Pergamon

Topology Vol. 34. No. 1. pp. 99–108. 1995
Copyright © 1994 Elsevier Science Ltd
Printed in Great Britain. All rights reserved
0040-9383/95 $9.50 + 0.00

0040-9383(94)E0018-F

SOME COMPACT CONTRACTIBLE MANIFOLDS CONTAINING DISJOINT SPINES

CRAIG R. GUILBAULT

(*Received* 18 *November* 1992; *in revised form* 29 *November* 1993)

0. INTRODUCTION

A compact subpolyhedron K in the interior of a PL manifold M is called a PL spine of M if M collapses to K. A long standing question in geometric topology is whether the PL spines of a class of compact contractible 4-manifolds constructed by Mazur may be "pushed off themselves". Equivalently one may ask whether these Mazur manifolds contain a disjoint pair of spines. Recently, focus on this question has been broadened to include all compact contractible manifolds not homeomorphic to a ball. In our main theorem (Theorem 4.1) we prove that one standard method of constructing compact contractible manifolds with non-simply concentrated boundary—as the closure of the complement of a regular neighborhood of an acyclic k-complex in an n-sphere with large enough dimension—always yields manifolds with disjoint spines if $n > 4k$. Note that these disjoint spines are necessarily of dimension $\geq n - 2$. As an application (Theorem 6.1) of our main theorem we prove that in many cases the suspension circle in the double suspension of a homology $(n - 2)$-sphere can be moved off itself by an arbitrarily small homeomorphism of the resulting n-sphere.

In Section 1 and Section 2 we present our basic definitions and some elementary lemmas. In the next section we describe a large class of compact contractible manifolds which contains the examples to which our results apply. Section 4 contains the main results of this paper. In Section 5 we discuss the notion spine in the topological category. The last two sections contain some applications and open questions.

1. PRELIMINARIES

The main result of this paper is obtained in the PL category. In Sections 2–4, all manifolds are assumed to be combinatorial, complexes are simplicial, and maps are piecewise linear. In these sections, $\approx$ indicates PL homeomorphic spaces. When we switch to the topological category in Section 5 and Section 6, the distinction will be made clear.

Throughout this paper, homology is with $\mathbb{Z}$-coefficients. Unless noted to the contrary, an unlabeled map between homology or homotopy groups is induced by inclusion. A space with the homology of a point is called *acyclic*. A group is *perfect* if its abelianization is the trivial group. In particular, the fundamental group of an acyclic space is perfect. A compact acyclic n-manifold is called a *homology n-cell*. An n-manifold with the same homology as S^n is called a *homology n-sphere*. The following well-known lemma follows from straightforward calculations involving the Mayer–Vietoris, Universal Coefficient, duality, Hurewicz, and Whitehead theorems from algebraic topology.

LEMMA 1.1. (a) *The boundary of a homology n-cell is a homology* $(n-1)$*-sphere.*

(b) *If* $\Sigma^{n-1} \subset \Gamma^n$ *is a bicollared homology* $(n-1)$*-sphere in a homology n-sphere and* V_1 *and* V_2 *are the components of* $\Gamma^n - \Sigma^{n-1}$, *then* $\bar{V}_1$ *and* $\bar{V}_2$ *are homology n-cells.*

(c) *The union of two homology n-cells along a common boundary is a homology n-sphere.*

(d) *A simply connected homology n-cell is contractible, and a simply connected homology n-sphere is homotopy equivalent to* S^n.

2. PL SPINES

A compact subpolyhedron in the interior of a manifold M is called a PL spine of M if M collapses to K. The standard example occurs when M is a regular neighborhood of a finite complex, K, lying in the interior of a manifold. In fact, if K is a PL spine of M then M is a regular neighborhood of K in the manifold M' obtained by adding an exterior collar to M along its boundary. A compactum P in the interior of a manifold M is a *PL pseudo-spine* of M if $M - P$ is (PL) homeomorphic to $\partial M \times [0,1)$. By regular neighborhood theory ([15] or [11]) a PL spine is a PL pseudo-spine. While the converse is not true, we have the following "near converse".

LEMMA 2.1. *Let M be a compact manifold containing a PL pseudo-spine P, and let N be a neighborhood of P in M. Then there is a spine K of M contained in N.*

Proof. Let $h:\partial M \times [0,1) \to M - P$ be a homeomorphism. Choose $0 < r < 1$ so that $h(\partial M \times [r,1)) \subset N$, and let $K = h(\partial M \times [r,1)) \cup P$. □

COROLLARY 2.2. *A compact manifold M contains a disjoint pair of PL spines iff it contains a disjoint pair of PL pseudo-spines.*

Remark. The proof of Lemma 2.1 also shows that any pseudo-spine (or spine) may be "thickened" (within an arbitrary neighborhood) to a spine which is a codimension 0 submanifold having locally flat boundary in M. If P is a spine, this can also be accomplished by letting K be a regular neighborhood of P. However, if P is just a pseudo-spine of M, then a regular neighborhood of P is just a pseudo-spine. See [11, pp. 36–41].

The following lemma, which is stated for PL spines, is also true if the word "spine" is changed to "pseudo-spine" anywhere in its statement.

LEMMA 2.3. *For any compact manifold M, the following are equivalent:*

(a) *M contains a pair of disjoint PL spines,*

(b) *for any PL spine K of M there is a PL spine of M disjoint from K,*

(c) *For any PL spine K of M there is an isotopy* $H: M \times I \to M \times I$, *fixed on* ∂M, *with* $H_0 = id_M$ *and* $H_1(K) \cap K = \emptyset$.

Proof. To see that (a)$\Rightarrow$(b), suppose K, K_1 and K_2 are PL spines of M and $K_1 \cap K_2 = \emptyset$. Since $M - K_1$ is homeomorphic to $\partial M \times [0,1)$ we may construct an ambient isotopy of M, fixed on an arbitrarily small neighborhood N of K_1 and also on ∂M, which pushes points "outwards" along collar lines towards ∂M. By choosing N to miss K_2, we may use this procedure to push K_2 arbitrarily close to ∂M. Since K is a compactum in int M, we may arrange that the image of K_2, which is a PL spine of M, misses K.

To check (b) $\Rightarrow$ (c), choose a PL spine K_1 of M disjoint from K and let N be a neighborhood of K_1 disjoint from K. Construct an ambient isotopy $H: M \times I \to M \times I$ in the manner described above which fixes $N \cup \partial M$ and moves K sufficiently close to ∂M that $H(K \times \{1\})$ misses K.

(c) $\Rightarrow$ (a) is immediate. □

Note. Condition (c) defines the notion of *pushing K off itself.*

The next lemma is standard. Since both the lemma and the technique are used often in this paper, we sketch a proof.

LEMMA 2.4. *If M is an n-dimensional compact PL manifold containing a k-dimensional polyhedron K with regular neighborhood N then* $\pi_j(M - \text{int}(N)) \to \pi_j(M)$ *is a surjection for* $j < n - k$ *and an isomorphism for* $j < n - k - 1$.

Proof. Let $j \geq 0$, $p \in \partial N$ and consider the inclusion induced homomorphism $i_\#: \pi_j(M - \text{int}(N), p) \to \pi_j(M, p)$. If $\alpha: (S^j, \{q\}) \to (M, \{p\})$ represents an element of $\pi_j(M, p)$ and if $j < n - k$, we may use general position to homotope $\alpha(\text{rel } \{q\})$ to a map α' with image missing K. Since $N - K \approx \partial N \times [0, 1)$, we may use these collar lines to homotope α' into $M\text{-int}(N)$. Hence, $i_\#$ is surjective.

Now suppose $\beta: (S^j, \{q\}) \to (M - \text{int}(N), \{p\})$ represents an element of $\ker(i_\#)$. Then there exists $\hat{\beta}: (B^{j+1}, \{q\}) \to (M, \{p\})$ extending β. If $j < n - k - 1$, we may use general position to homotope $\hat{\beta}(\text{rel } S^j)$ to a map $\hat{\beta}'$ with image $(\hat{\beta}') \cap K = \emptyset$. Again using the collar structure on $N - K$, we may homotope $\hat{\beta}'(\text{rel } S^q)$ into $M - \text{int}(N)$. Hence, $\hat{\beta}$ is trivial in $\pi_j(M - \text{int}(N))$, and $i_\#$ is injective. □

COROLLARY 2.5. *If K is a k-dimensional PL spine of M then* $\pi_j(\partial M) \to \pi_j(M)$ *is a surjection for* $j < n - k$ *and an isomorphism for* $j < n - k - 1$.

3. NEWMAN COMPACT CONTRACTIBLE MANIFOLDS

A classical technique developed by M. H. A. Newman can be used to produce compact contractible manifolds not homeomorphic to n-balls. Let L be a finite acyclic simplicial complex with $\pi_1(L) = G \neq \{1\}$. Then $k \geq 2$, and there is an embedding $e: L \to S^n$ for any $n \geq 2k + 1$. If N is a regular neighborhood of $e(L)$, then $H_*(N) \cong H_*(L)$, so N is a homology n-cell with a k-dimensional PL spine. By Lemma 1.1 and Corollary 2.5, ∂N is a homology $(n - 1)$-sphere with $\pi_1(\partial N) \cong G$. Moreover, Lemmas 1.1 and 2.4 combine to show that $M = S^n - \text{int}(N)$ is contractible. Since $\pi_1(\partial M) \cong G$, M is not an n-ball. We shall refer to a compact contractible manifold created in this manner as a *Newman compact contractible manifold.* When $2k + 1 < n$ any two embeddings of L into S^n are equivalent (see [6, Theorem 5]), hence, there is a unique Newman contractible n-manifold associated with L which we denote by $\text{New}(L, n)$.

Remarks. (1) Finite acyclic simplicial complexes with non-trivial fundamental groups are plentiful. Any finite presentation of a non-trivial perfect group with equal numbers of generators and relators gives such a complex. One easy example is obtained by building a CW complex with one 0-cell, two 1-cells and two 2-cells and $\pi_1(L) \cong \langle a, b \mid a^3 = b^5 = (ab)^2 \rangle$ (the *Poincaré dodecahedral group*). Inspection of the cellular chain complex reveals that L is acyclic. Triangulate L to complete the process.

(2) By Corollary 2.5, any PL spine of a Newman (or any other) compact contractible n-manifold with non-simply connected boundary will have dimension $\geq n-2$.

(3) In order for $New(L, n)$ to be well defined, it is not always necessary that $2\,dim(L)+1<n$. For example, techniques found in [1] may be used to show that there is a unique Newman compact contractible $(2k+1)$-manifold associated with any given finite acyclic k-complex. Since the main results of this paper require that $4\,dim(L)<n$, we do not concern ourselves with this matter.

4. MAIN RESULT

THEOREM 4.1. *Let L be a k-dimensional finite acyclic complex. For any $n>4k$ the compact contractible n-manifold $M = New(L, n)$ contains a disjoint pair of PL spines.*

Proof. For notational simplicity, we break the proof into two cases.

Case (a). n is odd.

Step 1 (*Set up*): Let $m=(n+1)/2$, $e: L \to S^m$ be a PL embedding, X be a regular neighborhood of $e(L)$ and $p \in \partial X$. Then $L_1 = e(L)\times\{p\}$ and $L_2=\{p\}\times e(L)$ are copies of L in $\partial(X\times X)$. Choose disjoint regular neighborhoods N_1 and N_2 of L_1 and L_2, respectively, in $\partial(X\times X)$. Being a regular neighborhood of L_i, each N_i is a homology cell, and by Corollary 2.5, $\pi_1(\partial N_i)\to\pi_i(N_i)$ is an isomorphism.

Let Δ_X and Δ_L be the diagonals of $X\times X$ and $e(L)\times e(L)$, respectively. Then $\Delta_X \approx X$, $\Delta_L\approx L$ and Δ_X collapses to Δ_L. Construct an embedding $H:\Delta_L\times[0,1]\to X\times X$, with $H_0 = id$ and $H_1(\Delta_L)\subset\partial(X\times X)-(N_1\cup N_2)$, as follows. Use general position to homotope Δ_L off of $L\times L$ in $X\times X$. Since $L\times L$ is a spine of $X\times X$ (thus, $X\times X-(L\times L)\approx\partial(X\times X)\times[0,1))$, we may further homotope Δ_L into $\partial(X\times X)$. General position within $\partial(X\times X)$ allows us to move the image of Δ_L off from $L_1\cup L_2$ and the collar structures on the $N_i - L_i$ allow us to push the image out of $N_1\cup N_2$. With a final application of general position we may adjust the union of these homotopies to the desired embedding. Let L_3 denote $H_1(\Delta_L)$ and let N_3 be a regular neighborhood of L_3 in $\partial(X\times X)$ disjoint from $N_1\cup N_2$.

By Kervaire [8], PL homology spheres of dimension $\neq 3$ bound (PL) contractible manifolds. Hence, there are (homeomorphic) contractible $(n-1)$-manifolds C_1 and C_2 with boundaries homeomorphic to $\partial N_1\approx\partial N_2$. Attach C_1 and C_2 to $X\times X$ via these homeomorphisms. We will denote this adjunction space by $(X\times X)\cup_\partial(C_1\cup C_2)$. See Fig. 1.

Step 2 (*Recognition of M*): Let $\Sigma=[\partial(X\times X)-\mathrm{int}(N_1\cup N_2)]\cup_\partial(C_1\cup C_2)$. We claim that Σ is homeomorphic to S^n. To this end, note that Σ is obtained from the homology n-sphere $\partial(X\times X)$ by removing the interiors of homology n-cells N_1 and N_2 and filling in with homology cells C_1 and C_2; therefore, by several applications of Lemma 1.1, Σ is a homology n-sphere. Next we observe that Σ is simply connected. Consider the standard inclusion induced isomorphism $\pi_1(X\times\{p\})\times\pi_1(\{p\}\times X)\to\pi_1(X\times X)$, and the isomorphism $\pi_1(\partial(X\times X))\to\pi_1(X\times X)$ promised by Corollary 2.5. Together these show that $(X\times\{p\})\cup(\{p\}\times X)$ carries $\pi_1(\partial(X\times X))$. Since L_1 and L_2 are spines of $X\times\{p\}$ and $\{p\}\times X$, respectively and since, $\pi_1(\partial N_i)\to\pi_1(N_i)$ is an isomorphism for each i, it is clear that $\pi_1(\partial(X\times X))$ is carried by $\partial N_1\cup\partial N_2\cup\alpha$ where α is any properly embedded arc in $\partial(X\times X)-\mathrm{int}(N_1\cup N_2)$ connecting ∂N_1 to ∂N_2. By Lemma 2.4, $\pi_1(\partial(X\times X)-(\mathrm{int}(N_1\cup N_2))$ is also carried by $\partial N_1\cup\partial N_2\cup\alpha$. Since loops in ∂N_i contract in C_i, an

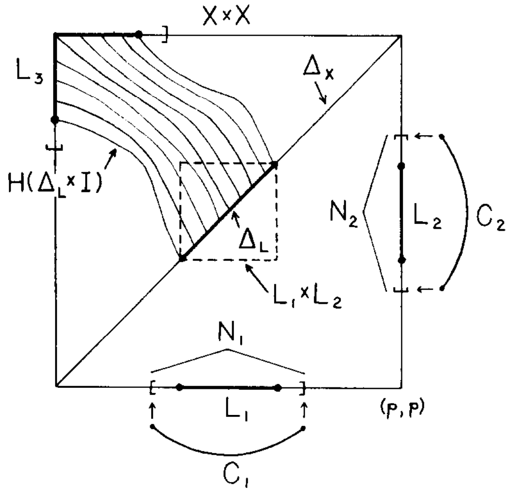

Fig. 1.

easy application of VanKampen's theorem implies the simple connectivity of Σ. By Lemma 1.1(d) and the PL Generalized Poincaré Conjecture [12], $\Sigma \approx S^n$.

Let $M = \Sigma - \text{int}(N_3)$. Since N_3 is a regular neighborhood of an embedded copy of L, M is by definition New(L, n). See Fig. 2.

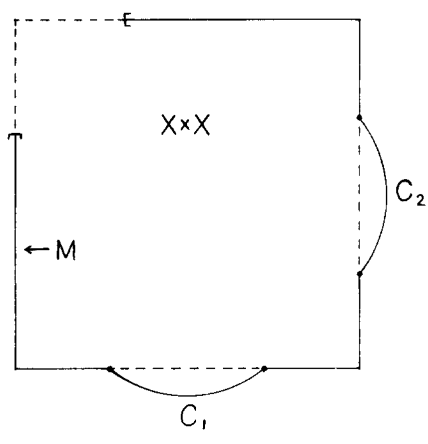

Fig. 2.

Step 3 (*Recognition of disjoint pseudo-spines*): We will show that C_1 and C_2 are PL pseudo-spines of M. Let $W_1 = M - \text{int}(C_1)$ and $W_2 = M - \text{int}(C_2)$. Consider the cobordisms $(W_1, \partial M, \partial C_1)$ and $(W_2, \partial M, \partial C_2)$. If these are h-cobordisms, it follows from the "weak h-cobordism theorem" (Stallings [14, Theorem 4]) that W_1-∂C_1 and W_2-∂C_2 are (PL) homeomorphic to $\partial M \times [0, 1)$, and thus, C_1 and C_2 are PL pseudo-spines of M.

To verify that $(W_i, \partial M, \partial C_i)$ is an h-cobordism for each i, we will show that $\pi_j(W_i, \partial C_i) = 0 = \pi_j(W_i, \partial M)$ for $j \leq (n-1)/2$ and $i = 1, 2$. Sufficiency of these conditions can be seen in a couple of ways. We outline a geometric proof based on a well-known engulfing strategy due to Stallings.

Fix i and choose disjoint collars A and B on ∂M and ∂C_i, respectively, in W_i. Let A' and B' be subcollars of A and B. Choose a triangulation $\mathscr{J}$ of W_i for which A, A', B and B'

correspond to subcomplexes. By our assumption, $\pi_j(W_i, A) = 0$ for $j \le (n-1)/2$, so by [13] there is a homeomorphism $h: W_i \to W_i$, fixed on A', so that $h(A)$ contains the $[(n-1)/2]$-skeleton of $W_i - \text{int}(B')$. Similarly, if $\mathscr{J}^*$ denotes the cell structure on W_i dual to $\mathscr{J}$, there is a PL homeomorphism $k: W_i \to W_i$, fixed on B', so that $k(A)$ contains the $[(n-1)/2]$-skeleton (relative to $\mathscr{J}^*$) of W_i-int(A'). By exploiting the joint structure between $\mathscr{J}^{(n-1)/2}$ and $(\mathscr{J}^*)^{(n-1)/2}$, we may alter h so that $h(A) \cup k(B) = W_i$. Now, W_i may be deformed into $k(B)$ along the collar lines of $h(A)$, and then onto ∂C_i along the collar lines of $k(B)$. Similarly, W_i deforms into ∂M, hence, $(W_i, \partial M, \partial C_i)$ is an h-cobordism.

We now verify the necessary homotopy conditions for $(W_1, \partial M, \partial C_1)$. The argument for $(W_2, \partial M, \partial C_2)$ is identical.

Claim (i). $\pi_j(W_1, \partial C_1) = 0$ for $j \le (n-1)/2$.

Consider the following diagram of inclusion induced maps.

$$\begin{array}{ccc} \pi_j(W_1, \partial C_1) \to \pi_j(\partial(X \times X) \cup_\partial C_2, N_1) \to \pi_j((X \times X) \cup_\partial C_2, N_1) & & \\ & \uparrow & \\ & \pi_j((X \times X) \cup_\partial C_2, L_1) \to \pi_j((X \times X) \cup_\partial C_2, X \times \{p\}). & \end{array}$$

Clearly the vertical map and the map in the lower row are isomorphisms for all j. Arguments like that used in Lemma 2.4 show the maps in the first row to be isomorphisms when $j \le (n-1)/2$. Note that the full force of the restriction on the size of k first becomes necessary in verifying the injectivity of the second map. In this verification, we use general position to move the $(j+1)$-dimensional image of a homotopy off the $2k$-dimensional spine, $L \times L$, of the $(n+1)$-dimensional manifold $X \times X$. This requires that $(j+1) + 2k < n+1$. Since, by hypothesis, $4k < n$, this condition holds.

Utilizing the isomorphisms in the above diagram, it suffices to show that $\pi_j((X \times X) \cup_\partial C_2, X \times \{p\}) = 0$ for all $j \le (n-1)/2$. Let $\tilde{X}$ be the universal cover of X, and $p: \tilde{X} \to X$ be the covering projection. Then $(p, id_X): \tilde{X} \times X \to X \times X$ is a covering map, and $(p, id_X)^{-1}(X \times \{p\}) = \tilde{X} \times \{p\}$. Note, since $\pi_1(L_1)$ injects into $\pi_1(X \times X)$, that the preimage of N_2 under this covering projection is a discrete collection $\{N_\alpha\}$ of copies of N_2, one for each element of $\pi_1(X)$. For each N_α, attach a copy C_α of C_2 to $\tilde{X} \times X$ along its boundary in the obvious way. The resulting space, $(\tilde{X} \times X) \cup_\partial \{C_\alpha\}$, is simply connected by repeated use of Van Kampen's Theorem and thus is the universal cover of $(X \times X) \cup_\partial C_2$ with the obvious covering projection, call it ρ, extending (p, id_X). Since $\rho^{-1}(X \times \{p\}) = \tilde{X} \times \{p\}$, then $\pi_j((X \times X) \cup_\partial C_2, X \times \{p\}) \cong \pi_j((X \times X) \cup_\partial \{C_\alpha\}, \tilde{X} \times \{p\})$ for all j. By the Hurewicz theorem, it suffices to show that $H_j((\tilde{X} \times X) \cup_\partial \{C_\alpha\}, \tilde{X} \times \{p\}) = 0$ for $j \le (n-1)/2$. By the long exact sequence for pairs, this may be accomplished by showing that $H_j(\tilde{X} \times \{p\}) \to H_j((\tilde{X} \times X) \cup_\partial \{C_\alpha\})$ is an isomorphism for all $j \le (n-1)/2$. Since both spaces are simply connected, this is immediate for $j = 0,1$. *For* $2 \le j \le (n-1)/2$, factor the map as follows:

$$H_j(\tilde{X} \times \{p\}) \to H_j((\tilde{X} \times X)) \to H_j((X \times X) \cup_\partial \{C_\alpha\}).$$

Since X is acyclic, it is clear from the Künneth formula that the first map is an isomorphism for all j. Now consider the Mayer–Vietoris sequence

$$\cdots \to H_j(\cup(\partial C_\alpha)) \to H_j(\tilde{X} \times X) \oplus H_j(\cup C_\alpha) \to H_j((\tilde{X} \times X) \cup_\partial \{C_\alpha\}) \to H_{j-1}(\cup(\partial C_\alpha)) \to \cdots$$

Since each C_α is contractible, and each ∂C_α is a homology n-sphere, $H_j(\cup C_\alpha)$ and $H_j(\cup \partial C_\alpha)$ are trivial for $1 \le j < n$. Plugging this information into the sequence shows that $H_j((\tilde{X} \times X)) \to H_j((\tilde{X} \times X) \cup_\partial \{C_\alpha\})$ is an isomorphism for all $2 \le j < n$. Hence, Claim (i) holds.

CLAIM (ii). $\pi_j(W_1, \partial M) = 0$ for $j \leq (n-1)/2$.

Consider the following diagram.

$$\begin{array}{ccc}
\pi_j(W, \partial M) \to \pi_j(\partial(X \times X) \cup_\partial C_2, N_3 & \to & \pi_j((X \times X) \cup_\partial C_2, N_3) \\
 & & \uparrow \\
 & & \pi_j((X \times X) \cup_\partial C_2, L_3) \\
 & & \downarrow{\scriptstyle v_1} \\
 & & \pi_j((X \times X) \cup_\partial C_2, H(\Delta_L \times I)) \\
 & & \uparrow{\scriptstyle v_0} \\
 & & \pi_j((X \times X) \cup_\partial C_2, \Delta_L) \to \pi_j((X \times X) \cup_\partial C_2, \Delta_X).
\end{array}$$

All maps in this diagram are inclusion induced and can be shown to be isomorphisms for $j \leq (n-1)/2$. With the exception of v_0 and v_1, verifications are like those made in Case (i). Recalling from Step 1 that $H: \Delta_L \times I \to X \times X$ is an embedding with $H(\Delta_L \times \{0\}) = \Delta_L$ and $H(\Delta_L \times \{1\}) = L_3$, it is easy to show that v_0 and v_1 are isomorphisms for all j. We leave this to the reader.

Applying the above diagram, if suffices to show that $\pi_j((X \times X) \cup_\partial C_2, \Delta_X) = 0$ for all $j \leq (n-1)/2$. Again consider the covering projection $\rho: (\tilde{X} \times X) \cup_\partial \{C_\alpha\} \to (X \times X) \cup_\partial C_2$. Notice that $\rho^{-1}(\Delta_X) = \{(x, p(x)) \mid x \in \tilde{X}\} \subset \tilde{X} \times X \subset (\tilde{X} \times X) \cup_\partial \{C_\alpha\}$. In particular, $\tau: \tilde{X} \to (\tilde{X} \times X) \cup_\partial \{C_\alpha\}$ defined by $\tau(x) = (x, p(x))$ takes $\tilde{X}$ homeomorphically onto $\rho^{-1}(\Delta_X)$. We denote $\rho^{-1}(\Delta_X)$ by $\tilde{\Delta}_X$ and observe that it is the universal cover of Δ_X. Arguing as in Case (i), it suffices to show that $H_j(\tilde{\Delta}_X) \to H_j((\tilde{X} \times X) \cup_\partial \{C_\alpha\})$ is an isomorphism for each $2 \leq j \leq (n-1)/2$. Factor this map into the following inclusion induced maps.

$$H_j(\tilde{\Delta}_X) \to H_j((\tilde{X} \times X)) \to H_j((\tilde{X} \times X) \cup_\partial \{C_\alpha\}).$$

The Mayer–Vietoris argument used in Case (i) shows that the second map is an isomorphism for all $2 \leq j < n$. To see that the first map in an isomorphism, define $\phi: \tilde{X} \times X \to \tilde{X}$ to be the projection map. Then the composition $\tilde{\Delta}_X \to \tilde{X} \times X \xrightarrow{\phi} \tilde{X}$ is a homeomorphism. Hence we have a commutative diagram

$$\begin{array}{ccc}
 & H_j((\tilde{X} \times X)) & \\
\nearrow & & \searrow{\scriptstyle \phi *} \\
H_j(\tilde{\Delta}_X) & \xrightarrow{\cong} & H_j(\tilde{X}).
\end{array}$$

Since X is acyclic, the Künneth formula shows that ϕ_* is an isomorphism for all j. Thus, $H_j(\tilde{\Delta}_L) \to H_j((\tilde{X} \times X))$ is also isomorphism.

Step 4 (*Existence of disjoint spines*). By Step 3 and Corollary 2.2, M contains a disjoint pair of spines.

Case (*b*). *n is even.*

Let $e: L \to S^{n/2}$ be a PL embedding, let X be a regular neighborhood of $e(L)$, and choose $p \in \partial X$. Instead of working in $X \times X$, we now work in $X \times X \times I$. Let $L_1 = e(L) \times \{p\} \times \{1/2\}$, $L_2 = \{p\} \times e(L) \times \{1/2\}$, $\Delta_X = \{(x, x, 1/2) \in X \times X \times I \mid x \in X\}$ and $\Delta_L = \{(x, x, 1/2) \in X \times X \times I \mid x \in e(L)\}$. Make similar obvious changes when necessary. Notation becomes more tedious in this case, but the proof requires no significant changes from that of Case (a) □

In the above proof the PL spines and pseudo-spines detected are codimension 0 submanifolds with locally flat boundaries. By the remark following Lemma 2.1, any PL spine or pseudo-spine may be "thickened" within a given neighborhood to a PL spine of this type. Moreover, as suggested by the proof of Theorem 4.1, a codimension 0 submanifold $C \subset \text{int } M$ with locally flat boundary, is a PL pseudo-spine of M if and only if (M-int C, ∂M, ∂C) is an h-cobordism. This allows a slight extension of our main result.

THEOREM 4.2. *Suppose Γ_1, Γ_2 are h-cobordant homology $(n-1)$-spheres $(n \geq 5)$ and let M_1, M_2 be the (unique) compact contractible n-manifolds bounded by Γ_1 and Γ_2, respectively. Then M_1 contains a disjoint pair of PL spines iff M_2 contains a disjoint pair of PL spines. In particular, if $M = \text{New}(L, n)$ for an acyclic k-complex L with $4k < n$, then any homology $(n-1)$-sphere h-cobordant to ∂M bounds a compact contractible n-manifold containing a pair of disjoint PL spines.*

Proof. Existence of M_1 and M_2 is by [8]. For uniqueness, suppose M_i and M_i' are compact contractible n-manifolds each with boundaries homeomorphic to Γ_i. Then $M_i \cup_\partial M_i'$ is a homology sphere (Lemma 1.1) which, again by [8] bounds a compact contractible $(n+1)$-manifold, N. By the h-cobordism theorem (see e.g. [11]), $(N, M_i, M_i') \approx (M_i \times I, M_i \times \{0\}, M_i \times \{1\})$, so $M_i \approx M_i'$.

Now let (W, Γ_1, Γ_2) be an h-cobordism and suppose M_1 contains a pair of disjoint spines, C and C'. By the remarks preceding this theorem, we may assume that C and C' are codimension-0 submanifolds with locally flat boundaries. $W \cup_{\Gamma_1} M_1$ is a compact contractible $(n+1)$-manifold with Γ_2 boundary. By uniqueness, $W \cup_{\Gamma_1} M_1 \approx M_2$. Moreover, $(W \cup_{\Gamma_1} M_1)$-int C and $(W \cup_{\Gamma_1} M_1)$-int C' are homeomorphic to W. Again by earlier remarks, this implies that C and C' are pseudo-spines of $W \cup_{\Gamma_1} M_1 \approx M_2$. By Corollary 2.2, M_2 contains disjoint spines. The converse holds by symmetry.

Combining the initial assertion of this result with Theorem 4.1 yields the final assertion. □

Note. A version of Theorem 4.2 with $n = 4$ is valid if we work with "topological spines" (see following section).

5. TOPOLOGICAL SPINES

A compactum A in the interior of a topological manifold X is a (*topological*) *spine* of X if there is a map $f: \partial X \to A$ and a homeomorphism of X onto the mapping cylinder, $\text{Cyl}(f)$, which takes ∂X identically onto the domain end and A identically onto the range end of $\text{Cyl}(f)$. By Whitehead [15], regular neighborhoods of compact polyhedra in combinatorial manifolds are mapping cylinders of this type; hence, the notion of topological spine generalizes the notion of PL spine. A compactum B in the interior of a topological manifold X is a (*topological*) *pseudo-spine* of X if $X - B$ is topologically homeomorphic to $\partial X \times [0, 1)$.

Most results in this paper remain true when the term "PL spine (or pseudo-spine)" is replaced by "topological spine (or pseudo-spine)". For example the proofs of Lemma 2.1 and its corollary are still valid, and of course the PL spines provided by Theorem 4.1 are also topological spines. An exception is the topological analogue of Corollary 2.5 which fails spectacularly, as is shown in [1].

6. APPLICATIONS TO TOPOLOGICAL EMBEDDING THEORY

Results presented in this section are topological in nature. We no longer assume that manifolds are combinatorial or that maps are PL. The symbol $\approx$ now denotes topological homeomorphism.

A subset K of a metric space X is said to be *slippery* if, for any $\varepsilon > 0$, there is a homeomorphism $h: X \to X$ such that $d(x, h(x)) < \varepsilon$ for all $x \in X$, and $h(K) \cap K = \emptyset$; otherwise, we say that K is *sticky*. For example, if $k < n/2$ then any k-dimensional subpolyhedron of a PL n-manifold is slippery. By contrast, Wright [16] has shown that for all $n \geq 4$, there exist sticky (topologically embedded) arcs in S^n. Furthermore, it is still unknown whether or not all Cantor sets in S^n ($n \geq 4$) are slippery (see [4]).

The sticky arcs produced by Wright are based on an extremely complicated construction by McMillan [9]. For some time it has seemed reasonable that simpler examples might lie on the "suspension circles" of doubly suspended non-simply connected homology spheres. If H is a homology sphere and $\Sigma^2(H)$ denotes the double suspension of H, then by Cannon–Edwards [2], $\Sigma^2(H) \approx S^n$; moreover, if H is not simply connected, the circle of suspension points is wildly embedded as a subset of S^n. Combined with Theorem 4.1, the following result shows that, in many cases, this double suspension circle (and, thus, any arc contained within) is actually slippery.

THEOREM 6.1. *Let H^{n-2} be a homology sphere, $\Sigma^2(H) \approx S^n$ be the double suspension of H, and $J \subset S^n$ correspond to the suspension circle. If H bounds a compact contractible $(n-1)$-manifold C which contains a pair of disjoint spines, then J is slippery in S^n.*

Proof. Let pH denote the cone over H viewed as $H \times [0, 1]/H \times \{1\}$ with p denoting the cone point. For each $0 \leq t < 1$, let $p_t H \subset pH$ denote the subcone; $H \times [t, 1]/H \times \{1\}$. If K is a spine of C, then there is an obvious map $\pi: C \to pH$ which sends K to $\{p\}$ and is a homeomorphism off K. Let C_t denote $\pi^{-1}(p_t H)$, and notice that C_t is homeomorphic to C.

Since $J \subset S^n$ has a neighborhood N homeomorphic to $pH \times S^1$, with J corresponding to $\{p\} \times S^1$, it will suffice to show that for any $\varepsilon > 0$, there exists an ε-homeomorphism $h: pH \times S^1 \to pH \times S^1$ which fixes $(H \times \{0\}) \times S^1$, and for which $h(\{p\} \times S^1) \cap (\{p\} \times S^1) = \emptyset$. Without loss of generality we assume that the metric on $pH \times S^1$ is of the form, $d((x_1, y_1), (x_2, y_2)) = max\{d(x_1, x_2), d(y_1, y_2)\}$.

Consider the cell-like map $\pi \times id: C \times S^1 \to pH \times S^1$. Choose t sufficiently close to 1 that $p_t H \times \{q\}$ has diameter less than $\varepsilon/2$ for all $q \in S^1$. By Cannon–Edwards (see [2]), there is a homeomorphism $f: C \times S^1 \to pH \times S^1$ such that $d(f(x, y), \pi \times id(x, y)) < \varepsilon/2$ for all $(x, y) \in C \times S^1$ and $f = \pi \times id$ on $(C - C_t) \times S^1$. Then $f^{-1}(\{p\} \times S^1) \subset \text{int}(C_t \times S^1)$ and since C_t contains disjoint spines, there is a spine, K', of C_t sufficiently close to ∂C_t (see proof of Lemma 2.3) that $(K' \times S^1) \cap f^{-1}(\{p\} \times S^1) = \emptyset$. By pushing out (towards ∂C_t) along collar lines of $C_t - K'$, we may construct a homeomorphism $g: C_t \times S^1 \to C_t \times S^1$ which fixes $\partial C_t \times S^1$, sends $C_t \times \{q\}$ into $C_t \times \{q\}$ for all q, and for which $g(f^{-1}(\{p\} \times S^1)) \cap f^{-1}(\{p\} \times S^1) = \emptyset$. Extend to all of $C \times S^1$ via the identity. Letting $h = fgf^{-1}: pH \times S^1 \to pH \times S^1$ gives the desired map. □

As mentioned earlier, the question of whether all Cantor sets in S^n ($n \geq 4$) are slippery is open. Many who have considered this question regard a collection of examples produced by Daverman in [4] as the most likely known candidates for counterexamples. We close this section by noting, without proof, that techniques similar to those used above can be employed to show that many of the Daverman examples are in fact slippery.

7. QUESTIONS

Numerous questions regarding the existence of disjoint spines of compact contractible manifolds remain open. The most obvious is:

Question 1. Does there exist a compact contractible manifold which does not contain a pair of disjoint (PL or topological) spines?

It seems that dimension may play a role in this question, hence we pose the following separately.

Question 2. Do there exist dimensions $n \geq 3$ for which all compact contractible n-manifolds contain disjoint (PL or topological) spines?

Question 3. Does there exist a dimension $n \geq 3$ for which no compact contractible n-manifold other than B^n contains a pair of disjoint spines?

Remarks (1) These questions are most appropriate when $n \geq 4$, since when $n = 3$, they are directly tied to the Poincaré conjecture. To see this, observe that any compact contractible 3-manifold has a 2-sphere boundary (apply Lemma 1.1); hence, there is a compact contractible 3-manifold not homeomorphic to B^3 iff the Poincaré conjecture fails. If a fake 3-cell, H^3, exists it does not contain a pair of disjoint spines. Indeed, if a pair of disjoint spines existed, we could apply the techniques of Lemma 2.3, to produce an infinite collection of pairwise disjoint spines of H^3. Since the boundary of each is a 2-sphere, this would violate the existence of a prime decomposition of H^3 as promised by Kneser (see [7]).

(2) By Freedman [5], there exist compact contractible 4-manifolds which allow no combinatorial structure. Hence, the consideration of topological spines is essential in dimension 4.

Acknowledgements—The author wishes to acknowledge Ric Ancel for several helpful remarks and observations.

REFERENCES

1. F.D. ANCEL and C.R. GUILBAULT: Compact contractible n-manifolds have arc spines ($n \geq 5$), to appear in *Pacific J. Math.*
2. J.W. CANNON: Shrinking cell-like decompositions of manifolds: codimension three, Ann. of Math. **110** (1979), 83–112.
3. R.J. DAVERMAN, Embedding phenomena based upon decomposition theory: Wild Cantor sets satisfying strong homogeneity properties, *Proc. Amer. Math. Soc.* **75** (1979), 177–182.
4. R.J. DAVERMAN, Problems about finite dimensional manifolds, *Open Problems in Topology*, J. vanMill and G.M. Reed, Eds., North-Holland, Amsterdam, 1990, 431–455.
5. M.H. FREEDMAN, The topology of four-dimensional manifolds, J. Diff. Geom. **17** (1982), 357–453.
6. V.K.A.M. GUGENHEIM, Piecewise linear isotopy and embedding of elements and spheres (I), *Proc. London Math. Soc.* **3** (1953), 29–53.
7. J. HEMPEL, 3-manifolds, *Ann. of Math.* Stud., (86) (1976).
8. M.A. KERVAIRE, Smooth homology spheres and their fundamental groups, *Trans. Amer. Math. Soc.* **144** (1969), 67–72.
9. D.R. MCMILLAN, An arc in a PL n-manifold with no neighborhood that embeds in S^n, $n \geq 4$, *Michigan Math. J.* **25** (1978), 29–43.
10. M.H.A. NEWMAN, Boundaries of ULC sets in Euclidean n-space, *Proc. Acad. Sci. U.S.A* **34** (1948), 193–196.
11. C.P. ROURKE and B.J. SANDERSON, *Introduction to Piecewise-Linear Topology*, Springer Berlin (1982).
12. S. SMALE, Generalized Poincaré conjecture in dimensions greater than four, Ann. Math. **74** (1961), 391–406.
13. J.R. STALLINGS, Polyhedral homotopy-spheres, *Bull. Amer. Math. Soc.*, **66** (1960), 485–488.
14. J.R. STALLINGS, On infinite processes leading to differentiability in the complement of a point, in: Differential and combinatorial topology, S. Cairns, Ed., Princeton Univ. Press, Princeton, 1965, pp. 245–254.
15. J.H.C. WHITEHEAD, Simplicial spaces, nuclei, and m-groups, *Proc. London Math. Soc.* **45** (1939), 243–327.
16. D.G. WRIGHT, A Sticky arc in S^n ($n \geq 3$), *Proc. of the Fifth Annual Western Workshop in Geometric Topology*, Colorado Springs, June 16–18, 1988, 36–37 (informal proceedings).

Department of Mathematical Sciences,
University of Wisconsin-Milwaukee,
Milwaukee,
Wisconsin 53201, *U.S.A*
EMS Building, PO Box 413.