Compressible spaces and $\mathcal{EZ}$-structures

by

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Abstract. Bestvina introduced a $\mathcal{Z}$-structure for a group G to generalize the boundary of a CAT(0) or hyperbolic group. A refinement of this notion, introduced by Farrell and Lafont, includes a G-equivariance requirement, and is known as an $\mathcal{EZ}$-structure. A recent result of the first two authors with Tirel put $\mathcal{EZ}$-structures on Baumslag–Solitar groups and $\mathcal{Z}$-structures on generalized Baumslag–Solitar groups. We generalize this to higher dimensions by showing that fundamental groups of graphs of closed nonpositively curved Riemannian n-manifolds (each vertex and edge manifold is of dimension n) admit $\mathcal{Z}$-structures, and graphs of negatively curved or flat Riemannian n-manifolds admit $\mathcal{EZ}$-structures.

1. Introduction. Suppose that a discrete group G acts properly and cocompactly by isometries on a CAT(0) space X. The space X can be naturally compactified by attaching the visual boundary $\partial_\infty X$, and $\overline{X} = X \cup \partial_\infty X$ is homotopy equivalent to X. In fact, $\partial_\infty X$ is a $\mathcal{Z}$-set in $\overline{X}$, meaning that $\partial_\infty X$ can be instantly homotoped ([1]) into X. Furthermore, large translates of compact sets in X "look small", i.e. if K is a compact set and $x_0 \in X$, then the visual diameter of $\{g_n K\}_{n=1}^\infty$ as viewed at x_0 goes to 0 if $d(1, g_n) \to \infty$ in G. It follows that for any open cover $\mathcal{U}$ of $\overline{X}$, all but finitely many translates of K are contained in an element $U \in \mathcal{U}$.

In [3], Bestvina introduced the concept of a $\mathcal{Z}$-structure on a group G in order to generalize the boundary of a CAT(0) (or word hyperbolic) group:

([1]) More precisely, there is a homotopy $h_t : \overline{X} \to \overline{X}$ such that $h_0 = \mathrm{Id}_{\overline{X}}$ and $h_t(\overline{X}) \subset X$ for all $t > 0$.

DEFINITION 1.1. A $\mathcal{Z}$-*structure* on a group G is a pair of spaces $(\overline{X}, Z)$ satisfying the following four conditions:

(1) $\overline{X}$ is a compact absolute retract,
(2) Z is a $\mathcal{Z}$-set in $\overline{X}$,
(3) $X = \overline{X} - Z$ is a proper metric space on which G acts properly and cocompactly by isometries,
(4) X satisfies the *nullity condition* with respect to the G-action: for every compact $K \subset X$ and any open cover $\mathcal{U}$ of X, all but finitely many G-translates of K lie in an element of $\mathcal{U}$.

If the group action of G on X extends to $\overline{X}$, this is called an $\mathcal{EZ}$-*structure* [17]. In Bestvina's original definition, the action on X was required to be free and $\overline{X}$ was required to be a Euclidean retract (finite-dimensional absolute retract). The modified definition here is due to Dranishnikov [13]. Among other things, this modification allows for groups with torsion. It is still open whether all groups of type F (or even type F^* or F^*_{AR}) admit $\mathcal{Z}$-structures (^{2}).

Many properties of CAT(0) or hyperbolic boundaries transfer over to this general setting. For example: the dimension of a $\mathcal{Z}$-boundary is equal to the global cohomological dimension of that boundary; the Čech cohomology of Z determines $H^*(G; \mathbb{Z}G)$; and, when G is torsion-free, the cohomological dimension of G is $1 + \dim Z$. See [3], [13] and [20].

In this paper, we are concerned with the following question:

QUESTION. *Suppose a group G acts properly and cocompactly on a* CAT(0) *space X by homeomorphisms. Under what conditions is $(\overline{X}, \partial_\infty X)$ part of an* $(\mathcal{E})\mathcal{Z}$-*structure for G?*

We note that the change to an action by homeomorphisms does not affect property (2) of Definition 1.1. In fact, in [1], it is shown that in this case there is a topologically equivalent metric on X for which the action is by isometries. Thus, in this situation, properties (1)–(3) are always satisfied. The key property to check then is the nullity condition. The latter can certainly fail for some actions, for example, the Baumslag–Solitar group

$$\mathrm{BS}(1, 2) = \langle a, t \mid tat^{-1} = a^2 \rangle$$

acts properly and cocompactly on the product $T \times \mathbb{R}$, where T is a trivalent tree. Certain translates of a fundamental domain have exponentially growing height (as measured by the distance to a fixed basepoint) in $T \times \mathbb{R}$, and this implies $\overline{X}$ fails the nullity condition.

To get around this problem, the first two authors with Tirel [19] modified the action of $\mathrm{BS}(1, 2)$ (or more generally $\mathrm{BS}(m, n)$) on $T \times \mathbb{R}$ by "compress-

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(^{2}) See [20] for definitions.

ing the $\mathbb{R}$-direction". The rough idea is to conjugate the given action by a homeomorphism of $T \times \mathbb{R}$ which is the identity on T and shrinks distances in the $\mathbb{R}$-direction. This changes the exponential growth of the diameter of translates of a fundamental domain to sublinear growth (the diameters are still forced to go to infinity), which is enough to ensure the nullity condition.

In the present paper, we give a more general picture of when this strategy works. Our setup requires that the CAT(0) space splits as a Cartesian product $X \times Y$. Then, given a fundamental domain K, we insist that all G-translates of K have uniformly bounded diameter in the X-direction, and have "properly controlled" diameter in the Y-direction. Finally, we assume Y is *compressible*, which roughly means that Y admits homeomorphisms which uniformly shrink all compact sets $K \subset Y$ (see Definition 4.1). With this setup, we can perform the same conjugating trick as in [19] to produce a $\mathcal{Z}$-structure.

The compressibility hypothesis on Y is rather strong since it requires that compressing be done via homeomorphisms. This rules out a variety of important candidates for the space Y. Remark 4.6 and Section 8 address some of the limitations imposed by this hypothesis. Nevertheless, we show that all simply connected nonpositively curved Riemannian manifolds are compressible. This allows us to take on an important and well-studied class of groups. One of our main theorems is the following:

THEOREM 1.2. *Fundamental groups of graphs of closed nonpositively curved Riemannian n-manifolds have $\mathcal{Z}$-structures.*

Here, a *graph of closed nonpositively curved n-manifold groups* means a finite connected graph of groups, where each vertex and edge group is the fundamental group of a closed nonpositively curved Riemannian n-manifold (where n is the same for each vertex and edge). Equivalently, we could have required that each vertex group is the fundamental group of a closed nonpositively curved n-manifold and each edge group is finite index in its vertex groups. The coarse geometric structure of these groups has been studied by Farb–Mosher [15], and more generally by Mosher–Sageev–Whyte [21]. In those papers, graphs of groups with the finite index property for edge groups are referred to as *geometrically homogeneous.*

The beauty of the geometric homogeneity property is that the fundamental group frequently acts properly and cocompactly on a Cartesian product space $T \times \widetilde{M_v}$, where $\widetilde{M_v}$ is the universal cover of a vertex space (in our case and in the case of [15], a Riemannian manifold homeomorphic to $\mathbb{R}^n$) and T is the associated Bass–Serre tree. Even though the action is usually not by isometries, the CAT(0) geometry of $T \times \widetilde{M_v}$ under the ℓ_2-metric, with its corresponding visual boundary, turns out to be quite useful.

In addition to the above theorem, we give conditions under which our (new) G-action extends to $\overline{X}$. Roughly speaking, we need to know that certain covering maps lift to maps on the universal cover which extend to visual boundaries; we also need a compressing homeomorphism that does not change much under linear reparametrization of the domain. If each vertex manifold is negatively curved or flat, our conditions will be satisfied, so we have the following:

THEOREM 1.3. *Fundamental groups of graphs of closed negatively curved or flat Riemannian n-manifolds have $\mathcal{EZ}$-structures.*

As in the proof of Theorem 1.2 the CAT(0) geometry of $T \times \widetilde{M_v}$ plays a vital role in the proof of this theorem.

A somewhat surprising ingredient in some part of the proofs of our main theorems is the use of the theory of Hilbert cube manifolds as pioneered by Anderson, Chapman, West, Toruńczyk, Edwards, and others. See [10] and [25] for overviews of the subject. Most—but not all—of our results can be obtained without the use of Hilbert cube manifolds, the key exception being the $n = 4$ cases of our main theorems. Perhaps more importantly, the Hilbert cube manifolds approach (which is valid in all dimensions) has the potential for proving theorems beyond the scope of this paper. We expand upon that thought in Section 9. The authors thank the referee, who pushed us to include this approach in the current article. Prior to the requested revisions, we had overlooked a gap in our proofs when $n = 4$.

2. Boundaries of CAT(0) spaces. We will assume the reader is familiar with the definitions and basic facts about CAT(0) spaces; see [6] for complete details.

Let X be a proper CAT(0) space, and let ∂X be the boundary of X. Fix a base point $x_0 \in X$. Each equivalence class of rays in ∂X contains exactly one representative emanating from x_0. We may endow $\overline{X} = X \cup \partial X$ with the *cone topology*, described below, under which ∂X is a closed subspace of $\overline{X}$ and $\overline{X}$ is compact. Equipped with the topology induced by the cone topology on $\overline{X}$, the boundary is called the *visual boundary* of X; we denote it by $\partial_\infty X$.

The cone topology on $\overline{X}$, for $x_0 \in X$, is generated by the basis $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_\infty$, where $\mathcal{B}_0$ consists of all open balls $B(x, r) \subset X$ and $\mathcal{B}_\infty$ is the collection of all sets of the form

$$U(c, r, \varepsilon) = \{x \in \overline{X} \mid d(x, x_0) > r \text{ and } d(p_r(x), c(r)) < \varepsilon\},$$

where $c : [0, \infty) \to X$ is any geodesic ray based at x_0, $r > 0$, $\varepsilon > 0$, and p_r is the natural projection of $\overline{X}$ onto $\overline{B}(x_0, r)$.

Now, for each pair $x \in X$ and $\varepsilon > 0$, let

$$V(x, \varepsilon) = \{y \in \overline{X} \mid d(x_0, y) > d(x_0, x) \text{ and } d(x, p_{d(x_0, x)}(y)) < \varepsilon\}.$$

LEMMA 2.1. *If $\mathcal{B}_0$ is the set of all open balls in a proper* CAT(0) *space X and $\mathcal{V}_{x_0}$ is the collection of all $V(x, \varepsilon)$ as defined above, then $\mathcal{B}_0 \cup \mathcal{V}_{x_0}$ is a basis for the usual cone topology on $\overline{X}$.*

Proof. Clearly each set $U(\xi, r, \varepsilon)$ can be expressed as $V(\xi(r), \varepsilon)$; so the cone topology is at least as fine as the proposed topology. For the reverse containment, suppose $y \in V(x, \varepsilon) \in \mathcal{V}_{x_0}$. Let $\delta = \varepsilon - d(x, p_{d(x_0, x)}(y))$. If $y \in \partial X$ then $y \in U(y, d(x_0, x), \delta) \subseteq V(x, \varepsilon)$. If $y \in X$, define $W = B(y, \delta) \setminus \overline{B(x_0, d(x_0, x))}$. Since projection onto $\overline{B(x_0, d(x_0, x))}$ does not increase distances, $W \subseteq V(x, \varepsilon)$. It follows that $V(x, \varepsilon)$ is open in the cone topology, so the proposed topology is at least as fine as the cone topology. ∎

The following lemma is similar in spirit to the Lebesgue covering lemma and is a generalization of [19, Lemma 2.3].

LEMMA 2.2. *Let (X, d) be a proper* CAT(0) *space and let $\mathcal{U}$ be an open cover of $\overline{X}$. Then there exist $R \gg 0$ and $\delta > 0$ such that for every $x \in X \setminus B(x_0, R)$, $V(x, \delta)$ lies in an element of $\mathcal{U}$.*

Proof. Without loss of generality we may assume $\mathcal{U}$ consists entirely of elements from the basis $\mathcal{B}_0 \cup \mathcal{V}_{x_0}$. Since $\partial_\infty X$ is compact, there exists $\{U_1, \ldots, U_k\} \subseteq \mathcal{U}$ that covers $\partial_\infty X$. For each i, write $U_i = V(x_i, \varepsilon_i)$. Since $X \setminus \bigcup_{i=1}^k U_i$ is a closed subset of X which contains no infinite rays, an Arzelà–Ascoli argument shows that $X \setminus \bigcup_{i=1}^k U_i$ is bounded. Choose $R \gg 0$ so that

$$R > \max\{d(x_0, x_i) \mid 1 \le i \le k\} \quad \text{and} \quad X \setminus \bigcup_{i=1}^k U_i \subseteq B(x_0, R).$$

Note that if an open ball $B(x, \varepsilon)$ lies in U_i then $V(x, \varepsilon) \subseteq U_i$. It follows that, for each $x \in S(x_0, R)$, there exists some $\varepsilon_x > 0$ such that $V(x, \varepsilon_x)$ is contained in some U_i. For each $i \in \{1, \ldots, k\}$, define a function $\eta_i : S(x_0, R) \to [0, \infty)$ by $\eta_i(x) = \sup \{\varepsilon \mid V(x, \varepsilon) \subseteq V_i\}$. Note that η_i is continuous, and $\eta_i(x) > 0$ if and only if $x \in V_i$. Thus, $\eta : S(x_0, R) \to [0, \infty)$ defined by $\eta(x) = \max \{\eta_i(x)\}_{i=1}^k$ is continuous and strictly positive. Let δ' be the minimum value of η and set $\delta = \min \{\delta'/2, 1/R\}$. Clearly $V(x, \delta)$ lies in some U_i for all $x \in S(x_0, R)$. Moreover, if $d(x_0, x) > R$ then $V(x, \delta) \subseteq V(p_R(x), \delta)$; so again $V(x, \delta)$ lies in some U_i. ∎

DEFINITION 2.3. A function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is *sublinear* if

$$\lim_{x \to \infty} \frac{\phi(x)}{x} = 0.$$

A function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is *uniformly sublinear* if

$$\bar{\phi}(t) := \max_{|x-y|=t} |\phi(x) - \phi(y)|$$

is sublinear.

For example, $\log(x+1) : \mathbb{R}^+ \to \mathbb{R}^+$ is a uniformly sublinear homeomorphism.

LEMMA 2.4. *Suppose* (X,d) *is a proper* CAT(0) *space. If* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *is sublinear and* $\mathcal{U}$ *is an open cover of* $\overline{X}$ *then there exists* $T > 0$ *such that whenever* $d(x_0, x) > T$, $B(x, \phi(d(x_0, x)))$ *lies in some* $U \in \mathcal{U}$.

Proof. Choose $R \gg 0$ and $\delta > 0$ as in the previous lemma. We can assume that $\delta < 1/R$. By sublinearity, choose $T > 0$ so that $\frac{\phi(t)}{t - \phi(t)} < \delta^2$ and $t - \phi(t) > R$ for all $t \geq T$. It suffices to prove:

CLAIM. *If* $d(x_0, x) > T$, *then* $B(x, \phi(d(x_0, x))) \subseteq V(p_R(x), \delta)$.

Let $y \in B(x, \phi(d(x_0, x)))$. Then $d(x, y)) < \phi(d(x_0, x))$ and $d(x_0, x) - \phi(d(x_0, x)) > R$. Let $x' = p_{d(x_0, x) - \phi(d(x_0, x))}(x)$ and $y' = p_{d(x_0, x) - \phi(d(x_0, x))}(y)$. Since projection does not increase distances, $d(x', y') < \phi(d(x_0, x))$. Then

$$\frac{d(p_R(x'), p_R(y'))}{1/\delta} \leq \frac{d(p_R(x'), p_R(y'))}{R}$$
$$\leq \frac{d(x', y')}{d(x_0, x) - \phi(d(x_0, x))} \text{ (CAT(0) inequality for } \triangle x_0 x' y')$$
$$\leq \frac{\phi(d(x_0, x))}{d(x_0, x) - \phi(d(x_0, x))} < \delta^2$$

Consequently, $d(p_R(x'), p_R(y')) < \delta$; and since $p_R(x') = p_R(x)$ and $p_R(y') = p_R(y)$, we conclude that $y \in V(p_R(x), \delta)$. ∎

3. The Hilbert cube and Hilbert cube manifolds. To begin this section, we provide a brief discussion of a simple CAT(0) space that will play a useful role in our main theorems. Let I^ω denote the infinite product $\prod_{i=0}^{\infty} [0, 1/2^i]$ endowed with the metric $d((x_i), (y_i)) = (\sum |x_i - y_i|^2)^{1/2}$. This metric induces the standard product topology, so I^ω is just a metrized version of the *Hilbert cube*. For each nonnegative integer n, endow $I^n = \prod_{i=0}^{n-1} [0, 1/2^i] \subseteq \mathbb{R}^n$ with the subspace metric, where $\mathbb{R}^n$ denotes n-dimensional Euclidean space. Clearly I^n is CAT(0) and the obvious inclusion $I^n \hookrightarrow I^\omega$ is an isometric embedding. It follows that I^ω is CAT(0) since a "fat triangle" in I^ω could otherwise be projected into I^n (for large n) to obtain a fat triangle in I^n.

If (X, d) is a proper CAT(0) space, then so is $X \times I^\omega$, under the ℓ_2-metric. Moreover, $\partial_\infty(X \times I^\omega) \approx \partial_\infty X$; in fact, for arbitrary $p \in I^\omega$, $X \times \{p\} \hookrightarrow$

$X \times I^\omega$ extends continuously to an inclusion map $\widehat{X} \times \{p\} \hookrightarrow \widehat{X \times I^\omega}$ which is a homeomorphism between visual boundaries.

A *Hilbert cube manifold* is a separable metric space with the property that every point has a neighborhood homeomorphic to I^ω. A surprising fact about I^ω is that it is homogeneous, i.e., for any $x, y \in I^\omega$ there exists a homeomorphism of I^ω taking x to y. From this, it is easy to see that (as in finite dimensions) all connected Hilbert cube manifolds are homogeneous.

In Section 7, we will make essential use of some classical theorems from the topology of Hilbert cube manifolds. We state those results here for easy access.

THEOREM 3.1 ([27]). *If X is a locally finite CW-complex then $A \times I^\omega$ is a Hilbert cube manifold.*

THEOREM 3.2 ([9]). *A map $f : A \to B$ between finite CW-complexes is a simple homotopy equivalence (in the sense of [28] and [11]) if and only if $f \times \mathrm{id}_{I^\omega} : A \times I^\omega \to B \times I^\omega$ is homotopic to a homeomorphism.*

Building upon his own work, along with work by West [26] and others, Chapman [8] used Hilbert cube technology to extend simple homotopy theory to the category of compact ANRs. From there, Edwards [14], building on work by Toruńczyk [23], [24], obtained the following generalization of the above theorems. Strictly speaking, we do not need this generalization for our main theorem, but believe it could be useful in attacking the problems posed in Section 9.

THEOREM 3.3. *The product of a locally compact ANR with I^ω is a Hilbert cube manifold. Moreover, a map $f : A \to B$ between compact ANRs is a simple homotopy equivalence if and only if the map $f \times \mathrm{id}_{I^\omega} : A \times I^\omega \to B \times I^\omega$ between Hilbert cube manifolds is homotopic to a homeomorphism.*

4. $\mathcal{Z}$-structures and compressible spaces. The following is our main definition. Recall that a function $f : X \to Y$ is *proper* if preimages of compact sets are compact.

DEFINITION 4.1. A proper metric space Y is *compressible* if for any proper function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ there is a homeomorphism $h_\psi : Y \to Y$ such that for every compact set $K \subset Y$ with $\mathrm{diam}(K) < \psi(R)$, we have $\mathrm{diam}(h_\psi(K)) < \phi(R)$ for $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ a sublinear function. We say h_ψ is a *compressing homeomorphism*.

Of course, it suffices to check compressibility for K being two points, but it is convenient to state it for all compact sets. For applications to geometric group theory, the following is obvious, but useful.

PROPOSITION 4.2. *If a map $f : X \to Y$ between proper metric spaces is both a homeomorphism and a quasi-isometry, and X is compressible, then so is Y.*

We can now state our main technical theorem, which generalizes the main result of [19].

THEOREM 4.3. *Suppose G acts properly and cocompactly on $X \times Y$, where X and Y are CAT(0) and Y is compressible. Let (x_0, y_0) be a basepoint in $X \times Y$. Let $\pi_X : X \times Y \to X \times y_0$ and $\pi_Y : X \times Y \to x_0 \times Y$ be the projections onto each factor. Assume that for a fixed compact K in $X \times Y$ with $GK = X \times Y$,*

(1) *there is $S > 0$ such that $\mathrm{diam}(\pi_X(gK)) < S$ for all $g \in G$,*
(2) *there is a proper function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that if $\pi_X(gK) \subset B_X(x_0, R)$, then $\mathrm{diam}(\pi_Y(gK)) < \psi(R)$.*

Then there exists a proper cocompact action of G on $X \times Y$ such that $(X \times Y, \partial_\infty(X \times Y))$ is a $\mathcal{Z}$-structure for G.

Proof. Let $h_\psi : Y \to Y$ be a compressing homeomorphism for ψ, and extend this to the homeomorphism $H_\psi := (\mathrm{id}_X, h_\psi) : X \times Y \to X \times Y$. Let $f : G \to \mathrm{Homeo}(X \times Y)$ be the given action. Now, modify the action by conjugating with H_ψ, i.e. define a new action by

$$f^{H_\psi}(g) : X \times Y \to X \times Y, \quad f^{H_\psi}(g) = H_\psi f(g) H_\psi^{-1}.$$

This conjugated action is again proper and cocompact. We only need to verify the nullity condition. If K is a fundamental domain for the original G-action, then $H_\psi(K)$ is a fundamental domain for the conjugated action. Let $g \in G$, and consider a translate $f^{H_\psi}(g).K$. By construction, this is contained in a product of balls $B(x, 2S) \times B(y, \phi(d(x_0, x)))$ for sublinear ϕ and some $x \in X$. By Lemma 2.4, the collection $\mathcal{B} = B(x, 2S) \times B(y, \phi(d(x_0, x)))$ satisfies the nullity condition. ∎

We now give examples of compressible spaces.

LEMMA 4.4. $\mathbb{R}^+$ *with the standard metric is compressible.*

Proof. By choosing a larger function, we can assume the proper function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is increasing and $\psi(x + y) \geq \psi(x) + \psi(y)$ for all $x, y \in \mathbb{R}^+$. Since ψ is increasing, we can define the inverse $\psi^{-1} : [\psi(0), \infty) \to \mathbb{R}^+$. Given an interval $[a, a + \psi(R)]$ with $a > \psi(0)$, we have

$$\psi^{-1}([a, a + \psi(R)]) \subset [\psi^{-1}(a), \psi^{-1}(a) + R].$$

Now, let the homeomorphism $\widehat{h}_\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by

$$\widehat{h}_\psi(x) = \begin{cases} x, & 0 \leq x \leq \psi(0), \\ \phi(\psi^{-1}(x)) + \psi(0), & x \geq \psi(0), \end{cases}$$

where ϕ is a uniformly sublinear homeomorphism with $\phi(0) = 0$. It follows that for any a, a' with $d(a, a') < \psi(R)$, $d(h_\psi(a), h_\psi(a')) < \phi(R) + \psi(0)$. ∎

Note that we can also assume $\widehat{h}_\psi(x)/x$ is decreasing. Our main examples of compressible spaces come from the following theorem.

THEOREM 4.5. *Let M be a simply connected, nonpositively curved, Riemannian manifold. Then M is compressible.*

Proof. We assume that the proper function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is increasing, $\psi(x + y) \geq \psi(x) + \psi(y)$ for all $x, y \in \mathbb{R}^+$, and $\widehat{h}_\psi$ is defined as in Lemma 4.4. Fix a basepoint $m_0 \in M$. Let

$$\exp_{m_0} : T_{m_0} M \cong \mathbb{R}^n \to M$$

be the exponential map which, by the Cartan–Hadamard Theorem, is a diffeomorphism taking geodesic rays in $\mathbb{R}^n$ emanating from the origin to geodesic rays in M emanating from m_0. Consider the homeomorphism

$$h_\psi := \exp_{m_0} \circ h'_\psi \circ \exp_{m_0}^{-1} : M \to M$$

where h'_ψ restricts to $\widehat{h}_\psi$ on geodesic rays in $\mathbb{R}^n$ emanating from the origin. Roughly speaking, h_ψ is the homeomorphism that restricts to $\widehat{h}_\psi$ on geodesic rays in M emanating from m_0.

CLAIM. *h_ψ is a compressing homeomorphism for M.*

Suppose that x and y are two points in M with $d(x, y) < \psi(R)$. If $d(x, m_0) = d(y, m_0) = D$, then $d(h_\psi(x), h_\psi(y)) \leq \widehat{h}_\psi(D)d(x, y)/D$ by the CAT(0) inequality. Since $D > d(x, y)/2$, by our assumption that $\widehat{h}_\psi(x)/x$ is decreasing we deduce that

$$\frac{\widehat{h}_\psi(D)d(x, y)}{D} < \frac{\widehat{h}_\psi(d(x, y)/2)d(x, y)}{d(x, y)/2} = 2\widehat{h}_\psi\left(\frac{d(x, y)}{2}\right) < 2\phi(R)$$

In general, assume that $d(x, m_0) < d(y, m_0)$, and choose z such that $d(x, m_0) = d(z, m_0)$ and z lies on the same ray emanating from m_0 as y. The projection of x to the geodesic between y and m_0 is less than $d(x, m_0)$ from m_0. By convexity of the distance function we have $d(x, z) < d(x, y)$. Similarly, since metric balls are convex and z is the projection of y onto the $d(x, m_0)$-ball around m_0, we have $d(y, z) < d(x, y)$. So, by assumption we have $d(x, z)$ and $d(z, y) < \psi(R)$. By the above, we see that

$$d(h_\psi(x), h_\psi(z)) \leq 2\widehat{h}_\psi(\psi(R)) \leq 2\phi(R),$$

and by Lemma 4.4,

$$d(h_\psi(z), h_\psi(y)) < 2\widehat{h}_\psi(\psi(R)) \leq 2\phi(R),$$

so we have

$$d(h_\psi(x), h_\psi(y)) \leq d(h_\psi(x), h_\psi(y)) + d(h_\psi(y), h_\psi(z)) \leq 4\phi(R)$$

and we are done. ∎

REMARK 4.6. A number of comments regarding compressible spaces are in order.

(1) For examples of compressible spaces not homeomorphic to $\mathbb{R}^n$, let X be a compact metric space and $\text{Cone}_\infty(X) := X \times [0, \infty)/X \times \{0\}$. There are natural "warped product" metrics that one can put on $\text{Cone}_\infty(X)$ and, for suitable choices, there are natural homeomorphisms which move points towards the cone point along cone lines to produce a compressing homeomorphism. For example, X could be three points and $\text{Cone}_\infty(X)$ the infinite tripod equipped with the natural path metric.

(2) Compressing homeomorphisms like the ones described above and in the proof of Theorem 4.5 are called *radial compressions*. More specifically, if Y is homeomorphic to an open cone and $\widehat{h} : \mathbb{R}^+ \to \mathbb{R}^+$ is a homeomorphism, then the map $h : Y \to Y$ which acts as $\widehat{h}$ on each cone line is called a *radial homeomorphism*. When a CAT(0) space Y admits an open cone structure where the cone lines are geodesic rays emanating from a fixed point $y_0 \in Y$, the proof of Theorem 4.5 shows that, for any proper function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ there is a homeomorphism $\widehat{h}_\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that the corresponding radial homeomorphism $h_\psi : Y \to Y$ is a compressing homeomorphism for ψ. We call such a space *radially compressible*.

(3) For an example of a compressible space that is not radially compressible, consider $\mathbb{R}^n \times X$, where $\mathbb{R}^n$ has the Euclidean metric and X is compact. More generally, the product of a compressible space with a compact metric space will always be compressible, when given the ℓ_2-metric.

(4) For a noncompressible CAT(0) space, let T be the universal cover of the wedge of two circles. (For further discussion of this example, see Section 8.)

(5) We are particularly interested in compressibility of universal covers of closed aspherical manifolds. As shown above, universal covers of nonpositively curved closed Riemannian n-manifolds are always compressible. Our proof generalizes to CAT(0) universal covers only when there are no points from which geodesic rays bifurcate. In Section 8 we will show that the exotic universal covers constructed by Davis in [12], many of which are CAT(0), are noncompressible. Knowing that a universal cover is homeomorphic to $\mathbb{R}^n$ does not appear to be enough; in fact, we suspect it is a rare phenomenon that a proper metric on $\mathbb{R}^n$ yields a compressible space.

(6) For a concrete example which exhibits our (lack of) knowledge outside the nonpositively curved case, we do not know if the geometries NIL and SOL are compressible, nor do we know if closed graph 3-manifolds have compressible universal cover.

5. $\mathcal{EZ}$-structures. In this section we identify conditions on a compressing homeomorphism h_ψ and on the initial action of G on $X \times Y$ which allow us to improve the $\mathcal{Z}$-structure $(\overline{X \times Y}, \partial_\infty(X \times Y))$ from Theorem 4.3 to an $\mathcal{EZ}$-structure. In other words, we are looking to extend the conjugated action of G on $X \times Y$ to $\partial_\infty(X \times Y)$. Rather than striving for the most general result, we prove a theorem that is sufficiently general for all applications presented in this paper.

Recall that, for proper CAT(0) spaces X and Y, $X \times Y$ (with the ℓ_2-metric) is CAT(0) with $\partial_\infty(X \times Y) \approx \partial_\infty X * \partial_\infty Y$. One of the conditions we will impose on the G-action on $X \times Y$ is that it splits as a product of G-actions. Neither of those actions is expected to be geometric, but another hypothesis will ensure that they extend over $\partial_\infty X$ and $\partial_\infty Y$. Our action on $\partial_\infty X * \partial_\infty Y$ will be the join of those actions.

For the purposes of this section, join lines of $\partial_\infty X * \partial_\infty Y$ are parameterized by $[0, \infty]$, so as to indicate slopes in $X \times Y$. The following lemma is based on standard CAT(0) geometry. We leave its proof to the reader.

LEMMA 5.1. *Let (X, d_X) and (Y, d_Y) be proper CAT(0) metric spaces; $\alpha : [0, \infty) \to X$ and $\beta : [0, \infty) \to Y$ be proper topological embeddings emanating from x_0 and y_0, respectively, and converging to points $\overline{z} \in \partial_\infty X$ and $\overline{w} \in \partial_\infty Y$. Let $Q = \{(\alpha(t), b(t)) \mid t \in [0, \infty)\}$. Then the closure of Q in $(X \times Y, d_2)$ is $\overline{Q} = Q \cup A_{\overline{zw}}$ where $A_{\overline{zw}}$ is the join line in $\partial_\infty(X \times Y)$ connecting $\overline{z}$ to $\overline{w}$. Furthermore, a proper topological ray (embedded or otherwise) $\gamma = (\gamma_1, \gamma_2) : [0, \infty) \to Q \subseteq X \times Y$ converges to a point $m \in A_{\overline{zw}}$ (of slope $m \in [0, \infty]$) if and only if*

$$\lim_{t \to \infty} \frac{d_Y(\gamma_2(t), y_0)}{d_X(\gamma_1(t), x_0)} = m.$$

Let X and Y be proper CAT(0) spaces and assume

(i) $h = (h_1, h_2) : X \times Y \to X \times Y$ is a factor-preserving homeomorphism,
(ii) $h_1 : X \to X$ is an isometry,
(iii) $h_2 : Y \to Y$ is a homeomorphism and a quasi-isometry which extends to a homeomorphism on $\overline{Y}$, and
(iv) Y is radially compressible toward a basepoint y_0.

Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a proper function, and let $h_\psi : Y \to Y$ be a corresponding radial compression function based on a homeomorphism $\widehat{h}_\psi : \mathbb{R}^+ \to \mathbb{R}^+$ (see Remark 4.6). We say that $\widehat{h}_\psi$ is *linearly controlled* if

$$\lim_{t \to \infty} \frac{\widehat{h}_\psi g \widehat{h}_\psi^{-1}(t)}{t} = 1$$

for all linear maps $g : \mathbb{R}^+ \to \mathbb{R}^+$.

REMARK 5.2. By precomposing $\widehat{h}_\psi$ with $\log(x+1)$, we can always assume that $\widehat{h}_\psi$ is a linearly controlled compressing homeomorphism for ψ.

Let $\eta_m = \left(\frac{1}{m}\eta_1, \eta_2\right)$ be a ray in $X \times Y$, where η_1 and η_2 are geodesic rays emanating from x_0 and y_0, and $\frac{1}{m}\eta_1(t) \equiv \eta_1(t/m)$. Viewing η_1 and η_2 as elements of $\partial_\infty X$ and $\partial_\infty Y$, respectively, η_m represents the point of the join line $A_{\eta_1\eta_2} \subseteq \partial_\infty X * \partial_\infty Y = \partial_\infty(X \times Y)$ at slope m (a generic point of $\partial_\infty X * \partial_\infty Y$). Next let $H_\psi = (\mathrm{id}_X, h_\psi) : X \times Y \to X \times Y$ (as in the proof of Theorem 4.3), and consider the topological ray

$$\eta' = H_\psi h H_\psi^{-1} \eta_m = \left(h_1\left(\frac{1}{m}\eta_1\right), h_\psi h_2 h_\psi^{-1} \eta_2 \right).$$

We wish to apply Lemma 5.1 with $h_1\eta_1$ and $h_\psi h_2 h_\psi^{-1}\eta_2$ playing the roles of α and β. Clearly the radial homeomorphisms h_ψ and h_ψ^{-1} extend via the identity to $\partial_\infty Y$. Since it is an isometry, h_1 extends to a homeomorphism of $\overline{X}$; and by hypothesis, h_2 extends to a homeomorphism of $\overline{Y}$. Therefore $h_1\eta_1$ and $h_\psi h_2 h_\psi^{-1}\eta_2$ satisfy the hypothesis on α and β, with $h_1\eta_1$ converging to $h_1(\eta_1) \in \partial_\infty X$ and $h_\psi h_2 h_\psi^{-1}\eta_2$ converging to $h_2(\eta_2) \in \partial_\infty Y$. Let $x_0' := h_1\eta_1(0) = h_1(x_0)$ and $y_0' := h_\psi h_2 h_\psi^{-1}\eta_2(0) = h_\psi h_2(y_0)$.

The role of $\gamma = (\gamma_1, \gamma_2)$ in our application of Lemma 5.1 is played by $\eta_m = \left(\frac{1}{m}\eta_1, \eta_2\right)$. Hence, consider

$$\lim_{t\to\infty} \frac{d_Y(h_\psi h_2 h_\psi^{-1}\eta_2(t), y_0')}{d_X\left(h_1\left(\frac{1}{m}\eta_1\right)(t), x_0'\right)} = \lim_{t\to\infty} \frac{d_Y(h_\psi h_2 h_\psi^{-1}\eta_2(t), y_0')/t}{d_X\left(h_1\left(\frac{1}{m}\eta_1\right)(t), x_0'\right)/t}.$$

It is easy to see that

$$\lim_{t\to\infty} \frac{d_X\left(h_1\left(\frac{1}{m}\eta_1\right)(t), x_0'\right)}{t} = \frac{1}{m}.$$

Since h_2 is a quasi-isometry, choose $K \geq 1$ and $\varepsilon \geq 0$ such that

$$\frac{1}{K}d(y, y') - \varepsilon \leq d(h_2(y), h_2(y')) \leq K d(y, y') + \varepsilon$$

for all $y, y' \in Y$. In particular,

$$\frac{1}{K} \cdot \widehat{h}_\psi^{-1}(t) - \varepsilon \leq d(h_2 h_\psi^{-1}\eta_2(t), h_2(y_0)) \leq K \cdot \widehat{h}_\psi^{-1}(t) + \varepsilon.$$

So, letting $C = d(h_2(y_0), y_0)$, we have

$$\frac{1}{K} \cdot \widehat{h}_\psi^{-1}(t) - C \leq d(h_2 h_\psi^{-1}\eta_2(t), y_0) \leq K \cdot \widehat{h}_\psi^{-1}(t) + C.$$

Since $\widehat{h}_\psi$ can be taken to be increasing, we have

$$\widehat{h}_\psi\left(\frac{1}{K}\cdot\widehat{h}_\psi^{-1}(t)-(\varepsilon+C)\right)\leq\widehat{h}_\psi\big(d(h_2h_\psi^{-1}\eta_2(t),y_0)\big)$$
$$\leq\widehat{h}_\psi(K\cdot\widehat{h}_\psi^{-1}(t)+\varepsilon+C)$$

for t sufficiently large.

Notice now that

$$d(h_\psi h_2 h_\psi^{-1}\eta_2(t),y_0)=h_\psi\big(d(h_2h_\psi^{-1}\eta_2(t),y_0)\big),$$

so, for sufficiently large t, we have

$$\widehat{h}_\psi\left(\frac{1}{K}\cdot\widehat{h}_\psi^{-1}(t)-(\varepsilon+C)\right)\leq d(h_\psi h_2 h_\psi^{-1}\eta_2(t),y_0)$$
$$\leq\widehat{h}_\psi(K\cdot\widehat{h}_\psi^{-1}(t)+\varepsilon+C).$$

Divide all three terms by t and let $t\to\infty$. By the linear control assumption on $\widehat{h}_\psi$, the corresponding left- and right-hand limits are both 1, hence the middle limit is 1. Finally, note that

$$d(h_\psi h_2 h_\psi^{-1}\eta_2(t),y_0)-d(y_0,y_0')\leq d(h_\psi h_2 h_\psi^{-1}\eta_2(t),y_0')$$
$$\leq d(h_\psi h_2 h_\psi^{-1}\eta_2(t),y_0)+d(y_0,y_0').$$

Again divide all three terms by t and let $t\to\infty$. Apply the above work to once more conclude that the left- and right-hand limits are 1, so the middle limit is 1 as well. Putting these pieces together and applying Lemma 5.1, we see that the ray $\eta'=H_\psi hH_\psi^{-1}\eta_m$ converges to the point of slope m in $\partial_\infty X*\partial_\infty Y$ on the join line between $h_1(\eta_1)\in\partial_\infty X$ and $h_2(\eta_2)\in\partial_\infty Y$.

Therefore, we have the following theorem.

THEOREM 5.3. *Let X,Y,h,h_ψ be as above. Then the compressed homeomorphism $h_\psi hh_\psi^{-1}$ extends to the boundary $\partial_\infty X*\partial_\infty Y$. The induced homeomorphism on $\partial_\infty X*\partial_\infty Y$ is the join of the induced homeomorphisms on $\partial_\infty X$ and $\partial_\infty Y$.*

6. Graphs of closed aspherical n-manifolds. We begin to focus on our main class of examples, which can be thought of as higher-dimensional analogues of generalized Baumslag–Solitar groups.

Suppose that G is the fundamental group of a finite connected graph of groups $(\mathcal{G},\Gamma)$, with the property that each vertex group G_v is the fundamental group of a closed aspherical manifold M_v and, for each edge e, the monomorphisms $G_e\xrightarrow{\phi_e^-}G_{i(e)}$ and $G_e\xrightarrow{\phi_e^+}G_{t(e)}$ are of finite index. The primary goal in this section is to realize $(\mathcal{G},\Gamma)$ with a graph of covering spaces (as defined and developed in the appendix). The assumptions on $(\mathcal{G},\Gamma)$ ensure that each G_e can be realized as the fundamental group

of both a finite-sheeted cover M_e^- of $M_{i(e)}$ and a finite-sheeted cover M_e^+ of $M_{t(e)}$. These covers are homotopy equivalent, but a priori, not homeomorphic. Being closed and aspherical, all vertex manifolds and their covers are necessarily of the same dimension. To proceed, we need a single edge space, for each edge e, which covers *both* of its vertex spaces (possibly the same space, in cases where e is a loop in Γ). We will describe two useful approaches. Each approach requires an additional hypothesis that is conjecturally satisfied in all cases. The first is conceptually simpler; it chooses one of the above-mentioned covers as the edge space and leaves the chosen vertex manifolds in place. The second approach is more drastic, but has some important benefits.

APPROACH I. Assume the Borel Conjecture holds for all edge groups.

At each vertex v of Γ, place the aspherical manifold M_v chosen above, then realize ϕ_e^- and ϕ_e^+ by finite-sheeted covering projections $q_e^- : M_e^- \to M_{i(e)}$ and $q_e^+ : M_e^+ \to M_{t(e)}$. By hypothesis, there is a homeomorphism $f : M_e^- \to M_e^+$. Let $M_e = M_e^-$ and define $p_e^- = q_e^-$ and $p_e^+ = q_e^+ \circ f$ to be the edge maps.

APPROACH II. Assume that the Whitehead group of each edge group G_e is trivial.

Using the same notation as above and the new hypothesis, M_e^- and M_e^+ are simple homotopy equivalent, so by the results discussed in Section 3, there is a homeomorphism $f : M_e^- \times I^\omega \to M_e^+ \times I^\omega$. Replace each M_v with $M_v \times I^\omega$ and, for each edge e, let $M_e^- \times I^\omega$ be the edge space (denoted simply by $M_e \times I^\omega$ from now on). Then insert the covering maps $p_e^- = q_e^- \times \mathrm{id}_{I^\omega}$ and $p_e^+ = (q_e^+ \times \mathrm{id}_{I^\omega}) \circ f$ to complete the realization of $(\mathcal{G}, \Gamma)$ as a graph of covering spaces.

Given the hypothesis and setup in Approach I, we can form the *total space*

$$X = \left(\bigcup_v M_v \right) \cup \left(\bigcup_e M_e \times [0,1] \right),$$

where $M_e \times \{0\}$ and $M_e \times \{1\}$ are glued to $M_{i(e)}$ and $M_{t(e)}$ using covering maps $p_e^- : M_e \to M_{i(e)}$ and $p_e^+ : M_e \to M_{t(e)}$ as defined above. From there we pass to the universal cover $\widetilde{X}$ to get the desired G-space. Any geodesic metric on X lifts to a G-invariant metric on $\widetilde{X}$. This cover together with the G-invariant metric satisfy the following properties (see [15] and the appendix to the present paper for details):

• There is a distance nonincreasing projection map $p_T : \widetilde{X} \to T$, where T is the Bass–Serre tree for $(\mathcal{G}, \Gamma)$.

- There is a homeomorphism $H : \widetilde{X} \to T \times \widetilde{M_v}$, where $\widetilde{M_v}$ is the universal cover of an arbitrary vertex space. Furthermore, $p_T^{-1}(t)$ maps to $t \times \widetilde{M_v}$ under H and, for all $m \in \widetilde{M_v}$, the map $T \to T \times m \to \widetilde{X}$ is a locally isometric embedding.
- There exists $C \geq 1$ such that for all edges e of T and $v \in e$, the retraction $r : e \to v$ induces a projection

$$p_T^{-1}(e) \xrightarrow{H} e \times \widetilde{M_v} \to v \times \widetilde{M_v} \xrightarrow{H^{-1}} p_t^{-1}(v)$$

which is C-Lipschitz.

The reader can compare these statements with the well-known picture of the Cayley complex of the Baumslag–Solitar group $BS(m, n)$ (homeomorphic to $T \times \mathbb{R}$, where T is the Bass–Serre tree of the splitting and $\mathbb{R} = \widetilde{S^1}$).

In the case of Approach II, everything works the same as above, except that the vertex and edge spaces are now the aspherical Hilbert cube manifolds $M_v \times I^\omega$ and $M_e \times I^\omega$ and their universal covers are $\widetilde{M_v} \times I^\omega$ and $\widetilde{M_e} \times I^\omega$. The spaces X and $\widetilde{X}$ are now Hilbert cube manifolds.

The above may be summarized as follows.

THEOREM 6.1. *Let $(\mathcal{G}, \Gamma)$ be a finite connected graph of groups $(\mathcal{G}, \Gamma)$, with the property that each vertex group G_v is the fundamental group of a closed aspherical manifold M_v and, for each edge e, the monomorphisms $G_e \xrightarrow{\phi_e^-} G_{i(e)}$ and $G_e \xrightarrow{\phi_e^+} G_{t(e)}$ are of finite index. Then:*

(I) *if the Borel Conjecture holds for each edge group G_e, then $(\mathcal{G}, \Gamma)$ can be realized by a graph of covering spaces with vertex spaces M_v;*

(II) *if the Whitehead group $\mathrm{Wh}(G_e)$ vanishes for each G_e, then $(\mathcal{G}, \Gamma)$ can be realized by a graph of covering spaces where the vertex spaces are the Hilbert cube manifolds $M_v \times I^\omega$.*

REMARK 6.2. For the purposes of this paper, Approaches (I) and (II) lead to nearly identical places. That is largely due to our reliance on a compressibility hypothesis, which we can verify only for nonpositively curved Riemannian manifolds. Farrell and Jones [16] have shown that, with one significant exception, both the Borel Conjecture and the triviality of the Whitehead group hold for (fundamental groups of) closed nonpositively curved Riemannian manifolds. The exception occurs when $n = 4$, where their surgery-theoretic proof of the Borel Conjecture does not apply. As such, Approach (II) is essential for obtaining the $n = 4$ case of our main results. Approach (II) also holds promise for proving more general theorems, but that is likely to require a different method—one that does not involve compressibility. Further discussion of that idea is included in Section 9.

7. Graphs of nonpositively curved Riemannian n-manifolds. We are now ready to present our main results, in which we apply Theorems 6.1, 4.3 and 5.3 to provide new classes of groups that admit $(\mathcal{E})\mathcal{Z}$-structures.

THEOREM 7.1. *Suppose G is the fundamental group of a finite graph of groups, where each vertex group is the fundamental group of a closed, non-positively curved Riemannian manifold, and each edge group is finite index in the corresponding vertex groups. Then G admits a $\mathcal{Z}$-structure. If the lifts to universal covers of all covering maps $p_e^- : M_e \to M_{i(e)}$ and $p_e^+ : M_e \to M_{t(e)}$ (discussed above) extend over the visual boundaries, then G admits an $\mathcal{E}\mathcal{Z}$-structure.*

Proof. For the sake of simplicity, begin by assuming that $n \neq 4$. Then, by [16], the Borel Conjecture holds for each edge group, so we may use the graph of covering spaces described in Approach I above.

Our initial task is to check the conditions found in Theorem 4.3. Using the homeomorphism $\widetilde{X} \to T \times \widetilde{M}_v$ noted above, G acts properly and cocompactly on the product $T \times \widetilde{M}_v$. (For a more detailed discussion of this action, see the appendix.) By Theorem 4.5 and Proposition 4.2, $\widetilde{M}_v$ is compressible for any $\pi_1(M_v)$-equivariant metric. Choose any nonpositively curved Riemannian metric. We fix a basepoint $t_0 \in T$; make $\widetilde{M}_v$ isometric to $p_T^{-1}(t_0)$; put the usual metric on T; and give $T \times \widetilde{M}_v$ the product metric. Choose a compact set K in $\widetilde{X}$ so that $GK = \widetilde{X}$. Note that $\operatorname{diam}(p_T(gK)) \leq \operatorname{diam}(K)$, so condition (1) of Theorem 4.3 is satisfied. Let $p_{\widetilde{M}_v}$ be the projection $\widetilde{X} \to p_T^{-1}(t_0) \cong t_0 \times \widetilde{M}_v$ and let $D = \operatorname{diam}(p_{\widetilde{M}_v} K)$. Now, suppose $p_T(gK) \subset B_T(t_0, R)$. The projection

$$p_T^{-1}(B_T(t_0, R)) \to B_T(t_0, R) \times \widetilde{M}_v \to t_0 \times \widetilde{M}_v \to p_T^{-1}(t_0)$$

is C^R-Lipschitz, so $p_{\widetilde{M}_v}(gK)$ has diameter $< DC^R$. Thus, condition (2) of Theorem 4.3 is satisfied. It follows that G admits a $\mathcal{Z}$-structure.

Now assume that all lifts $\widetilde{M}_{i(e)} \xleftarrow{\widetilde{p_e^-}} \widetilde{M}_e \xrightarrow{\widetilde{p_e^+}} \widetilde{M}_{t(e)}$ of our finite-sheeted coverings extend over their corresponding visual boundaries. To obtain an $\mathcal{E}\mathcal{Z}$-structure, it suffices to verify the conditions in Theorem 5.3 for all elements of G, viewed as self-homeomorphisms of $T \times \widetilde{M}_v$. By the proof of Theorem 4.5, we know that $\widetilde{M}_v$ is radially contractible, so it suffices to check (i)–(iii). Items (i) and (ii) are discussed in detail in Section 10.1 of the appendix, with the action on the first factor being the standard Bass–Serre action. As is discussed in Remark 10.6, each element of G acting on $\widetilde{M}_v$ is a finite composition of lift homeomorphisms, inverses of those homeomorphisms, and isometries of vertex spaces. Each of those is a quasi-isometric homeomorphism, and by hypothesis, each extends over the corresponding boundaries. Therefore, condition (iii) holds as well.

Next, in order to cover the $n = 4$ case (and to offer an alternative proof in all other dimensions), let us switch to the setup described in Approach II. Again, [16] confirms the necessary hypothesis. In order to apply Theorem 4.3, we need a $\pi_1(M_v)$-equivariant CAT(0) metric on $\widetilde{M}_v \times I^\omega$. This can be accomplished by using the ℓ_2-metric described in Section 3. We also need to know that $\widetilde{M}_v \times I^\omega$ is compressible—a fact that was noted in item (4.6) of Remark 4.6. Everything else in the above proof now goes through without changes. ∎

Note that if each manifold is negatively curved, the lift of any finite covering map is a quasi-isometry between Gromov hyperbolic spaces, and hence extends to the visual boundaries.

COROLLARY 7.2. *Graphs of nonpositively curved closed Riemannian n-manifolds admit $\mathcal{Z}$-structures. Graphs of negatively curved Riemannian n-manifolds admit $\mathcal{EZ}$-structures.*

For generic nonpositively curved Riemannian manifolds, we cannot be sure that the lifts of all the covering maps constructed in Section 6 extend over visual boundaries. The problem is the possibly nongeometric nature of the lifts of the homeomorphisms $f : M_e^- \to M_e^+$ (or $f : M_e^- \times I^\omega \to M_e^+ \times I^\omega$) used in defining $p_e^+ : M_e \to M_{t(e)}$ (or $p_e^+ : M_e \times I^\omega \to M_{t(e)} \times I^\omega$). By applying the Bieberbach Theorems [22], we can avoid this problem in the extreme (but important) special case where all vertex manifolds are flat.

COROLLARY 7.3. *Graphs of closed flat n-manifolds admit $\mathcal{EZ}$-structures.*

Proof. The third Bieberbach Theorem, as described in [22, Ch. 2], ensures that, for all n and any pair of closed flat n-manifolds with isomorphic fundamental groups, there is a corresponding affine homeomorphism; in other words, a homeomorphism that lifts to an affine homeomorphism $\mathbb{R}^n \to \mathbb{R}^n$. Among other things, this will allow us to use Approach I from Section 6, even when $n = 4$.

Choose a flat metric on each vertex manifold M_v, then use the covering maps $q_e^- : M_e^- \to M_{i(e)}$ and $q_e^+ : M_e^+ \to M_{t(e)}$ to lift those metrics to the finite-sheeted covers M_e^- and M_e^+. As such, the lifts to universal covers $\widetilde{q}_e^-$ and $\widetilde{q}_e^+$ become isometries of $\mathbb{R}^n$. The Bieberbach Theorem then allows us to choose a homeomorphism $f : M_e^- \to M_e^+$ that lifts to an affine isomorphism of $\mathbb{R}^n$. Since isometries and affine isomorphisms of $\mathbb{R}^n$ all extend over visual boundaries, our corollary follows. ∎

Using these results, we also obtain a strengthening of the result from [19]:

COROLLARY 7.4. *Generalized Baumslag–Solitar groups admit $\mathcal{EZ}$-structures.*

REMARK 7.5. A few comments are in order as we close this section.

(1) The groups addressed in Corollaries 7.2 and 7.3 have been previously studied by a number of people; see [15] and [21]. Among other things, these fundamental groups are quasi-isometrically rigid in the sense that any group quasi-isometric to such a group is itself the fundamental group of a finite graph of groups with vertex/edge groups quasi-isometric to the original vertex/edge groups.

(2) The proof of Theorem 7.1 is valid for graphs of (non-Riemannian) nonpositively curved manifolds, provided they are compressible (to get a $\mathcal{Z}$-structure) or radially compressible (to get an $\mathcal{EZ}$-structure). At this time, we do not know any examples of that type.

(3) A primary motivation for studying $\mathcal{EZ}$-structures is that a torsion-free group with a $\mathcal{EZ}$-structure satisfies the Novikov Conjecture. See [17] and also [7]. All of the examples covered by Corollaries 7.2–7.4 were previously known to satisfy the Novikov Conjecture for other reasons. For example, hyperbolic and free abelian groups have finite asymptotic dimension, hence by work of Bell and Dranishnikov [5], so do graphs of groups with these as vertex and edge groups. It is an open question whether fundamental groups of all nonpositively curved manifolds (Riemannian or otherwise) have finite asymptotic dimension. Hence, it is possible that Theorem 7.1 contains new examples of groups which satisfy the Novikov Conjecture.

8. Noncompressible spaces. In this section, we highlight the delicate nature of compressibility by looking at some noncompressible CAT(0) spaces which occur as universal coverings of compact aspherical CW-complexes and manifolds. We begin with the simplest such example.

EXAMPLE 8.1. Let T_4 be the tree with valence 4 at each vertex and standard path length metric, i.e., the universal cover of a wedge of two circles. If we view T_4 as a 1-manifold with singularites at the vertices, it is clear that large balls have more singular points than small balls. This is an obstruction to the existence of compressing homeomorphisms.

EXAMPLE 8.2. Now consider $T_4 \times I^\omega$ with the ℓ_2-metric. By [27], this is a Hilbert cube manifold, therefore a homogeneous space. This nullifies the above argument, but compressibility still fails since large balls in $\mathbb{T}_4 \times I^\omega$ have more complementary components than small balls.

Next we examine an example of more direct relevance to this paper. In particular, we identify a family of closed, nonpositively curved (locally CAT(0)) finite-dimensional manifolds whose universal covers are not compressible.

By a *standard Davis example* we are referring to the special case of the construction in [12]. This begins with a compact contractible q-manifold Q^q with a mirror structure $\{Q_v\}_{v \in V}$ consisting of tame $(q-1)$-cells in ∂Q^q, and

a Coxeter system (Γ, V), consisting of a Coxeter group Γ and a preferred generating set V in one-to-one correspondence with the mirrors. We assume that $\partial Q_q = \bigcup Q_v$ and the mirror structure is "Γ-finite". For any compact contractible q-manifold Q^q, such an arrangement exists: begin with a flag triangulation K of ∂Q^q and let the mirrors be the top-dimensional cells of the corresponding dual cell-structure on ∂Q^q; they are indexed by the vertex set $V = K^0$. A corresponding (right-angled) Coxeter system (Γ, V) is obtained by declaring $v_i^2 = 1$ for all $v_i \in V$ and $(v_i v_j)^2 = 1$ when v_i and v_j bound an edge in K.

Roughly speaking, Γ provides instructions for gluing together members of the discrete collection $\Gamma \times Q^q$ of copies of Q^q, to obtain a contractible open manifold X^q that admits a proper cocompact Γ-action. Within X^q, the individual copies of Q^q are referred to as *chambers*, with the chamber corresponding to $\{g\} \times Q^q$ denoted by gQ^q, and the identity chamber $\{e\} \times Q^q$ denoted as Q^q. By passing to a torsion-free finite index subgroup $\Gamma' \leq \Gamma$, one obtains a covering projection $X^q \to \Gamma' \backslash X^q$ with quotient a closed aspherical manifold.

From the perspective of this paper, the key facts about X^q are contained in [12, Lemma 8.2 and Remark 10.6]. It is observed that if the elements of Γ are ordered $1 = g_1, g_2, g_3, \ldots$ so that $\text{length}(g_{j+1}) \geq \text{length}(g_j)$ and $T_i = \bigcup_{j=1}^{i} g_j Q^q$, then for all i, T_i is a connected q-manifold with boundary, and $T_i \cap g_{i+1} Q^q$ is a tame $(q-1)$-cell in the boundary of each (made up of a finite union of panels). Since Q^q is simply connected, it is orientable; so assume now that Q^q is an oriented manifold, and give each chamber $g_i Q^q$ that same orientation when $\text{length}(g_i)$ is even and the opposite orientation when $\text{length}(g_i)$ is odd. A quick look at the gluing instructions in [12] for assembling the chambers into X^q confirms that the orientations on the chambers fit together to provide appropriate orientations on the T_i. All of this implies that T_i is a boundary connected sum of i copies of $\pm Q$, hence ∂T_i is a connected sum of i copies of $\pm \partial Q$. This is most interesting when $q \geq 4$ and $\pi_1(\partial Q^q) = G \neq 1$, in which case $\pi_1(\partial T_i)$ is the free product $*_{k=1}^{i} G$. A key observation of Davis is that, for the corresponding neighborhood of infinity $N_i = X^q - \text{int}\, T_i$, we know that $\partial T_i \hookrightarrow N_i$ is a homotopy equivalence, therefore $\pi_1(N_i) = *_{k=1}^{i} G$. Davis used this fact to show that X^q is not simply connected at infinity (hence, is not homeomorphic to $\mathbb{R}^q$). We will use it for a similar, but different, reason.

Place a geodesic metric d' on Q^q; let $R = \text{diam}(Q^q)$; and give X^q the corresponding path length metric d. As such, (X^q, d) is a proper geodesic metric space, and the action of Γ on X^q is geometric. Give Γ the word length metric ρ corresponding to the generating set V. Choose $x_0 \in \text{int}\, Q^q$ and let

$f : (\Gamma, \rho) \to (X^q, d)$ be defined by $f(g) = gx_0$. Then f has R-dense image, and by Švarc–Milnor, is a (K, ε)-quasi-isometry for some $K \geq 1$ and $\varepsilon \geq 0$. Note that, by our choice of x_0, f is injective.

Let $\beta : \mathbb{N} \to \mathbb{N}$ be the growth function for (Γ, ρ), i.e., $\beta(n) = |B_\rho[e, n]|$, and for $r > 0$ and $A \subseteq X^q$, let $N_d[A; r]$ denote the closed r-neighborhood of A in X^q. Let S be the smallest integer such that $B_d[x_0; R] \subseteq T_{\beta(S)}$.

LEMMA 8.3. *For each $n \in \mathbb{N}$,*

(1) $B_d[x_0, n/K - \varepsilon - R] \subseteq N_d[f(B_\rho[e, n]), R]$,
(2) $N_d[f(B_\rho[e, n]), R] \subseteq T_{\beta(n+S)}$, *and*
(3) $T_{\beta(n)} \subseteq B_d[x_0, Kn + \varepsilon + R]$.

Proof. To prove (1), suppose $y \notin N_d[f(B_\rho[e, n]), R]$. Then there exists $g \in \Gamma$ such that $\rho(e, g) > n$ and $d(y, f(g)) \leq R$. Therefore

$$\frac{1}{K}\rho(e, g) - \varepsilon \leq d(x_0, f(g)) \leq d(x_0, y) + R.$$

So

$$\frac{n}{K} - \varepsilon - R < d(x_0, y).$$

For item (2), let $gx_0 \in f(B_\rho[e, n])$. Then $\mathrm{length}(g) \leq n$ and

$$B_d[gx_0 : R] \subseteq gT_{\beta(S)} = \bigcup_{h \in B_\rho[e; S]} ghQ^q.$$

By definition, $\mathrm{length}(g) \leq n$ and $\mathrm{length}(h) \leq S$, so gh in the above equality has length $\leq n + S$. Therefore the right-hand set is contained in $T_{\beta(n+S)}$.

For the final item, suppose gQ^q is a summand in $T_{\beta(n)}$. Then $\mathrm{length}(g) \leq n$, so $gx_0 \in f(B_\rho[e, n]) \subseteq B_d[x_0, Kn + \varepsilon]$. By the triangle inequality, $gQ^q \subseteq B_d[x_0, Kn + \varepsilon + R]$. ∎

THEOREM 8.4. *Let X^q be a Davis manifold with chamber a compact contractible q-manifold Q^q with non-simply-connected boundary. Let Γ be the corresponding Coxeter group and d a metric on X^q such that Γ acts geometrically on X^q. Then X^q is noncompressible under the metric d.*

Note that Proposition 4.2 implies that X^q is noncompressible for any quasi-isometric metric. The following corollary is immediate from [2].

COROLLARY 8.5. *For all $q \geq 5$, there exist closed locally $\mathrm{CAT}(0)$ q-manifolds with noncompressible universal covers.*

Proof of Theorem 8.4. We will use the metric d and the constants K, ε, R, and S defined above.

By Lemma 8.3,

$$B_d\left[x_0, \frac{n}{K} - \varepsilon - R\right]$$
$$\subseteq T_{\beta(n+S)} \subsetneq T_{\beta(n+S+1)} \subseteq B_d[x_0, K(n + S + 1) + \varepsilon + R]$$

for all $n \in \mathbb{N}$. Suppose that X^q is compressible, and let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be the identity function. Then there exists a homeomorphism $h_\psi : X^q \to X^q$ and a sublinear function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\text{diam}(h_\psi(C)) \leq \phi(\text{diam}(C))$$

for all bounded $C \subseteq X^q$. By composing with an isometry from Γ, we may assume that $d(x_0, h_\psi(x_0)) \leq R$ and by the above arrangement,

$$\text{diam}\big(h_\psi(B_d[x_0, K(n + S + 1) + \varepsilon + R])\big) \leq \phi\big(2(K(n + S + 1) + \varepsilon + R)\big)$$

for all $n \in \mathbb{N}$. Since

$$\lim_{n\to\infty} \frac{\phi(n)}{n} = 0,$$

we have

$$\lim_{n\to\infty} \frac{\phi(2(K(n + S + 1) + \varepsilon + R))}{2(n/K - \varepsilon - R)} = 0.$$

By choosing n so large that $\text{diam}(h_\psi(B_d[x_0, K(n + S + 1) + \varepsilon + R])) < \frac{1}{2}(n/K - \varepsilon - R)$ and $R < \frac{1}{2}(n/K - \varepsilon - R)$, we obtain

$$h_\psi(B_d[x_0, K(n + S + 1) + \varepsilon + R]) \subseteq B_d(x_0, n/K - \varepsilon - R).$$

As a result, $h_\psi(T_{\beta(n+S+1)}) \subseteq \text{int}\, T_{\beta(n+S)}$.

Let $W = T_{\beta(n+S)} - \text{int}(h_\psi(T_{\beta(n+S+1)}))$ and consider the cobordism $(W, \partial T_{\beta(n+S)}, h_\psi(\partial T_{\beta(n+S+1)}))$. By Lemma 8.6 below, W deformation retracts onto $h_\psi(\partial T_{\beta(n+S+1)})$ and the restriction of this deformation is a degree ± 1 map $d : \partial T_{\beta(n+S)} \to h_\psi(\partial T_{\beta(n+S+1)})$. It is a standard fact that degree ± 1 maps induce π_1-surjections, so we have a surjection $d_* : *_{k=1}^{\beta(n+S)} G \to *_{k=1}^{\beta(n+S+1)} G$. But then the rank domain is at least as large as the rank of the range, violating Grushko's Theorem. ∎

LEMMA 8.6. *Let M^n be an orientable open n-manifold containing closed neighborhoods of infinity N and N', each a codimension 0 submanifold with tame (bicollared) boundary. Suppose also that $N' \subseteq \text{int}\, N$ and both $\partial N \hookrightarrow N$ and $\partial N' \hookrightarrow N'$ are homotopy equivalences. Let $W = N - \text{int}\, N'$. Then*

(1) *W is a compact n-manifold with $\partial W = \partial N \sqcup \partial N'$,*
(2) *W deformation retracts onto ∂N, and*
(3) *the resulting retraction $r : W \to \partial N$ restricts to a degree ± 1 map $\partial N' \to \partial N$.*

Proof. Assertion (1) is immediate. For assertion (2), let H_t and J_t be deformation retractions of N onto ∂N and N' onto $\partial N'$, respectively. Then $H_1 : N \to \partial N$ and $J_1 : N' \to \partial N'$ are retractions, and $J_1 \circ H_t$ is a deformation retraction of W onto ∂N.

For assertion (3), note that since ∂N is a connected orientable $(n-1)$-manifold, $H_{n-1}(\partial N) \cong \mathbb{Z}$; and since $r_* : H_{n-1}(W) \to H_{n-1}(\partial N)$ is an isomorphism, $H_{n-1}(W) \cong \mathbb{Z}$. By duality $H_*(W, \partial N') = 0$, so $\mathrm{incl}_* : H_{n-1}(\partial N') \to H_{n-1}(W)$ is also an isomorphism. It follows that $(r|_{\partial N'})_* = r_* \circ \mathrm{incl}_*$ is an isomorphism, so $|\deg(r|_{\partial N'})| = 1$. ∎

Unfortunately, crossing a Davis manifold with the Hilbert cube does not improve its compressibility properties.

THEOREM 8.7. *Let (X^q, d) be a Davis manifold of the type described in Theorem 8.4. Then $X^q \times I^\omega$ is noncompressible under the corresponding ℓ_2-metric or any metric quasi-isometric to it.*

Proof. If we assume compressibility, the same sort of argument used above leads to a cobordism of Hilbert cube manifolds $(W \times I^\omega, \partial T_{\beta(n+S)} \times I^\omega, h_\psi(\partial T_{\beta(n+S+1)}) \times I^\omega)$ which deformation retracts onto $h_\psi(\partial T_{\beta(n+S+1)}) \times I^\omega$. Projection yields a deformation retraction of W onto $h_\psi(\partial T_{\beta(n+S+1)})$ and the same contradiction obtained earlier. ∎

9. Some open questions. The results presented in this paper raise several questions. The most obvious revolve around the compressibility hypothesis. Roughly speaking, compressibility allowed us to take advantage of the CAT(0) geometry of the product spaces $T \times \widetilde{M_v}$ and $T \times (\widetilde{M_v} \times I^\omega)$, even when the corresponding proper cocompact action is not by isometries. In the absence of compressibility, a different strategy is clearly needed. Nonetheless, the questions remain.

QUESTION 9.1. *Suppose G is the fundamental group of a finite connected graph of groups $(\mathcal{G}, \Gamma)$ with the property that each vertex group G_v is the fundamental group of a closed aspherical manifold M_v and, for each edge e, the monomorphisms*

$$G_e \xrightarrow{\phi_e^-} G_{i(e)} \quad and \quad G_e \xrightarrow{\phi_e^+} G_{t(e)}$$

are of finite index. Does G admit a $\mathcal{Z}$-structure? An $\mathcal{EZ}$-structure?

In attacking the above question, one is likely to bump up against the unresolved nature of the Borel Conjecture or the Whitehead Group Conjecture. For that and other reasons, the following is an appealing special case.

QUESTION 9.2. *Suppose G is the fundamental group of a finite connected graph of groups $(\mathcal{G}, \Gamma)$ with the property that each vertex group G_v is the fundamental group of a closed, locally CAT(0) manifold M_v and, for each*

edge e, the monomorphisms $G_e \xrightarrow{\phi_e^-} G_{i(e)}$ and $G_e \xrightarrow{\phi_e^+} G_{t(e)}$ are of finite index. Does G admit a $\mathcal{Z}$-structure? An $\mathcal{EZ}$-structure?

We expect a positive answer. The point here is that [4] assures us that the Borel Conjecture holds and the Whitehead group vanishes for these edge groups. In addition, we still have CAT(0) geometry to work with.

If we are going to give up the compressibility hypothesis anyway, Approach (II) provides a method for attacking the above questions in even greater generality. In particular, there may be no need to confine ourselves to closed aspherical manifolds as vertex groups.

QUESTION 9.3. *Suppose G is the fundamental group of a finite connected graph of groups $(\mathcal{G}, \Gamma)$ with the property that each vertex group G_v is the fundamental group of a finite aspherical CW-complex Y_v and, for each edge e, the monomorphisms $G_e \xrightarrow{\phi_e^-} G_{i(e)}$ and $G_e \xrightarrow{\phi_e^+} G_{t(e)}$ are of finite index. Does G admit a $\mathcal{Z}$-structure? An $\mathcal{EZ}$-structure? Does it help to assume that $\mathrm{Wh}(G_e) = 0$ for all edge groups? that each Y_v is nonpositively curved?*

The point here is that, with the help of Hilbert cube technology, we can often conclude that G acts properly and cocompactly on a product space or even a CAT(0) space—now of the form $T \times (\widetilde{Y}_v \times I^\omega)$.

10. Appendix: Graphs of covering spaces and actions on products.
In this appendix, we expand upon the notion of a *graph of covering spaces* as introduced in Section 7.

A *graph of pointed topological spaces* is a system $(\mathcal{T}, \Gamma)$ consisting of

- a connected oriented graph Γ with vertex set E_0 and edge set E_1,
- a collection $\mathcal{T}$ of pointed path-connected topological spaces (Y_s, y_s) indexed by $E_0 \cup E_1$, and
- for each $e \in E_1$, a pair of continuous *edge maps*

$$(Y_{i(e)}, y_{i(e)}) \xleftarrow{p_e^-} (Y_e, y_e) \xrightarrow{p_e^+} (Y_{t(e)}, y_{t(e)}),$$

each inducing a π_1-monomorphism.

The *total space* of $(\mathcal{T}, \Gamma)$, denoted $\mathrm{Tot}(\mathcal{T}, \Gamma)$, is the adjunction space

$$\mathrm{Tot}(\mathcal{T}, \Gamma) = \Big(\bigcup_{v \in E_0} Y_v \Big) \cup \Big(\bigcup_{e \in E_1} Y_e \times [0, 1] \Big),$$

where $Y_e \times [0, 1]$ is glued onto $Y_{o(e)}$ and $Y_{t(e)}$ using p_e^- and p_e^+ respectively. There is a natural projection map $\pi : \mathrm{Tot}(\mathcal{T}, \Gamma) \to \Gamma$ for which the preimage of each $v \in E_0$ is a copy of Y_v and for each point y lying on the interior of an edge e, $\pi^{-1}(y)$ is a copy of Y_e. There is a copy of Γ sitting in $\mathrm{Tot}(\mathcal{T}, \Gamma)$ made up of the images of $y_e \times [-1, 1]$ under the quotient map q. Under this realization of Γ, π may be viewed as a retraction. When each (Y_s, y_s)

is a CW-pair and each p_e^- and p_e^+ is cellular, $\mathrm{Tot}(\mathcal{T}, \Gamma)$ inherits a natural CW-structure with Γ a subcomplex and π a cellular map. Call $(\mathcal{T}, \Gamma)$ a *compact* graph of pointed topological spaces if Γ is a finite graph and each edge and vertex space is compact. This is equivalent to requiring $\mathrm{Tot}(\mathcal{T}, \Gamma)$ to be compact.

Given a graph of pointed topological spaces $(\mathcal{T}, \Gamma)$, there is an *induced graph of groups* $(\mathcal{G}, \Gamma)$ with vertex and edge groups $G_s = \pi_1(Y_s, y_s)$, and edge monomorphisms $\phi_e^- = (p_e^-)_\#$ and $\phi_e^+ = (p_e^+)_\#$. Moreover, given a graph of groups $(\mathcal{G}, \Gamma)$, it is possible to realize $(\mathcal{G}, \Gamma)$ as a graph of pointed topological spaces $(\mathcal{T}, \Gamma)$; if desired, the (Y_s, y_s) can be chosen to be CW-pairs and the maps to be cellular. Since numerous choices are involved, there is a great deal of flexibility in choosing a graph of topological spaces realizing a given graph of groups.

Suppose each G_s has presentation

$$G_s = \langle A_s \,|\, R_s \rangle.$$

DEFINITION 10.1. Given a maximal tree $\Gamma_0 \subseteq \Gamma$, the *fundamental group of* $(\mathcal{G}, \Gamma)$ *based at* Γ_0, denoted $\pi_1(\mathcal{G}, \Gamma; \Gamma_0)$, has generators

$$\Big(\bigcup_{v \in E_0} A_v \Big) \cup \{ t_e \,|\, e \in E_1 \}$$

and relations

$$\bigcup_{v \in R_v} R_v \cup \{ t_e^{-1} \phi_e^-(g) t_e = \phi_e^+(g) \,|\, g \in G_e, e \in E_1 \} \cup \{ t_e = 1 \,|\, e \in \Gamma_0 \}.$$

A notable property of $\pi_1(\mathcal{G}, \Gamma; \Gamma_0)$ is that it contains canonical copies of each G_v, and—up to isomorphism—it does not depend on the choice of Γ_0 (although the canonical copies of G_v do). Furthermore, if $(\mathcal{T}, \Gamma)$ is a corresponding graph of pointed topological spaces, there is a natural isomorphism between $\pi_1(\mathcal{G}, \Gamma; \Gamma_0)$ and $\pi_1(\mathrm{Tot}(\mathcal{T}, \Gamma), \Gamma_0)$. Here we use the fundamental group of $\mathrm{Tot}(\mathcal{T}, \Gamma)$ based at Γ_0 rather than the usual fundamental group based at a point. This is a matter of convenience; if v is any of the vertices of Γ_0, there is a natural isomorphism between $\pi_1(\mathrm{Tot}(\mathcal{T}, \Gamma), v)$ and $\pi_1(\mathrm{Tot}(\mathcal{T}, \Gamma), \Gamma_0)$. See [18] for details.

Of particular interest to us are a pair of spaces on which $\widetilde{\pi_1(\mathcal{G}, \Gamma; \Gamma_0)}$ act: the Bass–Serre tree for $(\mathcal{G}, \Gamma)$, and the universal cover $\widetilde{\mathrm{Tot}(\mathcal{T}, \Gamma)}$ of a corresponding total space:

- The *Bass–Serre tree* T for $(\mathcal{G}, \Gamma)$ has vertex set $\widehat{E}_0$ containing one element for each left coset of each $G_v \leq \pi_1(\mathcal{G}, \Gamma; \Gamma_0)$, and edge set $\widehat{E}_1$ containing one element for each left coset of each $G_e \leq \pi_1(\mathcal{G}, \Gamma; \Gamma_0)$. The edge corresponding to a coset aG_e connects the two vertices whose corresponding cosets contain aG_e. The left action on T is the obvious one, a key fact

being that the stabilizer of a vertex corresponding to a coset aG_v is the group $aG_v a^{-1}$, and the stabilizer of the edge corresponding to a coset aG_e is $aG_e a^{-1}$. The quotient of this action is the original graph Γ; let $q : T \to \Gamma$ be that quotient map.

- The *universal cover* $\widetilde{\mathrm{Tot}(\mathcal{T}, \Gamma)}$, on the other hand, admits a proper and free $\pi_1(\mathcal{G}, \Gamma; \Gamma_0)$-action (by covering transformations); it is cocompact if and only if $(\mathcal{T}, \Gamma)$ is a compact graph of spaces.

The spaces T and $\widetilde{\mathrm{Tot}(\mathcal{T}, \Gamma)}$ and their actions are closely related. The space $\widetilde{\mathrm{Tot}(\mathcal{T}, \Gamma)}$ can be viewed as $\mathrm{Tot}(\mathcal{U}, T)$ where

- the Bass–Serre tree T is oriented so that $q : T \to \Gamma$ is orientation-preserving,
- for each $s \in \widehat{E}_0 \cup \widehat{E}$ the vertex/edge space is $(\widetilde{Y}_s, \widetilde{y}_s)$ where $\widetilde{Y}_s$ is the universal cover of $\widetilde{Y}_{q(s)}$ and $\widetilde{y}_s$ is a preimage of $y_{q(S)}$,
- the edge maps

$$(\widetilde{Y}_{i(e)}, \widetilde{y}_{i(e)}) \xleftarrow{\widetilde{p_e^-}} (\widetilde{Y}_e, \widetilde{y}_e) \xrightarrow{\widetilde{p_e^+}} (Y_{t(e)}, y_{t(e)})$$

are the (unique) pointed lifts of the edge maps

$$(Y_{o(q(e))}, y_{o(q(e))}) \xleftarrow{p_{q(e)}^-} (Y_{q(e)}, y_{q(e)}) \xrightarrow{p_{q(e)}^+} (Y_{t(q(e))}, y_{t(q(e))}).$$

Hence, there is a $\pi_1(\mathcal{G}, \Gamma; \Gamma_0)$-equivariant projection $\pi : \widetilde{\mathrm{Tot}(\mathcal{T}, \Gamma)} \to T$ such that for each $v \in \widehat{E}_0$ corresponding to the coset aG_v, $\pi^{-1}v) \approx \widetilde{Y}_v$, and for each point y in the interior of $e \in \widehat{E}_1$ corresponding to the coset aG_e, $\pi^{-1}(y) \approx \widetilde{Y}_e$. (An alternative construction of the Bass–Serre tree is as the quotient of $\widetilde{\mathrm{Tot}(\mathcal{T}, \Gamma)}$ obtained by identifying these covering spaces to points; see [18].) The equivariance of π means that, for a vertex v of T stabilized by $aG_v a^{-1}$, the set $\pi^{-1}(v) \approx \widetilde{Y}_v$ is stabilized (setwise) by $aG_v a^{-1}$.

10.1. Graphs of covering spaces. If each map in a graph of pointed topological spaces $(\mathcal{T}, \Gamma)$ is a covering projection, we call $(\mathcal{T}, \Gamma)$ a *graph of covering spaces.* By covering space theory, every group monomorphism can be realized as a covering projection, but realizing an arbitrary graph of groups $(\mathcal{G}, \Gamma)$ as a graph of covering spaces requires compatibility between these projections. To obtain such a realization, choices are required under which each edge space Y_e *simultaneously* covers $Y_{i(e)}$ and $Y_{t(e)}$ (in a manner that realizes the given group monomorphisms). When this is possible, we say that $(\mathcal{G}, \Gamma)$ is *realizable by covering spaces.*

EXAMPLE 10.2. Suppose that each vertex and edge group is isomorphic to $\mathbb{Z}$, so that $\pi_1(\mathcal{G}, \Gamma; \Gamma_0)$ is a *generalized Baumslag–Solitar group*. By placing a copy of S^1 at each vertex and noting that every finite-sheeted cover of S^1

is homeomorphic to S^1, we see that $(\mathcal{G}, \Gamma)$ is realizable by compact graph of finite-sheeted covering spaces.

EXAMPLE 10.3. As a generalization of the above, place a copy of $\mathbb{Z}^n$ on each vertex and edge of Γ. For edge maps, choose arbitrary monomorphisms of varying index. This graph can be realized by a compact graph of finite-sheeted covering spaces, where the space on each vertex and edge is the n-torus $\mathbb{T}^n$.

EXAMPLE 10.4. Suppose $(\mathcal{G}, \Gamma)$ is a graph of finitely generated free groups, and all monomorphisms are finite index. If we restrict ourselves to graphs as vertex and edge spaces, there are instances where it is impossible to realize $(\mathcal{G}, \Gamma)$ by covering spaces. However, if we allow the use of 3-dimensional orientable handlebodies, where genus determines topological type, we can realize any such $(\mathcal{G}, \Gamma)$ as a compact graph of finite-sheeted covering spaces. (Unfortunately, in the context of our purposes, the corresponding universal covers will not be compressible.)

A key ingredient in our main theorems is the following general fact.

LEMMA 10.5. *Let $(\mathcal{T}, \Gamma)$ be a graph of covering spaces and $v_0 \in \Gamma$ be a vertex. Then $\widetilde{\mathrm{Tot}(\mathcal{T}, \Gamma)}$ is homeomorphic to $T \times \widetilde{Y}_{v_0}$, where T is the Bass–Serre tree corresponding to the induced graph of groups.*

Proof. Since each map $p_e^- : Y_e \to Y_{i(e)}$ [resp., $p_e^+ : Y_e \to Y_{t(e)}$] is a covering map, so is each $\widetilde{p_e^-} : \widetilde{Y}_e \to \widetilde{Y}_{i(e)}$ [resp., $\widetilde{p_e^+} : \widetilde{Y}_e \to \widetilde{Y}_{t(e)}$]. By uniqueness of universal covers, these latter maps are necessarily homeomorphism. Thus, all of the double mapping cylinders in the construction of $\widetilde{\mathrm{Tot}(\mathcal{T}, \Gamma)}$ are actual products. It follows easily that $\pi : \widetilde{\mathrm{Tot}(\mathcal{T}, \Gamma)} \to T$ is a fiber bundle. Since T is contractible, it is a trivial bundle. ∎

Next, for graphs of covering spaces, we give a concrete description of the action of $\pi_1(\mathrm{Tot}(\mathcal{T}, \Gamma), \Gamma_0)$ on $\widetilde{\mathrm{Tot}(\mathcal{T}, \Gamma)}$, where the latter is viewed as $T \times \widetilde{Y}_{v_0}$. Specifically, we will define a $\pi_1(\mathrm{Tot}(\mathcal{T}, \Gamma), \Gamma_0)$-action on $\widetilde{Y}_{v_0}$ which, when paired diagonally with the Bass–Serre action on T, gives the desired covering space action on $T \times \widetilde{Y}_{v_0}$.

To define the desired action on $\widetilde{Y}_{v_0}$ it is enough to:

- define, for each $v \in E_0$, a homomorphism $\theta_v : G_v \to \mathrm{Homeo}(\widetilde{Y}_{v_0})$,
- define a homomorphism $\theta_F : F(E_1) \to \mathrm{Homeo}(\widetilde{Y}_{v_0})$,
- let $\overline{\Theta} : (*_{v \in E_0} G_v) * F(E_1) \to \mathrm{Homeo}(\widetilde{Y}_{v_0})$ be the union of the above homomorphisms, and
- check that all relators described in Definition 10.1 are sent to $\mathrm{id}_{Y_{v_0}}$.

Toward that end, inductively orient the edges of Γ_0 outward away from v_0. Then orient each edge not in Γ_0 arbitrarily. By changing some symbols, but without loss of generality, we may assume this is the orientation on Γ used in the basic definitions. As noted above, for each $e \in E_1$ we have homeomorphisms

$$(\widetilde{Y}_{i(e)}, \widetilde{y}_{i(e)}) \xleftarrow{\widetilde{p_e^-}} (\widetilde{Y}_e, \widetilde{y}_e) \xrightarrow{\widetilde{p_e^+}} (\widetilde{Y}_{t(e)}, \widetilde{y}_{t(e)}).$$

Let $f_e : (\widetilde{Y}_{i(e)}, \widetilde{y}_{i(e)}) \to (\widetilde{Y}_{t(e)}, \widetilde{y}_{t(e)})$ be the composition $\widetilde{p_e^+} \circ (\widetilde{p_e^-})^{-1}$, and for each $v \in E_0$, let $h_v = f_{e_k} \circ \cdots \circ f_{e_1} : Y_{v_0} \to Y_v$ where $e_1 * \cdots * e_k$ is the reduced edge path in Γ_0 from v_0 to v. (Let $h_{v_0} = \mathrm{id}_{\widetilde{Y}_{v_0}}$.) Since basepoints have been chosen, we have a well-defined G_v-action on each vertex space $\widetilde{Y}_v$. Viewing each $\alpha \in G_v$ as a self-homeomorphism of $\widetilde{Y}_v$, define $\theta_v : G_v \to \mathrm{Homeo}(\widetilde{Y}_{v_0})$ by $\theta_v(\alpha) = h_v^{-1} \alpha h_v$.

To define $\theta_F : F(E_1) \to \mathrm{Homeo}(\widetilde{Y}_{v_0})$ we need only specify the images of the generators. Do this by setting $\theta_F(e) = h_{t(e)}^{-1} \circ f_e \circ h_{i(e)}$. Note that if $e \in \Gamma_0$, then $f_e \circ h_{i(e)} = h_{t(e)}$, so $\theta_F(e) = \mathrm{id}_{Y_{v_0}}$; hence all type (ii) relators are sent to the identity element. The key to checking that type (i) relators $r = e \cdot (p_e^-)_\#(\beta) \cdot e^{-1} \cdot ((p_e^+)_\#(\beta))^{-1}$ are sent to the identity is the observation that $(p_e^-)_\#(\beta) = \widetilde{p_e^-} \circ \beta \circ (\widetilde{p_e^-})^{-1}$ and $(p_e^+)_\#(\beta) = \widetilde{p_e^+} \circ \beta \circ (\widetilde{p_e^+})^{-1}$.

It suffices to show that $\overline{\Theta}(e \cdot (p_e^-)_\#(\beta) \cdot e^{-1}) = \overline{\Theta}((p_e^+)_\#(\beta))$. We provide that calculation:

$$\overline{\Theta}(e \cdot (p_e^-)_\#(\beta) \cdot e^{-1}) = \theta_F(e) \cdot \theta_{i(e)}((p_e^-)_\#(\beta)) \cdot \theta_F(e)^{-1}$$

$$= (h_{t(e)}^{-1} \circ f_e \circ h_{i(e)}) \cdot (h_{i(e)}^{-1} \circ (p_e^-)_\#(\beta) \circ h_{i(e)}) \cdot (h_{i(e)}^{-1} \circ f_e^{-1} \circ h_{t(e)})$$

$$= (h_{t(e)}^{-1} \circ f_e \circ h_{i(e)}) \cdot (h_{i(e)}^{-1} \circ (p_e^-)_\#(\beta) \circ h_{i(e)}) \cdot (h_{i(e)}^{-1} \circ f_e^{-1} \circ h_{t(e)})$$

$$= h_{t(e)}^{-1} \circ f_e \circ (p_e^-)_\#(\beta) \circ f_e^{-1} \circ h_{t(e)}$$

$$= h_{t(e)}^{-1} \circ f_e \circ (\widetilde{p_e^-} \circ \beta \circ (\widetilde{p_e^-})^{-1}) \circ f_e^{-1} \circ h_{t(e)}$$

$$= h_{t(e)}^{-1} \circ (\widetilde{p_e^+} \circ (\widetilde{p_e^-})^{-1}) \circ (\widetilde{p_e^-} \circ \beta \circ (\widetilde{p_e^-})^{-1}) \circ (\widetilde{p_e^-} \circ (\widetilde{p_e^+})^{-1}) \circ h_{t(e)}$$

$$= h_{t(e)}^{-1} \circ \widetilde{p_e^+} \circ \beta \circ (\widetilde{p_e^+})^{-1} \circ h_{t(e)}$$

$$= h_{t(e)}^{-1} \circ (p_e^+)_\#(\beta) \circ h_{t(e)}$$

$$= \theta_{t(e)}((p_e^+)_\#(\beta)) = \overline{\Theta}((p_e^+)_\#(\beta)).$$

Since all relators of the relative presentation of $\pi_1(\mathrm{Tot}(\mathcal{T}, \Gamma), \Gamma_0)$ are sent to the identity element, $\overline{\Theta}$ induces a homomorphism

$$\Theta : \pi_1(\mathrm{Tot}(\mathcal{T}, \Gamma), \Gamma_0) \to \mathrm{Homeo}(\widetilde{Y}_{v_0}).$$

This is the desired action. Just as with the Bass–Serre action of the

$\pi_1(\mathrm{Tot}(\mathcal{T}, \Gamma), \Gamma_0)$-action on T, we do not expect the $\pi_1(\mathrm{Tot}(\mathcal{T}, \Gamma), \Gamma_0)$-action on $\widetilde{Y}_{v_0}$ to be proper or free. But combined, these two actions yield a proper free action on $T \times \widetilde{Y}_{v_0}$.

REMARK 10.6. In cases where $(\mathcal{T}, \Gamma)$ is a graph of finite-sheeted covering spaces, all of the lift homeomorphisms

$$(\widetilde{Y}_{i(e)}, \widetilde{y}_{i(e)}) \xleftarrow{\widetilde{p_e^-}} (\widetilde{Y}_e, \widetilde{y}_e) \xrightarrow{\widetilde{p_e^+}} (\widetilde{Y}_{t(e)}, \widetilde{y}_{t(e)})$$

are quasi-isometric homeomorphisms. Since the group action

$$\Theta : \pi_1(\mathrm{Tot}(\mathcal{T}, \Gamma), \Gamma_0) \to \mathrm{Homeo}(\widetilde{Y}_{v_0}),$$

as described above, takes each group element to a finite composition of lift homeomorphisms, inverses of those homeomorphisms, and isometries of vertex spaces, the action is by quasi-isometric homeomorphisms.

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