

COARSE \mathcal{Z} -BOUNDARIES FOR GROUPS

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ABSTRACT. We generalize Bestvina’s notion of a \mathcal{Z} -boundary for a group to that of a “coarse \mathcal{Z} -boundary.” We show that established theorems about \mathcal{Z} -boundaries carry over nicely to the more general theory, and that some wished-for properties of \mathcal{Z} -boundaries become theorems when applied to coarse \mathcal{Z} -boundaries. Most notably, the property of admitting a coarse \mathcal{Z} -boundary is a pure quasi-isometry invariant. In the process, we streamline both new and existing definitions by introducing the notion of a “model \mathcal{Z} -geometry.” In accordance with the existing theory, we also develop an equivariant version of the above—that of a “coarse $E\mathcal{Z}$ -boundary.”

The primary goal of this paper is an expansion of Bestvina’s notion of a \mathcal{Z} -boundary. His approach placed Gromov boundaries of torsion-free hyperbolic groups and visual boundaries of torsion-free CAT(0) groups in a general framework which allows other classes of groups G to admit a boundary. Later Dranishnikov relaxed that framework to allow for groups with torsion. Here we relax the requirements further: instead of a geometric action of G on a proper metric AR (absolute retract) X , we allow for a “coarse near-action” (a concept that will be developed here and which contains quasi-actions as special cases). Our boundaries will be called $c\mathcal{Z}$ -boundaries or “coarse \mathcal{Z} -boundaries.” With the new definition in place, we are able to accomplish a primary goal:

Theorem 0.1. *If a group G admits a $c\mathcal{Z}$ -boundary and H is quasi-isometric to G , then H admits a $c\mathcal{Z}$ -boundary. In fact, G and H admit the same $c\mathcal{Z}$ -boundaries.*

Of course, this generalization is only useful if the information carried by \mathcal{Z} -boundaries is also carried by $c\mathcal{Z}$ -boundaries. In addition to proving new theorems, we revisit established results and show that their analogs remain valid in the broader context. Along the way, we introduce the notion of a *model geometry* and a *model \mathcal{Z} -geometry* as part of an effort to streamline the axiom system for \mathcal{Z} -boundaries—both coarse and classical. We also expand upon the important notion of an $E\mathcal{Z}$ -boundary.

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1. INTRODUCTION

Bestvina [Bes96] and Dranishnikov [Dra06] developed a general theory of group boundaries which contains Gromov boundaries of hyperbolic groups and visual boundaries of CAT(0) groups as special cases. These generalized boundaries—known as \mathcal{Z} -boundaries—involve a mix of geometry, topology, and group theory. The necessary ingredients are:

- i) a geometric action of the group G on some proper metric AR (X, d) ; and
- ii) a \mathcal{Z} -set compactification $\overline{X} = X \sqcup Z$ in which compacta from X vanish in \overline{X} as they are pushed toward Z by the G -action.

When this can be arranged, Z is called a \mathcal{Z} -boundary and (\overline{X}, Z, d) a \mathcal{Z} -structure for G . In [FL05], Farrell and Lafont introduced an additional condition (also satisfied by all hyperbolic and CAT(0) groups):

- iii) the G -action on X extends to a G -action on \overline{X}

When all three conditions are satisfied, they call (\overline{X}, Z, d) an $E\mathcal{Z}$ -structure (equivariant \mathcal{Z} -structure) and Z an $E\mathcal{Z}$ -boundary for G .

Many non-hyperbolic, non-CAT(0) groups, for example, Baumslag-Solitar groups and systolic groups, are now known to admit \mathcal{Z} - or $E\mathcal{Z}$ -structures. But a full characterization of which groups admit (E) \mathcal{Z} -structures is an interesting open problem. Section 6 of [GM19] contains a survey of known results; original sources include [BM91], [Dah03], [OP09], [Tir11], [Mar14], [Gui14], [GMT19], and more recently [EW18], [GMS20], [CCGO20], and [Pie18].

The following theorem from [GM19] has its origins in [Bes96].

Theorem 1.1 (Generalized Boundary Swapping). *If G is quasi-isometric to a group H which admits a \mathcal{Z} -boundary Z , and G acts geometrically on some proper metric AR X , then G also admits Z as a \mathcal{Z} -boundary.*

It is tempting to conclude that the property of admitting a \mathcal{Z} -boundary is a quasi-isometry invariant of groups. Unfortunately, that conclusion would be premature. Consider, for example:

Question A. *If $H \leq G$ is of finite index, and H admits a \mathcal{Z} -boundary, does G admit a \mathcal{Z} -boundary?*

The issue is this: Application of Theorem 1.1 requires an AR X on which G acts geometrically. One might hope that the H -action on an AR Y , implicit in the hypothesis, can be extended to a G -action. Those familiar with an analogous (open) problem for CAT(0) groups will recognize the difficulty. The more general problem of finding a geometric action of G on an AR X , given only that G is quasi-isometric to a group H that admits such an action, appears even more difficult.

In this paper we provide a way around these questions as they pertain to group boundaries. The key is a relaxation of the main definition to that of a *coarse \mathcal{Z} -structure*. Under the new definition, the requirement of a geometric G -action is

relaxed to allow for a (proper and cobounded) “coarse near-action.”¹ When G is quasi-isometric to a group H which acts geometrically on a proper metric AR Y , obtaining a coarse near-action of G on Y is relatively easy: Use our Theorem 2.12 (Generalized Švarc-Milnor) to obtain a coarse equivalence $f : G \rightarrow Y$. Then use f and a coarse inverse $g : Y \rightarrow G$ to conjugate the action of G on itself to Y . The converse portion of Theorem 2.12 assures that the result is a coarse near-action on Y . When this and the supporting machinery have been established, we will have:

Theorem 1.2. *If groups G and H are quasi-isometric and (\overline{Y}, Z, d) is a \mathcal{Z} -structure for H , then there is a coarse near-action of G on Y under which (\overline{Y}, Z, d) is a coarse \mathcal{Z} -structure for G .*

The new approach is useful only if a coarse \mathcal{Z} -boundary carries information, comparable to that of an actual \mathcal{Z} -boundary. For example, work by Bestvina, Mess, Geoghegan, Ontaneda, Dranishnikov, and Roe [BM91], [Bes96], [Geo86], [Ont05], [GO07], [Dra06], [Roe03] has established that, for \mathcal{Z} -structures on a group G :

- the boundary Z is well-defined up to shape equivalence;
- $\dim Z$ is a group invariant; and
- the Čech cohomology of Z reveals the group cohomology of G with RG -coefficients (R a PID).

We will show that these relationships carry over to the realm of coarse \mathcal{Z} -structures. In fact, the added flexibility allows for a *strengthening* of some of these conclusions.

Here is a quick outline of the remainder of this paper. In Section 2, we develop the notion of a *coarse near-action* of a group G on a metric space (X, d) and prove some fundamental facts—most significantly, a coarse version of the classical Švarc-Milnor Lemma, along with a crucial converse. In Section 3 we introduce the notions of a *model geometry* and a *model \mathcal{Z} -geometry*. Those are used in Section 4 to define a coarse \mathcal{Z} -structure. That definition includes classical \mathcal{Z} -structures as a special case. In addition, we establish an equivalent—strikingly simple—formulation of a coarse \mathcal{Z} -structure which highlights the benefits of our approach. In Sections 5 and 6 we show that key theorems about groups and their \mathcal{Z} -structures remain valid for coarse \mathcal{Z} -structures, and that new, stronger versions are often possible. In Section 7 we define a *coarse $E\mathcal{Z}$ -structure* and prove a few basic theorems. In a final section, we discuss the fundamental questions (such as Question A above) which motivated this work. These questions go well beyond \mathcal{Z} -boundaries; some are well-known to the experts. Our goal is to shine a light on a family of interesting open problems and to highlight their connections to the study of group boundaries.

¹Coarse near-action is a straightforward generalization of *quasi-action*. The extra generality is useful when metric spaces are not necessarily geodesic (or quasi-geodesic). For those who prefer quasi-actions, nearly all of our definitions and theorems have analogs in that category at the expense of some mild additional hypotheses. See Section 2 for details.

2. COARSE NEAR-ACTIONS

Definition 2.1. Let $C \geq 0$. A C -near-action of a group G on a metric space (X, d) is a function ψ which assigns to each $\gamma \in G$ a function $\psi(\gamma) : X \rightarrow X$ satisfying the following conditions:

- (1) $d(\psi(1)(x), x) \leq C$, for all $x \in X$, and
- (2) $d(\psi(\gamma_1\gamma_2)(x), \psi(\gamma_1)\psi(\gamma_2)(x)) \leq C$, for all $x \in X$ and $\gamma_1, \gamma_2 \in G$.

When the constant C is unimportant, we simply refer to ψ as a *near-action*.

Definition 2.2. A function $f : (X, d) \rightarrow (Y, d')$ is

- (1) a *coarse embedding* if there exist nondecreasing functions $\rho_-, \rho_+ : [0, \infty) \rightarrow [0, \infty)$ with $\rho_-(r) \rightarrow \infty$ as $r \rightarrow \infty$ such that for all $x_1, x_2 \in X$,

$$\rho_-(d(x_1, x_2)) \leq d'(f(x_1), f(x_2)) \leq \rho_+(d(x_1, x_2))$$

- (2) *coarsely surjective* if there exists $C \geq 0$ such that, for all $y \in Y$, there exists $x \in X$ such that $d'(y, f(x)) \leq C$,
- (3) a *coarse equivalence* if it is a coarsely surjective coarse embedding.

If ρ_-, ρ_+ and C are specified, we sometimes call a function f which satisfies: (1) a (ρ_-, ρ_+) -coarse embedding; (2) C -surjective; and (3) a (ρ_-, ρ_+, C) -coarse equivalence.

Remark 2.3. A coarse embedding $f : (X, d) \rightarrow (Y, d')$ is a coarse equivalence if and only if f has a coarse inverse, that is, there is a coarse embedding $g : (Y, d') \rightarrow (X, d)$ such that gf and fg are boundedly close to id_X and id_Y , respectively.

Definition 2.4. Let $\rho_-, \rho_+ : [0, \infty) \rightarrow [0, \infty)$ be fixed nondecreasing functions with $\rho_-(r) \rightarrow \infty$ as $r \rightarrow \infty$ and let $C \geq 0$ be a fixed constant. A (ρ_-, ρ_+, C) -coarse near-action is a C -near-action ψ of a group G on a metric space (X, d) with the property that $\psi(\gamma)$ is a (ρ_-, ρ_+) -coarse embedding for all $\gamma \in G$.

When the specific control functions and constant are unimportant, we simply refer to ψ as a *coarse near-action*. Even when not specified, it is important that a single choice of (ρ_-, ρ_+) applies uniformly to all $\psi(\gamma)$.

Remark 2.5. Note that properties of a C -near-action ensure that each $\psi(\gamma)$ is $2C$ -surjective and $\psi(\gamma^{-1})$ is a coarse inverse for $\psi(\gamma)$. So each $\psi(\gamma)$ is a coarse equivalence.

The following generalizations of cocompact and proper actions are useful.

Definition 2.6. Let ψ be a coarse near-action of G on (X, d) .

- (1) ψ is *cobounded* if there exists $x_0 \in X$ and $R > 0$ such that for all $y \in X$, there exists $\gamma \in G$ such that $d(\psi(\gamma)(x_0), y) < R$.
- (2) ψ is *proper* if for each $R > 0$, there exists $M \in \mathbb{N}$ such that for all $x, y \in X$,

$$|\{\gamma \in G \mid \psi(\gamma)(B_d[x, R]) \cap B_d[y, R] \neq \emptyset\}| \leq M$$

where $B_d[x, R]$ is the closed metric ball.

Remark 2.7. As an exercise, one can show that Condition (1) in Definition 2.6 is equivalent to the statement that, for each $x \in X$ there exists an $R' > 0$ such that for every $y \in X$, there exists $\gamma \in G$ such that $d(\psi(\gamma)(x), y) < R'$. With some additional effort, one can identify a single $R' > 0$ which works for all x .

A geometric action (i.e., proper, cocompact, and by isometries) on a proper metric space is a proper, cobounded, coarse near-action (with $\rho_- = \rho_+ = \text{id}_{[0, \infty)}$ and $C = 0$), so previously studied \mathcal{Z} -structures fall within the new framework. Similarly, Definitions 2.2 and 2.4 generalize the notion of *quasi-isometry* and *quasi-action*, where control functions are required to be of the form $\frac{1}{K}r - \varepsilon$ and $Kr + \varepsilon$. (To reconcile the definitions, $\frac{1}{K}r - \varepsilon$ can be truncated below at 0.) Quasi-actions have been widely studied; see for example, [Tuk86], [Tuk94], [Nek97], [KL01], [MSW03], [Man06], [KL09], [MSW11], [Eis15], and [DK18]. In addition, uniform actions by coarse equivalences (called *coarse actions* in [BDM08]) have been studied. To the best of our knowledge, the notion of a coarse near-action has not appeared before.

Remark 2.8. It is a standard fact (see, for example, [NY12, Cor.1.4.14]) that a coarse equivalence between quasi-geodesic spaces is always a quasi-isometry. By the same reasoning, it can be shown that every coarse action on a quasi-geodesic space is a quasi-action. Since we do not require our geometric models to be quasi-geodesic spaces, it is more natural for us to work with coarse actions. The decision to work in this generality is both historical (see earlier papers on \mathcal{Z} -structures) and practical. Aside from the following minimal condition, which is immediate for path connected spaces, additional requirements on X would not lead to stronger conclusions; moreover the additional effort required to maintain our level of generality is minimal.

Definition 2.9. A metric space (X, d) is *coarsely connected* if there exists $R > 0$ such that, for all $x, x' \in X$, there is a finite sequence $x = x_0, x_1, \dots, x_n = x'$ of points (an R -chain) in X with $d(x_i, x_{i+1}) \leq R$ for all $0 \leq i \leq n - 1$.

The following proposition can be viewed as a generalization of the classical Švarc-Milnor Lemma as well as [BDM07, Cor.0.9] and the reverse implication of [Nek97, Th.8.4]. As usual, a finitely generated group is given the word metric with respect to some finite generating set.

Proposition 2.10. *Suppose a coarsely connected metric space (X, d) admits a proper, cobounded coarse near-action ψ by a group G . Then G is finitely generated, and for any $x_0 \in X$, $\gamma \mapsto \psi(\gamma)(x_0)$ is a coarse equivalence between G and X .*

Proof. Let ψ be a (ρ_-, ρ_+, C) -coarse near-action on (X, d) and $x_0 \in X$. For use later in this proof, note that we may replace ρ_- by an even smaller function (still called ρ_-), which is identically 0 on an interval $[0, B]$, and strictly increasing to infinity on $[B, \infty)$ for $B > 0$.

Choose a single constant $R > 0$ satisfying Definition 2.9 and such that $\{\psi(\gamma)(x_0) \mid \gamma \in G\}$ is R -dense in X . Let

$$S = \{\gamma \in G \mid d(\psi(\gamma)(x_0), x_0) \leq \rho_+(3R) + 3C\}$$

By properness, S is finite. Let $H \leq G$ be the subgroup generated by S . We claim that $H = G$.

Suppose otherwise. Let

$$A = \bigcup_{g \in H} B_d[\psi(g)(x_0), R], \text{ and}$$

$$B = \bigcup_{g \in G \setminus H} B_d[\psi(g)(x_0), R]$$

Note that $A \cup B = X$, and neither set is empty. So there exists an R -chain connecting a point of A to a point of B , and within that chain there exist points $a \in A$ and $b \in B$ with $d(a, b) \leq R$. Choose $g_1 \in H$ and $g_2 \in G \setminus H$ such that $d(a, \psi(g_1)(x_0)) \leq R$ and $d(b, \psi(g_2)(x_0)) \leq R$. Then

- $d(x_0, \psi(g_1^{-1})\psi(g_1)(x_0)) \leq 2C$,
- $d(\psi(g_1^{-1})\psi(g_1)(x_0), \psi(g_1^{-1})\psi(g_2)(x_0)) \leq \rho_+(3R)$, and
- $d(\psi(g_1^{-1})\psi(g_2)(x_0), \psi(g_1^{-1}g_2)(x_0)) \leq C$.

By the triangle inequality, $d(x_0, \psi(g_1^{-1}g_2)(x_0)) \leq \rho_+(3R) + 3C$, so $g_1^{-1}g_2 \in S$. But then $g_2 = g_1(g_1^{-1}g_2) \in H$, a contradiction. The claim follows.

Next define $\sigma_-, \sigma_+ : [0, \infty) \rightarrow [0, \infty)$ by

$$\sigma_-(r) = \inf \{d(x_0, \psi(\gamma)(x_0)) \mid d(1, \gamma) \geq r\}, \text{ and}$$

$$\sigma_+(r) = \sup \{d(x_0, \psi(\gamma)(x_0)) \mid d(1, \gamma) \leq r\}$$

Notice that both σ_- and σ_+ are nondecreasing, $\sigma_-(r) \leq \sigma_+(r)$ for all r , and (by properness), $\sigma_-(r) \rightarrow \infty$ as $r \rightarrow \infty$.

Let $\gamma_1, \gamma_2 \in G$. Then

$$\begin{aligned} \rho_-(d(\psi(\gamma_1)(x_0), \psi(\gamma_2)(x_0))) &\leq d(\psi(\gamma_1^{-1})\psi(\gamma_1)(x_0), \psi(\gamma_1^{-1})\psi(\gamma_2)(x_0)) \\ &\leq d(\psi(\gamma_1^{-1})\psi(\gamma_1)(x_0), x_0) + d(x_0, \psi(\gamma_1^{-1})\psi(\gamma_2)(x_0)) \\ &\leq 2C + [C + d(x_0, \psi(\gamma_1^{-1}\gamma_2)(x_0))] \\ &\leq 3C + \sigma_+(d(\gamma_1, \gamma_2)) \end{aligned}$$

Therefore

$$(2.1) \quad d(\psi(\gamma_1)(x_0), \psi(\gamma_2)(x_0)) \leq \rho_-^{-1}(3C + \sigma_+(d(\gamma_1, \gamma_2)))$$

where $\rho_-^{-1} : [0, \infty) \rightarrow [B, \infty)$ is defined to be the inverse of $\rho_-|_{[B, \infty)}$. Since ρ_-^{-1} strictly increases to infinity, we may define $\tau_+ : [0, \infty) \rightarrow [0, \infty)$ by

$$\tau_+(r) = \rho_-^{-1}(3C + \sigma_+(r))$$

to obtain an upper control function satisfying the inequality

$$d(\psi(\gamma_1)(x_0), \psi(\gamma_2)(x_0)) \leq \tau_+(d(\gamma_1, \gamma_2))$$

As for the lower control function, note that

$$\begin{aligned}
\rho_- \sigma_-(d(\gamma_1, \gamma_2)) &= \rho_- \sigma_-(d(1, \gamma_1^{-1} \gamma_2)) \\
&\leq \rho_-(d(x_0, \psi(\gamma_1^{-1} \gamma_2)(x_0))) \\
&\leq d(\psi(\gamma_1)(x_0), \psi(\gamma_1) \psi(\gamma_1^{-1} \gamma_2)(x_0)) \\
&\leq d(\psi(\gamma_1)(x_0), \psi(\gamma_2)(x_0)) + d(\psi(\gamma_2)(x_0), \psi(\gamma_1) \psi(\gamma_1^{-1} \gamma_2)(x_0)) \\
&\leq d(\psi(\gamma_1)(x_0), \psi(\gamma_2)(x_0)) + C
\end{aligned}$$

Therefore

$$(2.2) \quad \rho_- \sigma_-(d(\gamma_1, \gamma_2)) - C \leq d(\psi(\gamma_1)(x_0), \psi(\gamma_2)(x_0))$$

Define $\tau_- : [0, \infty) \rightarrow [0, \infty)$ by

$$\tau_-(r) = \begin{cases} 0 & \text{if } \rho_- \sigma_-(r) - C < 0 \\ \rho_- \sigma_-(r) - C & \text{otherwise} \end{cases}$$

Since the right-hand side of (2.2) is ≥ 0 , the inequality

$$\tau_-(d(\gamma_1, \gamma_2)) \leq d(\psi(\gamma_1)(x_0), \psi(\gamma_2)(x_0))$$

holds. \square

The next proposition generalizes [Eis15, Appendix A] and the forward implication of [Nek97, Th.8.4].

Proposition 2.11. *If a metric space (X, d) is coarsely equivalent to a countable² group G then X admits a proper cobounded coarse near-action by G .*

Proof. Let $f_1 : G \rightarrow X$ be a (ρ_-, ρ_+, C) -coarse equivalence, where, as in the proof of Proposition 2.10, ρ_- is identically 0 on some interval $[0, B]$ and strictly increasing on $[B, \infty]$; so ρ_-^{-1} is defined on $[B, \infty)$. Let $f_2 : X \rightarrow G$ be a coarse inverse of f_1 . Then f_2 is a (σ_-, σ_+) -coarse embedding with the property that $d(x, f_1 f_2(x)) \leq C$ for all $x \in X$ and $d(\gamma, f_2 f_1(\gamma)) \leq C$ for all $\gamma \in G$. We will show that $\psi(\gamma) : X \rightarrow X$, defined by $\psi(\gamma) = f_1 \circ \gamma \circ f_2$ (where γ simultaneously represents an element of G and the isometry of G defined by left multiplication by γ) determines a coarse near-action of G on X .

First note that, for each $\gamma \in G$, $f_1 \circ \gamma \circ f_2$ is a $(\rho_- \circ \sigma_-, \rho_+ \circ \sigma_+)$ -coarse embedding. Thus, to show that ψ is a coarse near-action, we only need to check the conditions of Definition 2.4.

We observe first that $\psi(1) = f_1 f_2$, which is of bounded distance $\leq C$ from id_X . Next let $\gamma_1, \gamma_2 \in G$ and $x \in X$. By the near inversive properties of f_1 and f_2 ,

$$d(f_2 f_1(\gamma_2 f_2(x)), \gamma_2 f_2(x)) \leq C$$

Left-multiplication by γ_1 yields

$$d(\gamma_1 f_2 f_1(\gamma_2 f_2(x)), \gamma_1 \gamma_2 f_2(x)) \leq C$$

²Countability is needed to endow G with a proper word length metric, well-defined up to coarse equivalence. See [NY12, §1.2].

Plugging both terms into f_1 gives

$$d(f_1\gamma_1 f_2 f_1(\gamma_2 f_2(x)), f_1\gamma_1\gamma_2 f_2(x)) \leq \rho_+(C)$$

In other words,

$$d(\psi(\gamma_1)\psi(\gamma_2)(x), \psi(\gamma_1\gamma_2)(x)) \leq \rho_+(C)$$

Letting $C' = \max\{C, \rho_+(C)\}$, we have shown that ψ is a $(\rho_- \circ \sigma_-, \rho_+ \circ \sigma_+, C')$ -coarse near-action of G on X .

Now, we show that this coarse near-action ψ of G on X is proper and cobounded. First, observe that for each $\gamma \in G$,

$$\begin{aligned} d(f_1(\gamma), \psi(\gamma)(f_1(1))) &= d(f_1(\gamma), f_1\gamma f_2(f_1(1))) \\ &\leq \rho_+(d(\gamma, \gamma f_2(f_1(1)))) \\ &= \rho_+(d(1, f_2(f_1(1)))) \\ &\leq \rho_+(C) \\ &\leq C' \end{aligned}$$

Set $x_0 = f_1(1)$ and $R = C' + C$. Let $y \in X$. Since f_1 is a coarse equivalence, choose $\gamma \in G$ such that $d(f_1(\gamma), y) \leq C$. Then,

$$\begin{aligned} d(\psi(\gamma)(x_0), y) &= d(\psi(\gamma)(f_1(1)), y) \\ &\leq d(\psi(\gamma)(f_1(1)), f_1(\gamma)) + d(f_1(\gamma), y) \\ &\leq C' + C \end{aligned}$$

proving that ψ is cobounded.

To show that ψ is proper, let $R > 0$ and $x, y \in X$. We wish to show that

$$\{\gamma \in G \mid \psi(\gamma)(B[x, R]) \cap B[y, R] \neq \emptyset\} \subset \{\gamma \in G \mid \gamma(B[\gamma_1, M]) \cap B[\gamma_2, M] \neq \emptyset\}$$

for some $\gamma_1, \gamma_2 \in G$ and $M = \rho_-^{-1}(2C' + R + \rho_+\sigma_+(C'))$. Since the latter set is finite (G acts properly on itself), we can conclude that ψ is proper.

Thus, choose a $\gamma \in G$ so that $\psi(\gamma)(B[x, R]) \cap B[y, R] \neq \emptyset$. Let $z \in B[x, R]$ so that $\psi(\gamma)(z) \in B[y, R]$. Since f_1 is a coarse equivalence, there is $\gamma_1, \gamma_2, \gamma_3 \in G$ so that $d(f_1(\gamma_1), x) \leq C'$, $d(f_1(\gamma_2), y) \leq C'$ and $d(f_1(\gamma_3), z) \leq C'$. From this we observe that

$$d(\psi(\gamma)(z), \psi(\gamma)(f_1(\gamma_3))) \leq \rho_+\sigma_+(C')$$

Moreover, we have:

$$\begin{aligned} d(f_1(\gamma_2), f_1(\gamma_3)) &\leq d(f_1(\gamma_2), y) + d(y, \psi(\gamma)(z)) + d(\psi(\gamma)(z), \psi(\gamma)(f_1(\gamma_3))) + d(\psi(\gamma)(f_1(\gamma_3)), f_1(\gamma_3)) \\ &\leq 2C' + R + \rho_+\sigma_+(C') \end{aligned}$$

Since f_1 is a coarse-equivalence, using the previous inequality, we have

$$\rho_-(d(\gamma_2, \gamma_3)) \leq 2C' + R + \rho_+\sigma_+(C')$$

Hence $d(\gamma_2, \gamma\gamma_3) \leq \rho^{-1}(2C' + R + \rho_+\sigma_+(C')) \leq M$, proving $\gamma\gamma_3 \in B[\gamma_2, M]$. Furthermore, we know

$$\begin{aligned} d(f_1(\gamma_1), f_1(\gamma_3)) &\leq d(f_1(\gamma_1), x) + d(x, z) + d(z, f_1(\gamma_3)) \\ &\leq 2C' + R \end{aligned}$$

Once again, using the previous inequality we see that $d(\gamma_1, \gamma_3) \leq \rho^{-1}(2C' + R) \leq M$ proving that $\gamma\gamma_3 \in \gamma(B[\gamma_1, M])$. Thus, we have $\gamma\gamma_3 \in \{\gamma \in G \mid \gamma(B[\gamma_1, M]) \cap B[\gamma_2, M] \neq \emptyset\}$. \square

Combining the previous two propositions and restricting to the cases of primary interest (for our purposes) gives us the following.

Theorem 2.12 (Generalized Švarc-Milnor with converse). *Let (X, d) be a coarsely connected metric space and G a finitely generated group. Then X admits a proper, cobounded coarse near-action by G if and only if X is coarsely equivalent to G .*

Corollary 2.13. *A quasi-geodesic metric space (X, d) admits a proper, cobounded quasi-action by G if and only if G is finitely generated and quasi-isometric to X .*

Proof. This is [Nek97, Th.8.4]. Alternatively, use Theorem 2.12 and Remark 2.8. \square

3. MODEL GEOMETRIES AND MODEL \mathcal{Z} -GEOMETRIES

Throughout this paper, all spaces are assumed separable and metrizable. A metric space (X, d) is *proper* if every closed metric ball $B_d[x, r] \subseteq X$ is compact. It is *cocompact* if there exist $x_0 \in X$ and $R > 0$ so that $\{B_d[\gamma x_0, R] \mid \gamma \in \text{Isom}(X)\}$ covers X . Here $\text{Isom}(X)$ denotes the group of self-isometries of X . For later use, we review a few well-known properties that follow from properness and/or cocompactness.

A metric space (X, d) is *uniformly contractible* if for each $R > 0$, there exists $S > R$ so that every open metric ball $B_d(x, R)$ contracts in $B_d(x, S)$.

Lemma 3.1. *If a proper metric space (X, d) is cocompact and contractible, then it is uniformly contractible.*

Proof. See, for example, [GM19, Lemma 4.8]. \square

An open cover \mathcal{U} of a metric space (X, d) is *uniformly bounded* if $\{\text{diam}(U) \mid U \in \mathcal{U}\}$ is bounded above; in that case the supremum of this set is the *mesh* of \mathcal{U} . The *order* of \mathcal{U} is the largest $k \in \mathbb{N} \cup \{\infty\}$ such that some $x \in X$ is contained in k elements of \mathcal{U} . We say that (X, d) has *macroscopic dimension* $\leq n$ if there exists a uniformly bounded open cover of X having order $\leq n + 1$. A cover \mathcal{V} *refines* \mathcal{U} if, for every $V \in \mathcal{V}$ there exists some $U \in \mathcal{U}$ such that $V \subseteq U$; in that case we write $\mathcal{U} \succ \mathcal{V}$.

Lemma 3.2. *Let (X, d) be proper and cocompact. Then X has finite macroscopic dimension; in fact, X admits a sequence of finite order, uniformly bounded, open covers $\mathcal{U}_0 \succ \mathcal{U}_1 \succ \mathcal{U}_2 \succ \dots$ such that $\text{mesh}(\mathcal{U}_i) \rightarrow 0$.*

Proof. The argument provided in [Mor16, Lemma 3.1] proves the existence of some $x_0 \in X$, $r > 0$, and $\{\lambda_i\}_{i \in \mathbb{N}} \subseteq \text{Isom}(X)$ such that $\mathcal{U}_0 = \{B_d(\lambda_i(x_0), r)\}$ is a finite order cover of X . The following method produces a refinement \mathcal{V}_ε of \mathcal{U}_0 of mesh $\leq \varepsilon$ for arbitrary $\varepsilon > 0$: Use properness to choose a finite cover \mathcal{W}'_ε of $B_d[x_0, r]$ by open ε -balls, and let \mathcal{W}_ε be the collection of intersections of the elements of \mathcal{W}'_ε with $B_d(x_0, r)$. Then let

$$\mathcal{V}_\varepsilon = \{\lambda_i(W) \mid W \in \mathcal{W}_\varepsilon \text{ and } i \in \mathbb{N}\}$$

Since \mathcal{W}_ε is finite and \mathcal{U}_0 has finite order, \mathcal{V}_ε has finite order.

To produce the sequence of open covers promised in the lemma, we apply the above procedure inductively. Let ε_1 be arbitrary and $\mathcal{U}_1 = \mathcal{V}_{\varepsilon_1}$. Next let $\varepsilon_2 > 0$ be a Lebesgue number for the cover $\mathcal{W}'_{\varepsilon_1}$ to obtain $\mathcal{U}_2 = \mathcal{V}_{\varepsilon_2}$ which refines \mathcal{U}_1 and has mesh $\leq \varepsilon_2$. Continue inductively, making sure that the chosen Lebesgue numbers converge to 0. \square

Lemma 3.3. *Suppose (X, d) is uniformly contractible; \mathcal{U} is a uniformly bounded, finite order, open cover of X ; K is the nerve of \mathcal{U} ; and $\psi : X \rightarrow K$ is a corresponding barycentric map. Then there is a map $s : K \rightarrow X$ and a bounded homotopy $H : X \times I \rightarrow X$ joining $s \circ \psi$ with the identity id_X .*

Proof. Since K is finite-dimensional and X is uniformly contractible, it is straightforward to build a map $s : K \rightarrow X$ inductively over the skeleta of K such that $s \circ \psi$ is bounded distance from id_X . (This is an easier version of [GM19, Prop.5.2].) From there one can apply [GM19, Cor.5.3] to obtain the desired homotopy. \square

A locally compact space X is an *absolute neighborhood retract* (ANR for short) if, whenever X is embedded as a closed subset of another space Y , some neighborhood of X retracts onto X . A contractible ANR is called an *absolute retract* or simply an *AR*. The category of ANRs provides a common ground of “nice” spaces which includes manifolds, locally finite complexes, and proper CAT(0) spaces—the spaces most commonly encountered in geometric group theory. For a quick introduction to ANRs, see [GM19, §2]. It is worth noting that some authors do not require ANRs to be locally compact (or separable and metrizable). For our purposes, we consider those conditions to be part of the definition. We use the term *metric AR* (or *metric ANR*) when a specific metric plays a role.

Definition 3.4. A *model geometry* is a proper, cocompact, metric AR (X, d) .

Given a model geometry, we often seek a nice compactification. A closed subset A of a space Y , is a \mathcal{Z} -set if there exists a homotopy $H : Y \times [0, 1] \rightarrow Y$ such that $H_0 = \text{id}_Y$ and $H_t(Y) \subset Y - A$ for every $t > 0$. In this case we say H *instantly homotopes Y off from A* . A \mathcal{Z} -compactification of a space X is a compactification $\overline{X} = X \sqcup Z$ such that Z is a \mathcal{Z} -set in \overline{X} . If X is separable and metrizable then so is \overline{X} ; and if X is an AR then so is \overline{X} . For these and other facts about \mathcal{Z} -compactifications, see [GM19, §3].

A *controlled \mathcal{Z} -compactification* of a proper metric space (X, d) is a \mathcal{Z} -compactification \overline{X} satisfying the additional condition:

(‡) For every $R > 0$ and open cover \mathcal{U} of \overline{X} , there is a compact set $C \subset X$ so that if $A \subseteq X - C$ and $\text{diam}_d A < R$, then $A \subseteq U$ for some $U \in \mathcal{U}$.

If we choose a metric \overline{d} for \overline{X} , condition (‡) is equivalent to:

(‡') For every $R > 0$ and $\epsilon > 0$, there is a compact set $C \subset X$ so that if $A \subseteq X - C$ and $\text{diam}_d A < R$, then $\text{diam}_{\overline{d}} A < \epsilon$.

Definition 3.5. A *model \mathcal{Z} -geometry* (\overline{X}, Z, d) is a controlled \mathcal{Z} -compactification $\overline{X} = X \sqcup Z$ of a model geometry (X, d) . In this case we refer to (X, d) as the *underlying geometry* and the space Z as the *\mathcal{Z} -boundary* of (\overline{X}, Z, d) .

It is important to note that the space $\overline{X} = X \sqcup Z$ does not, by itself, determine the model \mathcal{Z} -geometry; the metric d is a crucial ingredient.³ It is also worth noting that a given model geometry (X, d) can admit any number of distinct model \mathcal{Z} -geometries (possibly none at all). It is useful to think about the following simple examples.

Example 3.6. The Euclidean plane (\mathbb{R}^2, d_E) and hyperbolic plane (\mathbb{H}^2, d_H) are model geometries. Adding the standard visual circles at infinity gives model \mathcal{Z} -geometries $(\overline{\mathbb{R}^2}, S^1, d_E)$ and $(\overline{\mathbb{H}^2}, S^1, d_H)$. In each case, all isometries of the original space extend to homeomorphisms of the compactification. By contrast, if we quotient out the upper half-circle in either of these boundaries, we get a new model \mathcal{Z} -geometry for which the boundary is still a circle, but now many isometries do not extend. We will return to the issue of extendability in Section 7.

Example 3.7. As above, we can obtain model \mathcal{Z} -geometries $(\overline{\mathbb{R}^n}, S^{n-1}, d_E)$ and $(\overline{\mathbb{H}^n}, S^{n-1}, d_H)$ by adding the visual $(n - 1)$ -sphere at infinity to Euclidean and hyperbolic n -space. If $A \subseteq S^{n-1}$ is a non-cellular cell-like set (such as a Fox-Artin arc or the Whitehead continuum in S^3), then quotienting out by A produces model \mathcal{Z} -geometries with boundaries not homeomorphic to S^{n-1} .

We require cocompactness in our model geometries but we do not assume they admit a geometric group action—or even proper, cobounded coarse near action. Heintze [Hei74] has observed the existence of homogeneous negatively curved Riemannian manifolds which, by virtue of not being symmetric spaces, do not admit quotients of finite volume. Cornulier [Cor18] shows that these spaces are not even quasi-isometric to a finitely generated group, and hence admit no proper, cobounded coarse near action. More recently, Healy and Pengitore [HP] have constructed higher rank CAT(0) spaces with similar properties. As such, there are model geometries and \mathcal{Z} -geometries not relevant to the group theoretic applications that are the focus of this paper. Nevertheless, several theorems in the coming sections *can* be applied to those spaces.

³By contrast, the choice of a metric \overline{d} realizing the topology on \overline{X} , while convenient, is of little additional significance; any such metric will do.

Model geometries are not required to be finite-dimensional but the argument presented in [Mor16], or more explicitly [GM16, Th.2], gives the following.

Theorem 3.8. *The \mathcal{Z} -boundary of every model \mathcal{Z} -geometry (\overline{X}, Z, d) has finite Lebesgue covering dimension. More specifically, if the macroscopic dimension of X is n , then $\dim Z \leq n - 1$.*

We close this section with an observation that will be used in Section 6 and can be found in [GM19]. We repeat it here for easy access and to emphasize the difference between the metrics d and \overline{d} mentioned above.

Lemma 3.9. *Suppose (\overline{X}, Z, d) is a model \mathcal{Z} -geometry. For each $z \in Z$, neighborhood \overline{U} of z in \overline{X} and $r > 0$, there is a neighborhood \overline{V} of z in \overline{X} such that $d(V, X - U) \geq r$ where $V = \overline{V} - Z$ and $U = \overline{U} - Z$.*

4. DEFINING \mathcal{Z} -STRUCTURES AND COARSE \mathcal{Z} -STRUCTURES

We can now formulate one of our main definitions—that of a “coarse \mathcal{Z} -structure”. The task is made simpler by using the notion of a model \mathcal{Z} -geometry. First we reformulate the classical notion of a \mathcal{Z} -structure in this way (see [Bes96, Definition 1.1], [Dra06, Definition 1], or [GMT19, Definition 6.1] for versions of this classical definition). Lemma 6.4 of [GM19] assures that this formulation is equivalent to the original.

Definition 4.1. A \mathcal{Z} -structure on a group G consists of a model \mathcal{Z} -geometry (\overline{X}, Z, d) and a geometric action of G on (X, d) . In this case, we call Z a \mathcal{Z} -boundary for G .

Remark 4.2. Equivalently, a \mathcal{Z} -structure on G is a homomorphism $\phi : G \rightarrow \text{Isom}(X)$ such that $\ker \phi$ is finite and $\phi(G)$ is both cocompact and proper.

We are now ready to generalize.

Definition 4.3. A coarse \mathcal{Z} -structure on a group G ($c\mathcal{Z}$ -structure for short) consists of a model \mathcal{Z} -geometry (\overline{X}, Z, d) and a proper, cobounded, coarse near-action of G on X . In this case we call Z a coarse \mathcal{Z} -boundary (or $c\mathcal{Z}$ -boundary) for G .

The “if and only if” nature of Theorem 2.12 allows for a simple equivalent definition.

Definition 4.4 (alternative formulation). A $c\mathcal{Z}$ -structure on a finitely generated group G consists of a model \mathcal{Z} -geometry (\overline{X}, Z, d) and a coarse equivalence $f : G \rightarrow (X, d)$.

Remark 4.5. Proposition 2.10 implies that a group G admitting a $c\mathcal{Z}$ -structure is finitely generated. In that context, we always give G a standard word length metric.

Since every \mathcal{Z} -structure is a $c\mathcal{Z}$ -structure, we immediately have many groups that admit $c\mathcal{Z}$ -structures. As for new examples, those are produced primarily by applications of Theorem 0.1. We will look more closely at specific cases in the next section.

For those who prefer working with quasi-isometries and quasi-actions, we formulate a definition in that category. The situation is complicated slightly by the extra hypothesis required in the Švarc-Milnor Theorem for quasi-actions (see Remark 2.8).

Definition 4.6. A *quasi- \mathcal{Z} -structure* on a group G (*$q\mathcal{Z}$ -structure* for short) consists of a model \mathcal{Z} -geometry (\overline{X}, Z) and a proper, cobounded quasi-action of G on X . In that case we call Z a *quasi- \mathcal{Z} -boundary* (or *$q\mathcal{Z}$ -boundary*) for G .

Proposition 4.7. *A finitely generated group G admits a $q\mathcal{Z}$ -structure based on a model \mathcal{Z} -geometry (\overline{X}, Z, d) if G is quasi-isometric to (X, d) . If (X, d) is quasi-geodesic space the converse is true.*

Proof. Instead of Theorem 2.12, apply Theorem 8.4 of [Nek97]. \square

For the sake of simplicity, we will focus primarily on the coarse category in the remainder of this paper.

Before moving on, we discuss the general class of groups that are candidates for \mathcal{Z} - and $c\mathcal{Z}$ -structures. It is well-known that, for a torsion-free group G to admit a \mathcal{Z} -structure, it must be of *Type F*, meaning that there exists a finite $K(G, 1)$ complex. This is true for the following reason: Since G is torsion-free, the assumed proper, cocompact action on an AR, X , is free; so the quotient map $q : X \rightarrow G \backslash X$ is a covering map. As a result, $G \backslash X$ is a compact aspherical ANR. A theorem of West [Wes77] assures that $G \backslash X$ is homotopy equivalent to a finite CW complex K , which is our $K(G, 1)$ complex. By passing to universal covers, we can give an alternative definition: G is Type F if there exists a contractible CW complex admitting a proper, free, cocompact, rigid cellular G -action.⁴

All Type F groups are torsion-free, but there are many groups with torsion that admit \mathcal{Z} -structures. To aid in discussing those groups in the coming sections, we introduce the following definition.

Definition 4.8. A group G is *Type F^** if there exists a contractible CW complex admitting a proper, cocompact, rigid cellular G -action. Similarly, G is *Type F_{AR}^** if there exists an AR admitting a proper, cocompact G -action.

Remark 4.9. Unfortunately, the trick used above (showing that Type $F =$ Type F_{AR}) relies on a covering space argument not applicable when G has torsion. Whether there is a group of Type F_{AR}^* that is not of Type F^* is an open question. We will return to this and related questions in Section 8.

There is also room for a definition of *Type \underline{E}* , by which we mean a group admitting a cocompact $\underline{E}G$ complex, and *Type \underline{E}_{AR}* , meaning a group that admits a proper cocompact action on an AR such that stabilizers of finite subgroups are contractible. These variations are not needed in this paper.

5. UNIQUENESS AND BOUNDARY SWAPPING FOR $c\mathcal{Z}$ -STRUCTURES

We now begin justifying the definitions of the previous section by extending key theorems about geometric actions, \mathcal{Z} -structures, and \mathcal{Z} -boundaries to the realm of proper cobounded coarse near-actions, $c\mathcal{Z}$ -structures, and $c\mathcal{Z}$ -boundaries. We start

⁴By a *rigid* cellular action, we mean that the stabilizer of each cell, e , acts trivially on e . See [Geo08].

with generalized versions of three fundamental theorems about a given group G . See [GM19] for the classical analogs. For a brief review of the notion of proper homotopy equivalence, see page 302 of [GM19].

Theorem 5.1 (Coarse uniqueness of geometric models). *If a group G admits proper, cobounded, coarse near-actions on model geometries (X_1, d_1) and (X_2, d_2) , then there exists a continuous coarse equivalence $f : X_1 \rightarrow X_2$. As a consequence, X_1 is proper homotopy equivalent to X_2 .*

Theorem 5.2 ($c\mathcal{Z}$ -boundary swapping). *If G admits a $c\mathcal{Z}$ -structure based on a model \mathcal{Z} -geometry (\overline{X}, Z, d) and (Y, d') is a model geometry on which there is a proper, cobounded, coarse near-action by G , then there is a model \mathcal{Z} -geometry of the form (\overline{Y}, Z, d') underlying a $c\mathcal{Z}$ -structure on G .*

Corollary 5.3 ($c\mathcal{Z}$ -boundary swapping—alternate version). *If G admits a $c\mathcal{Z}$ -structure based on a model \mathcal{Z} -geometry (\overline{X}, Z, d) and (Y, d') is a model geometry coarsely equivalent to (X, d) or G , then there is a model \mathcal{Z} -geometry of the form (\overline{Y}, Z, d') underlying a $c\mathcal{Z}$ -structure on G .*

Proof. Combine Theorem 5.2 with Theorem 2.12. □

Corollary 5.4. *If G admits a $c\mathcal{Z}$ -structure with boundary Z and G is type F_{AR}^* , then G admits a \mathcal{Z} -structure with boundary Z .*

Theorem 5.5 (Shape uniqueness of $c\mathcal{Z}$ -boundaries). *If Z_1 and Z_2 are $c\mathcal{Z}$ -boundaries for a group G , then Z_1 is shape equivalent to Z_2 .*

Each of the above theorems is implied by the following collection of more general theorems, which involve pairs of quasi-isometric groups. This is where the benefits of our generalization scheme become clear. For example, Corollary 5.9 is significantly cleaner and more general than its analog for \mathcal{Z} -boundaries.

Remark 5.6. Under word length metrics, finitely generated groups are quasi-geodesic spaces, so there is no difference between quasi-isometric and coarse equivalent finitely generated groups. For that reason, we stick with the more common notion of quasi-isometry **when comparing groups**.

Theorem 5.7 (Generalized coarse uniqueness of geometric models). *If quasi-isometric groups G and H admit proper, cobounded, coarse actions on model geometries (X_1, d_1) and (X_2, d_2) , respectively, then there exists a continuous coarse equivalence $f : X_1 \rightarrow X_2$. In particular, X_1 is proper homotopy equivalent to X_2 .*

Proof. By Theorem 2.12, X_1 and X_2 are coarsely equivalent; so we may apply [GM19, Cor.5.4]. Lemmas 3.1 and 3.2 assure the necessary hypotheses. □

Theorem 5.8 (Generalized $c\mathcal{Z}$ -boundary swapping). *Suppose G and H are quasi-isometric groups; H admits a $c\mathcal{Z}$ -structure based on a model \mathcal{Z} -geometry (\overline{X}, Z, d) ; and (Y, d_Y) is a model geometry which admits a proper, cobounded, coarse near-action by G . Then there is model \mathcal{Z} -geometry of the form (\overline{Y}, Z, d_Y) underlying a $c\mathcal{Z}$ -structure on G .*

Proof. Apply [GM19, Th.7.1], with Theorem 5.7 providing the setup. \square

For emphasis, we state the following corollaries, the first of which is a restatement of Theorem 0.1:

Corollary 5.9. *If G and H are quasi-isometric groups and H admits a $c\mathcal{Z}$ -structure, then so does G . Moreover, if Z is a $c\mathcal{Z}$ -boundary for H , then Z is also a $c\mathcal{Z}$ -boundary for G .*

Corollary 5.10. *If G is quasi-isometric to a group H which admits a $c\mathcal{Z}$ -structure with boundary Z , and G is type F_{AR}^* , then G admits a \mathcal{Z} -structure with boundary Z .*

Example 5.11. If G contains a finite index subgroup H which is $\text{CAT}(0)$, i.e., G is *virtually* $\text{CAT}(0)$, it is unclear whether G is also $\text{CAT}(0)$ (or even Type F^*). As such, generalized boundary swapping ([GM19]) does not guarantee that G admits a \mathcal{Z} -structure. However, since $H \hookrightarrow G$ is a quasi-isometry, Theorem 5.8 shows that G admits a $c\mathcal{Z}$ -structure. By similar reasoning, virtual Baumslag-Solitar groups and virtual systolic groups admit $c\mathcal{Z}$ -structures. By combining this strategy with [Pie18] in an inductive manner, we can deduce that every poly-(finite or cyclic) group admits a $c\mathcal{Z}$ -structure. By Corollary 5.10, the moment any of these groups can be shown to act properly and cocompactly on an AR, the $c\mathcal{Z}$ -structure can be upgraded to a \mathcal{Z} -structure.

Theorem 5.12 (Uniqueness of coarse- \mathcal{Z} -boundaries up to shape). *If Z_1 and Z_2 are $c\mathcal{Z}$ -boundaries for quasi-isometric groups G and H , then Z_1 is shape equivalent to Z_2 .*

Proof. Let (\overline{X}, Z_1, d_X) and (\overline{Y}, Z_2, d_Y) be $c\mathcal{Z}$ -structures for G and H . By Theorem 2.12, G is coarsely equivalent to (X, d_X) so, by hypothesis, H is coarsely equivalent to (X, d_X) . Theorem 2.12 provides a proper cobounded coarse near-action of H on X so, by Theorem 5.8, there is a model \mathcal{Z} -geometry (underlying a $c\mathcal{Z}$ -structure on H) of the form $(\overline{X}', Z_2, d_X)$, where $\overline{X}' = X \sqcup Z_2$ is a second controlled \mathcal{Z} -compactification of X . From here, the argument provided in [GM19, §8] goes through unchanged. \square

We close this section by placing Theorem 3.8 into a group-theoretic context, where it generalizes results from [Gro87], [Swe99], and [Mor16].

Theorem 5.13. *Every $c\mathcal{Z}$ -boundary of a group G has finite Lebesgue covering dimension.*

6. FURTHER APPLICATIONS OF $c\mathcal{Z}$ -STRUCTURES

In this section, we show that many of the theorems from [BM91], [Bes96], [Ont05], [Dra06], and [GO07] remain valid in the more general context of coarse \mathcal{Z} -structures. We begin with some definitions, notation, and basic facts needed to describe these results.

Aside from some appeals to *group cohomology* (see [Bro82]) and *coarse cohomology* (see [Roe03]), all cohomology used here is Čech-Alexander-Spanier cohomology (see

[Mas78]) with coefficients in a PID R . This is a compactly supported cohomology theory which agrees with classical Čech cohomology on compact metric spaces. As such, it agrees with singular cohomology on compact ANRs. For locally compact ANRs it is isomorphic to singular cohomology with compact supports. For that reason, we will use the notation $\check{H}^*(\ ; R)$ when applying this functor to compact metric spaces and $\check{H}_c^*(\ ; R)$ for spaces that are not necessarily compact. Since many of the spaces of interest are non-ANRs, the choice of cohomology theories is crucial. For relative cohomology, this theory requires subspaces to be closed. A useful property is that, for every compact metric pair (Y, A) , $\check{H}^*(Y, A; R) \cong \check{H}_c^*(Y - A; R)$. More generally, if (Y, A) is a closed pair of locally compact metric spaces, $\check{H}_c^*(Y, A; R) \cong \check{H}_c^*(Y - A; R)$. As usual, a tilde indicates reduced cohomology.

Lemma 6.1. *For any model \mathcal{Z} -geometry (\overline{X}, Z, d) , $\check{H}^n(Z; R) \cong \check{H}_c^{n+1}(X; R)$ for all n .*

Proof. Since $X = \overline{X} - Z$ is an AR, then so is \overline{X} and hence \overline{X} is contractible. Then the isomorphism follows from the above remarks and the exact sequence for the pair (\overline{X}, Z) . \square

For a locally compact metric space Y the *cohomological dimension* with respect to R is defined by

$$\begin{aligned} \dim_R X &= \max \{n \mid \check{H}_c^n(U; R) \neq 0 \text{ for some } U \subseteq_{\text{open}} Y\} \\ &= \max \{n \mid \check{H}_c^n(Y, A; R) \neq 0 \text{ for some } A \subseteq_{\text{closed}} Y\} \end{aligned}$$

We let $\dim Y$ denote Lebesgue covering dimension. It is a classical fact that $\dim_{\mathbb{Z}} Y = \dim Y$ whenever the latter is finite [Wal81]. The *global cohomological dimension* with respect to R of a space X is defined by

$$\text{gcd}_R Y = \max \{n \mid \check{H}_c^n(Y; R) \neq 0\}$$

6.1. The group cohomology theorem for $c\mathcal{Z}$ -boundaries. One of Bestvina and Mess's initial applications of \mathcal{Z} -set technology [BM91] was to reveal a connection between the group cohomology of a torsion-free hyperbolic group G and topological properties of its Gromov boundary ∂G . A particularly striking assertion is that

$$H^{n+1}(G, RG) \cong \check{H}^n(\partial G; R)$$

for all integers n and PID R . In [Bes96], Bestvina applied the same reasoning to extend this result to all torsion-free groups admitting a \mathcal{Z} -structure; here the \mathcal{Z} -boundary plays the role of ∂G . The following result extends that theorem to $c\mathcal{Z}$ -boundaries and also allows for groups with torsion. The proof relies on coarse cohomology of metric spaces as developed by Roe in [Roe03].

Theorem 6.2. *If G admits a $c\mathcal{Z}$ -structure based on a model \mathcal{Z} -geometry (\overline{X}, Z, d) , then $H^{n+1}(G, RG) \cong \check{H}^n(Z; R)$ for all integers n and coefficients R .*

Proof. To avoid confusion, we let G denote the group and $|G|$ the corresponding metric space. By [Roe03, Example 5.21], $HX^*(|G|; R) \cong H^*(G; RG)$ and, since $|G|$ is coarsely equivalent to (X, d) , $HX^*(|G|; R) \cong HX^*(X; R)$. By [Roe03, Theorem 5.28], $HX^*(X; R) \cong \check{H}_c^*(X; R)$, so an application of Lemma 6.1 completes the proof. \square

6.2. The Bestvina-Mess formula for $c\mathcal{Z}$ -boundaries and model \mathcal{Z} -geometries.

Another key insight from [BM91] is that, for a torsion-free hyperbolic group G , ∂G satisfies:

$$(6.1) \quad \dim_R \partial G = \gcd_R \partial G$$

and since ∂G is always finite-dimensional:

$$(6.2) \quad \dim \partial G = \gcd_{\mathbb{Z}} \partial G$$

In [Bes96], Bestvina extended these observations to \mathcal{Z} -boundaries of torsion-free groups admitting a slightly more restrictive version of \mathcal{Z} -structure. Theorem 3.8 makes that restriction unnecessary. Later, Dranishnikov [Dra06] expanded the notion of \mathcal{Z} -structure to allow for groups with torsion and showed that the analog of (6.1) still holds. Another application of Theorem 3.8 implies (6.2). Independently, Geoghegan and Ontaneda [GO07] verified (6.1) and (6.2) for CAT(0) groups, with the visual boundary standing in for ∂G . Here we extend these results still further.

Theorem 6.3. *Let (\bar{X}, Z, d) be a $c\mathcal{Z}$ -structure on a group G . Then for every PID R , $\dim_R Z = \gcd_R Z$ and, since Z is finite-dimensional, $\dim Z = \gcd_{\mathbb{Z}} Z$.*

Corollary 6.4. *For groups admitting a $c\mathcal{Z}$ -structure, the dimension of the boundary and the cohomological dimension of that boundary over each PID R are quasi-isometry invariants.*

Remark 6.5. The point of Corollary 6.4 is that, since a $c\mathcal{Z}$ -structure on G guarantees the existence of a $c\mathcal{Z}$ -structure on all groups quasi-isometric to G , these are invariants of the entire quasi-isometry class of G .

Corollary 6.4 follows from a combination of Theorem 6.3, Lemma 6.1, and any one of several results from Section 5. Theorem 6.3 is implied by the following more general result that does not require a proper coarse near-action.

Theorem 6.6. *Let (\bar{X}, Z, d) be a model \mathcal{Z} -geometry. Then for every PID R , $\dim_R Z = \gcd_R Z$. Since Z is finite-dimensional, we also have $\dim Z = \gcd_{\mathbb{Z}} Z$.*

To obtain the main assertion of Theorem 6.6 we appeal to the argument presented in [Dra06]. A little additional care must be taken since our hypotheses are weaker than his: rather than a geometric action on X , we only assume that X is cocompact. Fortunately, that presents only minor challenges. A few adjustments are necessary, but mostly it suffices to reexamine each step of the earlier proof and observe that the weaker hypothesis is sufficient. As for the adjustments, those were anticipated when we stated and proved Lemmas 3.1-3.3.

In [Dra06], the reader will come across frequent uses of inclusion induced homomorphisms $\check{H}_c^n(V; R) \rightarrow \check{H}_c^n(U; R)$ where $V \subseteq U$ are open sets of a metric space X . At first counterintuitive, since cohomology is a contravariant functor⁵, these homomorphisms are discussed briefly in [Hat02, §3.3] and in more detail in [GH81, p.216] and [Mas78]. The following lemma is a rephrased and slightly generalized version of the key lemma in Dranishnikov’s proof.

Lemma 6.7 (see [Dra06, Lemma 3]). *Let (X, d) be a model geometry and R a PID. Suppose $\check{H}_c^i(X; R) = 0$ for all $i > n$. Then there is a number r such that for every open set $U \subseteq X$, the inclusion induced map $\check{H}_c^i(U; R) \rightarrow \check{H}_c^i(N_r(U); R)$ is trivial for all $i > n$.*

The necessary generalization is accomplished by inserting Lemmas 3.1-3.3 in the appropriate places. From there the proof of Theorem 1 in [Dra06] carries over to provide a proof of Theorem 6.6. The idea is to use Lemma 6.7 to conclude that any corresponding controlled \mathcal{Z} -boundary would necessarily have dimension $\leq n - 1$. Combined with Lemma 6.1, this provides the key inequality, $\dim_R Z \leq \gcd_R Z$. In the lowest dimension, this argument provides more than is contained in the statement of Theorem 6.6. This difference is significant since it allows us to obtain extensions of [BM91, Corollary 1.3 (d)] and the $d = 0$ case of [GO07, Main Theorem] (which is already covered by Theorem 6.6 when $d > 0$).

Proposition 6.8. *Suppose (\overline{X}, Z, d) is a model \mathcal{Z} -geometry and $\dim_R Z = 0$. Then*

- (1) $\check{H}_c^1(X; R) \neq 0$, and
- (2) Z is not a one-point set.

Proof. First note that $\check{H}_c^i(X; R) = 0$ for all $i > 1$, otherwise $\gcd_R Z \geq 1$, contradicting the assumption that $\dim_R Z = 0$. Since X is connected and noncompact, $\check{H}_c^0(X; R) = 0$. So, if $\check{H}_c^1(X; R) = 0$, then $\check{H}_c^i(X; R) = 0$ for all i , and the above argument would imply that $Z = \emptyset$, a contradiction. Assertion 1) follows.

For assertion 2), apply Lemma 6.1 to conclude that $\check{H}^{\sim 0}(Z; R) \neq 0$. □

Corollary 6.9. *The boundary of a model \mathcal{Z} -geometry never has the Čech cohomology of a point.*

6.3. Generalization of Ontaneda’s Almost Geodesic Completeness Theorem. Recall that a metric space (X, d) is *almost geodesically complete* if there exists $R > 0$ such that, for every $x, y \in X$, there is a geodesic ray $r : [0, \infty) \rightarrow X$ such that $r(0) = x$ and $d(r, y) \leq R$. Our next goal is an application of Theorem 6.6 which can be viewed as a generalization of the “Almost Geodesic Completeness Theorems” found in [Ont05] and [GO07] for CAT(0) spaces, and in [BM91] (attributed to M. Mihalik) for hyperbolic spaces.

Let (\overline{X}, Z, d) be a model \mathcal{Z} -geometry and $H : \overline{X} \times [0, 1] \rightarrow \overline{X}$ be a homotopy that instantly pulls \overline{X} off from Z . Using the contractibility of X , we can assume further

⁵To make cohomology with compact supports a functor, one must restrict to proper maps.

that H contracts \overline{X} to a point $x_0 \in X$. Let

$$CZ = Z \times [0, \infty] / \sim \quad \text{and} \quad OZ = Z \times [0, \infty) / \sim$$

be the *cone* and *open cone* on Z , where \sim identifies $Z \times \{0\}$ to a point. We view Z as a subset CZ by identifying it with $Z \times \{\infty\}$. By reversing and reparameterizing H , we obtain a map $\overline{F} : (CZ, Z) \rightarrow (\overline{X}, Z)$ whose restriction to the open cone $F : OZ \rightarrow X$ is proper. These maps, and any others with the same properties, are the topic of the next theorem.

Theorem 6.10. *Given a model \mathcal{Z} -geometry (\overline{X}, Z, d) and a map $\overline{F} : (CZ, Z) \rightarrow (\overline{X}, Z)$ which takes Z identically onto Z and OZ into X , there exists $R > 0$ such that $\overline{F}(OZ)$ is R -dense in X .*

Proof. The map \overline{F} induces a commuting diagram of long exact sequences

$$\begin{array}{ccccccccccc} \leftarrow & \check{H}^k(Z) & \leftarrow & \check{H}^k(\overline{X}) & \leftarrow & \check{H}^k(\overline{X}, Z) & \leftarrow & \check{H}^{k-1}(Z) & \leftarrow & \check{H}^{k-1}(\overline{X}) & \leftarrow \\ & \cong \downarrow \text{id} & & \cong \downarrow & & \downarrow \overline{F}^* & & \cong \downarrow \text{id} & & \cong \downarrow & \\ \leftarrow & \check{H}^k(Z) & \leftarrow & \check{H}^k(CZ) & \leftarrow & \check{H}^k(CZ, Z) & \leftarrow & \check{H}^{k-1}(Z) & \leftarrow & \check{H}^{k-1}(CZ) & \leftarrow \end{array}$$

By the Five-Lemma, \overline{F}^* is an isomorphism for all k . From this we may conclude that the restriction map $F : OZ \rightarrow X$, which is necessarily proper, induces isomorphisms $F^* : \check{H}_c^i(X) \rightarrow \check{H}_c^i(OZ)$ for all i .

For each integer i , let $B_i = B_d(x_0, i)$ to obtain an exhaustion $B_1 \subseteq B_2 \subseteq \dots$ of X by open sets with compact closures, and let $C_i = F^{-1}(B_i)$ to get a corresponding exhaustion $C_1 \subseteq C_2 \subseteq \dots$ of OZ . Then

$$\check{H}_c^k(X) = \varinjlim \check{H}^k(X, X - B_i) \quad \text{and} \quad \check{H}_c^k(OZ) = \varinjlim \check{H}^k(OZ, OZ - C_i)$$

and there is a commuting diagram between direct sequences

$$\begin{array}{ccccccc} \rightarrow & \check{H}^k(X, X - B_{j-1}) & \rightarrow & \check{H}^k(X, X - B_j) & \rightarrow & \check{H}^k(X, X - B_{j+1}) & \rightarrow \\ & \downarrow F^* & & \downarrow F^* & & \downarrow F^* & \\ \rightarrow & \check{H}^k(OZ, OZ - C_{j-1}) & \rightarrow & \check{H}^k(OZ, OZ - C_j) & \rightarrow & \check{H}^k(OZ, OZ - C_{j+1}) & \rightarrow \end{array}$$

By Theorem 6.6, there exist $k \geq 1$ for which $\check{H}_c^k(X)$ contains a nontrivial element α , and by properties of direct limits, j can be chosen sufficiently large that there exists $\alpha_j \in \check{H}^k(X, X - B_j)$ which projects to $\alpha \in \check{H}_c^k(X)$. Then $F^*(\alpha_j)$ likewise projects to the nontrivial element $F^*(\alpha) \in \check{H}_c^k(OZ)$.

Suppose now that our theorem fails. Then, for all $R > 0$, there exists $x_R \in X$ such that $B_d(x_R; R) \cap F(OZ) = \emptyset$. By choosing $R > 2j$ and using the cocompactness of X , we may choose $g \in \text{Isom}(X)$ such that $gB_j \subseteq B_d(x_R, R)$. For each integer i , let $B'_i = gB_i = B_d(gx_0, i)$ and let $D_i = F^{-1}(B'_i)$ to get a commuting diagram

$$\begin{array}{ccccccc} \rightarrow & \check{H}^k(X, X - B'_{j-1}) & \rightarrow & \check{H}^k(X, X - B'_j) & \rightarrow & \check{H}^k(X, X - B'_{j+1}) & \rightarrow \\ & \downarrow F^* & & \downarrow F^* & & \downarrow F^* & \\ \rightarrow & \check{H}^k(OZ, OZ - D_{j-1}) & \rightarrow & \check{H}^k(OZ, OZ - D_j) & \rightarrow & \check{H}^k(OZ, OZ - D_{j+1}) & \rightarrow \end{array}$$

Since g is a homeomorphism, $\alpha' = (g^*)^{-1}(\alpha)$ is a nontrivial element of $\check{H}_c^k(X) = \varinjlim \check{H}^k(X, X - B'_i)$ and $\alpha'_j = (g^*)^{-1}(\alpha_j)$ is an element of $\check{H}^k(X, X - B'_j)$ that projects

to α' . By commutativity, $F^*(\alpha')$ projects to the corresponding nontrivial element of $\check{H}_c^k(OZ) = \varinjlim \check{H}^k(OZ, OZ - D_i)$. But, since B'_j is disjoint from $F(OZ)$, $D_j = \emptyset$, so $\check{H}^k(OZ, OZ - D_j) = 0$, giving a contradiction. \square

Remark 6.11. Given $\bar{F} : (CZ, Z) \rightarrow (\bar{X}, Z)$ as above, we can (by restriction) associate to each $z \in Z$ the proper (not necessarily embedded) ray $r_z : [0, \infty) \rightarrow X$ which emanates from x_0 and limits to z in \bar{X} . Theorem 6.10 can be interpreted as saying that every $x \in X$ is at a distance $< R$ from one of these rays. By cocompactness we have:

Corollary 6.12. *Given a model \mathcal{Z} -geometry (\bar{X}, Z, d) and a map $\bar{F} : (CZ, Z) \rightarrow (\bar{X}, Z)$ which takes Z identically onto Z and OZ into X , there exists $S > 0$ such that, for every $x_1, x_2 \in X$, there exists $z \in Z$ and $g \in \text{Isom}(X)$ such that $d(gx_0, x_1) < S$ and $d(gr_z, x_2) < S$.*

Proof. Let $R > 0$ be the constant promised in Theorem 6.10 and note that for each $g \in \text{Isom}(X)$, every $x \in X$ is at a distance $< R$ from a ray gr_z for some $z \in Z$. Choose $R' > 0$ so that $\{gx_0 \mid g \in \text{Isom}(X)\}$ is R' -dense in X and let $S = \max\{R, R'\}$. \square

Corollary 6.13 (after [GO07, Corollary 3]). *Every cocompact proper CAT(0) space (X, d) is almost geodesically complete.*

Proof. Let $(\bar{X}, \partial_\infty X, d)$ be the model \mathcal{Z} -geometry obtained by attaching to X its visual boundary, and let the contraction $H : \bar{X} \times [0, 1] \rightarrow X$ be the geodesic retraction to x_0 . Then the rays r_z are precisely the geodesic rays in X emanating from x_0 and the rays gr_z are the geodesic rays emanating from gx_0 .

Let $S > 0$ be the constant promised by Corollary 6.12 and let $x, y \in X$. Then there exists $g \in \text{Isom}(X)$ such that $d(gx_0, x) < S$ and $d(gr_z, y) < S$. By a standard construction in CAT(0) geometry there exists a geodesic ray r emanating from x and asymptotic to gr_z . That same construction assures that $d(gr_z(t), r(t)) \leq S$ for all $t > 0$, so $d(y, r) \leq 2S$. \square

The nature of Theorem 6.10 brings to mind another theorem about geodesic rays in CAT(0) spaces. Geoghegan and Swenson [GS19] (also see their arXiv update which contains slightly stronger conclusions) showed that a 1-ended proper CAT(0) space (X, d) is semistable (all proper maps $r : [0, \infty) \rightarrow X$ are properly homotopic) if and only if all *geodesic* rays in X emanating from a common $x_0 \in X$ are properly homotopic. After analyzing their proof, we conclude that we have nothing new to offer, except the observation that their proof already implies the following generalization.

Theorem 6.14. *Let $\bar{X} = X \sqcup Z$ be a (not necessarily controlled) \mathcal{Z} -compactification of a (not necessarily cocompact) proper metric 1-ended AR (X, d) . Let $x_0 \in X$ and $\{r_z\}_{z \in Z}$ be the family of (singular) proper rays described above. Then X is semistable if and only if r_z is properly homotopic to $r_{z'}$ for all $z, z' \in Z$.*

7. $E\mathcal{Z}$ -STRUCTURES AND COARSE- $E\mathcal{Z}$ -STRUCTURES

A discussion of \mathcal{Z} -structures would be incomplete without some mention of $E\mathcal{Z}$ -structures. Here we show how the concepts introduced in this paper can be extended to allow for coarse $E\mathcal{Z}$ -structures.

Definition 7.1. Each model \mathcal{Z} -geometry (\overline{X}, Z, d) determines a corresponding *uniform subgroup* of $\text{Isom}(X)$ defined by:

$$\mathcal{U}(\overline{X}, Z, d) = \{ \gamma \in \text{Isom}(X) \mid \gamma \text{ extends to a homeomorphism } \overline{\gamma} : \overline{X} \rightarrow \overline{X} \}$$

A coarse analog of this is the following.

Definition 7.2. Each model \mathcal{Z} -geometry (\overline{X}, Z, d) determines a corresponding *coarse uniform subset* of $\text{Coarse}(X)$ defined by:

$$\begin{aligned} \mathcal{cU}(\overline{X}, Z, d) = \{ \gamma \in \text{Coarse}(X) \mid \gamma \text{ extends to a map } \overline{\gamma} : \overline{X} \rightarrow \overline{X} \\ \text{that is continuous at all points of } Z \} \end{aligned}$$

Example 7.3. Recall the model \mathcal{Z} -geometries $(\overline{\mathbb{R}^2}, S^1, d_E)$ and $(\overline{\mathbb{H}^2}, S^1, d_H)$ discussed in Example 3.6. In each case, the corresponding uniform subgroup is the entire isometry group. By contrast, if we quotient out the upper half-circle in either boundary, we get a new model \mathcal{Z} -geometry where the \mathcal{Z} -boundary is still a circle, but now many isometries do not extend. Similar comments can be made regarding Example 3.7.

Recall from Section 4 that a \mathcal{Z} -structure on G is a homomorphism $\phi : G \rightarrow \text{Isom}(X)$ such that $\ker \phi$ is finite and $\phi(G)$ is both cocompact and proper. If, in addition, $\phi(G) \subseteq \mathcal{U}(\overline{X}, Z, d)$, we call this an $E\mathcal{Z}$ -structure and Z an $E\mathcal{Z}$ -boundary for G . We generalize this established definition as follows:

Definition 7.4. A *coarse $E\mathcal{Z}$ -structure* on a group G (*$cE\mathcal{Z}$ -structure* for short) is a $c\mathcal{Z}$ -structure $(\overline{X}, Z, d, \psi)$ with the additional property that, for each $\gamma \in G$, the coarse equivalence $\psi(\gamma) : X \rightarrow X$ extends to a map $\overline{\psi}(\gamma) : \overline{X} \rightarrow \overline{X}$ that is continuous at all points of Z . More succinctly, we require $\psi(G) \subseteq \mathcal{cU}(\overline{X}, Z, d)$.

Remark 7.5. By properness, each $\overline{\psi}(\gamma)$ in the above definition maps Z into Z ; moreover, by upgrading the $c\mathcal{Z}$ -structure to a coarse near-action by continuous maps (see [GM19, §5]), we can require that $\overline{\psi}(\gamma) : \overline{X} \rightarrow \overline{X}$ be continuous. Lemma 7.4 of [GM19] ensures that every coarse self-equivalence of X that is boundedly close to id_X extends (continuously and uniquely) over Z via the identity. As a result $\overline{\psi}(1)|_Z = \text{id}_Z$, and for each $\gamma \in G$, both $\overline{\psi}(\gamma) \circ \overline{\psi}(\gamma^{-1})$ and $\overline{\psi}(\gamma^{-1}) \circ \overline{\psi}(\gamma)$ are the identity when restricted to Z . It follows that each $\overline{\psi}(\gamma)|_Z : Z \rightarrow Z$ is a homeomorphism and, with a little more effort, that restriction gives an actual G -action on Z . So, if desired, this requirement could be added to Definition 7.4 without a loss of generality.

Theorem 7.6. *Suppose G admits a cEZ -structure (\overline{Y}, Z, d) and (X, d') is another model geometry coarsely equivalent to G . Then G admits a cEZ -structure of the form (\overline{X}, Z, d') .*

Proof. Corollary 5.3 ensures that there is a cZ -structure of the form (\overline{X}, Z, d') for G . By Remark 7.5, we can ensure that $\overline{\psi(\gamma)}|_Z : Z \rightarrow Z$ is a homeomorphism that gives rise to an actual G -action on Z . An application of Proposition 7.5 from [GM19] guarantees that the coarse near action of G on X extends to an action by homeomorphisms on Z . \square

Corollary 7.7. *Suppose G admits a cEZ -structure. If G is type F_{AR}^* then G admits an EZ -structure. If, in addition, G is torsion-free, the Novikov Conjecture holds for G .*

Proof. The first assertion is clear. From there one can make the conclusion regarding the Novikov Conjecture by applying [FL05], which requires that G be torsion-free. \square

8. OPEN QUESTIONS

We close by shining a light on some open questions. To streamline the discussion, we introduce a few more definitions.

Definition 8.1. A group G is of *Type Z* [resp., *Type EZ*] if it admits a Z -structure [resp., EZ -structure]. It is of *Type cZ* [resp., *Type cEZ*] if it admits a cZ -structure [resp., cEZ -structure].

The best-know questions about $[E]Z$ structures have been raised by Bestvina and Farrell-Lafont. We supplement their questions with our own variations.

Question 1. *Does every Type F group have Type Z ? Type EZ ? Does every Type F^* or Type F_{AR}^* group have Type Z ? Type EZ ?*

With the benefit of Definition 8.1, Theorem 0.1 can be rephrased as follows: *Type cZ is a quasi-isometry invariant.* That begs the question.

Question 2. *Are any of the following quasi-isometry invariants: Type F^* , Type F_{AR}^* , Type Z , Type EZ , Type cEZ ?*

As noted in Remark 4.9, a beautifully simple question from ANR theory asks:

Question 3. *Are Type F^* and F_{AR}^* equivalent?*

Another purely topological question asks:

Question 4. *Under what conditions does a model geometry admit a controlled Z -compactification? Any Z -compactification?*

The reader interested in Question 4 might want to look at [CS76] and [Gui01].

Finally, the usefulness of EZ -structures in attacks on the Novikov Conjecture makes the following question natural.

Question 5. *Does every group G which admits an EZ -structure satisfy the Novikov Conjecture? [Yes by [FL05] when G is torsion-free, and in some additional cases by [Ros06]; but the general question seems to be open.]. Does every [torsion-free] group that admits a cEZ -structure satisfy the Novikov conjecture?*

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