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We continue our study of ends of noncompact manifolds, with a focus on the inward tameness condition. For manifolds with compact boundary, inward tameness, by itself, has significant implications. For example, such manifolds have stable homology at infinity in all dimensions. Here we show that these manifolds have "almost perfectly semistable" fundamental group at each of their ends. That observation leads to further analysis of the group-theoretic conditions at infinity, and to the notion of a "near pseudocollar" structure. We obtain a complete characterization of n-manifolds ($n \geq 6$) admitting such a structure, thereby generalizing our previous work (*Geom. Topol.* 10 (2006), 541–556). We also construct examples illustrating the necessity and usefulness of the new conditions introduced here. Variations on the notion of a perfect group, with corresponding versions of the Quillen plus construction, form an underlying theme of this work.

1. Introduction

In [Guilbault 2000; Guilbault and Tinsley 2003; 2006] we carried out a program to generalize L. C. Siebenmann's famous manifold collaring theorem [1965] in ways applicable to manifolds with nonstable fundamental group at infinity. Motivated by some important examples of finite-dimensional manifolds and a seminal paper by

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T. A. Chapman and Siebenmann [1976] on Hilbert cube manifolds, we chose the following definitions.

A manifold N^n with compact boundary is called a *homotopy collar* if $\partial N^n \hookrightarrow N^n$ is a homotopy equivalence. If N^n contains arbitrarily small homotopy collar neighborhoods of infinity, we call N^n a *pseudocollar*. Clearly, an actual open collar N^n, i.e., $N^n \approx \partial N^n \times [0, \infty)$, is a special case of a pseudocollar. Fundamental to [Siebenmann 1965; Chapman and Siebenmann 1976] and our earlier work is the notion of inward tameness.

A manifold M^n is *inward tame* if each of its clean neighborhoods of infinity is finitely dominated; it is *absolutely inward tame* if those neighborhoods all have finite homotopy type. An alternative formulation of this definition (see p. 95) justifies the adjective "inward" — a term that helps distinguish this version of tameness from a similar, but inequivalent, version found elsewhere in the literature.

In [Guilbault and Tinsley 2006] a classification of pseudocollarable n-manifolds for $6 \le n < \infty$ was obtained. In simplified form, it says:

Theorem 1.1 (pseudocollarability characterization — simple version). *A* 1*-ended n-manifold M^n ($n \ge 6$) with compact boundary is pseudocollarable if and only if*

(a) *M^n is absolutely inward tame*, *and*

(b) *the fundamental group at infinity is $\mathcal{P}$-semistable.*

A "$\mathcal{P}$-semistable (or perfectly semistable) fundamental group at infinity" indicates that an inverse sequence of fundamental groups of neighborhoods of infinity can be arranged so that bonding homomorphisms are surjective with perfect kernels.

By way of comparison, the simple version of Siebenmann's collaring theorem is obtained by replacing (b) with the stronger condition of π_1-stability, while Chapman and Siebenmann's pseudocollarability characterization for Hilbert cube manifolds is obtained by omitting (b) entirely. Thus, the differences among these three results lie entirely in the fundamental group at infinity.

In this paper we take a close look at n-manifolds satisfying only the inward tameness hypothesis. By necessity, our attention turns to the group theory at the ends of those spaces. Unlike the case of infinite-dimensional manifolds, CW complexes, or even n-manifolds with noncompact boundary, inward tameness has major implications for the fundamental group at the ends of n-manifolds with compact boundary. Unfortunately, inward tameness (ordinary or absolute) does not imply $\mathcal{P}$-semistability — an example from [Guilbault and Tinsley 2003] attests to that — but it comes remarkably close. One of the main results of this paper is the following.

Theorem 1.2. *Let M^n be an inward tame n-manifold with compact boundary. Then M^n has an $\mathcal{AP}$-semistable* (*almost perfectly semistable*) *fundamental group at each of its finitely many ends.*

The initial goals of this paper are developing the appropriate group theory (including the definition of $\mathcal{AP}$-semistable) and proving the above theorem. After that is accomplished, we apply those investigations by proving a structure theorem for manifolds that are inward tame, but not necessarily pseudocollarable.

Theorem 1.3 (near pseudocollarability characterization — simple version).
A 1*-ended n-manifold* M^n $(n \geq 6)$ *with compact boundary is nearly pseudocollarable if and only if*

(a) M^n *is absolutely inward tame*, *and*

(b) *the fundamental group at infinity is* $\mathcal{SAP}$*-semistable.*

The notion of near pseudocollarability will be defined and explored in Section 4. For now, we note that nearly pseudocollarable manifolds admit arbitrarily small clean neighborhoods of infinity N, containing compact codimension 0 submanifolds A for which $A \hookrightarrow N$ is a homotopy equivalence. Obtaining a near pseudocollar structure requires a slight strengthening of $\mathcal{AP}$-semistability to $\mathcal{SAP}$-semistability (strong almost perfect semistability). The essential nature of this stronger condition is verified by a final result, in which our group-theoretic explorations come together in a concrete set of examples.

Theorem 1.4. *For all* $n \geq 6$, *there exist* 1*-ended open n-manifolds that are absolutely inward tame but do not have* $\mathcal{SAP}$*-semistable fundamental group at infinity*, *and thus*, *are not nearly pseudocollarable.*

In Section 7, we close with a discussion of some open questions.

Remark 1.5. Throughout this paper attention is restricted to noncompact manifolds with compact boundaries. When a boundary is noncompact, its end topology gets entangled with that of the ambient manifold, leading to very different issues. In the study of noncompact manifolds, a focus on those with compact boundaries is analogous to a focus on closed manifolds in the study of compact manifolds. An investigation of manifolds with noncompact boundaries is planned for [Guilbault and Gu $\geq$ 2017].

2. Definitions and background

Variations on the notion of a perfect group. In this subsection we review the definition of perfect group and discuss some variations.

Given elements a and b of a group K, the *commutator* $a^{-1}b^{-1}ab$ will be denoted by $[a, b]$. The *commutator subgroup* of K, denoted by $[K, K]$, is the subgroup generated by all commutators. It is a standard fact that $[K, K]$ is normal in K and is the smallest such subgroup with an abelian quotient. We call K *perfect* if $K = [K, K]$.

Now suppose K and J are normal subgroups of G. Define $[K, J]$ to be the subgroup of G generated by the set of commutators

$$[k, j] = \{k^{-1} j^{-1} k j \mid k \in K \text{ and } j \in J\}.$$

The following is standard and easy to verify.

Lemma 2.1. *For normal subgroups K and J of a group G,*

(1) $[K, J] \trianglelefteq G$,

(2) $[K, J] \trianglelefteq K$ *and* $[K, J] \trianglelefteq J$, *and*

(3) $[K, J] = [J, K]$.

Given the above setup, we say that K is *J-perfect* if $K \subseteq [J, J]$, and that K is *strongly J-perfect* if $K \subseteq [K, J]$. By Lemma 2.1, both of these conditions imply that $K \trianglelefteq J$; so we customarily begin with that as an assumption.

The following two lemmas are immediate. We state them explicitly for the purpose of comparison.

Lemma 2.2. *Let $K \trianglelefteq J$ be normal subgroups of G.*

(1) *K is perfect if and only if each element of K can be expressed as $\prod_{i=1}^{k} [a_i, b_i]$, where $a_i, b_i \in K$ for all i.*

(2) *K is J-perfect if and only if each element of K can be expressed as $\prod_{i=1}^{k} [a_i, b_i]$, where $a_i, b_i \in J$ for all i.*

(3) *K is strongly J-perfect if and only if each element of K can be expressed as $\prod_{i=1}^{k} [a_i, b_i]$, where $a_i \in K$ and $b_i \in J$ for all i.*

Lemma 2.3. *Let $K \trianglelefteq J \trianglelefteq L$ be normal subgroups of G.*

(1) *If K is* [*strongly*] *J-perfect, then K is* [*strongly*] *L-perfect for every normal subgroup L containing J.*

(2) *K is* [*strongly*] *K-perfect if and only if K is a perfect group.*

Remark 2.4. Lemma 2.3 suggests a key theme: the smaller the group L for which K is [strongly] L-perfect, the closer K is to being a genuine perfect group.

The various levels of perfectness can be nicely characterized using *group homology*. The $\mathbb{Z}$-homology of a group G may be defined as the $\mathbb{Z}$-homology of a $K(G, 1)$ space K_G. If $\lambda : G \to H$ is a homomorphism, there is a map $f_\lambda : K_G \to K_H$, unique up to basepoint-preserving homotopy, inducing λ on fundamental groups. Define $\lambda_* : H_*(G; \mathbb{Z}) \to H_*(H; \mathbb{Z})$ to be the homomorphism induced by f_λ.

Lemma 2.5. *Let $K \trianglelefteq J$, $i : K \hookrightarrow J$ be inclusion, and $q : J \to J/K$ be projection.*

(1) *K is perfect if and only if $H_1(K; \mathbb{Z}) = 0$.*

(2) *K is J-perfect if and only if* $i_* : H_1(K;\mathbb{Z}) \xrightarrow{0} H_1(J;\mathbb{Z})$ *if and only if* $q_* : H_1(J;\mathbb{Z}) \xrightarrow{\cong} H_1(J/K;\mathbb{Z})$.

(3) *K is strongly J-perfect if and only if K is J-perfect and* $q_* : H_2(J;\mathbb{Z}) \to H_2(J/K;\mathbb{Z})$ *is surjective.*

Proof. Claim (1) is clear from the standard fact that $H_1(K) \cong K/[K,K]$. Claim (2) can be verified with elementary group theory. Claim (3) follows from a well-known 5-term exact sequence due to Stallings [1965] and Stammbach [1966]. Due to its importance in this paper, we state it as a separate lemma. □

Lemma 2.6 (5-term exact sequence for group homology). *Given a normal subgroup K of a group J, there is a natural exact sequence*:

$$H_2(J;\mathbb{Z}) \to H_2(J/K;\mathbb{Z}) \to K/[K,J] \to H_1(J;\mathbb{Z}) \to H_1(J/K;\mathbb{Z}) \to 0.$$

The following elementary facts about group homology will be useful.

Lemma 2.7. *Let $f : X \to Y$ be a map between connected CW complexes and $\lambda : \pi_1(X) \to \pi_1(Y)$ the induced homomorphism. Then*

(1) $H_1(X;\mathbb{Z}) \cong H_1(\pi_1(X,*);\mathbb{Z})$;

(2) $f_* : H_1(X;\mathbb{Z}) \to H_1(Y;\mathbb{Z})$ *realizes* $\lambda_* : H_1(\pi_1(X);\mathbb{Z}) \to H_1(\pi_1(Y);\mathbb{Z})$; *and*

(3) *if* $f_* : H_2(X;\mathbb{Z}) \to H_2(Y;\mathbb{Z})$ *is surjective*, *then* $\lambda_* : H_2(\pi_1(X);\mathbb{Z}) \to H_2(\pi_1(Y);\mathbb{Z})$ *is also surjective.*

Proof. Build a $K(\pi_1(X),1)$ complex X' by attaching cells of dimension ≥ 3 to X and a $K(\pi_1(Y),1)$ complex Y' by attaching cells of dimension ≥ 3 to Y. Both $X \overset{i}{\hookrightarrow} X'$ and $Y \overset{j}{\hookrightarrow} Y'$ induce isomorphisms on π_1 and H_1, so (1) follows. Use the asphericity of Y' to extend f to $f' : X' \to Y'$, also inducing λ on π_1. Clearly $i_* : H_2(X;\mathbb{Z}) \to H_2(X';\mathbb{Z})$ and $j_* : H_2(Y;\mathbb{Z}) \to H_2(Y';\mathbb{Z})$ are surjective.

This gives a commutative diagram

$$\begin{array}{ccc} H_2(X;\mathbb{Z}) & \xrightarrow{f_*} & H_2(Y;\mathbb{Z}) \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle j_*} \\ H_2(\pi_1(X);\mathbb{Z}) & \xrightarrow{f'_*} & H_2(\pi_1(Y);\mathbb{Z}) \end{array}$$

Since the other maps are all surjective, so is f'_*. □

Lastly we offer a topological characterization of the various levels of perfectness. For the purposes of this paper, these are possibly the most useful.

Let S_g denote a compact orientable surface of genus g with a single boundary component. A collection of oriented simple closed curves $\{\alpha_1,\beta_1,\alpha_2,\beta_2,\dots,\alpha_g,\beta_g\}$ on S_g with the property that each α_i intersects β_i transversely at a single point, and each of $\alpha_i \cap \alpha_j$, $\beta_i \cap \beta_j$, and $\alpha_i \cap \beta_j$ is empty when $i \neq j$, is called a *complete*

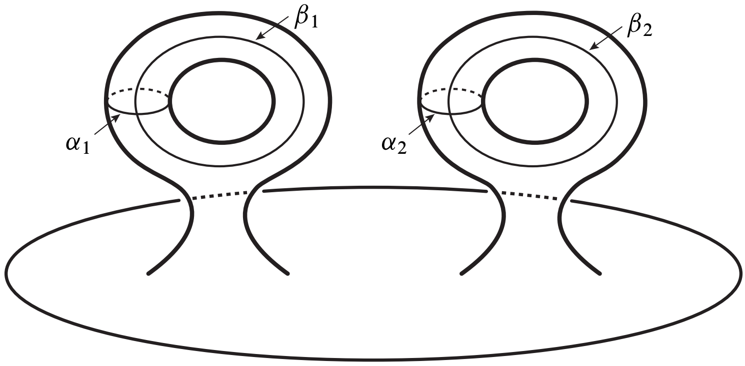


Figure 1. Complete set of handle curves ($g = 2$ case).

set of handle curves for S_g. A complete set of handle curves on S_g is not unique; however, given any such set, there exists a homeomorphism of S_g to the "disk with g handles" pictured in Figure 1 taking each α_i and β_i to the corresponding curves in the diagram.

Given a (not necessarily embedded) loop γ in a topological space X, we say that γ *bounds a compact orientable surface in* X if, for some g, there exists a map $f : S_g \to X$ such that $f|_{\partial S_g} = \gamma$. Notice that we do not require that f be an embedding. We often abuse terminology slightly by saying that γ bounds the surface S_g in X. Similarly, we often do not distinguish between a set of handle curves on S_g and their images in X.

Lemma 2.8. *Let X be a space with $\pi_1(X, x_0) \cong G$ and let $K \trianglelefteq J$ be normal subgroups of G. Then*:

(1) *K is perfect if and only if each loop γ in X representing an element of K bounds a surface S_g in X containing a complete set of handle curves $\{\alpha_1, \beta_1, \alpha_2, \beta_2, \dots, \alpha_g, \beta_g\}$ with each α_i and β_i belonging to K.*

(2) *K is J-perfect if and only if each loop γ in X representing an element of K bounds a surface S_g in X containing a complete set of handle curves $\{\alpha_1, \beta_1, \alpha_2, \beta_2, \dots, \alpha_g, \beta_g\}$ with each α_i and β_i belonging to J.*

(3) *K is strongly J-perfect if and only if each loop γ in X representing an element of K bounds a surface S_g in X containing a complete set of handle curves $\{\alpha_1, \beta_1, \alpha_2, \beta_2, \dots, \alpha_g, \beta_g\}$ with each α_i belonging to K and each β_i belonging to J.*

Remark 2.9. We are being informal in the statement of Lemma 2.8. Since the handle curves are not based, we should also choose, for each pair (α_i, β_i), an arc τ_i in S_g from x_0 to $p_i = \alpha_i \cap \beta_i$. The element of $\pi_1(X, x_0)$ represented by α_i is then $\tau_i * \alpha_i * \tau_i^{-1}$, and similarly for β_i. Notice that, by normality, the question of whether one of these loops belongs to K or J is independent of the choice of τ_i.

Algebra of inverse sequences. Understanding the fundamental group at infinity requires the language of inverse sequences. We briefly review the necessary definitions and terminology.

Throughout this subsection all arrows denote homomorphisms, while those of type $\twoheadrightarrow$ or $\twoheadleftarrow$ specify surjections. The symbol $\cong$ denotes isomorphisms.

Let

$$G_0 \xleftarrow{\lambda_1} G_1 \xleftarrow{\lambda_2} G_2 \xleftarrow{\lambda_3} \cdots$$

be an inverse sequence of groups and homomorphisms. A *subsequence* is an inverse sequence of the form

$$G_{i_0} \xleftarrow{\lambda_{i_0+1}\circ\cdots\circ\lambda_{i_1}} G_{i_1} \xleftarrow{\lambda_{i_1+1}\circ\cdots\circ\lambda_{i_2}} G_{i_2} \xleftarrow{\lambda_{i_2+1}\circ\cdots\circ\lambda_{i_3}} \cdots .$$

In the future we denote a composition $\lambda_i \circ \cdots \circ \lambda_j$ $(i \le j)$ by $\lambda_{i,j}$.

Sequences $\{G_i, \lambda_i\}$ and $\{H_i, \mu_i\}$ are *pro-isomorphic* if, after passing to subsequences, there exists a commuting diagram:

$$\begin{array}{ccccccccccc}
G_{i_0} & & \xleftarrow{\lambda_{i_0+1,i_1}} & & G_{i_1} & & \xleftarrow{\lambda_{i_1+1,i_2}} & & G_{i_2} & & \xleftarrow{\lambda_{i_2+1,i_3}} \quad \cdots \\
& \nwarrow & & \swarrow & & \nwarrow & & \swarrow & & \nwarrow & & \swarrow \\
& & H_{j_0} & & \xleftarrow{\mu_{j_0+1,j_1}} & & H_{j_1} & & \xleftarrow{\mu_{j_1+1,j_2}} & & H_{j_2} & \quad \cdots
\end{array}$$

Clearly an inverse sequence is pro-isomorphic to each of its subsequences. To avoid tedious notation, we often do not distinguish $\{G_i, \lambda_i\}$ from its subsequences. Instead we assume $\{G_i, \lambda_i\}$ has the properties of a preferred subsequence — prefaced by the words "after passing to a subsequence and relabeling".

An inverse sequence $\{G_i, \lambda_i\}$ is *stable* if it is pro-isomorphic to an $\{H_i, \mu_i\}$ for which each μ_i is an isomorphism. A more usable formulation is that $\{G_i, \lambda_i\}$ is stable if, after passing to a subsequence and relabeling, there is a commutative diagram of the form

$$(*) \qquad \begin{array}{ccccccccccccc}
G_0 & & \xleftarrow{\lambda_1} & & G_1 & & \xleftarrow{\lambda_2} & & G_2 & & \xleftarrow{\lambda_3} & & G_3 \xleftarrow{\lambda_4} \cdots \\
& \nwarrow & & \swarrow & & \nwarrow & & \swarrow & & \nwarrow & & \swarrow & \\
& & \operatorname{im}(\lambda_1) & & \xleftarrow{\cong} & & \operatorname{im}(\lambda_2) & & \xleftarrow{\cong} & & \operatorname{im}(\lambda_3) & \xleftarrow{\cong} & \cdots
\end{array}$$

where all unlabeled maps are obtained by restriction. If $\{H_i, \mu_i\}$ can be chosen so that each μ_i is an epimorphism, we call our sequence *semistable* (or *Mittag-Leffler*, or *pro-epimorphic*). In that case, it can be arranged that the maps in the bottom row of $(*)$ are epimorphisms. Similarly, if $\{H_i, \mu_i\}$ can be chosen so that each μ_i is a monomorphism, we call our sequence *pro-monomorphic*; it can then be arranged that the restriction maps in the bottom row of ($*$) are monomorphisms. It is easy to show that an inverse sequence that is semistable and pro-monomorphic is stable.

An inverse sequence is *perfectly semistable* if it is pro-isomorphic to an inverse sequence

$$G_0 \xleftarrow{\lambda_1} G_1 \xleftarrow{\lambda_2} G_2 \xleftarrow{\lambda_3} \cdots$$

of finitely presentable groups and surjections, where each $\ker(\lambda_i)$ is perfect. A straightforward argument [Guilbault 2000, Corollary 1] shows that sequences of this type behave well under passage to subsequences.

Augmented inverse sequences and almost perfect semistability. An *augmentation* of an inverse sequence $\{G_i, \lambda_i\}$ is a sequence $\{L_i\}$, where $L_i \trianglelefteq G_i$ and $\lambda_i(L_i) \leq L_{i-1}$ for each i. The corresponding *augmentation sequence* is the sequence $\{L_i, \lambda|_{L_i}\}$.

The *minimal augmentation* (or the *unaugmented case*) occurs when $L_i = \{1\}$; the *maximal augmentation* is the case where $L_i = G_i$; and the *standard augmentation* occurs when $L_i = \ker \lambda_i$ for each i. Any augmentation where $L_i \leq \ker \lambda_i$ for each i is called a *small augmentation*. For each subsequence $\{G_{k_i}\}$ of a sequence $\{G_i, \lambda_i\}$ augmented by $\{L_i\}$, there is a corresponding augmentation $\{L_{k_i}\}$.

We say that $\{G_i, \lambda_i\}$ satisfies the *$\{L_i\}$-perfectness property* if, for each i, $\ker \lambda_i$ is $\lambda_i^{-1}(L_{i-1})$-perfect; it satisfies the *strong $\{L_i\}$-perfectness property* if each $\ker \lambda_i$ is strongly $\lambda_i^{-1}(L_{i-1})$-perfect. More concisely, if $K_i = \ker \lambda_i$ and $J_i = \lambda_i^{-1}(L_{i-1})$, these conditions require that each K_i be [strongly] J_i-perfect.

Employing the above terminology, we can restate the definition of perfect semistability (abbreviated *$\mathcal{P}$-semistable*) by requiring that the sequence be pro-isomorphic to an inverse sequence of finitely presented groups and surjections satisfying the $\{L_i\}$-perfectness property for the minimal augmentation $\{L_i\} = \{1\}$. More generally, we call an inverse sequence of groups

- *$\mathcal{AP}$-semistable* (for almost perfectly semistable) if it is pro-isomorphic to an inverse sequence $\{G_i, \lambda_i\}$ of finitely presentable groups and surjections, satisfying the $\{L_i\}$-perfectness property for some small augmentation $\{L_i\}$, and
- *$\mathcal{SAP}$-semistable* (for strongly almost perfectly semistable) if it is pro-isomorphic to an inverse sequence $\{G_i, \lambda_i\}$ of finitely presentable groups and surjections satisfying the strong $\{L_i\}$-perfectness property for some small augmentation $\{L_i\}$.

Remark 2.10. Note that an inverse sequence satisfies the [strong] $\{L_i\}$-perfectness property for some small augmentation $\{L_i\}$ if and only if it satisfies that property for the standard augmentation.

When applying sequences of the above types to geometric constructions, it is frequently desirable to pass to subsequences without losing the defining property of the sequence. For that reason, the following observation is crucial.

Proposition 2.11. *If an inverse sequence $\{G_i, \lambda_i\}$ of surjections augmented by $\{L_i\}$ satisfies the* [*strong*] *$\{L_i\}$-perfectness property, then any subsequence $\{G_{k_i}\}$ satisfies the corresponding* [*strong*] *$\{L_{k_i}\}$-perfectness property.*

Proof. Since the proofs for perfectness and strong perfectness are similar, we prove only the latter. Assume $\{G_i, \lambda_i\}$ augmented by $\{L_i\}$ satisfies strong $\{L_i\}$-perfectness. Simplifying notation, a portion of the given subsequence becomes

$$G_a \xleftarrow{\lambda_{a+1,b}} G_b \xleftarrow{\lambda_{b+1,c}} G_c,$$

where $-1 \leq a < b < c$. We must show that

$$\ker(\lambda_{b+1,c}) \subseteq [\ker(\lambda_{b+1,c}), \lambda_{b+1,c}^{-1}(L_b)].$$

Suppose the proposition holds for $j < c$. If $c = b + 1$, then $\lambda_{b+1,c} = \lambda_c$, and the result follows by hypothesis. Now, assume $c \geq b + 2$ and write

$$\lambda_{b+1,c} = \lambda_{b+1,c-1} \circ \lambda_c : G_c \to G_{c-1} \to G_b.$$

Let $\omega \in \ker(\lambda_{b+1,c})$; then $\lambda_c(\omega) \in \ker(\lambda_{b+1,c-1})$. By induction, $\ker(\lambda_{b+1,c-1}) \subseteq [\ker(\lambda_{b+1,c-1}), \lambda_{b+1,c-1}^{-1}(L_b)]$; so, $\lambda_c(\omega)$ is a product of commutators $[\alpha_m, \beta_m]$, where $\beta_m \in \lambda_{b+1,c-1}^{-1}(L_b)$ and $\alpha_m \in \ker(\lambda_{b+1,c-1})$. Since λ_c is surjective over G_{c-1} we identify for each m a pair of elements $\alpha'_m, \beta'_m \in G_c$ that map to α_m and β_m, respectively. Thus, $\beta'_m \in \lambda_{b+1,c}^{-1}(L_b)$, $\alpha'_m \in \ker(\lambda_{b+1,c})$, and $[\alpha'_m, \beta'_m] \in [\ker(\lambda_{b+1,c}), \lambda_{b+1,c}^{-1}(L_b)]$.

Now, let ν be the product of the commutators with $[\alpha'_m, \beta'_m]$ replacing $[\alpha_m, \beta_m]$. By construction, $\lambda_c(\omega) = \lambda_c(\nu)$ and $\nu \in [\ker(\lambda_{b+1,c}), \lambda_{b+1,c}^{-1}(L_b)]$. Thus,

$$\omega\nu^{-1} \in \ker(\lambda_c) \subseteq [\ker(\lambda_c), \lambda_c^{-1}(L_{c-1})] \subseteq [\ker(\lambda_{b+1,c}), \lambda_{b+1,c}^{-1}(L_b)].$$

Consequently, $\omega \in [\ker(\lambda_{b+1,c}), \lambda_{b+1,c}^{-1}(L_b)]$ as well. □

Topology of ends of manifolds. Next we supply some topological definitions and background. Throughout the paper, $\approx$ represents homeomorphism and $\simeq$ indicates homotopic maps or homotopy equivalent spaces. The word *manifold* means *manifold with* (*possibly empty*) *boundary*. A manifold is *open* if it is noncompact and has no boundary. As noted earlier, we restrict our attention to manifolds with compact boundaries.

For convenience, all manifolds are assumed to be PL; analogous results may be obtained for smooth or topological manifolds in the usual ways. Our standard resource for PL topology is [Rourke and Sanderson 1972]. Some of the results presented here are valid in all dimensions. Others are valid in dimensions ≥ 4 or ≥ 5, but require the purely topological 4-dimensional techniques found in [Freedman and Quinn 1990] for the 4- and/or 5-dimensional cases; there the conclusions are only topological. The main focus of this paper is on dimensions ≥ 6.

Let M^n be a manifold with compact (possibly empty) boundary. A set $N \subseteq M^n$ is a *neighborhood of infinity* if $\overline{M^n - N}$ is compact. A neighborhood of infinity N is *clean* if

- N is a closed subset of M^n,
- $N \cap \partial M^n = \varnothing$, and
- N is a codimension 0 submanifold of M^n with bicollared boundary.

It is easy to see that each neighborhood of infinity contains a clean neighborhood of infinity.

We say that M^n *has k ends* if it contains a compactum C such that, for every compactum D with $C \subseteq D$, $M^n - D$ has exactly k unbounded components, i.e., k components with noncompact closures. When k exists, it is uniquely determined; if k does not exist, we say M^n *has infinitely many ends*. If M^n is k-ended, then it contains a clean neighborhood of infinity N consisting of k connected components, each of which is a 1-ended manifold with compact boundary. Thus, when studying manifolds with finitely many ends, it suffices to understand the 1-ended situation. That is the case in this paper, where our standard hypotheses ensure finitely many ends. (See Theorem 3.1.)

A connected clean neighborhood of infinity with connected boundary is called a *0-neighborhood of infinity*. A 0-neighborhood of infinity N for which $\partial N \hookrightarrow N$ induces a π_1-isomorphism is called a *generalized 1-neighborhood of infinity*. If, in addition, $\pi_j(N, \partial N) = 0$ for $j \leq k$, then N is a *generalized k-neighborhood of infinity*.

A nested sequence $N_0 \supset N_1 \supset N_2 \supset \cdots$ of neighborhoods of infinity is *cofinal* if $\bigcap_{i=0}^{\infty} N_i = \varnothing$. We will refer to any cofinal sequence $\{N_i\}$ of closed neighborhoods of infinity with $N_{i+1} \subseteq \operatorname{int} N_i$, for all i, as an *end structure* for M^n. Descriptors will be added to indicate end structures with additional properties. For example, if each N_i is clean we call $\{N_i\}$ a *clean end structure*; if each N_i is clean and connected we call $\{N_i\}$ a *clean connected end structure*; and if each N_i is a generalized k-neighborhood of infinity, we call $\{N_i\}$ a *generalized k-neighborhood end structure*.

Remark 2.12. The word "generalized" in the above definitions is in deference to the terminology in [Siebenmann 1965], where the ambient manifold M^n is assumed to have stable fundamental group at infinity. There a (nongeneralized) k-neighborhood of infinity N is also required to satisfy $\pi_1(\varepsilon(M^n)) \xrightarrow{\cong} \pi_1(N)$.

Building upon the above terminology, the primary goal of this paper is to identify, construct, and detect the existence of various end structures for manifolds. A central example: the *pseudocollar* can be described as an end structure $\{N_i\}$ where each N_i is a homotopy collar.

We say M^n is *inward tame* if, for arbitrarily small neighborhoods of infinity N, there exist homotopies $H : N \times [0,1] \to N$ such that $H_0 = \mathrm{id}_N$ and $\overline{H_1(N)}$ is compact. Thus inward tameness means each neighborhood of infinity can be pulled into a compact subset of itself. In this case we refer to H as a *taming homotopy*.

In [Guilbault 2000], the existence of generalized $(n-3)$-neighborhood end structures is shown for all inward tame M^n $(n \geq 5)$.

Recall that a space X is *finitely dominated* if there exists a finite complex K and maps $u : X \to K$ and $d : K \to X$ such that $d \circ u \simeq \mathrm{id}_X$. The following lemma uses this notion to offer equivalent formulations of inward tameness.

Lemma 2.13 [Guilbault and Tinsley 2003, Lemma 2.4]. *For a manifold M^n, the following are equivalent.*

(1) *M^n is inward tame.*

(2) *Each clean neighborhood of infinity in M^n is finitely dominated.*

(3) *For each clean end structure $\{N_i\}$, the inverse sequence*

$$N_0 \xleftarrow{j_1} N_1 \xleftarrow{j_2} N_2 \xleftarrow{j_3} \cdots$$

is pro-homotopy equivalent to an inverse sequence of finite polyhedra.

Given a clean connected end structure $\{N_i\}_{i=0}^{\infty}$, basepoints $p_i \in N_i$, and paths $\alpha_i \subseteq N_i$ connecting p_i to p_{i+1}, we obtain an inverse sequence:

$$\pi_1(N_0, p_0) \xleftarrow{\lambda_1} \pi_1(N_1, p_1) \xleftarrow{\lambda_2} \pi_1(N_2, p_2) \xleftarrow{\lambda_3} \cdots .$$

Here, each $\lambda_{i+1} : \pi_1(N_{i+1}, p_{i+1}) \to \pi_1(N_i, p_i)$ is the homomorphism induced by inclusion followed by the change-of-basepoint isomorphism determined by α_i. The singular ray obtained by piecing together the α_i is called the *base ray* for the inverse sequence. Provided the sequence is semistable, its pro-isomorphism class does not depend on any of the choices made above (see [Guilbault 2016] or [Geoghegan 2008, §16.2]). In the absence of semistability, the pro-isomorphism class of the inverse sequence depends on the base ray; hence, the ray becomes part of the data. The same procedure may be used to define $\pi_k(\varepsilon(M^n))$ for all $k \geq 1$. Similarly, but without need for a base ray or connectedness, we may define $H_k(\varepsilon(M^n))$.

Wall [1965] showed that each finitely dominated connected space X determines a well-defined $\sigma(X) \in \widetilde{K}_0(\mathbb{Z}[\pi_1 X])$ (the reduced projective class group) that vanishes if and only if X has the homotopy type of a finite complex. Given a clean connected end structure $\{N_i\}_{i=0}^{\infty}$ for an inward tame M^n, we have a Wall finiteness obstruction $\sigma(N_i)$ for each i. These may be combined into a single obstruction

$$\sigma_\infty(M^n) = (-1)^n(\sigma(N_0), \sigma(N_1), \sigma(N_2), \dots) \in \widetilde{K}_0(\pi_1(\varepsilon(M^n))) \equiv \varprojlim \widetilde{K}_0(\mathbb{Z}[\pi_1 N_i])$$

that is well defined and which vanishes if and only if each clean neighborhood of infinity in M^n has finite homotopy type. See [Chapman and Siebenmann 1976] or [Guilbault 2000] for details.

We now state the full version of the main theorem of [Guilbault and Tinsley 2006].

Theorem 2.14 (pseudocollarability characterization — complete version). *A 1-ended n-manifold M^n ($n \geq 6$) with compact boundary is pseudocollarable if and only if*

(1) *M^n is inward tame,*

(2) *$\pi_1(\varepsilon(M^n))$ is $\mathcal{P}$-semistable, and*

(3) *$\sigma_\infty(M^n) = 0 \in \widetilde{K}_0(\pi_1(\varepsilon(M^n)))$.*

3. Some consequences of inward tameness

In this section we show that, for manifolds with compact boundary, the inward tameness condition, by itself, has significant implications. The main goal is a proof of Theorem 1.2 — that every inward tame manifold with compact boundary has $\mathcal{AP}$-semistable fundamental group at each of its finitely many ends. Results in this section are valid in all (finite) dimensions and build upon an earlier theorem.

Theorem 3.1 [Guilbault and Tinsley 2003]. *If an n-manifold with compact (possibly empty) boundary is inward tame, then it has finitely many ends, each of which has semistable fundamental group and stable homology in all dimensions.*

Remark 3.2. Note that *none* of the above conclusions is valid for Hilbert cube manifolds, polyhedra, or manifolds with noncompact boundary. See, for example, [Guilbault 2016, §4.5].

As preparation for the proof of Theorem 1.2, we look at an easier result that follows directly from Theorem 3.1.

Let M^n be an inward tame n-manifold with compact boundary. Since M^n is finite-ended, there is no loss of generality in assuming that M^n is 1-ended. By taking a product with $\mathbb{S}^k$ ($k \geq 2$) if necessary, we may arrange that $n \geq 6$, without changing the fundamental group at infinity. So, by the semistability conclusion of Theorem 3.1 combined with the generalized 1-neighborhood theorem [Guilbault 2000, Theorem 4], we may choose a generalized 1-neighborhood end structure

$\{N_i\}$ for which each bonding map in the inverse sequence

$$\pi_1(N_0, p_0) \xleftarrow{\lambda_1} \pi_1(N_1, p_1) \xleftarrow{\lambda_2} \pi_1(N_2, p_2) \xleftarrow{\lambda_3} \cdots \tag{3-1}$$

is surjective. Abelianization gives an inverse sequence

$$H_1(N_0) \xleftarrow{\lambda_{1*}} H_1(N_1) \xleftarrow{\lambda_{2*}} H_1(N_2) \xleftarrow{\lambda_{3*}} \cdots, \tag{3-2}$$

which, by Theorem 3.1, is stable. It follows that all but finitely many of the epimorphisms in (3-2) are isomorphisms, so by omitting finitely many terms (then relabeling), we may assume all bonds in (3-2) are isomorphisms. A term-by-term application of Lemma 2.5 gives the following.

Proposition 3.3. *Every* 1*-ended inward tame manifold* M^n *with compact boundary admits a generalized* 1*-neighborhood end structure* $\{N_i\}$ *for which all bonding maps in the sequence* $\{\pi_1(N_i, p_i), \lambda_i\}$ *are surjective and each* $\ker \lambda_i$ *is* $\pi_1(N_i, p_i)$*-perfect*; *in other words*, *if* $\{L_i = \pi_1(N_i, p_i)\}$ *is the maximal augmentation*, *then* $\{\pi_1(N_i, p_i), \lambda_i\}$ *satisfies the* $\{L_i\}$*-perfectness property.*

Theorem 1.2 is a stronger version of Proposition 3.3. For clarity, we restate it in a similar form.

Proposition 3.4. *Every* 1*-ended inward tame manifold* M^n *with compact boundary admits a generalized* 1*-neighborhood end structure* $\{N_i\}$ *for which all bonding maps in the sequence* $\{\pi_1(N_i, p_i), \lambda_i\}$ *are surjective and*, *if we let* $K_i = \ker \lambda_i$ *for each* $i \geq 1$ (*the standard augmentation*), *then* K_i *is* $\lambda_i^{-1}(K_{i-1})$*-perfect for all* $i \geq 2$. *In other words*, $\{\pi_1(N_i, p_i), \lambda_i\}$ *satisfies the* $\{K_i\}$*-perfectness property*; *so* M^n *has* $\mathcal{AP}$*-semistable fundamental group at infinity.*

Proof. Assume the sequence $\{N_i\}$ was chosen so that, for each i, N_{i+1} is sufficiently small that a taming homotopy H^i pulls N_i into $A_i = N_i - \operatorname{int} N_{i+1}$, i.e., $H_1^i(N_i) \subseteq A_i$, and N_{i+3} is sufficiently small that $H^i(\partial N_{i+2} \times [0,1]) \cap N_{i+3} = \varnothing$. By compactness of $H_1^i(N_i)$ and $H^i(\partial N_{i+2} \times [0,1])$ those choices can be made.

Now let $i \geq 2$ be fixed and $q_{i-2} : \widetilde{N}_{i-2} \to N_{i-2}$ be the universal covering projection. Let $\widetilde{A}_{i-2} = q_{i-2}^{-1}(A_{i-2})$ and for $j > i-2$, $\widehat{N}_j = q_{i-2}^{-1}(N_j)$ and $\widehat{A}_j = p_{i-2}^{q-1}(A_j)$. Then

$$\widetilde{N}_{i-2} \supset \widehat{N}_{i-1} \supset \widehat{N}_i \supset \widehat{N}_{i+1};$$

and H^{i-2} lifts to a proper homotopy $\widetilde{H}^{i-2}$ that pulls $\widetilde{N}_{i-2}$ into $\widetilde{A}_{i-2}$ and for which $\widetilde{H}^i(\partial \widehat{N}_i \times [0,1])$ misses $\widehat{N}_{i+1}$.

We may associate $\lambda_i^{-1}(K_{i-1})$ with $\pi_1(\widehat{N}_i)$ and K_i with $\ker(\pi_1(\widehat{N}_i) \to \pi_1(\widehat{N}_{i-1}))$. Thus, an arbitrary element of K_i may be viewed as a loop α in $\partial \widehat{N}_i$ that bounds a disk D in $\widehat{A}_{i-1}$. To prove the proposition, it suffices to show that α bounds an orientable surface in $\widehat{N}_i$. By π_1-surjectivity and the fact that the N_j are generalized 1-neighborhoods, α may be homotoped within $\widehat{A}_i$ to a loop α_0 in $\partial \widehat{N}_{i+1}$. Let E be the cylinder in $\widehat{A}_i$ between α and α_0 traced out by that homotopy. Then the disk $D \cup E$ may be viewed as an element $[\beta] \in H_2(\widehat{A}_i \cup \widehat{A}_{i-1}, \partial \widehat{N}_{i+1})$. Let

$$\widehat{f} : \partial \widehat{N}_i \times [0,1] \cup_{\partial \widehat{N}_i \times \{0\}} \widehat{A}_i \to \widetilde{A}_{i-2} \cup \widehat{A}_{i-1} \cup \widehat{A}_i$$

be the identity on $\widehat{A}_i$ and $\widetilde{H}^{i-2}|$ on $\partial \widehat{N}_i \times [0,1]$. By PL transversality theory (see [Rourke and Sanderson 1968] or [Buoncristiano et al. 1976, §II.4]), we may — after a small proper adjustment that does not alter $\widehat{f}$ on $(\partial \widehat{N}_i \times \{0,1\}) \cup \widehat{A}_i$ — assume that $\widehat{f}^{-1}(\widehat{A}_{i-1} \cup \widehat{A}_i)$ is a manifold with boundary that is a homeomorphism over a collar neighborhood of $\partial \widehat{N}_{i+1}$. Let $\widehat{C}$ be the component of $\widehat{f}^{-1}(\widehat{A}_{i-1} \cup \widehat{A}_i)$ containing that neighborhood. Then, by local characterization of degree, $\widehat{f}| : \widehat{C} \to \widehat{A}_{i-1} \cup \widehat{A}_i$ is a proper degree 1 map, and $\widehat{f}|^{-1}(\partial \widehat{N}_{i+1}) = \partial \widehat{N}_{i+1}$. Thus we have a surjection

$$\widehat{f}|_* : H_2(\widehat{C}, \partial \widehat{N}_{i+1}) \to H_2(\widehat{A}_i \cup \widehat{A}_{i-1}, \partial \widehat{N}_{i+1}).$$

Let $[\beta']$ be a preimage of $[\beta]$. We may assume that β' is an orientable surface with boundary in $\widehat{C}$. Since $\widehat{f}$ is the identity on $\partial \widehat{N}_{i+1}$, $\partial\beta'$ is homologous in $\partial \widehat{N}_{i+1}$ to $\partial\beta = \alpha_0$. Without loss of generality, we may assume that $\partial\beta' = \alpha_0$. Since $\widehat{C}$ lies in $\partial \widehat{N}_i \times [0,1] \cup_{\partial \widehat{N}_i \times \{0\}} \widehat{A}_i$, we may push β', rel boundary, into $\widehat{A}_i$. This provides an orientable surface in $\widehat{A}_i$ with boundary α_0. Gluing the cylinder E to that surface along α_0 produces the bounding surface for α that we desire. □

Early attempts to prove $\mathcal{P}$-semistability (hence pseudocollarability) with only an assumption of absolute inward tameness were brought to a halt by the discovery of a key example presented in [Guilbault and Tinsley 2003]. Ideas contained in that example play an important role here, so we provide a quick review.

An easy way to denote normal subgroups will be helpful. Let G be a group and $S \subseteq G$. The *normal closure of S in G* is the smallest normal subgroup of G containing S. We denote it by $\mathrm{ncl}(S, G)$.

Example 3.5 (main example from [Guilbault and Tinsley 2003]). For all $n \geq 6$, there exist 1-ended absolutely inward tame open n-manifolds with fundamental group system

$$G_0 \overset{\lambda_1}{\twoheadleftarrow} G_1 \overset{\lambda_2}{\twoheadleftarrow} G_2 \overset{\lambda_3}{\twoheadleftarrow} \cdots,$$

where

$$G_i = \left\langle a_0, a_1, \ldots, a_i \;\middle|\; a_1 = [a_1, a_0],\ a_2 = [a_2, a_1],\ \ldots,\ a_i = [a_i, a_{i-1}] \right\rangle$$

and λ_i sends a_j to a_j for $0 \leq j \leq i-1$ and a_i to 1.

By a largely algebraic argument, it was shown that these examples do not have $\mathcal{P}$-semistable fundamental group at infinity, and thus are not pseudocollarable. Notice, however, that each $K_i = \ker \lambda_i$ is the normal closure of a_i and $a_i = [a_i, a_{i-1}]$ in G_i; so $K_i \trianglelefteq [K_i, \lambda_i^{-1}(K_{i-1})]$. In other words, $\{G_i, \lambda_i\}$ satisfies the strong $\{K_i\}$-perfectness property, and is therefore $\mathcal{SAP}$-semistable.

In addition to the above algebra, these examples have nice topological properties. Although they do not contain small homotopy collar neighborhoods of infinity, they do contain arbitrarily small generalized 1-neighborhoods of infinity N for which $\partial N \hookrightarrow N$ is $\mathbb{Z}$-homology equivalence. In fact, they contain a sequence $\{N_i\}$ of generalized 1-neighborhoods of infinity with $\pi_1(N_i) \cong G_i$ and $\partial N_i \hookrightarrow N_i$ a $\mathbb{Z}[G_{i-1}]$-homology equivalence.

The observation in Example 3.5 provides much of the motivation for the remainder of this paper.

4. Generalizing one-sided h-cobordisms, homotopy collars and pseudocollars

We begin developing ideas for placing Example 3.5 into a general context. We will see that end structures like those found in that example are possible only when kernels satisfy a strong relative perfectness condition. Conversely, we will show that whenever such a group-theoretic condition is present, a corresponding "near pseudocollar" structure is attainable.

We have already defined a pseudocollar structure on the end of a manifold M^n to be an end structure $\{N_i\}$ for which each N_i is a homotopy collar, i.e., each $\partial N_i \hookrightarrow N_i$ is a homotopy equivalence. The existence of such a structure allows us to express each N_i as a union

$$N_i = W_i \cup W_{i+1} \cup W_{i+2} \cup \cdots ,$$

where $W_i = N_i - \operatorname{int} N_{i+1}$, and each triple $(W_i, \partial N_i, \partial N_{i+1})$ is a compact *one-sided h-cobordism* in the sense that $\partial N_i \hookrightarrow W_i$ is a homotopy equivalence (and $\partial N_{i+1} \hookrightarrow W_i$ is probably not). One-sided cobordisms play an important role in manifold topology in general, and the topology of ends in particular. See [Guilbault 2000, §4] for details. For later use, we review a few key properties of one-sided h-cobordisms. See, for example, [Guilbault and Tinsley 2003, Theorem 2.5].

Theorem 4.1. *Let (W, P, Q) be a compact cobordism between closed manifolds with $P \hookrightarrow W$ a homotopy equivalence. Then*

(1) *$P \hookrightarrow W$ and $Q \hookrightarrow W$ are $\mathbb{Z}[\pi_1(W)]$-homology equivalences, i.e.,*

$$H_*(W, P; \mathbb{Z}[\pi_1(W)]) = 0 = H_*(W, Q; \mathbb{Z}[\pi_1(W)]);$$

(2) *$\pi_1(Q) \to \pi_1(W)$ is surjective; and*

(3) *$K = \ker(\pi_1(Q) \to \pi_1(W))$ is perfect.*

Moving forward, we require generalizations of the fundamental concepts of homotopy equivalence, homotopy collar, one-sided h-cobordism and pseudocollar:

- Let (X, A) be a CW pair for which $i : A \hookrightarrow X$ induces a π_1-isomorphism and let $L \trianglelefteq \pi_1(A)$. Call i a (mod L)-*homotopy equivalence* if $H_*(X, A; \mathbb{Z}[\pi_1(A)/L])$

is zero for all $*$. Extension to arbitrary maps is accomplished by use of mapping cylinders.

- A manifold N with compact boundary is a $(\operatorname{mod} L)$-*homotopy collar* if $L \trianglelefteq \pi_1(\partial N)$ and $\partial N \hookrightarrow N$ is a $(\operatorname{mod} L)$-homotopy equivalence.
- Let (W, P, Q) be a compact cobordism between closed manifolds and $L \trianglelefteq \pi_1(W)$. We call (W, P, Q) a $(\operatorname{mod} L)$-*one-sided h-cobordism* if $i : P \hookrightarrow W$ is a $(\operatorname{mod} L)$-homotopy equivalence and $j : Q \hookrightarrow W$ induces a surjection on fundamental groups.
- Let $\{N_i\}$ be a generalized 1-neighborhood end structure on a manifold M^n, chosen so that the bonding maps in
$$\pi_1(N_0) \overset{\lambda_1}{\twoheadleftarrow} \pi_1(N_1) \overset{\lambda_2}{\twoheadleftarrow} \pi_1(N_2) \overset{\lambda_3}{\twoheadleftarrow} \cdots .$$
are surjective, and let $\{L_i\}$ be an augmentation of this sequence. Call $\{N_i\}$ a $\operatorname{mod}(\{L_i\})$ *pseudocollar structure* if each $\partial N_i \hookrightarrow N_i$ is a $(\operatorname{mod} L_i)$-homotopy equivalence.

Remark 4.2. (i) Each of the above definitions reduces to its traditional counterpart when the subgroup(s) involved are trivial.

(ii) In the generalization of one-sided h-cobordism, we *require* $j_\# : \pi_1(Q) \to \pi_1(W)$ to be surjective — a condition that is automatic when $L = \{1\}$, but not in general. Analogs of the other two assertions of Theorem 4.1 will be shown to follow.

(iii) For the maximal augmentation, the generalization of pseudocollar requires only that each $\partial N_i \hookrightarrow N_i$ be a $\mathbb{Z}$-homology equivalence, whereas, for the trivial augmentation, we have a genuine pseudocollar. The key dividing line between those extremes occurs when $\{L_i\}$ is a small augmentation ($L_i \leq \ker \lambda_i$ for all i). In those cases, we call $\{N_i\}$ a *near pseudocollar* structure, and say that a 1-ended M^n with compact boundary is *nearly pseudocollarable* if it admits such a structure. The geometric significance of the small augmentation requirement will become clear in the proof of Theorem 5.1. Further discussion of that topic is contained in Section 7.

The following lemma adds topological meaning to the definition of $(\operatorname{mod} L)$-homotopy equivalence.

Lemma 4.3. *Let (X, A) be a CW pair for which $i : A \hookrightarrow X$ induces a π_1-isomorphism, $L \trianglelefteq \pi_1(A)$, and $S \subseteq L$ for which* $\operatorname{ncl}(S, \pi_1(A)) = L$. *Obtain A' from A by attaching a 2-disk D_s along each $s \in S$; let $X' = X \cup \left(\bigcup_{s\in S} D_s\right)$, and $i' : A' \hookrightarrow X'$. Then i is a* $(\operatorname{mod} L)$*-homotopy equivalence if and only if i' is a homotopy equivalence.*

Proof. Let $p : \widehat{X} \to X$ be the covering projection corresponding to L. Then $\widehat{A} = p^{-1}(A)$ is the cover of A corresponding to L. Viewing S as a collection of loops in A and $\widehat{S}$ the set of all lifts of those loops, then attaching 2-disks to $\widehat{A}$ (and simultaneously $\widehat{X}$) along $\widehat{S}$ produces universal covers $\widetilde{A}'$ of A' and $\widetilde{X}'$ of X'.

Assume now that $i : A \hookrightarrow X$ is a $(\bmod L)$-homotopy equivalence. Then by Shapiro's lemma [Davis and Kirk 2001, p. 100], $H_*(\widehat{X}, \widehat{A}; \mathbb{Z}) = 0$, so by excision $H_*(\widetilde{X}', \widetilde{A}'; \mathbb{Z}) = 0$. Because both spaces are simply connected, the relative Hurewicz theorem implies that $\pi_*(\widetilde{X}', \widetilde{A}') = 0$; therefore $\pi_*(X', A') = 0$. By Whitehead's theorem i' is a homotopy equivalence.

Conversely, if i' is a homotopy equivalence, then its lift $\widetilde{A}' \hookrightarrow \widetilde{X}'$ is a homotopy equivalence. Therefore $H_*(\widetilde{X}', \widetilde{A}'; \mathbb{Z}) = 0$, so by excision $H_*(\widehat{X}, \widehat{A}; \mathbb{Z}) = 0$, and by Shapiro's lemma $H_*(X, A; \mathbb{Z}[\pi_1(A)/L]) = 0$. □

The following is a useful corollary.

Lemma 4.4. *Let (X, A) be a CW pair for which $i : A \hookrightarrow X$ induces a π_1-isomorphism and suppose $L \trianglelefteq \pi_1(A)$. If $H_*(X, A; \mathbb{Z}[\pi_1(A)/L]) = 0$, then $H_*(X, A; \mathbb{Z}[\pi_1(A)/J]) = 0$ for any J with $L < J \trianglelefteq \pi_1(A)$. In particular, $H_*(X, A; \mathbb{Z}) = 0$.*

The next observation is a direct analog of Theorem 4.1.

Theorem 4.5. *Let (W, P, Q) be a compact* $(\bmod L)$*-one-sided h-cobordism between closed manifolds with $L \trianglelefteq \pi_1(W)$. Let $j : Q \hookrightarrow W$ and $L' = j_\#^{-1}(L)$. Then*

(1) *both $P \hookrightarrow W$ and $Q \hookrightarrow W$ are $\mathbb{Z}[\pi_1(W)/L]$-homology equivalences, i.e.,*

$$H_*(W, P; \mathbb{Z}[\pi_1(W)/L]) = 0 = H_*(W, Q; \mathbb{Z}[\pi_1(W)/L]);$$

and

(2) $K = \ker j_\# \trianglelefteq \pi_1(Q)$ *is strongly L'-perfect.*

Proof. First note that by the surjectivity of $j_\# : \pi_1(Q) \to \pi_1(W)$, there is a canonical isomorphism $\pi_1(Q)/L' \xrightarrow{\cong} \pi_1(W)/L$ that is assumed throughout. Let $p : \widehat{W}_L \to W$ be the covering projection corresponding to L, $\widehat{P} = p^{-1}(P)$ and $\widehat{Q} = p^{-1}(Q)$. Then both $\widehat{P}$ and $\widehat{Q}$ are connected, and their projections onto P and Q are the coverings corresponding to L and L'.

The assertion that $H_*(W, P; \mathbb{Z}[\pi_1(W)/L]) = 0$ is part of the hypothesis, and (by Shapiro's lemma [Davis and Kirk 2001, p. 100]) equivalent to the assumption that $H_*(\widehat{W}_L, \widehat{P}; \mathbb{Z}) = 0$. To show that $H_*(W, Q; \mathbb{Z}[\pi_1(W)/L])$ vanishes in all dimensions, it suffices to show that $H_*(\widehat{W}_L, \widehat{Q}; \mathbb{Z}) = 0$. This will follow from Poincaré duality for noncompact manifolds if we can verify:

Claim. $H^*_f(\widehat{W}_L, \widehat{P}; \mathbb{Z}) = 0$, *where the f indicates cellular cohomology based on finite cochains.* (See [Geoghegan 2008, Chapter 12].)

Applying Lemma 4.3, attach 2-cells to W along a collection S of loops in P to kill L, obtaining spaces P' and W', and a homotopy equivalence $P' \hookrightarrow W'$.

Since W is compact, any strong deformation retraction of W' onto P' is proper, and hence, lifts to a proper strong deformation retraction of universal covers $\widetilde{W}'$ onto $\widetilde{P}'$ [Geoghegan 2008, §10.1]. It follows that $H_f^*(\widehat{W}', \partial \widehat{N}'_{i-1}; \mathbb{Z}) = 0$. Both universal covers are obtained by attaching disks along the collection $\widehat{S}$ of lifts to $\widehat{P}$ and $\widehat{W}$ of the loops in S. By excising the interiors of those disks, we conclude that $H_f^*(\widehat{W}, \partial \widehat{N}; \mathbb{Z}) = 0$.

To verify assertion (2), consider the short exact sequence

$$1 \to K \to L' \xrightarrow{q} L'/K \to 1,$$

where L'/K may be identified with L. Lemma 2.6 provides the 5-term exact sequence

$$H_2(L'; \mathbb{Z}) \xrightarrow{q_{*2}} H_2(L'/K; \mathbb{Z}) \to K/[K, L'] \to H_1(L'; \mathbb{Z}) \xrightarrow{q_{*1}} H_1(L'/K; \mathbb{Z}) \to 0,$$

from which the L'-perfectness of K can be deduced by showing that q_{*2} is an epimorphism and q_{*1} an isomorphism.

Since $\widehat{Q} \hookrightarrow \widehat{W}_L$ induces $q : L' \to L$ and since $H_2(\widehat{W}_L, \widehat{Q}; \mathbb{Z}) = 0$, the long exact sequence for that pair ensures that $H_1(L'; \mathbb{Z}) \xrightarrow{\cong} H_1(L; \mathbb{Z})$. In addition, the surjectivity of $H_2(\widehat{Q}; \mathbb{Z}) \to H_2(\widehat{W}_L; \mathbb{Z})$ combines with Lemma 2.7 to imply the surjectivity of $H_2(L'; \mathbb{Z}) \to H_2(L; \mathbb{Z})$. □

5. Structure of inward tame ends

With all necessary definitions in place, we are ready to prove the second main theorem described in the introduction. We begin by stating a strong form of the theorem, written in the style of earlier characterization theorems from [Siebenmann 1965; Guilbault and Tinsley 2006].

Theorem 5.1 (near pseudocollarability characterization). *A 1-ended n-manifold M^n ($n \geq 6$) with compact boundary is nearly pseudocollarable if and only if*

(1) *M^n is inward tame*,

(2) *the fundamental group at infinity is $\mathcal{SAP}$-semistable*, *and*

(3) $\sigma_\infty(M^n) = 0 \in \widetilde{K}_0(\pi_1(\varepsilon(M^n)))$.

Recall that condition (2) calls for the existence of a representation of $\pi_1(\varepsilon(M^n)$ of the form

$$G_0 \xleftarrow{\lambda_1} G_1 \xleftarrow{\lambda_2} G_2 \xleftarrow{\lambda_3} \cdots \tag{5-1}$$

with a small augmentation $\{L_i\}$ ($L_i \trianglelefteq K_i = \ker \lambda_i$ for all i) so that each K_i is strongly J_i-perfect, where $J_i = \lambda_i^{-1}(L_{i-1})$.

Proof. First we verify that a nearly pseudocollarable 1-ended manifold with compact boundary must satisfy conditions (1)–(3).

The hypothesis provides a generalized 1-neighborhood end structure $\{N_i\}$ on M^n with group data

$$G_0 \overset{\lambda_1}{\twoheadleftarrow} G_1 \overset{\lambda_2}{\twoheadleftarrow} G_2 \overset{\lambda_3}{\twoheadleftarrow} \cdots \tag{5-2}$$

($G_i = \pi_1(N_i)$) and a small augmentation $\{L_i\}$ ($L_i \trianglelefteq K_i = \ker \lambda_i$) such that each N_i is a $\operatorname{mod}(L_i)$-homotopy collar.

To simultaneously verify (1) and (3), it suffices to exhibit a cofinal sequence of clean neighborhoods of infinity, each having finite homotopy type. Lemma 4.4 ensures that each N_i is a $\operatorname{mod}(K_i)$-homotopy collar, and since each λ_i is a surjection between finitely presented groups, each K_i is finitely generated as a normal subgroup of G_i. Let i be fixed and $A = \{\alpha_j\}$ be a finite collection of loops in ∂N_i that normally generates K_i in G_i. By Lemma 4.3, if we abstractly attach a 2-disk Δ_j^2 along each α_j, we obtain a homotopy equivalence

$$\partial N_i \cup \left(\bigcup \Delta_j^2\right) \hookrightarrow N_i \cup \left(\bigcup \Delta_j^2\right).$$

In particular, $N_i \cup \left(\bigcup \Delta_j^2\right)$ has the homotopy type of a finite complex. But, since each α_j represents an element of $\ker \lambda_i$, we may assume that each Δ_j^2 is properly embedded in $N_{i-1} - \operatorname{int} N_i$. By thickening these 2-disks to 2-handles, we obtain a clean neighborhood of infinity N_i^* with finite homotopy type, lying in N_{i-1}.

This leaves only $\mathcal{SAP}$-semistability to be checked. We will show that (5-2) satisfies the strong $\{L_i\}$-perfectness property; in other words, each K_i is strongly J_i-perfect, where $J_i = \lambda_i^{-1}(K_{i-1})$.

For each $i > 0$, let $W_{i-1} = N_{i-1} - \operatorname{int} N_i$.

Claim. *$(W_{i-1}, \partial N_{i-1}, \partial N_i)$ is a* $(\operatorname{mod} L_{i-1})$*-one-sided h-cobordism.*

Fix i and let $p : \widehat{N}_{i-1} \to N_{i-1}$ be the covering corresponding to $L_{i-1} \trianglelefteq G_{i-1} = \pi_1(N_{i-1}) \cong \pi_1(W_{i-1})$. Let $\widehat{W}_{i-1}$ denote $p^{-1}(W_{i-1})$ and let $\widehat{N}_i$ denote $p^{-1}(N_i)$. Then $\widehat{W}_{i-1}$ is the cover of W_{i-1} corresponding to J_{i-1}, and $\widehat{N}_i$ is the cover of N_i corresponding to $J_i \trianglelefteq G_i = \pi_1(N_i)$. By Lemma 4.4 and Shapiro's lemma

$$0 = H_*(N_i, \partial N_i; \mathbb{Z}[G_i/J_i]) \cong H_*(\widehat{N}_i, \partial \widehat{N}_i; \mathbb{Z}),$$

and from the long exact homology sequence for the triple $(\widehat{N}_{i-1}, \widehat{W}_{i-1}, \partial \widehat{N}_{i-1})$, excision and Shapiro's lemma

$$H_*(\widehat{W}_{i-1}, \partial \widehat{N}_{i-1}; \mathbb{Z}) \cong H_*(W_{i-1}, \partial N_{i-1}; \mathbb{Z}[G_{i-1}/L_{i-1}]) = 0.$$

The claim follows.

Finally, since the bonding map $G_{i-1} \overset{\lambda_i}{\twoheadleftarrow} G_i$ is represented by the inclusion $W_{i-1} \hookleftarrow \partial N_i$, K_i is strongly J_i-perfect by Theorem 4.5.

For the converse, we must show that conditions (1)–(3) imply the existence of a near pseudocollar structure on M^n. Though the proof is rather complicated, it follows the same outline as that in [Guilbault 2000], which followed the original proof in [Siebenmann 1965]. For a full understanding, the reader should be familiar with [Guilbault 2000]. The new argument presented here generalizes the final portions of that proof. A concise review of [Guilbault 2000] can be found in [Guilbault and Tinsley 2006, §4].

In [Guilbault 2000; Guilbault and Tinsley 2006] the goal was to improve arbitrarily small neighborhoods of infinity to homotopy collars. That is impossible with our weaker hypotheses; instead, the goal is to improve neighborhoods of infinity to homotopy collars modulo certain subgroups of their fundamental groups.

By condition (2) the pro-isomorphism class of $\pi_1(\varepsilon(M^n))$ may be represented by a sequence

$$G_0 \overset{\lambda_1}{\twoheadleftarrow} G_1 \overset{\lambda_2}{\twoheadleftarrow} G_2 \overset{\lambda_3}{\twoheadleftarrow} \cdots \tag{5-3}$$

of finitely presented groups, along with a small augmentation $\{L_i\}$ ($L_i \trianglelefteq K_i = \ker \lambda_i$ for all i) so that each K_i is strongly J_i-perfect, where $J_i = \lambda_i^{-1}(L_{i-1})$.

By [Guilbault 2000, Lemma 8] there is a sequence $\{N_i\}$ of generalized 1-neighborhoods of infinity whose inverse sequence of fundamental groups is isomorphic to a subsequence of $\{G_i\}$.

$$\begin{array}{ccccccccc}
G_{i_0} & \overset{\lambda_{i_0+1,i_1}}{\twoheadleftarrow} & G_{i_1} & \overset{\lambda_{i_1+1,i_2}}{\twoheadleftarrow} & G_{i_2} & \overset{\lambda_{i_2+1,i_3}}{\twoheadleftarrow} & G_{i_3} & \overset{\lambda_{i_3+1,i_4}}{\twoheadleftarrow} & \cdots \\
\updownarrow \cong & & \updownarrow \cong & & \updownarrow \cong & & \updownarrow \cong & & \\
\pi_1(N_0,p_0) & \overset{\text{inc}_\#}{\twoheadleftarrow} & \pi_1(N_1,p_1) & \overset{\text{inc}_\#}{\twoheadleftarrow} & \pi_1(N_2,p_2) & \overset{\text{inc}_\#}{\twoheadleftarrow} & \pi_1(N_3,p_3) & \overset{\text{inc}_\#}{\twoheadleftarrow} & \cdots
\end{array}$$

This diagram and Proposition 2.11 ensure that, for each j, $\ker(\lambda_{i_{j-1}+1,i_j})$ is strongly $\lambda^{-1}_{i_{j-1}+1,i_j}(L_{i_{j-1}})$-perfect. So by passing to this subsequence and relabeling, we may assume that sequence (5-1) and the corresponding subgroup data match the fundamental group data of $\{N_i\}$. Note here that the J-groups (which are not viewed as part of the original data) are not the same as the previous J-groups; they are now preimages of *compositions* of the original bonding maps.

Next we inductively improve the sequence $\{N_j\}$ to generalized k-neighborhoods of infinity for increasing values of k, up to $k = n-3$. We must frequently pass to subsequences; however, each improvement of a given N_j leaves its fundamental group and that of ∂N_i intact. So at each stage, the "new" fundamental group data will be a subsequence of the original (5-1), along with the subsequence augmentation. The J-groups will change as per their definition, but, by Proposition 2.11, we always maintain the appropriate strong relative perfectness condition.

This neighborhood improvement process uses only the hypothesis that M^n is inward tame; it is identical to that used in [Guilbault 2000, Theorem 5] and outlined in [Guilbault and Tinsley 2006, Theorem 3.2]. To save on notation we relabel the neighborhood sequences and their corresponding groups at each stage, designating the resulting cofinal sequence of generalized $(n-3)$-neighborhoods of infinity by $\{N_i\}$, with $G_i = \pi_1(N_i)$, $\lambda_i : G_i \to G_{i-1}$ the corresponding homomorphism, $L_i \trianglelefteq K_i = \ker\lambda_i$, and $J_i = \lambda_i^{-1}(L_{i-1})$.

For each i, let $R_i = N_i - \mathring{N}_{i+1}$ and consider the collection of cobordisms $\{(R_i, \partial N_i, \partial N_{i+1})\}$. The following summary comprises the contents of Lemmas 11 and 12 of [Guilbault 2000], along with new hypotheses regarding kernels.

(i) Each N_i is a generalized $(n-3)$-neighborhood of infinity.

(ii) Each induced bonding map $\pi_1(N_i) \twoheadleftarrow \pi_1(N_{i+1})$ is surjective.

(iii) Each inclusion $\partial N_i \hookrightarrow R_i \hookrightarrow N_i$ induces a π_1-isomorphism.

(iv) Each $\partial N_{i+1} \hookrightarrow R_i$ induces a π_1-epimorphism with kernel strongly J_i-perfect.

(v) $\pi_k(R_i, \partial N_i) = 0$ for all $k < n-3$ and all i.

(vi) Each $(R_i, \partial N_i, \partial N_{i+1})$ admits a handle decomposition based on ∂N_i containing handles only of index $n-3$ and $n-2$.

(vii) Each N_i admits an infinite handle decomposition with handles only of index $n-3$ and $n-2$.

(viii) Each $(N_i, \partial N_i)$ has the homotopy type of a relative CW pair $(K_i, \partial N_i)$ with $\dim(K_i - \partial N_i) \le n-2$.

The obvious next goal is attempting to improve the N_i to generalized $(n-2)$-neighborhoods of infinity, which by item (viii) would necessarily be homotopy collars. In previous work [Siebenmann 1965; Guilbault 2000; Guilbault and Tinsley 2006], that is the final (and most difficult and interesting) step. The same is true here, where the weakened hypotheses create greater difficulties and the strategy and end goal must eventually be altered. For now, we continue with the earlier strategies by turning our attention to $\pi_{n-2}(N_i, \partial N_i) \cong H_{n-2}(\widetilde{N}_i, \partial\widetilde{N}_i)$, which may be viewed as a $\mathbb{Z}[\pi_1 N_i]$-module $H_{n-2}(N_i, \partial N_i; \mathbb{Z}[\pi_1 N_i])$. The content of [Guilbault 2000, Lemma 13] is given by the next two items.

(ix) $H_{n-2}(\widetilde{N}_i, \partial\widetilde{N}_i)$ is a finitely generated projective $\mathbb{Z}[\pi_1 N_i]$-module.

(x) As an element of $\widetilde{K}_0(\mathbb{Z}[\pi_1 N_i])$, $[H_{n-2}(\widetilde{N}_i, \partial\widetilde{N}_i)] = (-1)^n \sigma(N_i)$, where $\sigma(N_i)$ is the Wall finiteness obstruction for N_i.

Together, these elements of $\widetilde{K}_0(\mathbb{Z}[\pi_1 N_i])$ determine the obstruction $\sigma_\infty(\varepsilon(M^n))$ found in condition (3). From now on we assume that $\sigma_\infty(M^n)$ vanishes. This is equivalent to assuming that each $\sigma(N_i)$ is the trivial element of $\widetilde{K}_0(\mathbb{Z}[\pi_1 N_i])$, in other words, each $H_{n-2}(\widetilde{N}_i, \partial\widetilde{N}_i)$ is a stably free $\mathbb{Z}[\pi_1 N_i]$-module. Therefore:

(xi) By carving out finitely many trivial $(n-3)$-handles from each N_i, we can arrange that $H_{n-2}(\widetilde{N}_i, \partial\widetilde{N}_i)$ is a finitely generated free $\mathbb{Z}[\pi_1 N_i]$-module.

Item (xi) can be done so that these sets remain a generalized $(n-3)$-neighborhood of infinity, and so that their fundamental groups and those of their boundaries are unchanged. Again, to save on notation, we denote the improved collection by $\{N_i\}$. See [Guilbault 2000, Lemma 14] for details.

The finite generation of $H_{n-2}(\widetilde{N}_i, \partial\widetilde{N}_i)$ allows us to, after again passing to a subsequence and relabeling, assume that

(xii) $H_{n-2}(\widetilde{R}_i, \partial\widetilde{N}_i) \twoheadrightarrow H_{n-2}(\widetilde{N}_i, \partial\widetilde{N}_i)$ is surjective for each i.

The long exact sequence for the triple $(\widetilde{N}_i, \widetilde{R}_i, \partial\widetilde{N}_i)$ from there shows that

(xiii) $H_{n-2}(\widetilde{R}_i, \partial\widetilde{N}_i) \xrightarrow{\cong} H_{n-2}(\widetilde{N}_i, \partial\widetilde{N}_i)$ is an isomorphism for each i (and hence, $H_{n-2}(\widetilde{R}_i, \partial\widetilde{N}_i)$ is a finitely generated free $\mathbb{Z}[\pi_1 R_i]$-module).

As above, we may choose handle decompositions for the R_i based on ∂N_i having handles only of index $n-3$ and $n-2$.

From now on, let i be fixed. After introducing some trivial $(n-3, n-2)$-handle pairs, an algebraic lemma and some handle slides allow us to obtain a handle decomposition of R_i based on ∂N_i with $(n-2)$-handles $h_1^{n-2}, h_2^{n-2}, \dots, h_r^{n-2}$ and an integer $s \le r$, such that the subcollection $\{h_1^{n-2}, h_2^{n-2}, \dots, h_s^{n-2}\}$ is a free $\mathbb{Z}[\pi_1 R_i]$-basis for $H_{n-2}(\widetilde{R}_i, \partial\widetilde{N}_i)$. So we have:

(xiv) The $\mathbb{Z}[\pi_1 R_i]$-cellular chain complex for $(R_i, \partial N_i)$ may be expressed as

$$0 \to \langle h_1^{n-2}, \dots, h_s^{n-2}\rangle \oplus \langle h_{s+1}^{n-2}, \dots, h_r^{n-2}\rangle \xrightarrow{\partial} \langle h_1^{n-3}, \dots, h_t^{n-3}\rangle \to 0, \tag{5-4}$$

where

- $\langle h_1^{n-2}, \dots, h_s^{n-2}\rangle$ and $\langle h_{s+1}^{n-2}, \dots, h_r^{n-2}\rangle$ represent free $\mathbb{Z}[\pi_1 R_i]$-submodules of $\widetilde{C}_{n-2}$ generated by the corresponding handles;
- $\langle h_1^{n-3}, \dots, h_t^{n-3}\rangle = \widetilde{C}_{n-3}$ is the free $\mathbb{Z}[\pi_1 R_i]$-module generated by the $(n-3)$-handles in R_i;
- $H_{n-2}(\widetilde{R}_i, \partial\widetilde{N}_i) = \ker\partial = \langle h_1^{n-2}, \dots, h_s^{n-2}\rangle \oplus \{0\}$; and
- ∂ takes $\{0\} \oplus \langle h_{s+1}^{n-2}, \dots, h_r^{n-2}\rangle$ injectively into $\langle h_1^{n-3}, \dots, h_t^{n-3}\rangle$.

Item (xiv) and the preceding paragraph are the content of Lemma 15 in [Guilbault 2000].

To this point, we have only used the hypotheses of inward tameness and triviality of the Wall obstruction to build the structure described by items (i)–(xiv). All arguments used thus far appear in [Guilbault 2000; Guilbault and Tinsley 2006], with simpler analogs in [Siebenmann 1965].

Under the π_1-stability hypothesis of [Siebenmann 1965], $H_{n-2}(\widetilde{R}_i, \partial\widetilde{N}_i)$ can now be killed by sliding the offending $(n-2)$-handles $\{h_1^{n-2}, \dots, h_s^{n-2}\}$ off the $(n-3)$-handles and carving out their interiors. Under the weaker $\mathcal{P}$-semistability hypothesis of [Guilbault and Tinsley 2006], a similar strategy works, but only after

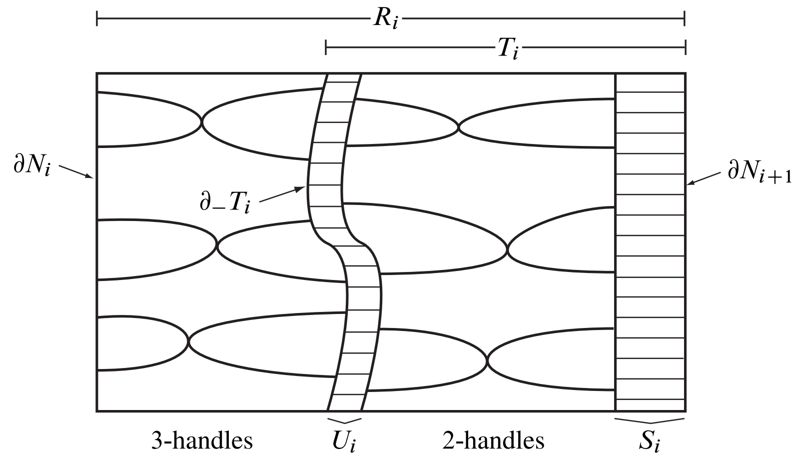


Figure 2. Schematic of R_i.

a significant preparatory step, made possible by perfect kernels. In [Guilbault 2000] an alternate strategy was employed. Instead of killing $H_{n-2}(\widetilde{R}_i, \partial \widetilde{N}_i) = \ker \partial$ by removing its generating handles $\{h_1^{n-2}, \dots, h_s^{n-2}\}$, the task was accomplished by introducing new $(n-3)$-handles, which became images of the $\{h_1^{n-2}, \dots, h_s^{n-2}\}$ under the resulting boundary map, thereby trivializing the kernel. Complete discussions of these approaches can be found in [Guilbault and Tinsley 2006, §3] and [Guilbault 2000, §8]; the strategy employed here is based on the latter.

It is helpful to change our perspective by switching to the dual handle decomposition of R_i. Let S_i be a closed collar neighborhood of ∂N_{i+1} in R_i, and for each $(n-2)$-handle h_k^{n-2} identified earlier, let $\bar{h}_k^2$ be its dual, attached to S_i. Similarly, for each $(n-3)$-handle h_k^{n-3}, let $\bar{h}_k^3$ be its dual. As is standard, the attaching and belt spheres of a given handle switch roles in its dual.

Let $T_i = S_i \cup (\bar{h}_1^2 \cup \dots \cup \bar{h}_s^2 \cup \bar{h}_{s+1}^2 \cup \dots \cup \bar{h}_r^2)$, $\partial_- T_i = \partial T_i - \partial N_{i+1}$, and U_i be a closed collar on $\partial_- T_i$ in T_i. Observe that $R_i = T_i \cup (\bar{h}_1^3 \cup \dots \cup \bar{h}_t^3)$. See Figure 2.

A simplified view of the next step is that we will find a collection of 3-handles $\{\bar{k}_1^3, \dots, \bar{k}_s^3\}$ attached to the left-hand boundary of R_i and lying in R_{i-1} so that the collection $\{\Gamma_j^2\}_{j=1}^s$ of attaching spheres of those 3-handles is algebraically dual to the belt spheres of $\{\bar{h}_1^2, \dots, \bar{h}_s^2\}$ and has trivial algebraic intersection with the belt spheres of $\{\bar{h}_{s+1}^2, \dots, \bar{h}_r^2\}$. Adding those 3-handles to the mix, then inverting the handle decomposition again, results in a cobordism with chain complex

$$(5\text{-}5) \quad 0 \to \langle h_1^{n-2}, \dots, h_s^{n-2} \rangle \oplus \langle h_{s+1}^{n-2}, \dots, h_r^{n-2} \rangle \xrightarrow{\partial} \langle k_1^{n-3}, \dots, h_s^{n-3} \rangle \oplus \langle h_1^{n-3}, \dots, h_t^{n-3} \rangle \to 0$$

in which $\ker \partial = 0$ as desired — but with a caveat. Although addition of the 3-handles does not change the fundamental group of the cobordism, the arranged algebraic intersections between the attaching spheres of $\{\bar{k}_1^3, \dots, \bar{k}_s^3\}$ and the belt spheres of the existing 2-handles are $\mathbb{Z}[\pi_1(R_i)/L_i]$-intersection numbers; this is the best the hypotheses will allow. Then, to arrive at the desired conclusion — that we have effectively killed the relative second homology — it is necessary to switch the coefficient ring to $\mathbb{Z}[\pi_1(R_i)/L_i]$ (in other words, mod out by L_i), and reinterpret (5-5) as a $\mathbb{Z}[\pi_1(R_i)/L_i]$-complex. Then, letting $V_i = N_i \cup (\bar{k}_1^3 \cup \cdots \cup \bar{k}_s^3)$, it follows that

$$\pi_1(V_i) \cong \pi_1(R_i) \cong \pi_1(N_i),$$

$\partial V_i \hookrightarrow V_i$ induces a π_1-isomorphism, and $H_*(V_i, \partial V_i; \mathbb{Z}[\pi_1(R_i)/L_i]) = 0$. In other words, V_i is a mod(L_i)-homotopy collar.

In order to carry out the above program, we first identify a collection $\{\Gamma_j^2\}_{j=1}^s$ of pairwise disjoint 2-spheres in $\partial_- T_i$ algebraically dual over $\mathbb{Z}[\pi_1(R_i)/L_i]$ to the collection $\{\beta_j^{n-3}\}_{j=1}^s$ of belt spheres of the 2-handles $\{\bar{h}_1^2, \dots, \bar{h}_s^2\}$ and having trivial $\mathbb{Z}[\pi_1(R_i)/L_i]$-intersections with the belt spheres $\{\beta_j^{n-3}\}_{j=s+1}^r$ of the remaining 2-handles $\{\bar{h}_{s+1}^2, \dots, \bar{h}_r^2\}$. Keeping in mind that $\pi_1(R_i)/L_i$ is canonically isomorphic to $\pi_1(R_{i+1})/J_{i+1}$, and using the hypothesis that K_{i+1} is strongly J_{i+1}-perfect, such a collection $\{\Gamma_j^2\}_{j=1}^s$ exists, as is shown in [Guilbault and Tinsley 2013, §5]. By general position, the collection can be made disjoint from the attaching tubes of the 3-handles $\{\bar{h}_1^3, \dots, \bar{h}_t^3\}$, so they may be viewed as lying in ∂N_i. If the collection $\{\Gamma_j^2\}_{j=1}^s$ bounds a pairwise disjoint collection of embedded 3-disks in R_{i-1}, regular neighborhoods of those disks would provide the desired 3-handles, and the proof is complete. (The argument from [Guilbault 2000, §8] provides details.)

For $n \geq 7$, the issue is just whether the 2-spheres $\{\Gamma_j^2\}_{j=1}^s$ contract in R_{i-1}. (In dimension 6, a special argument is needed to get pairwise disjoint embeddings.) Contractibility is not guaranteed; but with additional work it can be arranged. The additional work involves the *spherical alteration* of 2-handles developed in [Guilbault and Tinsley 2013]. The idea is to alter the 2-handles $\{\bar{h}_1^2, \dots, \bar{h}_s^2\}$ in a planned manner so that the correspondingly altered $\{\Gamma_j^2\}_{j=1}^s$ contract in the new R_{i-1}. Along the way it will be necessary to reconstruct the 3-handles $\{\bar{h}_1^3, \dots, \bar{h}_t^3\}$ as well; for later use, let $\{\Theta_j^2\}_{j=1}^t$ denote the attaching spheres of those handles.

All of the details were carefully laid out in [Guilbault and Tinsley 2013], with this application in mind. The tailor-made lemma, stated in the final section of that paper, is repeated here.

Lemma 5.2 [Guilbault and Tinsley 2013, Lemma 6.1]. *Let $R' \subseteq R$ be a pair of n-manifolds ($n \geq 6$) with a common boundary component B, and suppose there is a subgroup L' of* $\ker(\pi_1(B) \to \pi_1(R))$ *for which* $K = \ker(\pi_1(B) \to \pi_1(R'))$ *is strongly L'-perfect. Suppose further that there is a clean submanifold $T \subseteq R'$ consisting of a finite collection $\mathcal{H}^2$ of 2-handles in R' attached to a collar neighborhood S of B with $T \hookrightarrow R'$ inducing a π_1-isomorphism (the 2-handles precisely kill* *the group K) and a finite collection $\{\Theta_t^2\}$ of pairwise disjoint embedded 2-spheres in $\partial T - B$, each of which contracts in R'.*

Then on any subcollection $\{h_j^2\}_{j=1}^k \subseteq \mathcal{H}^2$, one may perform spherical alterations to obtain 2-handles $\{\dot{h}_j^2\}_{j=1}^k$ in R' so that in $\partial\dot{T} - B$ (where $\dot{T}$ is the correspondingly altered version of T) there is a collection of 2-spheres $\{\dot{\Gamma}_j^2\}_{j=1}^k$ algebraically dual over $\mathbb{Z}[\pi_1(B)/L']$ to the belt spheres $\{\beta_j^{n-3}\}_{j=1}^k$ common to $\{h_j^2\}_{j=1}^k$ and $\{\dot{h}_j^2\}_{j=1}^k$ with the property that each $\dot{\Gamma}_j^2$ contracts in R.

Furthermore, each correspondingly altered 2-sphere $\dot{\Theta}_t^2$ (now lying in $\partial\dot{T} - B$) has the same $\mathbb{Z}[\pi_1(B)/L']$-intersection number with those belt spheres and with any other oriented $(n-3)$-manifold lying in both $\partial T - B$ and $\partial\dot{T} - B$ as did Θ_t^2. Whereas the 2-spheres $\{\Theta_t^2\}$ each contracted in R', the $\dot{\Theta}_t^2$ each contract in R.

We apply Lemma 5.2 to the current setup, with the following substitutions:

Lemma 5.2	**Current situation**
R'	R_i
R	$R_i \cup R_{i-1}$
B	∂N_{i+1}
$\mathcal{H}^2$	$\{\bar{h}_1^2, \ldots, \bar{h}_s^2, \bar{h}_{s+1}^2, \ldots, \bar{h}_r^2\}$
L'	$J_{i+1} = \lambda_{i+1}^{-1}(L_i)$
T	$T_i = S_i \cup (\bar{h}_1^2 \cup \cdots \cup \bar{h}_s^2 \cup \bar{h}_{s+1}^2 \cup \cdots \cup \bar{h}_r^2)$
$k \in \mathbb{Z}$	$s \in \mathbb{Z}$
$\{h_j^2\}_{j=1}^k$	$\{\bar{h}_j^2\}_{j=1}^s$
$\{\Gamma_j^2\}_{j=1}^k$	$\{\Gamma_j^2\}_{j=1}^s$
$\{\Theta_t^2\}$	$\{\Theta_j^2\}_{j=1}^t$

After applying this lemma, the collection $\{\bar{h}_j^2\}_{j=1}^s$ is replaced by altered versions $\{\dot{\bar{h}}_j^2\}_{j=1}^s$ and the original collection $\{\bar{h}_j^2\}_{j=s+1}^r$ is retained. Let

$$\dot{T}_i = S_i \cup \left(\dot{\bar{h}}_1^2 \cup \cdots \cup \dot{\bar{h}}_s^2 \cup \bar{h}_{s+1}^2 \cup \cdots \cup \bar{h}_r^2\right)$$

and $\partial_-\dot{T}_i = \partial\dot{T}_i - \partial N_{i+1}$. The collections $\{\Gamma_j^2\}_{j=1}^s$ and $\{\Theta_j^2\}_{j=1}^t$ are replaced by altered versions $\{\dot{\Gamma}_j^2\}_{j=1}^s$ and $\{\dot{\Theta}_j^2\}_{j=1}^t$ which lie in $\partial_-\dot{T}_i$ and contract in

$$\overline{R_i \cup R_{i-1} - \dot{T}}.$$

The original 3-handles $\{\bar{h}_j^3\}_{j=1}^t$ must be discarded since their attaching tubes have been disrupted; replacements will be constructed shortly. When $n \geq 7$, use general position to choose a pairwise disjoint collection of properly embedded 3-disks in

$$\overline{R_i \cup R_{i-1} - \dot{T}}$$

with boundaries corresponding to the 2-spheres $\{\dot{\Gamma}_j^2\}_{j=1}^k \cup \{\dot{\Theta}_t^2\}$. Those 3-disks may be thickened to 3-handles by taking regular neighborhoods. With all of these handles finally in place, the argument described earlier completes the proof. When $n = 6$, the same is true, but the π-π argument used in [Guilbault and Tinsley 2013, Theorems 4.2 and 5.3] is needed in order to find pairwise disjoint embedded 3-disks. □

Remark 5.3. In reality, we have shown a stronger result than what is stated in Theorem 5.1. Specifically, the near pseudocollar structures obtained are as close to actual pseudocollars as the augmentation is to the trivial augmentation. For example, if $\{L_i\}$ is the trivial augmentation, the above argument contains an alternative proof of the main result of [Guilbault and Tinsley 2006] (stated here as Theorem 2.14). More generally, if $\{L_i\}$ lies somewhere between the trivial augmentation and the standard augmentation, then a near pseudocollar structure on M^n can be chosen to reflect that augmentation.

6. The examples: proof of Theorem 1.4

Introduction to the examples. The main examples of [Guilbault and Tinsley 2003], described here in Example 3.5, proved the existence of (absolutely) inward tame open manifolds that are not pseudocollarable. In this section we construct open manifolds that are absolutely inward tame but not nearly pseudocollarable. Since the examples from that paper are nearly pseudocollarable, the new examples fill a gap in the spectrum of known end structures.

The examples of [Guilbault and Tinsley 2003] began with algebra. The main theorems of that paper showed that all inward tame open manifolds have pro-finitely generated, semistable fundamental group, and stable $\mathbb{Z}$-homology, at infinity. The missing ingredient for detecting a pseudocollar structure was $\mathcal{P}$-semistability. With that knowledge, an inverse sequence of groups satisfying the necessary properties, but failing $\mathcal{P}$-semistability, became the blueprint for an example. A nontrivial handle-theoretic strategy was needed to realize the examples, but the heart of the matter was the group theory.

A similar story plays out here. We will begin with an inverse sequence of finitely presented groups with surjective bonding maps that become isomorphisms upon abelianization; but this time, in light of Theorems 1.2 and 1.3, we want an $\mathcal{AP}$-semistable sequence that is not $\mathcal{SAP}$-semistable. The first step is to identify such a sequence.

Let $\mathbb{F}_3 = \langle a_1, a_2, a_3 \mid \, \rangle$, the free group on the three generators; $r_{1,1} = [a_2, a_3]$, $r_{1,2} = [a_1, a_3]$, and $r_{1,3} = [a_1, a_2]$; $\mathbb{A}_1 = \operatorname{ncl}(\{r_{1,1}, r_{1,2}, r_{1,3}\}, \mathbb{F}_3)$; and $G_1 = \mathbb{F}_3/\mathbb{A}_1$. Notice that $\mathbb{A}_1$ is precisely the commutator subgroup $[\mathbb{F}_3, \mathbb{F}_3]$, so $G_1 \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$.

Suppose $r_{2,1} = [r_{1,2,}, r_{1,3}]$, $r_{2,2} = [r_{1,1}, r_{1,3}]$, and $r_{1,3} = [r_{1,1}, r_{1,2}]$; $\mathbb{A}_2 = \operatorname{ncl}(\{r_{2,1}, r_{2,2}, r_{2,3}\}, \mathbb{F}_3)$; and $G_2 = \mathbb{F}_3/\mathbb{A}_2$. Since $\mathbb{A}_2 \leq \mathbb{A}_1$, there is an induced epimorphism

$$G_1 \xleftarrow{\lambda_2} G_2$$

which abelianizes to the identity map on $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$.

Continue inductively, letting $r_{i+1,1} = [r_{i,2}, r_{i,3}]$, $r_{i+1,2} = [r_{i,1}, r_{i,3}]$, and $r_{i+1,3} = [r_{i,1}, r_{i,2}]$; $\mathbb{A}_{i+1} = \operatorname{ncl}(\{r_{i+1,1}, r_{i+1,2}, r_{i+1,3}\}, \mathbb{F}_3)$; and $G_{i+1} = \mathbb{F}_3/\mathbb{A}_{i+1}$. The result is a nested sequence of normal subgroups of $\mathbb{F}_3$, $\mathbb{A}_1 \geq \mathbb{A}_2 \geq \mathbb{A}_3 \geq \cdots$, and a corresponding inverse sequence of quotient groups

$$G_1 \overset{\lambda_2}{\twoheadleftarrow} G_2 \overset{\lambda_3}{\twoheadleftarrow} G_3 \overset{\lambda_4}{\twoheadleftarrow} \cdots \tag{6-1}$$

which abelianizes to the constant inverse sequence

$$\mathbb{Z}^3 \xleftarrow{\text{id}} \mathbb{Z}^3 \xleftarrow{\text{id}} \mathbb{Z}^3 \xleftarrow{\text{id}} \cdots .$$

A more delicate motivation for our choices is the following: For each $i > 1$, $\ker \lambda_i = \operatorname{ncl}(\{r_{i-1,1}, r_{i-1,2}, r_{i-1,3}\}, G_i)$; similarly, for each $i > 2$,

$$\ker(\lambda_{i-1}\lambda_i) = \operatorname{ncl}(\{r_{i-2,1}, r_{i-2,2}, r_{i-2,3}\}, G_i).$$

Moreover, since the elements of $\{r_{i-1,1}, r_{i-1,2}, r_{i-1,3}\}$ are precisely the commutators of the elements of $\{r_{i-2,1}, r_{i-2,2}, r_{i-2,3}\}$,

$$\ker(\lambda_i) \leq [\ker(\lambda_{i-1}\lambda_i), \ker(\lambda_{i-1}\lambda_i)].$$

So, for the standard augmentation, $L_i = \ker \lambda_i$, (6-1) is $\{L_i\}$-perfect, hence, $\mathcal{AP}$-semistable.

Two tasks remain:

- prove that (6-1) is not $\mathcal{SAP}$-semistable, and
- construct 1-ended absolutely inward tame open manifolds with fundamental groups at infinity representable by (6-1).

Since these tasks are independent, the ordering of the following two subsections is arbitrary.

The sequence* (6-1) *is not $\mathcal{SAP}$-semistable. Let $\mathbb{F}_n = \langle a_1, \dots, a_n \mid \ \rangle$, the free group on n generators. We will exploit two standard constructions from group theory. The *derived series of* $\mathbb{F}_n$ is defined by

$$\mathbb{F}_n^{(0)} = \mathbb{F}_n \qquad \text{and} \qquad \mathbb{F}_n^{(k+1)} = [\mathbb{F}_n^{(k)}, \mathbb{F}_n^{(k)}] \quad \text{for } k \geq 0.$$

The *lower central series of* $\mathbb{F}_n$ is given by $(\mathbb{F}_n)_1 = \mathbb{F}_n$ and then $(\mathbb{F}_n)_{k+1} = [(\mathbb{F}_n)_k, \mathbb{F}_n]$ for $k \geq 0$. By inspection

$$\mathbb{F}_n^{(k+1)} \leq \mathbb{F}_n^{(k)}, \quad (\mathbb{F}_n)_{k+1} \leq (\mathbb{F}_n)_k, \quad \mathbb{F}_n^{(k)} \leq (\mathbb{F}_n)_{k+1} \qquad \text{for all } k.$$

A well-known fact, similar in spirit to our goal in this subsection, is that

$$\bigcap_{k=0}^{\infty} \mathbb{F}_n^{(k)} = \{1\} = \bigcap_{k=1}^{\infty} (\mathbb{F}_n)_k .$$

The following representation of $\mathbb{F}_n$ was discovered by Magnus; our general reference is [Lyndon and Schupp 1977].

Proposition 6.1 [Lyndon and Schupp 1977, Proposition 10.1]. *Let $\mathcal{P}_n$ be the non-commuting power series ring in indeterminates $\{x_1, x_2, \dots, x_n\}$ with $x_j^2 = 0$ for $j = 1, 2, \dots, n$. Then the function $\beta(a_j) = 1 + x_j$ $(j = 1, 2, \dots, n)$ induces a faithful representation of $\mathbb{F}_n$ into $\mathcal{P}_n^*$, the multiplicative group of units of $\mathcal{P}_n$.*

In $\mathcal{P}_n$, the *fundamental ideal* Δ is the kernel of the homomorphism $\rho : \mathcal{P}_n \to \mathbb{Z}$ that takes each x_j to 0. The elements of Δ are all sums of the form $\sum_{\nu=1}^{\infty} \pi_\nu$ where each π_ν is a homogeneous polynomial of degree at least one. Consequently, for any positive integer k the ideal Δ^k is made of all sums of the form $\sum_{\nu=1}^{\infty} \pi_\nu$ where each π_ν is a homogeneous polynomial of degree at least k.

The next proposition and lemma are useful for monitoring the location of commutators in a group.

Proposition 6.2 [Lyndon and Schupp 1977, Proposition 10.2]. *Let $\beta : \mathbb{F}_n \to \mathcal{P}^*$ be the representation given above. If $w_1, w_2 \in \mathbb{F}_n$ such that $\beta(w_1) - 1 \in \Delta^r$ and $\beta(w_2) - 1 \in \Delta^s$, then $\beta([w_1, w_2]) - 1 \in \Delta^{r+s}$.*

By applying Proposition 6.2 inductively, we obtain the following useful facts.

Lemma 6.3. *For all integers $n, i \geq 1$,*

(1) $\{\beta(w) - 1 \mid w \in \mathbb{F}_n^{(i)}\} \subseteq \Delta^{2^i}$,

(2) $\{\beta(w) - 1 \mid w \in (\mathbb{F}_n)_i\} \subseteq \Delta^i$,

(3) $\bigcap_{k=1}^{\infty} \Delta^k = 0$, *and*

(4) $\bigcap_{k=1}^{\infty} \mathbb{F}_n^{(k)} = \{1\} = \bigcap_{k=1}^{\infty} (\mathbb{F}_n)_k$.

We now focus our attention on $\mathbb{F}_3$ and its subgroups $\mathbb{A}_i = \mathrm{ncl}(\{r_{i,1}, r_{i,2}, r_{i,3}\}, \mathbb{F}_3)$, as defined earlier.

Lemma 6.4. *For each $k \geq 1$ and $j \in \{1, 2, 3\}$,*

(1) *$r_{k,j}$ is a member of at least one free basis for $\mathbb{F}_3^{(k)}$, and*

(2) $r_{k,j} \in \mathbb{F}_3^{(k)} - \mathbb{F}_3^{(k+1)}$.

Proof. Assertion (1) can be obtained from an inductive argument using Schreier systems. A model argument can be found in [Massey 1967, Example 8.1].

Assertion (2) follows from (1), since the quotient map $\mathbb{F}_3^k \to \mathbb{F}_3^k / \mathbb{F}_3^{k+1}$ is the abelianization of $\mathbb{F}_3^k$. □

Since $\mathbb{A}_i \leq \mathbb{F}_3^{(i)}$, the following is an easy consequence of Lemmas 6.3 and 6.4.

Lemma 6.5. *For each $i \geq 1$ and $j \in \{1, 2, 3\}$,*

(1) $\beta(r_{i,j}) - 1 \neq 0$, *and*

(2) $\{\beta(h) - 1 \mid h \in \mathbb{A}_i\} \subseteq \Delta^{2^i}$.

The definitions of derived and lower central series are clearly applicable to arbitrary groups. To expand those notions further, the following definition is useful. For $H \trianglelefteq G$, let $\Omega_1(H, G) = H$ and $\Omega_k(H, G) = [\Omega_{k-1}(H, G), G]$ for $k > 1$. By normality, $H = \Omega_1(H, G) \geq \Omega_2(H, G) \geq \Omega_3(H, G) \geq \cdots$. When H is strongly G-perfect, $\Omega_k(H, G) = H$ for all k.

Proposition 6.6. *For each $i \geq 1$, there exists $p_i > 0$ and $q_i \geq p_i$ such that*

(1) *for each $j \in \{1, 2, 3\}$, $\beta(r_{i,j}) - 1 \notin \Delta^{2^i + p_i}$, and*

(2) $\{\beta(w) - 1 \mid w \in \Omega_{q_i}(\mathbb{A}_i, \mathbb{F}_3)\} \subseteq \Delta^{2^i + p_i}$.

Proof. Let i be fixed. Existence of p_i follows from item (3) of Lemma 6.3. Existence of q_i may be obtained from an inductive application of Proposition 6.2. □

We shift focus one more time, from $\mathbb{F}_3$ and its subgroups to the quotient groups $G_i = \mathbb{F}_3/\mathbb{A}_i$ and their subgroups. In doing so, we will allow a word in the generators of $\mathbb{F}_3$ to represent both an element of $\mathbb{F}_3$ and the corresponding element of a G_i. For example, recalling that $\lambda_{i+1,j} = \lambda_{i+1} \circ \cdots \circ \lambda_j : G_j \to G_i$, we say $\ker(\lambda_{i+1,j}) = \mathrm{ncl}(\{r_{i,1}, r_{i,2}, r_{i,3}\}, G_j)$.

The following result is simple but useful.

Lemma 6.7. *Suppose $\lambda : G \to G'$ is a surjective homomorphism, $H \trianglelefteq G$, and $q \geq 0$. Then $\lambda(\Omega_q(H, G)) = \Omega_q(\lambda(H), G')$.*

Lemma 6.7 ensures that, for each $i < k$ and all $q \geq 0$, the quotient maps $\mathbb{F}_3 \twoheadrightarrow G_k$ restrict to epimorphisms

$$\Omega_q(\mathbb{A}_i, \mathbb{F}_3) \twoheadrightarrow \Omega_q(\mathrm{ncl}(\{r_{i,1}, r_{i,2}, r_{i,3}\}, G_k), G_k). \tag{6-2}$$

Proposition 6.8. *For p_i and q_i as chosen in Proposition 6.6, and each $j \in \{1, 2, 3\}$, $r_{i,j} \notin \Omega_{q_i}(\ker(\lambda_{i+1,k}), G_k)$ whenever $2^k \geq 2^i + p_i$.*

Proof. Suppose $r_{i,j} \in \Omega_{q_i}(\ker(\lambda_{i+1,k}), G_k) = \Omega_{q_i}(\mathrm{ncl}(\{r_{i,1}, r_{i,2}, r_{i,3}\}, G_k), G_k)$. Surjection (6-2) provides a $w \in \Omega_{q_i}(\mathbb{A}_i, \mathbb{F}_3)$ with cosets $\mathbb{A}_k \cdot r_{i,j} = \mathbb{A}_k \cdot w$. Consequently, there is an $h \in \mathbb{A}_k$ with $r_{i,j} = hw$ in $\mathbb{F}_3$. Then

$$\begin{aligned} \beta(r_{i,j}) - 1 &= \beta(h)\beta(w) - 1 \\ &= \beta(h)\beta(w) - \beta(h) + \beta(h) - 1 \\ &= \beta(h)(\beta(w) - 1) + (\beta(h) - 1). \end{aligned}$$

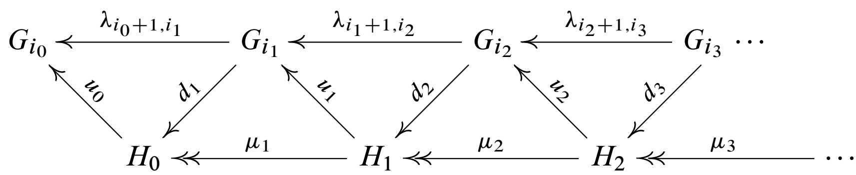
Since $\beta(w) - 1 \in \Delta^{2^i + p_i}$ and $\beta(h) - 1 \in \Delta^{2^k} \subseteq \Delta^{2^i + p_i}$, then $\beta(r_{i,j}) - 1 \in \Delta^{2^i + p_i}$, violating the choice of p_i. □

We are now ready for the main result of this subsection.

Theorem 6.9. *The inverse sequence* $\{G_i, \lambda_i\}_{i=0}^{\infty}$ *is not* $\mathcal{SAP}$*-semistable. In fact,* $\{G_i, \lambda_i\}_{i=0}^{\infty}$ *is not pro-isomorphic to any inverse sequence* $\{H_i, \mu_i\}$ *of surjections that satisfies the strong* $\{H_i\}$*-perfectness property.*

Proof. We proceed directly to the stronger assertion. Suppose $\{G_i, \lambda_i\}$ is pro-isomorphic to an inverse sequence $\{H_i, \mu_i\}$ of surjections that is strongly $\{H_i\}$-perfect; in other words, $\ker \mu_i = [\ker \mu_i, H_i]$ for all i.

By Proposition 2.11, each subsequence of $\{H_i, \mu_i\}$ satisfies the same essential property, so by our assumption, $\{G_i, \lambda_i\}$ contains a subsequence that fits into a commutative diagram of the following form:



Passing to a further subsequence if necessary, we may assume $2^{i_n} \geq 2^{i_{n-1}} + p_{i_{n-1}}$ for all n.

By Lemma 6.4, $1 \neq r_{i_1,j} \in \ker(\lambda_{i_1+1,i_2}) \leq G_{i_2}$. Choose $\alpha' \in H_2$ with $u_2(\alpha') = r_{i_1,j}$. Then, $\alpha' \in \ker(\mu_{1,2})$, and consequently $\alpha' \in [\ker(\mu_{1,2}), H_2]$, since $\ker(\mu_{1,2})$ is strongly H_2-perfect (again using Proposition 2.11). Thus $\alpha' \in \Omega_q(\ker(\mu_{1,2}), H_2)$ for all q. Moreover, since $u_2(\ker(\mu_{1,2})) \subseteq \ker(\lambda_{i_0+1,i_2})$,

$$r_{i_1,j} = u_2(\alpha') \in \Omega_q(u_2(\ker(\mu_{1,2})), G_{i_2}) \subseteq \Omega_q(\ker(\lambda_{i_0+1,i_2}), G_{i_2})$$

for all q, thereby contradicting Proposition 6.8. □

Construction of the examples. The goal of this subsection is to construct, for each $n \geq 6$, a 1-ended open manifold M^n that is absolutely inward tame and has fundamental group at infinity represented by the inverse sequence (6-1). By Theorem 1.3 or Theorem 5.1, such an example fails to be nearly pseudocollarable, thus completing the proof of Theorem 1.4.

Overview. We will construct M^n as a countable union of codimension 0 submanifolds

$$M^n = C_1 \cup A_1 \cup A_2 \cup A_3 \cup \cdots,$$

where C_1 is a compact "core" and $\{(A_i, \Gamma_i, \Gamma_{i+1})\}$ is a sequence of compact cobordisms between closed connected $(n-1)$-manifolds with $A_i \cap A_{i+1} = \Gamma_{i+1}$ for each $i \geq 1$, and $\partial C_1 = \Gamma_1$. Letting

$$N_i = A_i \cup A_{i+1} \cup A_{i+2} \cup \cdots$$

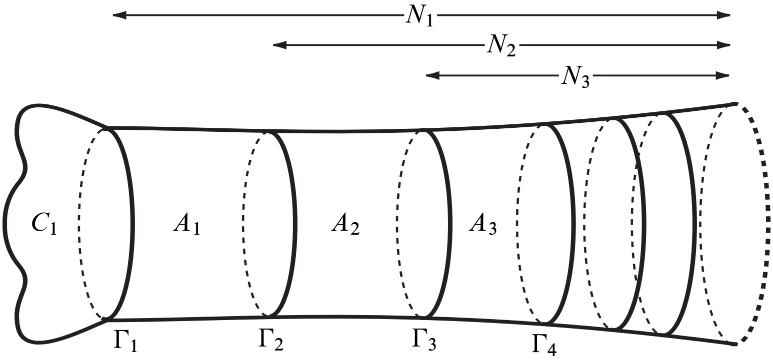


Figure 3. $M^n = C_1 \cup A_1 \cup A_2 \cup A_3 \cup \cdots$.

gives a preferred end structure $\{N_i\}$ with $\partial N_i = \Gamma_i$ for each i. See Figure 3.

So that pro-$\pi_1(\varepsilon(M^n))$ is represented by (6-1), the A_i will be constructed to satisfy:

(a) For all $i \geq 1$, $\pi_1(\Gamma_i, p_i) \cong G_i$ and $\Gamma_i \hookrightarrow A_i$ induces a π_1-isomorphism.

(b) The isomorphism between $\pi_1(\Gamma_i, p_i)$ and G_i may be chosen so that

$$\begin{array}{ccccc} G_i & & \xleftarrow{\quad\lambda_{i+1}\quad} & & G_{i+1} \\ \downarrow \cong & & & & \downarrow \cong \\ \pi_1(\Gamma_i, p_i) & \xleftarrow{\cong} & \pi_1(A_i, p_i) & \xleftarrow{\psi_{i+1}} & \pi_1(\Gamma_{i+1}, p_{i+1}) \end{array}$$

commutes. Here ψ_{i+1} is the composition

$$\pi_1(A_i, p_i) \xleftarrow{\widehat{\rho}_i} \pi_1(A_i, p_{i+1}) \xleftarrow{\iota_{i+1}} \pi_1(\Gamma_{i+1}, p_{i+1}),$$

where ι_{i+1} is induced by inclusion and $\widehat{\rho}_i$ is a change-of-basepoint isomorphism with respect to a path ρ_i in A_i between p_i and p_{i+1}.

From there it follows from Van Kampen's theorem that each $\Gamma_i = \partial N_i \hookrightarrow N_i$ induces a π_1-isomorphism, so by repeated application of (a) and (b), the inverse sequence

$$\pi_1(N_1, p_1) \xleftarrow{\mu_2} \pi_1(N_2, p_2) \xleftarrow{\mu_3} \pi_1(N_3, p_3) \xleftarrow{\mu_4}$$

is isomorphic to (6-1).

It will also be shown that each N_i has finite homotopy type; so M^n is absolutely inward tame. That argument requires specific details of the construction; it will be presented later.

Details of the construction. Recall that a p-handle h^p attached to an n-manifold P^n and a $(p+1)$-handle h^{p+1} attached to $P^n \cup h^p$ form a *complementary pair* if the attaching sphere of h^{p+1} intersects the belt sphere of h^p transversely in a single point. In that case $P^n \cup h^p \cup h^{p+1} \approx P^n$; moreover, we may arrange (by an isotopy of the attaching sphere of h^{p+1}) that $P^n \cap (h^p \cup h^{p+1})$ is an $(n-1)$-ball in ∂P^n. Conversely, for any ball $B^{n-1} \subseteq \partial P^n$, one may introduce a pair of complementary handles $P^n \cup h^p \cup h^{p+1}$ so that $P^n \cap (h^p \cup h^{p+1}) = B^{n-1}$. We call (h^p, h^{p+1}) a *trivial handle pair*. Note that the difference between a complementary pair and trivial pair is just a matter of perspective. In general, we say that h^p is *attached trivially* to P^n if it is possible to attach an h^{p+1} so that (h^p, h^{p+1}) is a complementary pair.

After a preliminary step where we construct the core manifold C_1, our proof proceeds inductively. At the i-th stage we construct the cobordism $(A_i, \Gamma_i, \Gamma_{i+1})$, along with a compact manifold C_{i+1} with $\partial C_{i+1} = \Gamma_{i+1}$, to be used in the following stage. Throughout the construction, we abuse notation slightly by letting $\partial C_i \times [0, \varepsilon]$ denote a small regular neighborhood of ∂C_i in C_i and $\Gamma_i \times [0, \varepsilon]$ to denote a small regular neighborhood of Γ_i in A_i.

Step 0 (preliminaries). Let C_0 be the n-manifold obtained by attaching three orientable 1-handles $\{h^1_{0,j}\}_{j=1}^3$ to the n-ball B^n. Choose a basepoint $p_0 \in \partial C_0$ and let a_1, a_2, and a_3 be embedded loops in ∂C_0, one through each 1-handle, intersecting only at p_0. Abuse notation slightly by writing

$$\pi_1(\partial C_0) = \pi_1(C_0) = \langle a_1, a_2, a_3 \mid \, \rangle.$$

A convenient way to arrange that the 1-handles are orientable is by attaching three trivial $(1,2)$-handle pairs $\{h^1_{0,j}, h^2_{0,j}\}_{j=1}^3$, then discarding the 2-handles.

Recall that

$$G_1 = \langle a_1, a_2, a_3 \mid r_{1,1}, r_{1,2}, r_{1,3} \rangle,$$

where $r_{1,1} = [a_2, a_3]$, $r_{1,2} = [a_1, a_3]$, and $r_{1,3} = [a_1, a_2]$. Attach a trio of 2-handles $\{h^2_{1,j}\}_{j=1}^3$ to C_0, where $h^2_{1,j}$ has attaching circle $r_{1,j}$. Choose the framings of these handles so that, if the 2-handles $\{h^2_{0,j}\}_{j=1}^3$ were added back in, then $\{h^2_{1,j}\}_{j=1}^3$ would be trivially attached (to an n-ball). Let

$$C_1 = C_0 \cup h^2_{1,1} \cup h^2_{1,2} \cup h^2_{1,3}$$

and note that $\pi_1(C_1) \cong \pi_1(\partial C_1) \cong G_1$.

Step 1 (constructing A_1 and C_2). Attach three trivial $(2,3)$-handle pairs to C_1, disjoint from the existing handles, then perform handle slides on each of the trivial 2-handles (over the handles $\{h^2_{1,j}\}_{j=1}^3$) so the resulting 2-handles $h^2_{2,1}$, $h^2_{2,2}$ and $h^2_{2,3}$ have attaching circles spelling out the words $r_{2,1}$, $r_{2,2}$ and $r_{2,3}$, respectively. This is possible since each $r_{2,k}$ can be viewed as a product of the loops $\{r_{1,j}\}_{j=1}^3$ and their inverses, which are the attaching circles of $\{h^2_{1,j}\}_{j=1}^3$. Sliding a 2-handle over $h^2_{1,j}$ inserts the loop $r^{\pm 1}_{1,j}$ into the new attaching circle of that 2-handle (with ± 1 depending on the orientation chosen).

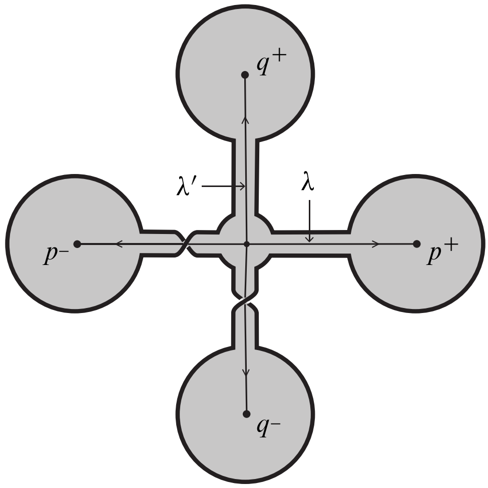


Figure 4. Attaching a $(2, 3)$-handle pair.

By keeping track of the attaching 2-spheres of the trivial 3-handles after the handle slides, it is possible to attach 3-handles $h^3_{2,1}$, $h^3_{2,2}$, and $h^3_{2,3}$ to $C_1 \cup h^2_{2,1} \cup h^2_{2,2} \cup h^2_{2,3}$ that are complementary to $h^2_{2,1}$, $h^2_{2,2}$, and $h^2_{2,3}$, respectively. Then

$$C_1 \cup \left(\bigcup_{j=1}^3 h^2_{2,j}\right) \cup \left(\bigcup_{j=1}^3 h^3_{2,j}\right) \approx C_1.$$

For later purposes, it is useful to have a schematic image of the attaching circles of $\{h^2_{1,j}\}_{j=1}^3$ and the attaching 2-spheres of the complementary handles $\{h^2_{1,j}\}_{j=1}^3$. Figure 4 provides such an image for one complementary pair. The outer loop represents the attaching circle for an $h^2_{2,j}$ and the shaded region represents the "lower hemisphere" of the attaching 2-sphere of $h^3_{2,j}$; the "upper hemisphere", which is not shown, is a parallel copy of the core of $h^2_{2,j}$. Within the lower hemisphere, the small central disk represents the lower hemisphere of the 2-sphere before handle slides. The arms are narrow strips whose centerlines are the paths along which the handle slides were performed; diametrically opposite paths lead to the same 2-handle, and are chosen to be parallel to a fixed path. We have indicated this by labeling one pair of centerlines λ and the other λ'. The four outer disks are parallel to the cores of the 2-handles over which the slides were made. A twist in the strip leading to an outer disk is used to reverse the orientation of the boundary of that disk. Thus, diametrically opposite outer disks are parallel to each other, but with opposite orientations. Center points of the outer disks represent transverse intersections with belt spheres of those handles; thus, p^+ and p^- are nearby intersections with the same belt sphere, and similarly for q^+ and q^-.

By rewriting

$$C_1 \cup \left(\bigcup_{j=1}^3 h^2_{2,j}\right) \cup \left(\bigcup_{j=1}^3 h^3_{2,j}\right)$$

as

$$C_0 \cup \big(\bigcup_{j=1}^3 h_{1,j}^2\big) \cup \big(\bigcup_{j=1}^3 h_{2,j}^2\big) \cup \big(\bigcup_{j=1}^3 h_{2,j}^3\big),$$

we may reorder the handles so that $h_{2,1}^2$, $h_{2,2}^2$, and $h_{2,3}^2$ are attached first. Define

$$C_2 = C_0 \cup \big(\bigcup_{j=1}^3 h_{2,j}^2\big)$$

and note that $\pi_1(C_2) \approx \pi_1(\partial C_2) \approx G_2$. Furthermore,

$$C_2 \cup \big(\bigcup_{j=1}^3 h_{1,j}^2\big) \cup \big(\bigcup_{j=1}^3 h_{2,j}^3\big) \approx C_1.$$

So, if we let

$$A_1 = (\partial C_2 \times [0,\varepsilon]) \cup \big(\bigcup_{j=1}^3 h_{1,j}^2\big) \cup \big(\bigcup_{j=1}^3 h_{2,j}^3\big),$$

(the result of excising the interior of a slightly shrunken copy of C_2), then $\partial A_1 \approx \partial C_2 \sqcup \partial C_1$. By letting $\Gamma_1 = \partial C_1$ and $\Gamma_2 = \partial C_2$ we obtain the first cobordism of the construction $(A_1, \Gamma_1, \Gamma_2)$. By avoiding the basepoint $p_0 \in \partial C_0$ in all of the above handle additions, we may let the arc $\rho_1 \subseteq A_1$ be the product line $p_0 \times [0,\varepsilon]$, with p_1 and p_2 its endpoints. Conditions (a) and (b) are then clear.

Inductive step (constructing A_i and C_{i+1}). Assume the existence of a cobordism $(A_{i-1}, \Gamma_{i-1}, \Gamma_i)$ satisfying (a) and (b) along with a compact manifold $C_i = C_0 \cup \big(\bigcup_{j=1}^3 h_{i,j}^2\big)$, with the attaching circle of each $h_{i,j}^2$ representing the relator $r_{i,j}$ in the presentation of G_i, and $\partial C_i = \Gamma_i$. Attach three trivial $(2,3)$-handle pairs to C_i, then perform handle slides on each of the trivial 2-handles (over the handles $\{h_{i,j}^2\}_{j=1}^3$) so that the resulting 2-handles $h_{i+1,1}^2$, $h_{i+1,2}^2$ and $h_{i+1,3}^2$ have attaching circles spelling out the words $r_{i+1,1}$, $r_{i+1,2}$ and $r_{i1,3}$, respectively. This is possible since each $r_{i+1,k}$ can be viewed as a product of the loops $\{r_{i,j}\}_{j=1}^3$ and their inverses, which are the attaching circles of $\{h_{i,j}^2\}_{j=1}^3$.

By keeping track of the attaching 2-spheres of the trivial 3-handles under the above handle slides, it is possible to attach 3-handles $h_{i+1,1}^3$, $h_{i+1,2}^3$, and $h_{i+1,3}^3$ to

$$C_i \cup h_{i+1,1}^2 \cup h_{i+1,2}^2 \cup h_{i+1,3}^2$$

that are complementary to $h_{i+1,1}^2$, $h_{i+1,2}^2$, and $h_{i+1,3}^2$, respectively. Then

$$C_i \cup \big(\bigcup_{j=1}^3 h_{i+1,j}^2\big) \cup \big(\bigcup_{j=1}^3 h_{i+1,j}^3\big) \approx C_i.$$

A picture like Figure 4, but with different indices, describes the current situation.

Rewrite

$$C_i \cup \big(\bigcup_{j=1}^3 h_{i+1,j}^2\big) \cup \big(\bigcup_{j=1}^3 h_{i+1,j}^3\big)$$

as

$$C_0 \cup \big(\bigcup_{j=1}^3 h_{i,j}^2\big) \cup \big(\bigcup_{j=1}^3 h_{i+1,j}^2\big) \cup \big(\bigcup_{j=1}^3 h_{i+1,j}^3\big),$$

then reorder the handles so that $h^2_{i+1,1}$, $h^2_{i+1,2}$, and $h^2_{i+1,3}$ are attached first. Define

$$C_{i+1} = C_0 \cup \left(\bigcup_{j=1}^{3} h^2_{i+1,j}\right)$$

and note that $\pi_1(C_{i+1}) \approx \pi_1(\partial C_{i+1}) \approx G_{i+1}$.

Furthermore,

$$C_{i+1} \cup \left(\bigcup_{j=1}^{3} h^2_{i,j}\right) \cup \left(\bigcup_{j=1}^{3} h^3_{i+1,j}\right) \approx C_i.$$

Excising the interior of a slightly shrunken copy of C_{i+1} gives

$$A_{i+1} = (\partial C_{i+1} \times [0,\varepsilon]) \cup \left(\bigcup_{j=1}^{3} h^2_{i,j}\right) \cup \left(\bigcup_{j=1}^{3} h^3_{i+1,j}\right);$$

then $\partial A_{i+1} \approx \partial C_{i+1} \sqcup \partial C_i$. Noting that $\Gamma_i = \partial C_i$ and letting $\Gamma_{i+1} = \partial C_{i+1}$, we obtain $(A_i, \Gamma_i, \Gamma_{i+1})$. By avoiding $p_i \in \partial C_i$ in all of the handle additions, letting $\rho_i \subseteq A_i$ be the product line $p_i \times [0,\varepsilon]$, and p_{i+1} the new endpoint, conditions (a) and (b) are clear.

Assembling the pieces in the manner described in Figure 3 completes the construction. In particular, we obtain a 1-ended open manifold

$$M^n = C_1 \cup A_1 \cup A_2 \cup A_3 \cup \cdots$$

whose fundamental group at infinity is represented by the inverse sequence (6-1).

Remark 6.10. In the construction of $(A_i, \Gamma_i, \Gamma_{i+1})$, we have written Γ_i on the left and Γ_{i+1} on the right to match the blueprint laid out in Figure 3. In that case, the handle decomposition of A_i implicit in the construction goes from right to left, with handles being attached to a collar neighborhood $\Gamma_{i+1} \times [0,\varepsilon]$ of Γ_{i+1}. Later, when our perspective becomes reversed, we will pass to the dual decomposition

$$A_i = (\Gamma_i \times [0,\varepsilon]) \cup \left(\bigcup_{j=1}^{3} \bar{h}^{n-3}_{1,j}\right) \cup \left(\bigcup_{j=1}^{3} \bar{h}^{n-2}_{2,j}\right),$$

where each $\bar{h}^{n-p}$ is the dual of an original h^p and $\Gamma_i \times [0,\varepsilon]$ is a thin collar neighborhood of Γ_i.

Absolute inward tameness of M^n. The following proposition will complete the proof of Theorem 1.4.

Proposition 6.11. *For the manifolds M^n constructed above, each clean neighborhood of infinity*

$$N_i = A_i \cup A_{i+1} \cup A_{i+2} \cup \cdots$$

has finite homotopy type. Thus, M^n is absolutely inward tame.

We will prove this by examining $H_*(N_i, \Gamma_i; \mathbb{Z}G_i)$ (equivalently, $H_*(\widetilde{N}_i, \widetilde{\Gamma}_i; \mathbb{Z})$ viewed as a $\mathbb{Z}G_i$-module), where $G_i = \pi_1(N_i) = \pi_1(\Gamma_i)$. In particular, we will prove:

Claim 6.12. *For each* i, $H_*(N_i, \Gamma_i; \mathbb{Z}G_i)$ *is trivial in all dimensions except for* $* = n-2$, *where it is isomorphic to the free module* $(\mathbb{Z}G_i)^3 = \mathbb{Z}G_i \oplus \mathbb{Z}G_i \oplus \mathbb{Z}G_i$.

Once this claim is established, Proposition 6.11 follows from [Siebenmann 1965, Lemma 6.2]. In Remark 6.13 at the conclusion of this section, we explain why this final observation is elementary, requiring no discussion of finite dominations or finiteness obstructions.

Proof. It is useful to consider compact subsets of the form

$$A_{i,k} = A_i \cup A_{i+1} \cup \cdots \cup A_k.$$

By repeated application of Remark 6.10, there is a handle decomposition of $A_{i,k}$ based on $\Gamma_i \times [0, \varepsilon]$ with handles only of indices $n-3$ and $n-2$. By reordering the handles, $(A_{i,k}, \Gamma_i)$ is seen to be homotopy equivalent to a finite relative CW complex $(K_{i,k}, \Gamma_i)$, where $K_{i,k}$ consists of Γ_i with an $(n-3)$-cell attached for each $(n-3)$-handle of $A_{i,k}$ followed by an $(n-2)$-cell for each $(n-2)$-handle. In the usual way, the $\mathbb{Z}G_i$-incidence number of an $(n-2)$-cell with an $(n-3)$-cell is equal to the $\mathbb{Z}G_i$-intersection number between the belt sphere of the corresponding $(n-3)$-handle and the attaching sphere of the corresponding $(n-2)$-handle. This process produces a sequence

$$K_{i,i} \subseteq K_{i,i+1} \subseteq K_{i,i+2} \subseteq \cdots$$

of relative CW complexes with direct limit a relative CW pair $(K_{i,\infty}, \Gamma_i)$ homotopy equivalent to (N_i, Γ_i). So we can determine $H_*(N_i, \Gamma_i; \mathbb{Z}G_i)$ by calculating $H_*(A_{i,k}, \Gamma_i; \mathbb{Z}G_i)$ and taking the direct limit as $k \to \infty$.

The $\mathbb{Z}G_i$-handle chain complex for $(A_{i,k}, \Gamma_i)$ (equivalently, the $\mathbb{Z}G_i$-cellular chain complex for $(K_{i,k}, \Gamma_i)$) looks like

$$0 \longrightarrow \mathcal{C}_{n-2} \xrightarrow{\partial} \mathcal{C}_{n-3} \longrightarrow 0,$$

where $\mathcal{C}_{n-2}$ and $\mathcal{C}_{n-3}$ are finitely generated free $\mathbb{Z}G_i$-modules generated by the handles of $A_{i,k}$, and the boundary map is determined by $\mathbb{Z}G_i$-intersection numbers between the belt spheres of $(n-3)$-handles and attaching spheres of the $(n-2)$-handles. These intersection numbers will be determined by returning to the construction.

Beginning with the compact manifold

$$C_i = C_0 \cup \big(\textstyle\bigcup_{j=1}^{3} h_{i,j}^2\big),$$

attach three trivial $(2,3)$-handle pairs, then perform handle slides on the 2-handles (over the handles $\{h_{i,j}^2\}_{j=1}^3$) to obtain $h_{i+1,1}^2$, $h_{i+1,2}^2$ and $h_{i+1,3}^2$ with attaching circles $r_{i+1,1}$, $r_{i+1,2}$ and $r_{i+1,3}$, respectively. Having kept track of the attaching

2-spheres of the trivial 3-handles under the handle slides, attach 3-handles $h^3_{i+1,1}$, $h^3_{i+1,2}$, and $h^3_{i+1,3}$ to

$$C_i \cup h^2_{i+1,1} \cup h^2_{i+1,2} \cup h^2_{i+1,3}$$

that are complementary to $h^2_{i+1,1}$, $h^2_{i+1,2}$, and $h^2_{i+1,3}$, respectively (all as described in inductive step above). This can all be done so that $h^3_{i+1,1}$, $h^3_{i+1,2}$, and $h^3_{i+1,3}$ do not touch the earlier 2-handles $h^2_{i,1}$, $h^2_{i,2}$ and $h^2_{i,3}$. Next attach a second trio of trivial $(2,3)$-handle pairs, taking care that they are disjoint from the existing handles, and slide the trivial 2-handles over the 2-handles $\{h^2_{i+1,j}\}_{j=1}^3$ so that the resulting 2-handles $\{h^2_{i+2,j}\}_{j=1}^3$ have attaching circles $r_{i+2,1}$, $r_{i+2,2}$ and $r_{i+2,3}$. Again, having kept track of the attaching 2-spheres of the trivial 3-handles under the handle slides, attach 3-handles $h^3_{i+2,1}$, $h^3_{i+2,2}$, and $h^3_{i+2,3}$ to

$$C_i \cup \big(\textstyle\bigcup_{j=1}^3 h^2_{i+1,j}\big) \cup \big(\bigcup_{j=1}^3 h^3_{i+1,j}\big) \cup \big(\bigcup_{j=1}^3 h^2_{i+2,j}\big)$$

that are complementary to $h^2_{i+2,1}$, $h^2_{i+2,2}$, and $h^2_{i+2,3}$, respectively, while taking care that these new 3-handles are completely disjoint from all 2- and 3-handles of lower index. Continue this process $k-i$ times, at each stage attaching three trivial $(2,3)$-handle pairs disjoint from the existing handles; sliding the trivial 2-handles over the 2-handles created in the previous step, in the manner prescribed above; and then attaching 3-handles complementary to these new 2-handles (and disjoint from earlier 2- and 3-handles) along the images of the attaching 2-spheres of the trivial 3-handles after the handle slides.

Since all of the 2- and 3-handles mentioned above, except for the original 2-handles $h^2_{i,1}$, $h^2_{i,2}$ and $h^2_{i,3}$, occur in complementary pairs, the manifold we just created is just a thickened copy of C_i; let us call it C_i'. By the standard reordering lemma, we may arrange that the 2-handles are pairwise disjoint, and all are attached before any of the 3-handles — which are also are attached in a pairwise disjoint manner. Then

$$\begin{aligned} C_i' &= C_i \cup \big(\textstyle\bigcup_{s=1}^k \big(\bigcup_{j=1}^3 h^2_{i+s,j}\big)\big) \cup \big(\bigcup_{s=1}^k \big(\bigcup_{j=1}^3 h^3_{i+s,j}\big)\big) \\ &= C_0 \cup \big(\textstyle\bigcup_{j=1}^3 h^2_{i,j}\big) \cup \big(\bigcup_{s=1}^k \big(\bigcup_{j=1}^3 h^2_{i+s,j}\big)\big) \cup \big(\bigcup_{s=1}^k \big(\bigcup_{j=1}^3 h^3_{i+s,j}\big)\big) \\ &= C_0 \cup \big(\textstyle\bigcup_{j=1}^3 h^2_{i+k,j}\big) \cup \big(\bigcup_{j=1}^3 h^2_{i,j}\big) \cup \big(\bigcup_{s=1}^{k-1} \big(\bigcup_{j=1}^3 h^2_{i+s,j}\big)\big) \\ &\qquad\qquad \cup \big(\textstyle\bigcup_{s=1}^k \big(\bigcup_{j=1}^3 h^3_{i+s,j}\big)\big) \\ &= C_k \cup \big(\textstyle\bigcup_{j=1}^3 h^2_{i,j}\big) \cup \big(\bigcup_{s=1}^{k-1} \big(\bigcup_{j=1}^3 h^2_{i+s,j}\big)\big) \cup \big(\bigcup_{s=1}^k \big(\bigcup_{j=1}^3 h^3_{i+s,j}\big)\big), \end{aligned}$$

where, going from the first to the second line, we apply the definition of C_i; going from the second to the third, we bring the last triple of 2-handles forward to the beginning; and in going from the third to the fourth, we apply the definition of C_k.

Excising a slightly shrunken copy of the interior of C_k from C_i' results in a cobordism between $\partial C_k = \Gamma_k$ and $\partial C_i' \approx \Gamma_i$, which has a handle decomposition

$$(\Gamma_k \times [0,\varepsilon]) \cup \big(\bigcup_{j=1}^{3} h_{i,j}^2\big) \cup \big(\bigcup_{s=1}^{k-1}\big(\bigcup_{j=1}^{3} h_{i+s,j}^2\big)\big) \cup \big(\bigcup_{s=1}^{k}\big(\bigcup_{j=1}^{3} h_{i+s,j}^3\big)\big).$$

Comparing this handle decomposition to our earlier construction reveals that this cobordism is precisely $A_i \cup A_{i+1} \cup \cdots \cup A_k = A_{i,k}$. In order to match the orientation of Figure 3, view Γ_k as the right-hand boundary and Γ_i as the left-hand boundary, with 2- and 3-handles being attached from right to left. Before switching to the dual handle decomposition, we analyze the $\mathbb{Z}G_i$-intersection numbers between the attaching spheres of the 3-handles and the belt spheres of the 2-handles. All should be viewed as submanifolds of the left-hand boundary of

$$(\Gamma_k \times [0,\varepsilon]) \cup \big(\bigcup_{j=1}^{3} h_{i,j}^2\big) \cup \big(\bigcup_{s=1}^{k-1}\big(\bigcup_{j=1}^{3} h_{i+s,j}^2\big)\big),$$

which has fundamental group G_i.

For each $1 \le s \le k$ and $j \in \{1,2,3\}$ let $\alpha_{i+s,j}^2$ denote the attaching 2-sphere of $h_{i+s,j}^3$; and for each $0 \le s' \le k-1$ and $j' \in \{1,2,3\}$ let $\beta_{i+s',j'}^{n-3}$ denote the belt $(n-3)$-sphere of $h_{i+s',j'}^2$. There are three cases to consider.

Case 1: $s = s'$. Then for each j, the pair $(h_{i+s,j}^2, h_{i+s,j}^3)$ is complementary; in other words $\alpha_{i+s,j}^2$ intersects $\beta_{i+s,j}^{n-3}$ transversely in a single point. Adjusting base paths, if necessary, and being indifferent to orientation (since it will not affect our computations), we have

$$\varepsilon_{\mathbb{Z}G_i}(\alpha_{i+s,j}^2, \beta_{i+s,j}^{n-3}) = \pm 1.$$

If $j \neq j'$, then $h_{i+s,j}^3$ does not intersect $h_{i+s,j'}^2$, so

$$\varepsilon_{\mathbb{Z}G_i}(\alpha_{i+s,j}^2, \beta_{i+s,j'}^{n-3}) = 0.$$

Case 2: $s = s'+1$. For each j, $\alpha_{i+s,j}^2$ can be split into a pair of disks. The "upper hemisphere" lies in the 2-handle $h_{i+s,j}^2$ and it intersects $\beta_{i+s,j}^{n-3}$ transversely in a single point; that point of intersection was accounted for in Case 1. The "lower hemisphere" is analogous to the one pictured in Figure 4. If $\{u,v\} = \{1,2,3\} - \{j\}$, then one pair of the diametrically opposite disks has boundaries labelled $r_{i+s-1,u}$ and $r_{i+s-1,u}^{-1}$ and the disks are parallel to the core of $h_{i+s-1,u}^2$, so each intersects $\beta_{i+s-1,u}^{n-3}$ transversely in points p_u^+ and p_u^-. Due to the flipped orientation of one of the disks, these points of intersection, between $\alpha_{i+s,j}^2$ and $\beta_{i+s-1,u}^{n-3}$, have opposite sign. Connecting p_u^+ and p_u^- by a path homotopic to $\lambda^{-1} * \lambda$ in $\alpha_{i+s,j}^2$ and a short path μ connecting p_u^+ and p_u^- in $\beta_{i+s-1,u}^{n-3}$ yields a loop that is contractible in the left-hand boundary of

$$(\Gamma_k \times [0,\varepsilon]) \cup \big(\bigcup_{j=1}^{3} h_{i,j}^2\big) \cup \big(\bigcup_{s=1}^{k-1}\big(\bigcup_{j=1}^{3} h_{i+s,j}^2\big)\big).$$

So together p_u^+ and p_u^- contribute 0 to the $\mathbb{Z}G_i$-intersection number of $\alpha^2_{i+s,j}$ and $\beta^{n-3}_{i+s-1,u}$; hence,

$$\varepsilon_{\mathbb{Z}G_i}(\alpha^2_{i+s,j}, \beta^{n-3}_{i+s-1,u}) = 0.$$

Similarly

$$\varepsilon_{\mathbb{Z}G_i}(\alpha^2_{i+s,j}, \beta^{n-3}_{i+s-1,v}) = 0.$$

Finally, $\alpha^2_{i+s,j}$ and $\beta^{n-3}_{i+s-1,j}$ do not intersect, so

$$\varepsilon_{\mathbb{Z}G_i}(\alpha^2_{i+s,j}, \beta^{n-3}_{i+s-1,j}) = 0$$

as well.

Case 3: $s \notin \{s', s'+1\}$. In this case, the handles $h^3_{i+s,j}$ and $h^2_{i+s',u}$ are disjoint, so $\varepsilon_{\mathbb{Z}G_i}(\alpha^2_{i+s,j}, \beta^{n-3}_{i+s,j'}) = 0$.

Now invert the above handle decomposition to obtain a handle decomposition of the cobordism $(A_{i,k}, \Gamma_i, \Gamma_k)$, based on Γ_i, containing only $(n-3)$- and $(n-2)$-handles. Specifically, we have

$$(\Gamma_i \times [0,\varepsilon]) \cup \Big(\bigcup_{s=1}^{k}\big(\bigcup_{j=1}^{3} \bar{h}^{n-3}_{i+s,j}\big)\Big) \cup \Big(\bigcup_{j=1}^{3} \bar{h}^{n-2}_{i,j}\Big) \cup \Big(\bigcup_{s=1}^{k-1}\big(\bigcup_{j=1}^{3} \bar{h}^{n-2}_{i+s,j}\big)\Big).$$

Since the belt sphere of each $\bar{h}^{n-3}$ is the attaching sphere of its dual h^3 and the attaching sphere of each $\bar{h}^{n-2}$ is the belt sphere of its dual h^2, the incidence numbers between these handles of this handle decomposition are determined (up to sign) by the earlier calculations. So the cellular $\mathbb{Z}G_i$-chain complex for the $(A_{i,k}, \Gamma_i)$ is isomorphic to

$$0 \to \bigoplus_{s=0}^{k-1} (\mathbb{Z}G_i)^3 \xrightarrow{\partial} \bigoplus_{s=1}^{k} (\mathbb{Z}G_i)^3 \to 0,$$

where the $(\mathbb{Z}G_i)^3$ summands on the left are generated by the handles $\{\bar{h}^{n-2}_{i+s,j}\}_{j=1}^3$ and those on the right by $\{\bar{h}^{n-3}_{i+s,j}\}_{j=1}^3$. Since $\varepsilon_{\mathbb{Z}G_i}(\alpha^2_{i+s,j}, \beta^{n-3}_{i+s,j}) = \pm 1$ for all $1 \le s \le k-1$ and all other intersection numbers are 0, the boundary map is trivial on the 0-th copy of $(\mathbb{Z}G_i)^3$; misses the k-th copy of $(\mathbb{Z}G_i)^3$ in the range; and restricts to an isomorphism $\bigoplus_{s=1}^{k-1} (\mathbb{Z}G_i)^3 \xrightarrow{\cong} \bigoplus_{s=1}^{k-1} (\mathbb{Z}G_i)^3$ elsewhere. Thus

$$H_{n-2}(A_{i,k}, \Gamma_i; \mathbb{Z}G_i) = \ker \partial \cong (\mathbb{Z}G_i)^3, \text{ and}$$

$$H_{n-3}(A_{i,k}, \Gamma_i; \mathbb{Z}G_i) = \operatorname{coker} \partial \cong (\mathbb{Z}G_i)^3,$$

where $H_{n-2}(K_{i,k}, \Gamma_i)$ is generated by the $s = 0$ summand and $H_{n-3}(K_{i,k}, \Gamma_i)$ is generated by the $s = k$ summand.

Now consider the inclusion $A_{i,k} \hookrightarrow A_{i,k+1}$ and the corresponding inclusion of $\mathbb{Z}G_i$-chain complexes. The chain complex of $A_{i,k+1}$ will contain an extra $(\mathbb{Z}G_i)^3$ summand in each dimension, generated by $\{\bar{h}^{n-2}_{i+k,j}\}_{j=1}^3$ and $\{\bar{h}^{n-3}_{i+k+1,j}\}_{j=1}^3$, respectively. The boundary map takes the new summand in the domain onto the previous cokernel, thereby killing $H_{n-3}(A_{i,k}, \Gamma_i; \mathbb{Z}G_i)$, and replacing it with a cokernel generated by $\{\bar{h}^{n-3}_{i+k+1,j}\}_{j=1}^{3}$. Said differently, the inclusion induced map

$$i_* : H_{n-3}(K_{i,k}, \Gamma_i; \mathbb{Z}G_i) \xrightarrow{0} H_{n-3}(K_{i,k+1}, \Gamma_i; \mathbb{Z}G_i)$$

is trivial. On the other hand, the expansion from $K_{i,k}$ to $K_{i,k+1}$ does not change $\ker \partial$, which is still generated by the handles $\{\bar{h}^{n-2}_{i,j}\}_{j=1}^{3}$. In other words, the inclusion induced map

$$i_* : H_{n-2}(K_{i,k}, \Gamma_i; \mathbb{Z}G_i) \xrightarrow{\cong} H_{n-2}(K_{i,k+1}, \Gamma_i; \mathbb{Z}G_i)$$

is an isomorphism.

Taking direct limits, we have

$$H_*(N_i, \Gamma_i; \mathbb{Z}G_i) \cong \begin{cases} (\mathbb{Z}G_i)^3 & \text{if } * = n-2, \\ 0 & \text{otherwise.} \end{cases} \qquad \square$$

Remark 6.13. The appeal to [Siebenmann 1965, Lemma 6.2] may give the impression that obtaining Proposition 6.11 from Claim 6.12 is complicated — that is not the case. The conclusion can be obtained directly as follows: If $\{e^{n-2}_{i,j}\}_{j=1}^{3}$ represents the cores of the $(n-2)$-handles $\{\bar{h}^{n-2}_{i,j}\}$, which generate $H_*(N_i, \Gamma_i; \mathbb{Z}G_i)$, abstractly attach $(n-2)$-disks $\{f^{n-2}_{i,j}\}_{j=1}^{3}$ to Γ_i along their boundaries. This does not affect fundamental groups, so by excision, the pair

$$\big(N_i \cup \{f^{n-2}_{i,j}\}_{j=1}^{3}, \Gamma_i \cup \{f^{n-2}_{i,j}\}_{j=1}^{3}\big)$$

has the same $\mathbb{Z}G_i$ -homology as (N_i, Γ_i), with the same generating set. Now attach an $(n-1)$-cell g^{n-1}_j along each sphere $e^{n-2}_{i,j} \cup f^{n-2}_{i,j}$ to obtain a pair

$$\big(N_i \cup \{f^{n-2}_{i,j}\}_{j=1}^{3} \cup \{g^{n-2}_{i,j}\}_{j=1}^{3}, \Gamma_i \cup \{f^{n-2}_{i,j}\}_{j=1}^{3}\big)$$

with trivial $\mathbb{Z}G_i$-homology in all dimensions. It follows that

$$\Gamma_i \cup \{f^{n-2}_{i,j}\}_{j=1}^{3} \hookrightarrow N_i \cup \{f^{n-2}_{i,j}\}_{j=1}^{3} \cup \{g^{n-1}_{i,j}\}_{j=1}^{3}$$

is a homotopy equivalence. But notice that each $g^{n-1}_{i,j}$ has a free face $f^{n-2}_{i,j}$, so

$$N_i \cup \{f^{n-2}_{i,j}\}_{j=1}^{3} \cup \{g^{n-1}_{i,j}\}_{j=1}^{3}$$

collapses onto N_i. Therefore, N_i is homotopy equivalent to $\Gamma_i \cup \{f^{n-2}_{i,j}\}_{j=1}^{3}$.

7. Remaining questions

In the introduction we commented that nearly pseudocollarable manifolds admit arbitrarily small clean neighborhoods of infinity N containing compact codimension 0 submanifolds A for which $A \hookrightarrow N$ is a homotopy equivalence. Call such a pair (N, A) a *wide homotopy collar*. The difference, of course, between a wide homotopy collar and a homotopy collar is that, in the latter, the subspace is required to be the (codimension 1) boundary of N. The fact that nearly pseudocollarable manifolds contain arbitrarily small wide homotopy collars is immediate from the following easy lemma.

Lemma 7.1. *Suppose N' is a* (mod J)*-homotopy collar neighborhood of infinity in a manifold M^n ($n \geq 5$), where J is a normally finitely generated subgroup of* $\ker(\pi_1(N') \to \pi_1(M^n))$. *Then M^n contains a wide homotopy collar neighborhood of infinity (N, A), where $N' \subseteq N \subseteq M^n$.*

Proof. Choose a finite collection of pairwise disjoint properly embedded 2-disks $\{D_i^2\}_{i=1}^k$ in $\overline{M^n - N'}$, with boundaries comprising a normal generating set for $\ker(\pi_1(N') \to \pi_1(M^n))$. Then let (N, A) be a regular neighborhood pair for

$$\bigl(N' \cup \bigl(\bigcup_{i=1}^k D_i^2\bigr), \partial N' \cup \bigl(\bigcup_{i=1}^k D_i^2\bigr)\bigr)$$

and apply Lemma 4.3. □

The following seem likely but, thus far, we have been unable to find proofs.

Questions. Must a manifold with compact boundary that contains arbitrarily small wide homotopy collar neighborhoods of infinity be nearly pseudocollarable? More specifically, can it be shown that the nonpseudocollarable examples in Section 6 do not contain arbitrarily small wide homotopy collar neighborhoods of infinity?

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