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Expo. Math. 37 (2019) 292–313

Expositiones Mathematicae

www.elsevier.com/locate/exmath

Proper homotopy types and $\mathcal{Z}$-boundaries of spaces admitting geometric group actions[☆]

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Received 24 July 2017; received in revised form 7 January 2018

Abstract

We extend several techniques and theorems from geometric group theory so they apply to geometric actions on arbitrary proper metric ARs (absolute retracts). Previous versions often required actions on CW complexes, manifolds, or proper CAT(0) spaces, or else included a finite-dimensionality hypothesis. We remove those requirements, providing proofs that simultaneously cover all of the usual variety of spaces. A second way that we generalize earlier results is by eliminating a freeness requirement often placed on the group actions. In doing so, we allow for groups with torsion.

The main theorems are new in that they generalize results found in the literature, but a significant aim is expository. Toward that end, brief but reasonably comprehensive introductions to the theories of ANRs (absolute neighborhood retracts) and $\mathcal{Z}$-sets are included, as well as a much shorter introduction to shape theory. Here is a sampling of the theorems proved here.

Theorem. *If quasi-isometric groups G and H act geometrically on proper metric ARs X and Y, resp., then X is proper homotopy equivalent to Y.*

Theorem. *If quasi-isometric groups G and H act geometrically on proper metric ARs X and Y, resp., and Y can be compactified to a $\mathcal{Z}$-structure $(\overline{Y}, Z)$ for H, then the same boundary can be added to X to obtain a $\mathcal{Z}$-structure for G.*

☆ This research was supported in part by Simons Foundation Grants 207264 and 427244, CRG.

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https://doi.org/10.1016/j.exmath.2018.03.004

Theorem. *If quasi-isometric groups G and H admit $\mathcal{Z}$-structures $(\overline{X}, Z_1)$ and $(\overline{Y}, Z_2)$, resp., then Z_1 and Z_2 are shape equivalent.*

MSC 2010: primary 20F65; secondary 20F69; 57N25; 55M15

Keywords: Absolute retract; Geometric action; Proper homotopy equivalence; Z-structure; Z-boundary; Shape equivalence

1. Introduction

In this paper all spaces are assumed separable and metrizable. A metric space (X, d) is *proper* if every closed ball in X is compact. Separability is automatic for proper metric spaces and every proper metric space is locally compact; conversely, every locally compact (separable and metrizable) space admits a proper metric. See Lemma 2.1 for details.

A locally compact space X is an *absolute neighborhood retract* (ANR) if, whenever X is embedded as a closed subset of a space Y, some neighborhood of X in Y retracts onto X. An ANR X is an *absolute retract* (AR) if, whenever X is embedded as a closed subset of Y, its image is a retract of Y.[1]

One of the most significant aspects of "ANR theory" is that it provides a common ground for studying a variety of nice spaces: manifolds; locally finite CW (including simplicial and cube) complexes; proper CAT(0) spaces; Hilbert cube manifolds; etc. This is particularly useful in subjects where it is desirable to move freely between categories. In this paper we will generalize several theorems and techniques from geometric group theory and metric geometry, previously restricted to one or more of these categories, to the full spectrum of ANRs. A second way that we improve upon known results is by extending several theorems that previously applied only to free geometric actions. In our versions, freeness is not required, so the theorems can be applied to groups with torsion.

For the reader with little or no experience working with abstract ANRs, we include a short, elementary introduction to ANR theory that is sufficient for a complete understanding of most of the work presented here. Indeed, a secondary goal of this paper is to provide the reader that background, and to provide a level of comfort with this useful category of spaces.

Here are the most notable theorems to be proved here. Versions of the first theorem and its corollary are well-known when X and Y are CW complexes (see [18, Ch.10]). The traditional proof is inductive over skeleta—sometimes called a "connect-the-dots strategy." Ours relies on a generalized version of that method, which does not require a CW structure.

Theorem 1.1. *If quasi-isometric groups G and H act geometrically on proper metric ARs X and Y, resp., then X is proper homotopy equivalent to Y. In fact, there exist continuous*

[1] Due to the applications presented here, local compactness is included as part of the definition of ANR. More general treatments are commonly found in the literature.

coarse equivalences $f : X \to Y$ *and* $g : Y \to X$ *such that* gf *and* fg *are boundedly (hence properly) homotopic to* id_X *and* id_Y.

The following corollary can be obtained via covering space theory and a deep result from ANR theory (see Theorem 2.11), when the G-actions are free, and for arbitrary CAT(0) groups by a theorem of Ontaneda [30]. Our proof is more elementary and the conclusion is more general.

Corollary 1.2. *If a group* G *acts geometrically on proper metric ARs* X *and* Y*, then* X *is proper homotopy equivalent to* Y *via continuous coarse equivalences* $f : X \to Y$ *and* $g : Y \to X$ *such that* gf *and* fg *are boundedly homotopic to* id_X *and* id_Y.

The next collection of theorems involves "$\mathcal{Z}$-boundaries" of groups—a notion introduced by Bestvina and expanded upon by Dranishnikov (see Section 6 for definitions). The idea is to provide an axiomatic treatment of group boundaries that includes Gromov boundaries of hyperbolic groups and visual boundaries of CAT(0) groups, but which can be applied more generally. Both [2] and [11] recognized ANR theory as the natural setting for such a theory. In order to obtain some of his most notable conclusions, Bestvina worked only with finite-dimensional ANRs and torsion-free groups. Dranishnikov relaxed those conditions, but some of Bestvina's conclusions were then lost. Here we extend several theorems from [2] to the more general setting suggested in [11].

The first of those theorems allows "boundary swapping" when a group admits a finite $K(G, 1)$ complex. More generally, Bestvina asserted a boundary swapping result for pairs of quasi-isometric groups, each of that type. We obtain generalizations which allow non-free actions on arbitrary ARs. In addition, we prove an equivariant version that applies to $E\mathcal{Z}$-boundaries, as defined by Farrell and Lafont [15].

Theorem 1.3 (*Boundary Swapping Theorem*)**.** *Suppose* G *acts geometrically on proper metric ARs* X *and* Y*, and* Y *can be compactified to a* $\mathcal{Z}$*-structure [resp.,* $E\mathcal{Z}$*-structure]* $(\overline{Y}, Z)$ *for* G*. Then* X *can be compactified, by addition of the same boundary, to a* $\mathcal{Z}$*-structure [resp.,* $E\mathcal{Z}$*-structure]* $(\overline{X}, Z)$ *for* G.

For a pair of quasi-isometric groups, the $E\mathcal{Z}$ conclusion no longer makes sense, but the rest of Theorem 1.3 goes through as follows.

Theorem 1.4 (*Generalized Boundary Swapping Theorem*)**.** *Suppose quasi-isometric groups* G *and* H *act geometrically on proper metric ARs* X *and* Y*, respectively, and* Y *can be compactified to a* $\mathcal{Z}$*-structure* $(\overline{Y}, Z)$ *for* H*. Then* X *can be compactified, by addition of the same boundary, to a* $\mathcal{Z}$*-structure* $(\overline{X}, Z)$ *for* G.

Remark 1. A *geometric action* is one that is proper, cocompact, and by isometries. By results to be discussed in Section 6, it is enough to assume that the action is proper and cocompact.

It is well-known that a word hyperbolic group G has a well-defined Gromov boundary, but a CAT(0) group can admit non-homeomorphic visual boundaries [9]. A version of well-definedness can be recovered using shape theory; one wishes to assert that any two

$\mathcal{Z}$-boundaries of a group G are shape equivalent. That assertion – implicit in [17] – was made explicitly in [2], for certain torsion-free groups. Later, Ontaneda [30] proved an analogous theorem, without a torsion-free hypothesis, for the special case of visual boundaries of CAT(0) groups. Here we show that all of these additional hypotheses are unnecessary.

Theorem 1.5. *If quasi-isometric groups G and H admit $\mathcal{Z}$-structures $\left(\overline{X}, Z_1\right)$ and $\left(\overline{Y}, Z_2\right)$, respectively, then Z_1 and Z_2 are shape equivalent.*

Corollary 1.6. *If a group G admits a $\mathcal{Z}$-boundary, then that boundary is well-defined up to shape equivalence.*

2. Some basics of ANR theory

In this section we cover the necessary background from ANR theory. Rather than simply quoting results from the literature, we provide a brief, elementary treatment of the topic—including proofs of most key facts used in this paper. More complete treatments can be found in [24] or [33].

We begin by proving a basic fact about metric spaces mentioned in the introduction.

Lemma 2.1. *Every proper metric space is separable and locally compact. Conversely, every locally compact separable metric space admits a proper metric.*

Proof. Let (X, d) be a proper metric space. Local compactness is clear. For separability, fix $x_0 \in X$ and note that each closed n-ball, $B[x_0, n]$, being a compact metric space, contains a countable dense subset A_n. Then $\cup_{n=1}^{\infty} A_n$ is a countable dense subset of X.

For the converse, let (X, d) be a locally compact separable metric space, where d in not necessarily proper. Separability assures that X is second countable, which together with local compactness allows us to construct a sequence of compact sets $\{C_i\}_{i=0}^{\infty}$ with $C_i \subseteq \operatorname{int}(C_{i+1})$ for all i, and $\cup_{n=0}^{\infty} C_n = X$. For each $i \geq 1$, let $D_i = C_i - \operatorname{int}(C_{i-1})$, a compact set containing the frontiers of C_{i-1} and C_i as disjoint closed subsets. By the Tietze Extension Theorem, there exist continuous functions $f_i : D_i \to [i-1, i]$ with $f_i(\operatorname{Fr}(C_{i-1})) = \{i-1\}$ and $f_i(\operatorname{Fr}(C_i)) = \{i\}$. By taking the union of the f_i and sending C_0 to 0, we obtain a proper map $f : X \to [0, \infty)$. The graph of f, $G_f \subseteq X \times [0, \infty)$ is an embedded copy of X, and if we give $X \times [0, \infty)$ the ℓ_2 metric, the subspace metric on G_f is proper. □

With that short diversion completed, we remind the reader that, in this paper, *all spaces* are assumed to be separable and metrizable.

A locally compact space X is an *absolute neighborhood retract* (ANR) if, whenever X is embedded as a closed subset of another space Y, some neighborhood of X retracts onto X; if the entire space Y always retracts onto X, we call X an *absolute retract* (AR).

Remark 2. A finite-dimensional ANR [resp., AR] is often called a *Euclidean neighborhood retract* (ENR) [resp. *Euclidean retract* (ER)]. A key aspect of this paper is the development of techniques that do not require a restriction to this subcategory.

Some of the most important properties of A[N]Rs involve extensions of maps. In fact, an alternative approach *defines* A[N]R using these very extension properties. In that setting, they are sometimes called A[N]Es (absolute [neighborhood] extensors). We will not use that terminology, but the following characterization is crucial.

Lemma 2.2 (*Extensor Characterization of ARs and ANRs*)**.** *A locally compact space X is an AR [resp., ANR] if and only if it satisfies the following extension property:*
(†) *If A is a closed subset of an arbitrary space Y and $f : A \to X$ is continuous, then there is a continuous extension $\overline{f} : Y \to X$ [resp., a continuous extension $f : U \to X$, where U is a neighborhood of A in Y].*

Proof. We begin with the reverse implications. Suppose X is embedded as a closed subset of a space Y and consider the identity map $\mathrm{id}_X : X \to X$. By viewing the domain copy of X as a closed subset of Y, the absolute extension property guarantees a map $\overline{f} : Y \to X$ that restricts to the identity on X, i.e., $\overline{f}$ is a retraction. If we assume the weaker extension property, we get a retraction $\overline{f} : U \to X$.

For the forward implications, choose a proper metric d on X, let $\{x_i\}_{i=1}^{\infty} \subseteq X$ be a dense subset, and for each i define $g_i(x) = d(x_i, x)$. Then $\mathbf{g} = (g_i)$ embeds X as a closed subset of $\mathbb{R}^{\infty}$ (with the product topology). In what follows, view X as a closed subset of $\mathbb{R}^{\infty}$ and let $j : X \hookrightarrow \mathbb{R}^{\infty}$ be the inclusion map.

Suppose A is a closed subset of a space Y and $f : A \to X$ is continuous. By applying the Tietze Extension Theorem coordinate-wise, extend $jf : A \to \mathbb{R}^{\infty}$ to a continuous map $F : Y \to \mathbb{R}^{\infty}$. If X is an AR, choose a retraction $r : \mathbb{R}^{\infty} \to X$ and let $\overline{f} = rF$. If X is an ANR, choose a retraction $r : V \to X$, where V is a neighborhood of X in $\mathbb{R}^{\infty}$; then let $U = F^{-1}(V)$ and $\overline{f} = rF|_U$. □

Proposition 2.3 (*Homotopy Extension Property*)**.** *Let A be a closed subset of a space Y, $T = (Y \times \{0\}) \cup A \times [0,1]$ and $h : T \to X$ a map into an ANR. Then there is an extension of h to a homotopy $H : Y \times [0,1] \to X$. If the homotopy $h|_{A\times[0,1]}$ is K-bounded*[2] *and Y is locally compact, then H can be chosen to be $(K+1)$-bounded.*

Proof. By Lemma 2.2 there is an extension of h to $\overline{h} : U \to X$, where U is a neighborhood of T in $Y \times [0,1]$. Let $V \subseteq Y$ be an open neighborhood of A such that $V \times [0,1] \subseteq U$, and choose a Urysohn function $\lambda : Y \to [0,1]$ with $\lambda(Y \setminus V) = 0$ and $\lambda(A) = 1$. Define

$$\overline{H}(y,t) = \begin{cases} \overline{h}(y, \lambda(y) \cdot t) & \text{if } y \in V \\ h(y,0) & \text{if } y \notin V. \end{cases}$$

Now suppose $h|_{A\times[0,1]}$ is K-bounded and Y is locally compact. Then each $a \in A$ has a compact neighborhood $N_a \subseteq V$; so by uniform continuity, there is an open neighborhood $V_a \subseteq N_a$ over which all tracks of $\overline{h}\big|_{V_a\times[0,1]}$ have diameter $< K+1$. Rechoose V to be $\cup_{a\in A} V_a$; then $\overline{H}$, as defined above, is $(K+1)$-bounded. □

Corollary 2.4. *If A is a closed subset of an arbitrary space Y and $f : A \to X$ is a null-homotopic map into an ANR, then f extends to a map $\overline{f} : Y \to X$.*

[2] Bounded homotopies are defined in Section 4.

Proof. Let $J : A \times [0, 1] \to X$ be a homotopy with $J_0(A) = \{p\}$ and $J_1 = f$. Extend J to a map of $h : (Y \times \{0\}) \cup A \times [0, 1] \to X$ by sending $Y \times \{0\}$ to p; then apply the Homotopy Extension Property and let $\overline{f} = H_1$. □

Corollary 2.5. *An ANR is an AR if and only if it is contractible.*

Proof. Suppose X is an AR. By the argument used in Lemma 2.2, we may assume X is a closed subset of $\mathbb{R}^\infty$; so by hypothesis, there is a retraction $r : \mathbb{R}^\infty \to X$. Composing any contraction of $\mathbb{R}^\infty$ with r gives a contraction of X.

For the converse, assume X is contractible and $X \hookrightarrow Y$ is an embedding as a closed subset of a space Y. Then $\mathrm{id}_X : X \to X$ is a null-homotopic map into an ANR, so by Corollary 2.4 it extends to a map of Y into X. That map is a retraction. □

Corollary 2.6. *Every open subset of an ANR is an ANR.*

Proof. Let V be an open subset of an ANR X, and $f : A \to V$ a continuous map, where A is a closed subset of a space Y. By Lemma 2.2, there is an open neighborhood U' of A in Y and an extension $f' : U' \to X$ of f. Let $U = U' \cap (f')^{-1}(V)$ and $\overline{f} = f'|_U$. □

With additional work, one can prove a similar, but different proposition.

Proposition 2.7. *A space X with the property that each $x \in X$ has an open neighborhood that is an ANR, is itself an ANR.*

The next proposition provides a wealth of examples.

Proposition 2.8. *Being an AR [ANR] is a topological property. Furthermore,*

(1) $[0, 1]$, $[0, 1)$, *and* $(0, 1)$ *are ARs,*
(2) *every finite product of A[N]Rs is an A[N]R,*
(3) *a countably infinite product of ARs is an AR, provided all but finitely many factors are compact,*
(4) *Every retract of an A[N]R is an A[N]R.*

Proof. That these properties are invariant under homeomorphism is clear. Assertion (1) follows from Lemma 2.2, Corollaries 2.5 and 2.6, and the Tietze Extension Theorem.

For assertion (2), let $A \subseteq Y$ be closed and $\mathbf{f} = (f_i) : A \to \prod_{i=1}^k X_i$ be continuous. If each X_i is an AR, choose extensions $\overline{f}_i : Y \to X_i$ to get an extension $\bar{\mathbf{f}} = \left(\overline{f}_i\right) : Y \to \prod_{i=1}^k X_i$. If each X_i is an ANR, choose extensions $\overline{f}_i : U_i \to X_i$, where U_i a neighborhood of A in Y, then let $U = \cap U_i$ and $\bar{\mathbf{f}} = \left(\overline{f}_i|_U\right) : U \to \prod_{i=1}^k X_i$.

For infinite products of ARs, we must restrict to countable products to ensure metrizability; and to ensure local compactness, only finitely many factors can be noncompact. With those caveats, the above proof remains valid for ARs (but fails for ANRs).

To prove assertion (4), first recall that if $r : X \to X_0$ is a retraction, then X_0 is closed in X, so X_0 is locally compact. Now suppose $f : A \to X_0$ is continuous, with A a closed subset of a space Y. If X is an AR, there is an extension $F : Y \to X$. Then $\overline{f} = rF : Y \to X_0$ is an extension, showing that X_0 is an AR. By the same approach, if X is an ANR, then so is X_0. □

Recall that a space X is *locally contractible at* $x \in X$ if every neighborhood U of x contains a neighborhood V, such that V contracts in U. A space that is locally contractible at each of its points is *locally contractible*.

Proposition 2.9. *Every ANR is locally contractible and every finite-dimensional locally compact and locally contractible space is an ANR.*

Sketch of Proof. It is easy to see that a retract of a locally contractible space is locally contractible. If X is an ANR then there is an embedding $X \hookrightarrow \mathbb{R}^\infty$ as a closed subset and a retraction $r : U \to X$ of an open neighborhood onto X. Since U is locally contractible, so is X.

If X is locally compact and finite-dimensional, there is a proper closed embedding of $X \hookrightarrow \mathbb{R}^n$ for some $n < \infty$. By Corollary 2.6 and Proposition 2.8, it suffices to exhibit a retraction of some neighborhood onto X. Choose a polyhedral neighborhood N_0 of X and an infinite triangulation of $N_0 - X$ that gets progressively finer near X. Define $r_0 : N_0^{(0)} \cup X \to X$ by sending each vertex of $N - X$ to a nearest point in X. Assume inductively that there is a polyhedral subneighborhood N_k of X and a retraction $r_k : N_k^{(k)} \cup X \to X$, where $N_k^{(k)}$ is the k-skeleton of $N_k - X$. By the local contractibility of X, there exists a polyhedral neighborhood $N_{k+1} \subseteq N_k$, sufficiently small, that for each $(k+1)$-simplex σ^{k+1} of N_{k+1}, $r_k|_{\partial\sigma^k}$ extends over σ^{k+1} to a map into X, thereby giving a retraction $r_{k+1} : N_{k+1}^{(k+1)} \cup X \to X$. The desired retraction follows by induction. □

Example 1. Using the above observations, one sees that: every finite-dimensional or Hilbert cube manifold; every locally finite CW, simplicial, or cube complex; and every finite-dimensional proper CAT(0) space is an ANR. (In fact, [30] shows that *every* proper CAT(0) space is an ANR, hence an AR.)

A few deeper facts about ANRs will play a role in this paper. We state them here without proofs.

Theorem 2.10 (*Hanner's Theorem [22]*)**.** *A space X is an ANR if for every open cover $\mathcal{U}$ of X there is an ANR Y and maps $f : X \to Y$ and $g : Y \to X$ such that gf is $\mathcal{U}$-homotopic to* id_X.

Theorem 2.11 (*West's Theorem [34]*)**.** *Every ANR is proper homotopy equivalent to a locally finite polyhedron; every compact ANR is homotopy equivalent to a finite polyhedron.*

Theorem 2.12 (*Edwards' Theorem [14]*)**.** *If X is a locally compact ANR, then $X \times [0, 1]^\infty$ is a Hilbert cube manifold.*

3. $\mathcal{Z}$-sets and $\mathcal{Z}$ -compactifications

A closed subset A of a space X, is a *$\mathcal{Z}$-set* if there exists a homotopy $H : X \times [0, 1] \to X$ such that $H_0 = \mathrm{id}_X$ and $H_t(X) \subset X - A$ for every $t > 0$. In this case we say that H *instantly homotopes X off from A*.

Remark 3. Notice that, if $A \subseteq X$ is a $\mathcal{Z}$-set, then A is nowhere dense in X.

Example 2. The prototypical $\mathcal{Z}$-set is the boundary of a manifold. More generally, a closed subset A of an n-manifold M^n is a $\mathcal{Z}$-set if and only if $A \subseteq \partial M^n$.

The following lemma generalizes [16, Prop.1.6]. It will play a crucial role in our boundary swapping theorems.

Lemma 3.1. *Let $f : (X, A) \to (Y, B)$ and $g : (Y, B) \to (X, A)$ be continuous maps with $f(X - A) \subseteq Y - B$, $g(Y - B) \subseteq X - A$, and $gf|_A = \mathrm{id}_A$. Suppose further that there is a homotopy $J : X \times [0, 1] \to X$ which is fixed on A and satisfies: $J_0 = \mathrm{id}_X$, $J_1 = gf$, and $J((X - A) \times [0, 1]) \subseteq X - A$. If B is a $\mathcal{Z}$-set in Y, then A is a $\mathcal{Z}$-set in X.*

Proof. Since $A = g^{-1}(B)$, A is closed in X. Choose $K : Y \times [0, 1] \to Y$ that instantly homotopes Y off from B. We will construct $H : X \times [0, 1] \to X$, which instantly homotopes X off from A, by describing the track of each $x \in X$.

For each $x \in X$ define $\alpha_x : [0, 1] \to X$ by $\alpha_x(t) = J_t(x)$ and $\beta_x : [0, 1] \to X$ by $\beta_x(t) = gK_t(f(x))$. Note that

(1) $\alpha_x(0) = x$ and $\alpha_x(1) = gf(x) = \beta_x(0)$,
(2) if $x \in X - A$ then $\alpha_x, \beta_x \subseteq X - A$,
(3) if $x \in A$ then $\alpha_x \equiv x$, $\beta_x(0) = x$, and $\beta_x((0, 1]) \subseteq X - A$, and
(4) $\mathrm{diam}(\alpha_x) \to 0$ as $d(x, A) \to 0$.

The track γ_x of x under H will follow the concatenation $\alpha_x \cdot \beta_x$, but in order for points to be instantly homotoped off from A, reparameterizations are necessary. Let $r_x = \min\left\{d(x, A), \frac{1}{2}\right\}$ and define $\gamma_x : [0, 1] \to X$ as follows:

$$\gamma_x(t) = \begin{cases} \alpha_x(t/r_x) & \text{if } 0 \le t < r_x \\ \beta_x(t - r_x/1 - r_x) & \text{if } r_x \le t \le 1. \end{cases}$$

Observation (4) and the fact that α_x, β_x, and r_x vary continuously with x combine to show that γ_x varies continuously with x. Define $H(x, t) = \gamma_x(t)$ and apply Observations (2) and (3) to deduce that H instantly homotopes X off from A. □

A $\mathcal{Z}$-*compactification* of a space Y is a compactification $\overline{Y}$ such that $Z \equiv \overline{Y} - Y$ is a $\mathcal{Z}$-set in $\overline{Y}$. In that case we call Z a $\mathcal{Z}$-*boundary* for Y. Here we follow standard convention for compactifications ([28] or [12]) by requiring $\overline{Y}$ to be Hausdorff and Y to be dense in $\overline{Y}$. Under that convention: Y must be locally compact and Hausdorff; Y is necessarily open in $\overline{Y}$; and Z is compact and nowhere dense.

Lemmas 3.2 and 3.3 allow us to stay within preferred categories of spaces when taking $\mathcal{Z}$-compactifications.

Lemma 3.2. *If $\overline{Y}$ is a $\mathcal{Z}$-compactification of a (separable metrizable) space Y, then $\overline{Y}$ is also separable and metrizable.*

Proof. For metrizability, we apply the Urysohn Metrization Theorem [28, p. 214]. Since $\overline{Y}$ is compact and Hausdorff it is a normal space, so we need only check that there is a countable basis for its topology. Being separable and metrizable, Y admits a countable basis $\mathcal{B}_0$. For each integer $i > 0$, let $\mathcal{B}_i = \left\{ H_{1/i}^{-1}(B) \mid B \in \mathcal{B}_0 \right\}$ and note that $\overline{\mathcal{B}} = \cup_{i \geq 0} \mathcal{B}_i$ is a countable basis for $\overline{Y}$.

Since $\overline{Y}$ is compact and metrizable, it is separable. □

Lemma 3.3. *If $\overline{Y}$ is a $\mathcal{Z}$-compactification of an ANR Y, then $\overline{Y}$ is an ANR. If Y is an AR, then so is $\overline{Y}$.*

Proof. The fact that $\overline{Y}$ is an ANR is a straight forward consequence of Theorem 2.10. For the latter observation, use the definition of $\mathcal{Z}$-set to homotope $\overline{Y}$ into Y, then follow that homotopy with a contraction of Y to obtain a contraction of $\overline{Y}$. Now apply Corollary 2.5. □

Lemma 3.4. *Let N be a neighborhood of a $\mathcal{Z}$-set A in an ANR X. Then A is a $\mathcal{Z}$ -set in N.*

Proof. Let $H : X \times [0, 1] \to X$ instantly homotope X off from A, and choose open sets $U, U' \subseteq X$ such that

$$A \subseteq U \subseteq \overline{U} \subseteq U' \subseteq \overline{U'} \subseteq \operatorname{int} N.$$

By truncating H at a time $0 < t_0 \leq 1$ then reparameterizing, we may obtain a homotopy $K : \overline{U} \times [0, 1] \to \operatorname{int} N$ such that $K\left(\overline{U} \times [0, 1]\right) \subseteq U'$. Extend K to the trivial (constantly identity) homotopy on $(N - U') \times [0, 1]$ and the identity map on $N \times \{0\}$. By Corollary 2.6, we can apply the Homotopy Extension Property to obtain $K : \operatorname{int} N \times [0, 1] \to \operatorname{int} N$, after which we can extend to the identity over the frontier of N. Moreover, by choosing a sufficiently small neighborhood V of $(\operatorname{int} N - U') \cup \overline{U}$ (as was used in the proof of the Homotopy Extension Property) we can arrange that no track of K passes through A. □

Remark 4. If one restricts attention to ANRs, Lemmas 3.3 and 3.4 allow a variety of equivalent formulations of $\mathcal{Z}$-set and $\mathcal{Z}$-compactification. For example, a closed subset A of an ANR X is a $\mathcal{Z}$-set if and only if $U - A \hookrightarrow U$ is a homotopy equivalence for every open set U in Y. See [23].

The next definition blends topology and geometry in that the specific metric on X plays a role.

Definition 3.5. A *controlled $\mathcal{Z}$-compactification* of a proper metric space (Y, d) is a $\mathcal{Z}$-compactification $\overline{Y}$ satisfying the additional condition:
(‡) For every $R > 0$ and every open cover $\mathcal{U}$ of $\overline{Y}$, there is a compact set $C \subset Y$ so that if $A \subseteq Y - C$ and $\operatorname{diam}_d A < R$, then $A \subseteq U$ for some $U \in \mathcal{U}$.

Example 3. The (standard) compactification of hyperbolic n-space, by addition of an end point to each ray emanating from the origin, is a controlled $\mathcal{Z}$-compactification; so is the analogous compactification of Euclidean n-space. More generally, adding the visual sphere at infinity, with the cone topology, to a proper CAT(0) space is a controlled $\mathcal{Z}$-compactification.

The compactification $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is a controlled $\mathcal{Z}$-compactification of $\mathbb{R}$. By contrast, $\overline{\mathbb{R}} \times \overline{\mathbb{R}}$ is a $\mathcal{Z}$-compactification of $\mathbb{R}^2$, but not a controlled $\mathcal{Z}$-compactification (under the Euclidean metric).

Remark 5. By Lemma 3.2, we may place a metric $\overline{d}$ on $\overline{Y}$. Then, using Lebesgue numbers, (‡) can be reformulated as:
(‡′) For every $R > 0$ and $\varepsilon > 0$, there is a compact $C \subset Y$ so that if $A \subseteq Y - C$ and $\mathrm{diam}_d A < R$, then $\mathrm{diam}_{\overline{d}} A < \varepsilon$.

Except for an impact on the size of C, the specific $\overline{d}$ chosen is unimportant; moreover, there is no direct relationship between d and $\overline{d}$. The following useful lemma highlights the difference between those metrics.

Lemma 3.6. *Suppose $\overline{Y} = Y \cup Z$ is a controlled $\mathcal{Z}$ -compactification of Y. For each $z \in Z$, each neighborhood $\overline{U}$ of z in $\overline{Y}$, and each $R > 0$, there is a neighborhood $\overline{V}$ of z so that $d_Y(V, Y - U) \geq R$.*

Proof. Place a metric $\overline{d}$ on $\overline{Y}$, then choose a neighborhood $\overline{U'}$ of z whose closure is contained in the interior of $\overline{U}$ and let $\epsilon = \overline{d}(\overline{U'}, \overline{Y} - \overline{U})$. From the control condition there is a $C \subseteq Y$ so that sets in $Y - C$ of d-diameter less than R have $\overline{d}$-diameter less than $\epsilon/2$. Set $\overline{V} = \overline{U'} - N_R(C)$, where $N_R(C)$ is the closed R-neighborhood of C, and suppose there exist $x \in V$ and $y \in Y - U$ with $d(x, y) < R$. Then $y \in Y - C$, so $\{x, y\} \subseteq Y - C$ and $\mathrm{diam}_d\{x, y\} < R$. By the control condition $\overline{d}(x, y) < \epsilon/2$, a contradiction. □

4. Preliminaries from geometric group theory and metric geometry

Through the remainder of this paper, functions (also called maps) are not always continuous. When continuity is assumed or required, it will be done explicitly. Since the concepts presented here are geometric, all spaces are assumed to come with a fixed metric. We use $B_d(x, r)$ and $B_d[x, r]$ to denote open and closed metric balls, respectively.

Definition 4.1. Let X be a space, $\mathcal{A}$ a collection of subsets of X, and $D \subseteq X$.

(1) $\mathcal{A}$ is *locally discrete* if each $x \in X$ has a neighborhood intersecting at most one element of $\mathcal{A}$
(2) $\mathcal{A}$ is *uniformly bounded* if the set $\{\mathrm{diam}(A) \mid A \in \mathcal{A}\}$ is bounded above, and
(3) D is *large-scale dense* (also called *quasi-dense*) if $\{d(x, D) \mid x \in X\}$ is bounded above.

Definition 4.2. Let $f, g : X \to Y$ be functions. Then

(1) f is *large-scale surjective* if $f(X)$ is large-scale dense in Y,
(2) f is *metrically proper* if the pre-image of every bounded subset of Y is a bounded in X,
(3) f is *large-scale uniform* (also called bornologous) if, for every $R > 0$, there is an $S > 0$ so that if $d_X(x, x') < R$, then $d_Y(f(x), f(x')) < S$,
(4) f and g are *boundedly close* if $\{\mathrm{diam}(f(x), g(x)) \mid x \in X\}$ is bounded above.

We reserve the term *proper* for its traditional meaning: a continuous function $f : X \to Y$ for which $f^{-1}(C)$ is compact whenever C is compact. Similarly, *homotopy* always indicates a continuous function. A *bounded homotopy* $H : X \times [0, 1] \to Y$ is one for which the collection $\{H(\{x\} \times [0, 1]) \mid x \in X\}$ is uniformly bounded. A continuous function $f : X \to Y$ is a *proper homotopy equivalence* if there exists a continuous $g : Y \to X$ such that gf and fg are homotopic, via proper homotopies, to id_X and id_Y, respectively. The following is immediate.

Lemma 4.3. *Let $f, g : X \to Y$ be functions between proper metric spaces. Then*

(1) *if f and g are boundedly close and f is metrically proper, then so is g,*
(2) *if f and g are large-scale uniform and boundedly close over a large-scale dense subset of X, then f and g are boundedly close,*
(3) *f is metrically proper and continuous if and only if f is proper, and*
(4) *if $H : X \times [0, 1] \to Y$ is a bounded homotopy, then H is proper if and only if H_t is proper for some [resp., all] t.*

The next set of definitions provides useful generalizations of "quasi-isometric embedding" and "quasi-isometry".

Definition 4.4. A map $f : X \to Y$ is:

(1) *coarse* if it is metrically proper and large-scale uniform;
(2) a *coarse equivalence* if it has a coarse inverse, i.e., a coarse map $g : Y \to X$ such that gf and fg are boundedly close to id_X and id_Y;
(3) a *coarse embedding* if $f : X \to f(X)$ is a coarse equivalence.

Remark 6. An equivalent formulation of coarse embedding is the existence of non-decreasing, metrically proper functions $\rho_-, \rho_+ : [0, \infty) \to [0, \infty)$ such that

$$\rho_-\left(d\left(x, x'\right)\right) \leq d\left(f(x), f\left(x'\right)\right) \leq \rho_+\left(d\left(x, x'\right)\right)$$

for all $x, x' \in X$, with f being a coarse equivalence if it is also large-scale surjective. Quasi-isometric embeddings and quasi-isometries are the special cases where ρ_- and ρ_+ can be chosen to be affine linear functions.

A group action on a metric space X is *geometric* if it is proper, cocompact, and by isometries. Here *cocompact* means that there exists a compact $K \subseteq X$ such that $GK = X$, and *proper* (sometimes called *properly discontinuous*) means that, for any compact $K \subseteq X$, the set $\{g \in G \mid gK \cap K \neq \varnothing\}$ is finite. A useful application of the notion of coarse equivalence is the following variation on the classical Švarc–Milnor Lemma.

Proposition 4.5 (*Generalized Švarc–Milnor*)**.** *Suppose G acts geometrically on a connected proper metric space X. Then G is finitely generated and, when G is endowed with a corresponding word metric and $x_0 \in X$, the map $g \mapsto gx_0$ is a coarse equivalence.*

Proof. This is a special case of [7, Cor.0.9]. Our version is simpler since the finite generation of G (which is standard [6, Th.I.8.10]), allows the use of a word metric, which (also standard) is well-defined up to quasi-isometry. □

Remark 7. For length spaces (hence, for finitely generated groups) coarsely equivalent and quasi-isometric are equivalent notions [29, p. 19]. A nice aspect of Proposition 4.5 is that X need not be a length space.

The *order* of an open cover $\mathcal{U}$ of a space X is the largest integer k such that some $x \in X$ is contained in k members of $\mathcal{U}$. Classical *Lebesgue covering dimension* of X looks at orders of open covers with arbitrarily small mesh; at the other extreme, asymptotic dimension of X considers orders of uniformly bounded covers with arbitrarily large Lebesgue numbers. The following definition is far less rigid than either of these, requiring a single uniformly bounded open cover of a given index.

Definition 4.6. A space X has *finite macroscopic dimension* if it admits a uniformly bounded open cover of finite order. If the order of that cover is $n+1$, we write mdim $X \leq n$; if n is the minimum such integer, we say mdim $X = n$.

Definition 4.7. A space X is *uniformly contractible* if for each $R > 0$, there exists $S > R$ so that every ball $B(x, R)$ contracts in $B(x, S)$.

The most fundamental examples of the above two definitions occur in the context of geometric group actions.

Lemma 4.8. *If a group G acts geometrically on a contractible proper metric space X, then X is uniformly contractible and has finite macroscopic dimension.*

Proof. Let $x_0 \in X$, and choose $T > 0$ sufficiently large that $GB(x_0, T) = X$. Applying properness to $B[x_0, T]$ shows that $\{gB(x_0, T) \mid g \in G\}$ is a finite order open cover.

For uniform contractibility, let $R > 0$ and $x \in X$ be arbitrary. Since $B[x_0, T + R]$ is compact, the contraction of X restricts to a contraction of $B[x_0, T + R]$ in $B(x_0, S)$ for some $S > 0$. Choose $g \in G$ such that $d(x, gx_0) < T$. Then $B(x, R) \subseteq B(gx_0, R + T)$. Since the latter contracts in $B(gx_0, S)$, which is contained in $B(x, T + S)$, then $B(x, R)$ contracts in $B(x, T + S)$. $\square$

5. Continuous approximations and proper homotopy equivalences

A crucial ingredient in this paper is an ability to approximate certain functions with continuous ones. In this section, we develop the necessary tools. Theorem 1.1 and Corollary 1.2 will be almost immediate consequences.

Let $\mathcal{U}$ be a locally finite open cover of a space X. The *nerve* of $\mathcal{U}$ is the abstract simplicial complex $N(\mathcal{U})$ with vertex set $\mathcal{U}$ and a k-simplex $\{U_0, U_1, \ldots, U_k\}$ whenever $\cap_{i=0}^{k} U_i \neq \varnothing$. Clearly $N(\mathcal{U})$ is a locally finite complex. When discussing the geometric realization $|N(\mathcal{U})|$ we will denote the vertex corresponding to $U \in \mathcal{U}$ by v_U. There is a partition of unity $\{\lambda_U\}_{U \in \mathcal{U}}$ where, for each $U_0 \in \mathcal{U}$, $\lambda_{U_0} : X \to [0, 1]$ is defined by

$$\lambda_{U_0}(x) = d(x, X \setminus U_0) / \sum_{U \in \mathcal{U}} d(x, X \setminus U).$$

The local finiteness assumption ensures that the sums are finite and continuous. Use these functions to define the barycentric map $\beta : X \to |N(\mathcal{U})|$ by

$$f(x) = \sum_{U \in \mathcal{U}} \lambda_U(x) v_U.$$

In other words, x is taken to the point in the geometric realization of the simplex $\{U_0, U_1, \ldots, U_k\}$ of all open sets containing x with barycentric coordinates $\lambda_{U_i}(x)$.

Our primary use of the above construction is the following.

Lemma 5.1. *Suppose X admits a uniformly bounded open cover of order $n+1$. Then X contains a collection $\mathcal{A}^0, \mathcal{A}^1, \ldots, \mathcal{A}^n$ of locally discrete families of disjoint, uniformly bounded, closed sets which together cover X.*

Proof. By a theorem of general topology (see Lemma 41.6 in [28]) we may assume that $\mathcal{U}$ is locally finite. Let $\beta : X \to |N(\mathcal{U})|$ be the barycentric map. Let N' and N'' be the first and second derived subdivision of $|N(\mathcal{U})|$; then for each i, let $\mathcal{A}^i$ be the collection of preimages of closed star neighborhoods in N'' of the vertices of N' which are barycenters of the i-simplices of $|N(\mathcal{U})|$. □

Proposition 5.2. *Let $f : X \to Y$ be a large-scale uniform function, where X has finite macroscopic dimension and Y is a uniformly contractible ANR. Then there is a continuous function $g : X \to Y$ that is a bounded distance from f. If $E \subseteq X$ is a closed set on which f is already continuous, then g may be chosen so that $g|_E = f|_E$.*

Proof. Let $\mathcal{A}^0, \mathcal{A}^1, \ldots, \mathcal{A}^n$ be a finite set of locally discrete collections of K-bounded closed sets which together cover X, and let $\mathcal{A}$ denote the collection $\cup \mathcal{A}^i$. From each $A \in \mathcal{A}$, choose a point p_A, and note that the set $\{p_A\}$ is discrete. By adjoining these points to E we may assume, without loss of generality, that E intersects each $A \in \mathcal{A}$.

Define closed sets $C_{-1} \subseteq C_0 \subseteq C_1 \subseteq \cdots \subseteq C_n$ by $C_{-1} = E$ and $C_i = C_{i-1} \cup \left(\cup_{A \in \mathcal{A}^i} A\right)$ for each $i \geq 0$. We will construct $g : X \to Y$ inductively over the C_i.

Choose $L > 0$ such that $d\left(f(x), f\left(x'\right)\right) < L$ whenever $d\left(x, x'\right) < K$. Then let $g_{-1} \equiv f|_E : C_{-1} \to Y$, and assume inductively that there exists $L_i > 0$ and a continuous function $g_i : C_i \to Y$ which is $2L_i$-close to $f|_{C_i}$ and agrees with f on E.

To extend g_i to $g_{i+1} : C_{i+1} \to Y$, let $A \in \mathcal{A}^{i+1}$. Since $f(A) \subseteq B(f(p_A), L)$, then $g^i\left(C^i \cap A\right) \subseteq B(f(p_A), L + 2L_i)$. Choose $L_{i+1} > L + 2L_i$ so that each $B(y, L + 2L_i) \subseteq Y$ contracts in $B(y, L_{i+1})$. By Corollaries 2.4 and 2.6, there is a continuous extension of $g_i|_{A \cap C^i}$ to $g^A_{i+1} : A \to B(y, L_{i+1})$; this map is necessarily $2L_{i+1}$-close to $f|_A$. Take the union of g_i with the g^A_{i+1}, over all $A \in \mathcal{A}^{i+1}$, to obtain a continuous map $g_{i+1} : C_{i+1} \to Y$ which is $2L_{i+1}$-close to $f|_{C_{i+1}}$ and is identical to f on E. The Proposition follows by induction. □

Corollary 5.3. *Suppose $f, g : X \to Y$ are continuous, boundedly close, large-scale uniform maps, where X has finite macroscopic dimension and Y is a uniformly contractible ANR. Then f and g are boundedly homotopic.*

Proof. Apply Proposition 5.2 to the situation $J_0 = f$ and $J_t = g$ for $0 < t \leq 1$ and $E = X \times \{0, 1\}$ to get a continuous approximation $K : X \times [0, 1] \to Y$. □

Corollary 5.4. *Let $f' : X \to Y$ be a coarse equivalence between uniformly contractible proper metric ANRs, each having finite macroscopic dimension. Then*

(1) *f' is boundedly close to a continuous coarse equivalence $f : X \to Y$,*
(2) *f (and hence f') has a continuous coarse inverse $g : Y \to X$,*
(3) *gf and fg are boundedly (hence properly) homotopic to id_X and id_Y, so f and g are proper homotopy equivalences.*

The final observation of this section provides strong versions Theorem 1.1 and Corollary 1.2.

Theorem 5.5. *Suppose quasi-isometric groups G and H act geometrically on proper metric ANRs X and Y, respectively. Then X and Y are proper homotopy equivalent via continuous coarse equivalences.*

Proof. Use Proposition 4.5 to conclude that X and Y are coarsely equivalent to their respective groups and, thus, to each other; then apply Lemma 4.8 and the previous corollary. □

6. $\mathcal{Z}$-boundaries of groups

In [2], Bestvina introduced the notion of "$\mathcal{Z}$-boundary of a group", to provide a framework that includes Gromov boundaries of word hyperbolic groups and visual boundaries of CAT(0) groups, and can also be applied to other types of groups, as well. To avoid some technical issues, he restricted attention to groups that act properly, freely, and cocompactly (i.e., by covering transformations) on finite-dimensional ARs (i.e., ERs). In [11], Dranishnikov modified the definition to allow for non-free actions and arbitrary (proper metric) ARs; but the added flexibility came with a loss of generality in some key theorems. Some of the lost generality was restored in [27]; most of the rest is taken care of in this paper. So, with recent progress taken into account, Dranishnikov's version seems to be the "right" definition. It is:

Definition 6.1. A *$\mathcal{Z}$-structure* on a group G is a pair of spaces $(\overline{X}, Z)$ satisfying the following four conditions:

(1) $\overline{X}$ is a compact AR,
(2) Z is a $\mathcal{Z}$-set in $\overline{X}$,
(3) $X = \overline{X} - Z$ is a proper metric space on which G acts geometrically, and
(4) $\overline{X}$ satisfies the following *nullity condition* with respect to the G-action on X: for every compact $C \subseteq X$ and any open cover $\mathcal{U}$ of $\overline{X}$, all but finitely many G translates of C lie in an element of $\mathcal{U}$.

When this definition is satisfied, Z is called a *$\mathcal{Z}$-boundary* for G. If, in addition to the above, the G-action on X extends to $\overline{X}$, the result is called an *$E\mathcal{Z}$-structure* (equivariant $\mathcal{Z}$-structure), and Z is called an *$E\mathcal{Z}$-boundary* for G.

Examples.

(1) If G acts geometrically on a proper CAT(0) space X, then $\overline{X} = X \cup \partial_\infty X$, with the cone topology, gives an $E\mathcal{Z}$-structure for G.
(2) In [3] it is shown that if G is a hyperbolic group, $P_\rho(G)$ is an appropriately chosen Rips complex, and ∂G is the Gromov boundary, then $\overline{P}_\rho(G) = P_\rho(G) \cup \partial G$ (appropriately topologized) gives an $E\mathcal{Z}$-structure for G.
(3) Osajda and Przytycki [31] have shown that systolic groups admit $E\mathcal{Z}$-structures.
(4) Guilbault, Moran, and Tirel [21] have shown that Baumslag–Solitar groups admit $E\mathcal{Z}$-structures.

More general classes of groups have been addressed by Tirel [32] (free and direct products), Dahmani [10] (relatively hyperbolic groups), and Martin [26] (complexes of groups).

Remark 8. Bestvina's definition of $\mathcal{Z}$-structure did not require G to act by isometries on X, but only cocompactly by covering transformations. By the following proposition, deduced from [1], there is no loss of generality in requiring a geometric action.

Proposition 6.2. *Suppose G acts properly and cocompactly on a locally compact space X. Then there is an equivalent proper metric for X under which the action is by isometries.*

Due to our use of coarse equivalences and the Generalized Švarc–Milnor Theorem, the following proposition (when it applies) is stronger than necessary. We include it because it is interesting and not widely known.

Proposition 6.3. *Suppose G acts cocompactly by covering transformations on a connected, locally connected space X. Then there is an equivalent proper geodesic metric for X under which the action is by isometries.*

Proof. Let $p : X \to G \setminus X$ be the resulting covering projection. By [4], there is a geodesic metric d' on $G \setminus X$ which generates the desired quotient topology. Lift that metric to X by defining

$$\rho(x, y) = \inf \{\text{length}(p\alpha) \mid \alpha \text{ is a path in } X \text{ from } x \text{ to } y\}.$$

Since p is a local homeomorphism, ρ generates the original topology on X. By similar reasoning, (X, ρ) is complete and locally compact; so by the Hopf–Rinow Theorem [6, Ch.I.3], ρ is a proper geodesic metric. Clearly the G-action on (X, ρ) is by isometries. □

Without the benefit of covering space theory, we do not know the answer to the following:

Question. *If a group G acts geometrically on a proper metric AR (X, d) does there exist an equivalent geodesic metric ρ under which the action is by isometries?*

We close this section with an easy, but useful, lemma linking controlled $\mathcal{Z}$-compactifications and $\mathcal{Z}$-structures.

Lemma 6.4. *If a group G admits a $\mathcal{Z}$-structure $(\overline{X}, Z)$, then $\overline{X}$ is a controlled $\mathcal{Z}$-compactification of $X = \overline{X} - Z$. Conversely, if G acts geometrically on a proper metric AR (X, d) which admits a controlled $\mathcal{Z}$ -compactification $\overline{X} = X \cup Z$, then $(\overline{X}, Z)$ is a $\mathcal{Z}$-structure on G.*

Proof. The initial observation can be found in [20]. For the converse, choose any compact set $C \subset X$ and any open cover $\mathcal{U}$ of $\overline{X}$. Set $R = \operatorname{diam} C + 1$. From the control condition on $\overline{X}$, there is a compact $K \subseteq X$ so that each subset of $X - K$ of diameter less than R is contained in element of $\mathcal{U}$. Since the action is by isometries $\operatorname{diam}(gC) < R$ for all $g \in G$, and since the action is proper only finitely many translates of C intersect K; thus, all but finitely many lie in an element of $\mathcal{U}$. □

7. Boundary swapping

We now obtain proofs of Theorems 1.3 and 1.4. Both follow from a more general theorem that does not involve group actions. Our approach expands upon one suggested by Ferry [16].

Theorem 7.1. *Let $f : (X, d_X) \to (Y, d_Y)$ be a coarse equivalence between uniformly contractible proper metric spaces, each having finite macroscopic dimension, and suppose Y admits a controlled $\mathcal{Z}$-compactification $\overline{Y} = Y \cup Z$. Then X admits a controlled $\mathcal{Z}$-compactification $\overline{X} = X \cup Z$. If f is continuous, this may be done so that f extends to a continuous map $\overline{f} : \overline{X} \to \overline{Y}$ which is the identity on Z.*

Proof. If necessary, use Corollary 5.4 to replace f with a continuous coarse equivalence and choose a continuous coarse inverse $g : Y \to X$ and a bounded homotopy $J : X \times [0, 1] \to X$ with $J_0 = \mathrm{id}_X$ and $J_1 = gf$.

Extend f to a function $\overline{f} : X \sqcup Z \to \overline{Y}$ by letting $\overline{f}$ be the identity on Z. Then give $X \sqcup Z$ the topology generated by the open subsets of X and sets of the form $\overline{f}^{-1}(U)$ where $U \subset \overline{Y}$ is open, and let $\overline{X}$ denote the resulting topological space. Clearly, $\overline{f} : \overline{X} \to \overline{Y}$ is continuous and $\overline{X}$ is compact, Hausdorff, and second countable. It follows that $\overline{X}$ is metrizable and separable. The theorem will be proved by showing that $\overline{X}$ is a controlled $\mathcal{Z}$-compactification of X.

Before proceeding, we establish some notation. Let $\overline{g} : \overline{Y} \to \overline{X}$ and $\overline{J} : \overline{X} \times [0, 1] \to \overline{X}$ be the obvious extensions which are the identity on Z, and fix metrics $\overline{d}_X$ and $\overline{d}_Y$ to $\overline{X}$ and $\overline{Y}$, respectively (these are *not* extensions of d_X and d_Y). Whenever $\overline{U}$ denotes a subset of $\overline{X}$ [resp., $\overline{Y}$], U will denote $\overline{U} \cap X$ [resp., $\overline{U} \cap Y$]. Finally, select $K > 0$ so that J is a K-homotopy and fg is K-close to id_Y.

Claim 1. *$\overline{X}$ is a $\mathcal{Z}$-compactification of X.*

This claim follows from Lemma 3.1, provided that $\overline{g}$ and $\overline{J}$ are continuous (all other hypotheses are immediate).

To see that $\overline{g}$ is continuous at $z \in Z$, let $\overline{f}^{-1}\left(\overline{U}\right)$ be a basic open neighborhood of $\overline{g}(z) = z$ in $\overline{X}$. Then $z \in \overline{U}$ and by Lemma 3.6, there is a smaller open neighborhood

$\overline{V}$ of z in $\overline{Y}$ such that $d_Y(V, Y - U) > K$. If $y \in V$ then $d_Y(y, f(g(y))) \leq K$, so $f(g(y)) \in U \subseteq \overline{U}$; therefore $g(y) \in \overline{f}^{-1}(\overline{U})$. It follows that $\overline{g}(\overline{V}) \subseteq \overline{f}^{-1}(\overline{U})$, so $\overline{g}$ is continuous at z.

To prove continuity of $\overline{J}$, we will need an analog of Lemma 3.6 that can be applied to $\overline{X}$.

Subclaim. *Given $z \in Z$, a neighborhood $\overline{U}$ of z in $\overline{X}$, and $R > 0$, there is a neighborhood $\overline{V}$ of z so that $d(V, X - U) \geq R$.*

Proof of subclaim. Since f is a coarse map, choose $S > 0$ so that whenever $d_X(x, x') < R$ then $d_Y(f(x), f(x')) < S$. Without loss of generality, we may assume that $\overline{U} = \overline{f}^{-1}(\overline{W})$, where $\overline{W}$ is open in $\overline{Y}$. By Lemma 3.6, z has an open neighborhood $\overline{W'} \subset \overline{W}$ $d_Y(W', Y - W) > S$. Let $\overline{V} = \overline{f}^{-1}(\overline{W'})$ and note that if $x \in V$, $y \in X - U$, and $d(x, y) < R$; then $f(x) \in W'$, $f(y) \in Y - W$, and $d_Y(f(x), f(y)) \leq S$, a contradiction which proves the subclaim.

To see that $\overline{J}$ is continuous at $(z, t) \in Z \times [0, 1]$, let $\overline{U}$ be a neighborhood of $\overline{J}(z, t) = z$ in $\overline{X}$. Apply the subclaim to find a neighborhood $\overline{V}$ of z in $\overline{X}$ such that $d_X(V, X - U) > K$. For any $x \in V$, $J(x \times [0, 1])$ intersects V and has diameter $\leq K$, so $J(V \times [0, 1]) \subseteq U$. It follows that $\overline{J}(\overline{V} \times [0, 1]) \subseteq \overline{U}$, so $\overline{J}$ is continuous at (x, t) and Claim 1 is complete.

Claim 2. *$\overline{X}$ is a controlled $\mathcal{Z}$-compactification of (X, d_X).*

Fix $R > 0$ and let $\mathcal{U}$ be a cover of $\overline{X}$ by basic open subsets. In particular, the elements of $\mathcal{U}$ that intersect Z are of the form $\overline{f}^{-1}(\overline{W})$ where $\overline{W}$ is an open subset of $\overline{Y}$ intersecting Z. Let

$$\mathcal{W} = \{\overline{W} \subseteq \overline{Y} \mid \overline{f}^{-1}(\overline{W}) \in \mathcal{U} \text{ and } \overline{W} \cap Z \neq \varnothing\}.$$

Since $\mathcal{W}$ covers Z, $C \equiv \overline{Y} - \cup\mathcal{W}$ is a compact subset of Y. Let D be a compact subset of Y such that $C \subseteq \operatorname{int} D$. Then $\mathcal{W}^* = \mathcal{W} \cup \{\operatorname{int} D\}$ is an open cover of $\overline{Y}$. Choose $S > 0$ so that, if $A \subseteq X$ and $\operatorname{diam}_{d_X}(A) \leq R$, then $\operatorname{diam}_{d_Y}(f(A)) \leq S$. Then choose compact $E \subseteq Y$ such that $E \supseteq D$ and subsets of $Y - E$ of d_Y-diameter $< S$ lie in an element of $\mathcal{W}^*$. Then $f^{-1}(E) \subseteq X$ is compact and if $A \subseteq X - f^{-1}(E)$ with $\operatorname{diam}_{d_X}(A) < R$, then $f(A)$ lies in an element of $\mathcal{W}^*$ which is clearly not $\operatorname{int} D$. So $f(A) \subseteq \overline{W}$ for some $\overline{W} \in \mathcal{W}$, and therefore $A \subseteq \overline{f}^{-1}(\overline{W}) \in \mathcal{U}$. □

Corollary 7.2 (*Generalized $\mathcal{Z}$-Boundary Swapping Theorem*). *If quasi-isometric groups G and H act geometrically on proper metric ARs X and Y, resp., and Y can be compactified to a $\mathcal{Z}$-structure $(\overline{Y}, Z)$ for H, then X can be compactified by addition of the same boundary to a $\mathcal{Z}$-structure $(\overline{X}, Z)$ for G.*

Proof. By Lemma 4.8 both X and Y are uniformly contractible with finite macroscopic dimension, and by Proposition 4.5 they are coarsely equivalent. Since $\overline{Y}$ is a controlled $\mathcal{Z}$ -compactification of Y (Lemma 6.4), Theorem 7.1 provides a corresponding controlled $\mathcal{Z}$-compactification of X. Another application of Lemma 6.4 assures the desired $\mathcal{Z}$-structure on G. □

The non-equivariant version of standard boundary swapping is now immediate.

Corollary 7.3 (*$\mathcal{Z}$-Boundary Swapping Theorem*)**.** *If G acts geometrically on proper metric ARs X and Y, and Y can be compactified to a $\mathcal{Z}$-structure $(\overline{Y}, Z)$ for G, then X can be compactified by addition of the same boundary to another $\mathcal{Z}$-structure $(\overline{X}, Z)$ for G.*

The $E\mathcal{Z}$-version of Corollary 7.3 requires some additional work. We already have a $\mathcal{Z}$-compactification $\overline{X} = X \sqcup Z$; moreover, G acts on X and Z, individually (the latter by restricting the action on $\overline{Y}$). The idea is to combine these into a single G-action on $\overline{X}$. For each $\gamma \in G$, if $\gamma_X : X \to X$ and $\gamma_Z : Z \to Z$ are the corresponding homeomorphisms under the sub-actions, we need $\gamma_X \cup \gamma_Z : \overline{X} \to \overline{X}$ to be a homeomorphism. For that, it is enough to verify continuity at points of Z.

A variation on the continuity argument for $\overline{g}$, used in Theorem 7.1, proves:

Lemma 7.4. *Let $\overline{X} = X \sqcup Z$ be a controlled $\mathcal{Z}$-compactification of a proper metric space X and suppose $f, f' : X \to X$ are boundedly close continuous functions. If $\overline{f} : \overline{X} \to \overline{X}$ is a continuous extension of f that takes Z into Z, then $\overline{f'} = f' \cup \overline{f}\big|_Z$ is a continuous extension of f'.*

In Theorem 7.1, we began with a continuous coarse equivalence $f : X \to Y$ and a continuous a coarse inverse $g : Y \to X$. Those were extended to continuous functions $\overline{f} : \overline{X} \to \overline{Y}$ and $\overline{g} : \overline{Y} \to \overline{X}$, both of which are the identity on Z. If, in addition, the G-action on Y extends to $\overline{Y}$, then for every $\gamma \in G$, $\overline{g}\gamma\overline{f} : \overline{X} \to \overline{X}$ is a continuous extension of $g\gamma f$ which agrees with γ on Z. By Lemma 7.4, if $g\gamma f$ is boundedly close to γ on X, we can extend γ to $\overline{X}$ using $\gamma|_Z$. So $E\mathcal{Z}$-boundary swapping is completed by the following proposition.

Proposition 7.5. *Suppose proper metric ARs X and Y each admit geometric G-actions. Then there exists a continuous coarse equivalence $f : X \to Y$, a continuous coarse inverse $h : Y \to X$, and a constant $K > 0$ such that, for every $\gamma \in G$, $h\gamma f$ is K-close to γ.*

Proof. Fix points $x_0 \in X$ and $y_0 \in Y$. Let $\alpha : G \to X$ be defined as $\alpha(g) = gx_0$ and $\beta : G \to Y$ be given by $\beta(g) = gy_0$. Since G acts geometrically on X and Y, Proposition 4.5 guarantees, when G is endowed with a word metric, that α and β are coarse equivalences. Define a coarse inverse $\alpha' : X \to G$ as follows: for each $gx_0 \in Gx_0$, let $\alpha'(gx_0) = g'$ where $g' \in G$ is such that $gx_0 = g'x_0$. Having defined α' on the orbit of x_0, we may now define α' on all of X. Since G acts cocompactly on X, there exists an $R > 0$ such that $X = GB(x_0, R)$. Then, for $x \in X$, choose a $g \in G$ such that $x \in gB(x_0, R)$ and let $\alpha'(x) = \alpha'(gx_0)$. Define a coarse inverse $\beta' : Y \to G$ in a similar fashion.

Now, let $f = \beta\alpha'$ and $h = \alpha\beta'$. As these are compositions of coarse equivalences, both f and h are coarse equivalences. We may further assume they are continuous from Corollary 5.4. We will show these are the desired coarse equivalences.

Since G acts geometrically on X and Y, stabilizers of points have uniformly bounded diameters (e.g., [6, I.8.5]). Choose $M > 0$ such that $\text{diam}_G G_z < M$ for all $z \in X \cup Y$. Then choose $L > 0$ so that whenever $d_G(g_1, g_2) < M$, $d_X(\alpha(g_1), \alpha(g_2)) = d_X(g_1x_0, g_2x_0) < L$, and set $K = R + 2L$. We will show that $h\gamma f$ is K-close to γ for all $\gamma \in G$.

Let $x \in X$ and $\gamma \in G$. Choose $g \in G$ so that $x \in gB(x_0, R)$ and note that $d_X(\gamma x, \gamma g x_0) < R$. Now, consider, $h\gamma f(x) = \alpha\beta'\gamma\beta\alpha'(x)$. By definition, $\alpha'(x) = \alpha'(gx_0) = g_1$ for some $g_1 \in G$ with $g_1x_0 = gx_0$. Notice that since $g^{-1}g_1 \in G_{x_0}$, then $d_G(g, g_1) < M$ and thus $d_X(gx_0, g_1x_0) < L$. Now, $\beta'\gamma\beta\alpha'(x) = \beta'\gamma\beta(g_1) = \beta'\gamma(g_1y_0) = \beta'(\gamma g_1 y_0) = g_2$ where $g_2 \in G$ is such that $\gamma g_1 y_0 = g_2 y_0$. Since $g_2^{-1}g_1 \in G_{y_0}$, $d_G(\gamma g_1, g_2) < M$ and thus $d_X(\gamma g_1 x_0, g_2 x_0) < L$.

Thus, we have $h\gamma f(x) = \alpha(g_2) = g_2x_0$ and we observe:

$$\begin{aligned} &d_X(\gamma x, h\gamma f(x)) = d_X(\gamma x, g_2x_0) \\ &\le d_X(\gamma x, \gamma g x_0) + d_X(\gamma g x_0, \gamma g_1 x_0) + d_X(\gamma g_1 x_0, g_2 x_0) < R + L + L = K. \quad \square \end{aligned}$$

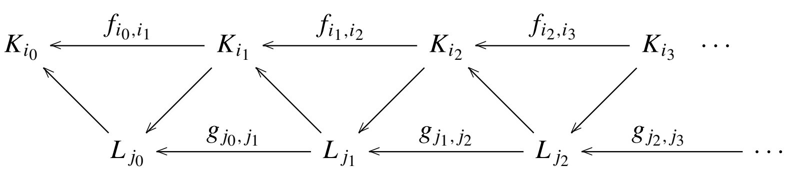
8. Shape equivalence of $\mathcal{Z}$-boundaries

We now prove Theorem 1.5 and Corollary 1.6. For readers familiar with shape theory, these are almost immediate consequences of the Boundary Swapping Theorems. That connection and the remaining technical details were worked out in [19, § 3.7]; but there the conclusions were less general since a full-blown boundary swapping theorem was not yet known. Here we complete the picture. We will begin with a brief review of shape theory and then outline the main argument. Additional details can be found in [19, § 3.7]. For comprehensive treatments of shape theory, see [5,13] or [25].

To define the *shape* of a compact metric space A, one first associates to A an inverse sequence of finite polyhedra and continuous maps

$$K_0 \xleftarrow{f_1} K_1 \xleftarrow{f_2} K_2 \xleftarrow{f_3} \cdots.$$

Such a sequence can be obtained in a variety of ways. For example: the K_i can be chosen to be nerves of progressively finer finite open covers of A; or, if A can be embedded in $\mathbb{R}^n$, the K_i can be progressively smaller polyhedral neighborhoods of A connected by inclusion maps. Clearly, these sequences are not uniquely determined by A. A pair of inverse sequences of finite polyhedra $\{K_i, f_i\}$ and $\{L_i, g_i\}$ are *pro-homotopy equivalent* if they contain subsequences that fit into a ladder diagram



in which each triangle of maps homotopy commutes. (Doubly subscripted maps are compositions.) It can be shown that any two inverse sequences associated with A are pro-homotopy equivalent. The pro-homotopy class of these sequences determines the *shape* of A, i.e., compact metric spaces A and A' are defined to be *shape equivalent* if their associated inverse sequences are pro-homotopy equivalent.

If A is a compact subset of an ANR X, one can always choose a nested sequence of compact neighborhoods

$$L_0 \hookleftarrow L_1 \hookleftarrow L_2 \hookleftarrow \cdots \tag{8.1}$$

with $\cap L_i = A$. If, in addition, the L_i themselves can be chosen to be ANRs (as is the case for the most commonly studied X), Theorem 2.11 allows us to use sequence (1.3) to represent the shape of A. From there, a proof of Theorem 1.1 is rather easy. For the general case, we will use just a bit of the theory of Hilbert cube manifolds (in particular Theorem 2.12) along with the boundary swapping techniques developed here, to replace X with a space for which that easy strategy can be applied.

Remark 9. For those who prefer to avoid Hilbert cube manifolds, a more general (but slightly more technical) approach to shape theory can be used to circumvent their need. That approach, developed in [25], allows the use of infinite CW complexes (or noncompact ANRs) in associated inverse sequences representing a compactum A. That means an allowable sequence of type (8.1) is immediately available—just choose the L_i to be *open* neighborhoods of A. (This approach was used by Ontaneda for similar purposes in [30].) Whichever approach one uses, boundary swapping helps to make the final conclusion straightforward, as the following argument shows.

Proof of Theorem 1.5. Suppose quasi-isometric groups G and H admit $\mathcal{Z}$-structures with boundaries Z and Z', respectively. Then, by Theorem 7.1, there exists a proper metric AR X and a pair of $\mathcal{Z}$-compactifications $\overline{X} = X \cup Z$ and $\overline{X}' = X \cup Z$. Assume for the moment that there is a sequence

$$\overline{N}_0 \hookleftarrow \overline{N}_1 \hookleftarrow \overline{N}_2 \hookleftarrow \cdots$$

of compact ANR neighborhoods of Z in $\overline{X}$ with $\cap \overline{N}_i = Z$. By Corollary 2.6 and Lemma 3.3, this is equivalent to the existence of a cofinal sequence

$$N_0 \hookleftarrow N_1 \hookleftarrow N_2 \hookleftarrow \cdots$$

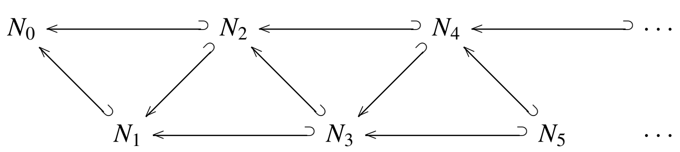
of closed ANR neighborhoods of infinity in X, where by Lemma 3.4, each $\overline{N}_i$ is a $\mathcal{Z}$-compactification of N_i. Since each $N_i \hookrightarrow \overline{N}_i$ is a homotopy equivalence, choose homotopy inverses $g_i : \overline{N}_i \to N_i$.

By the boundary swap, each N_i has an alternative $\mathcal{Z}$-compactification $\overline{N}'_i = N_i \cup Z' \subseteq \overline{X}'$, giving rise to a sequence

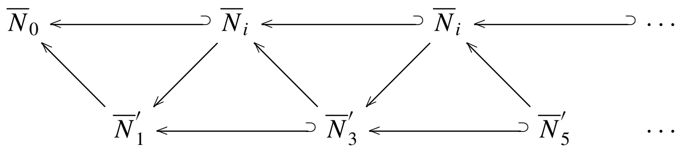
$$\overline{N}'_0 \hookleftarrow \overline{N}'_1 \hookleftarrow \overline{N}'_2 \hookleftarrow \cdots$$

of compact ANR neighborhoods of Z' with $\cap \overline{N}'_i = Z'$. For each i, let $g'_i : \overline{N}'_i \to N_i$ be a homotopy inverse for $N_i \hookrightarrow \overline{N}'_i$.

Begin with trivial ladder diagram



then compactify the top row by adding copies of Z and the bottom by adding copies of Z'. Use maps $\text{incl} \circ g'_i$ and $\text{incl} \circ g_i$ for up and down arrows to obtain a homotopy commuting diagram



which proves that Z and Z' are shape equivalent.

To complete the proof, we must address the situation where Z does not have arbitrarily small compact ANR neighborhoods in $\overline{X}$ (equivalently, X does not contain arbitrarily small closed ANR neighborhoods of infinity). In that case, we will replace X with the AR $X \times [0, 1]^\infty$. By Theorem 1.4 or Corollary 7.3, we can $\mathcal{Z}$-compactify $X \times [0, 1]^\infty$ with either Z or Z', and by Theorem 2.12, $X \times [0, 1]^\infty$ is a Hilbert cube manifold. By standard Hilbert cube manifold topology [8], $X \times [0, 1]^\infty$ contains arbitrarily small closed Hilbert cube manifold neighborhoods of infinity. Since these are ANRs, the general case follows from the earlier special case. □

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