(s)$, and $t' = \lambda(s')$. Choose $j \in \{1, 2, \dots, k+1\}$ so that $y \in U_j - U_{j-1}$. Then, since $d(y,y') < \delta'$, we have $y' \in$

$U_{j+1} - U_{j-2}$. This implies that $t_{j+1} < t < t_{j-1}$ and $t_{j+2} < t' < t_{j-2}$ by the choice of t_i and U_i, so that $|s - s'| < \frac{3}{k} < \epsilon$. Moreover, $d(z, z') \leqslant d(z, y) + d(y, y') + d(z', y') = d(z, F(z, s)) + d(y, y') + d(F(z', s'), z') = d(z, H(z, \lambda(s))) + d(y, y') + d(H(z', \lambda(s')), z') < \epsilon$ by definition of H and the fact that $\operatorname{diam} V < \delta \leqslant \frac{\epsilon}{3}$. □

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