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A comparison of large scale dimension of a metric space to the dimension of its boundary ☆

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ABSTRACT

Buyalo and Lebedeva have shown that the asymptotic dimension of a hyperbolic group is equal to the dimension of the group boundary plus one. Among the work presented here is a partial extension of that result to all groups admitting $\mathcal{Z}$-structures; in particular, we show that $\text{asdim}G \geq \dim Z + 1$ where Z is the $\mathcal{Z}$-boundary.

1. Introduction

The primary goal of this paper is to establish a connection between the asymptotic dimension of a group admitting a $\mathcal{Z}$-structure and the covering dimension of the group's boundary.

For hyperbolic G, the relationship is strong; Buyalo and Lebedeva [5] have shown that $\text{asdim}G = \dim\partial G + 1$. In [6], a partial extension to CAT(0) groups was attempted. Specifically, it was claimed that $\text{asdim}G \geq \dim\partial G + 1$, where ∂G is any CAT(0) boundary of G. However, in MathSciNet review MR3058238, X. Xie pointed out a critical error in the proof. Here we recover the same inequality as a special case of a more general theorem.

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Theorem 1. *Suppose a group G admits a $\mathcal{Z}$-structure, $(\overline{X}, Z)$. Then* $\dim Z + 1 \leq \operatorname{asdim} G$.

By a $\mathcal{Z}$-structure on G, we are referring to the axiomatized approach to group boundaries laid out in [2] and expanded upon in [8]. Groups known to admit $\mathcal{Z}$-structures include: hyperbolic groups (with X being a Rips complex and $Z = \partial G$) [3]; CAT(0) groups (with X being the CAT(0) space and Z its visual boundary) [2]; systolic groups [15], Baumslag–Solitar groups [10]; as well as various combinations of these classes, as described in [17,7,13]. Definitions of $\mathcal{Z}$-structure and other key terms used here will be provided in the next section. Theorem 1 will be obtained from a more general observation about metric spaces.

Theorem 2. *Suppose a proper metric space (X, d) admits a controlled $\mathcal{Z}$-compactification $\overline{X} = X \cup Z$. Then* $\dim Z + 1 \leq \dim_{\mathrm{mc}} X$.

Here, $\dim_{\mathrm{mc}}$ stands for Gromov's *macroscopic dimension*, a type of large scale dimension for metric spaces that is less restrictive than asymptotic dimension in that, for any (X, d), $\dim_{\mathrm{mc}} X \leq \operatorname{asdim} X$. To complete the proof of Theorem 1 it will then suffice to show that, for a $\mathcal{Z}$-structure $(\overline{X}, Z)$ on a group G, $\overline{X}$ is a controlled $\mathcal{Z}$-compactification and $\operatorname{asdim} X = \operatorname{asdim} G$.

Theorem 2 is inspired by the main argument in [11] together with the point of view presented in [14].

2. Background and definitions

We begin by providing a few definitions and results for the different dimension theories and then we discuss controlled $\mathcal{Z}$-compactifications and $\mathcal{Z}$-structures.

Given a cover $\mathcal{U}$ of a metric space X, $\operatorname{mesh}\mathcal{U} = \sup\{\operatorname{diam}(U) | U \in \mathcal{U}\}$. The cover is **uniformly bounded** if there exists some $D > 0$ such that $\operatorname{mesh}\mathcal{U} \leq D$. The **order** of $\mathcal{U}$ is the smallest integer n for which each element $x \in X$ is contained in at most n elements of $\mathcal{U}$.

Definition 3. The **covering dimension** of a space X is the minimal integer n such that every open cover of X has an open refinement of order at most $n + 1$.

There are various ways to show that a space has finite covering dimension. When working with compact metric spaces, we prefer the following.

Lemma 4. *For a compact metric space X,* $\dim X \leq n$ *if and only if, for every $\epsilon > 0$, there is an open cover $\mathcal{U}$ of X with* $\operatorname{mesh}\mathcal{U} < \epsilon$ *and* $\operatorname{order}\mathcal{U} \leq n + 1$.

Covering dimension can be thought of as a small-scale property. Gromov introduced asymptotic dimension as a large scale analog of covering dimension [9].

Definition 5. The **asymptotic dimension** of a metric space X is the minimal integer n such that for every uniformly bounded open cover $\mathcal{V}$ of X, there is a uniformly bounded open cover $\mathcal{U}$ of X with $\operatorname{order}\mathcal{U} \leq n+1$ so that $\mathcal{V}$ refines $\mathcal{U}$. In this case, we write $\operatorname{asdim} X = n$.

We note here that the covers need not be open in the definition of asymptotic dimension. We chose to follow conventional definitions here. For a nice survey of asymptotic dimension, see [1]. Although Theorem 1 is stated for asymptotic dimension, we will prove a stronger result using a weaker notion of large scale dimension known as *(Gromov) macroscopic dimension*.

Definition 6. The **Gromov macroscopic dimension** of a metric space X is the minimal integer n such that there exists a uniformly bounded open cover of X with order at most $n + 1$. In this case, we write $\dim_{\mathrm{mc}} X = n$.

Clearly $\dim_{\mathrm{mc}} X \leq \operatorname{asdim} X$ for every metric space X.

As noted in the introduction, Theorem 1 about groups and their boundaries will be deduced from a broader observation about certain $\mathcal{Z}$-compactifications of metric spaces. Recall that a closed subset, A, of an ANR, Y, is a ***$\mathcal{Z}$-set*** if there exists a homotopy $H : Y \times [0, 1] \to Y$ such that $H_0 = \mathrm{id}_Y$ and $H_t(Y) \subset Y - A$ for every $t > 0$.

Definition 7. A **controlled $\mathcal{Z}$-compactification** of a proper metric space X is a compactification $\overline{X} = X \cup Z$ satisfying the following two conditions:

- Z is a $\mathcal{Z}$-set in $\overline{X}$
- For every $\epsilon > 0$ and every $R > 0$, there exists a compact set $K \subset X$ so that every ball of radius R in X not intersecting K has diameter less than ϵ in $\overline{X}$.

In this case, Z is called a **$\mathcal{Z}$-boundary**, or simply a **boundary** for X.

There are a few things to take note of in the above definition. First, we have followed tradition and defined $\mathcal{Z}$-sets in ANRs; hence the compactification $\overline{X}$ must be an ANR. Furthermore, since open subsets of ANRs are also ANRs, X must be an ANR to be a candidate for a controlled $\mathcal{Z}$-compactification.[1] Secondly, it is important to distinguish between the (proper) metric d on X and the metric $\overline{d}$ on $\overline{X}$. The second condition, which we call the *control condition*, says balls of radius R in (X, d) get arbitrarily small near the boundary, when viewed in $(\overline{X}, \overline{d})$. The metric d is crucial; it is given in advance and determines the geometry of X. For our purposes the metric on $\overline{X}$ is arbitrary; any $\overline{d}$ determining the appropriate topology can be used.

Example 8. The addition of the visual boundary to a proper CAT(0) space is a prototypical example in Geometric Group Theory of a controlled $\mathcal{Z}$-compactification.

In the presence of nice group actions, controlled Z-compactifications arise rather naturally. As a result, our discussion can be extended to asymptotic dimension of groups and covering dimension of group boundaries. The following definition is key.

Definition 9. A **$\mathcal{Z}$-structure on a group G** is a pair of spaces $(\overline{X}, Z)$ satisfying the following four conditions:

(1) $\overline{X}$ is a compact AR,
(2) Z is a $\mathcal{Z}$-set in $\overline{X}$,
(3) $X = \overline{X} - Z$ is a proper metric space on which G acts properly, cocompactly, by isometries, and
(4) $\overline{X}$ satisfies a *nullity condition* with respect to the action of G on X: for every compact $C \subseteq X$ and any open cover $\mathcal{U}$ of $\overline{X}$, all but finitely many G translates of C lie in an element of $\mathcal{U}$.

Remark 1. This definition of $\mathcal{Z}$-structure is due to Dranishnikov [8]. It generalizes Bestvina's original definition from [2] by allowing $\overline{X}$ to be infinite-dimensional and G to have torsion. We have added an explicit requirement that the metric on X be proper; a quick review of [8] reveals that this requirement was assumed there as well.

Remark 2. If $\overline{d}$ is a metric on $\overline{X}$ and $\epsilon > 0$, we can consider the open cover $\mathcal{U}$ of $\overline{X}$ that consists of all open balls of radius ϵ (in the $\overline{d}$ metric). The nullity condition in (4) can then be restated as follows: for every compact $C \subset X$, all but finitely many G translates of C have diameter less than ϵ.

[1] See Remark 3.

3. Proofs

We begin with a proof of Theorem 2, as the other results will be obtained from it. A key ingredient is the following classical fact about covering dimension.

Lemma 10. *([12]) For any nonempty locally compact metric space X, $\dim(X \times [0,1]) = \dim X + 1$.*

Proof of Theorem 2. Suppose X admits a controlled $\mathcal{Z}$-compactification, $\overline{X} = X \cup Z$, and let $\epsilon > 0$. Assume that $\dim_{\mathrm{mc}} X = n$ and let $\mathcal{U}$ be a uniformly bounded open cover of X with order$\mathcal{U} \leq n+1$.

Using the control condition, we may choose a compact set K_0 such that $\mathrm{diam}_{\overline{d}} U \leq \frac{\epsilon}{3}$ for every $U \in \mathcal{U}$ with the property that $U \cap K_0 = \emptyset$. Let $\mathcal{U}' = \{U \in \mathcal{U} | U \cap K_0 = \emptyset\}$.

Since Z is a $\mathcal{Z}$-set, there is a homotopy $J : \overline{X} \times [0,1] \to \overline{X}$ such that $J_0 = \mathrm{id}_{\overline{X}}$ and $J_t(\overline{X}) \cap Z = \emptyset$ for all $t > 0$. By compactness there is some $T > 0$ such that $\overline{d}(z, J_t(z)) < \frac{\epsilon}{3}$ for all $z \in Z$ and $t \in [0,T]$. Furthermore, we may choose $T' > 0$ so that $J(Z \times (0,T']) \subset \bigcup_{U \in \mathcal{U}'} U$. Set $t_0 = \min\{T, T'\}$.

Define $H : \overline{X} \times [0,1] \to \overline{X}$ by setting $H(x,t) = J(x, t_0 \cdot t)$. Restrict H to $Z \times [0,1]$. We will reparametrize $H : Z \times [0,1] \to \overline{X}$ in a manner similar to [11], so that pre-images of the open sets in $\mathcal{U}'$ have small mesh. After one additional adjustment, those pre-images will form the desired cover of $Z \times [0,1]$. For convenience we will use the ℓ_∞ metric on $Z \times [0,1]$, $d_\infty = \max\{\overline{d}, |\cdot|\}$, where $|\cdot|$ is the standard metric on $[0,1]$.

Pick $n \in \mathbb{Z}^+$ so that $\frac{3}{n} < \frac{\epsilon}{3}$. Choose $t_1 > t_2 > \cdots > t_{n+1} \in [0,1]$ and compact sets $K_1, K_2, \ldots K_{n+1} \subset X$ as follows:

- let $t_1 = 1$ and choose K_1 so that $H(Z \times \{1\}) \subset K_1$
- for $i = 2, 3, \ldots, n$, choose t_i so that $H(Z \times [0,t_i]) \cap K_{i-1} = \emptyset$ and $K_i \subset X$ so that $H(Z \times [t_i, 1]) \cup K_{i-1} \subset K_i$ and K_i contains all elements of $\mathcal{U}'$ that intersect K_{i-1}. (By properness, elements of $\mathcal{U}'$ have compact closures in X.)
- let $t_{n+1} = 0$ and $K_{n+1} = \overline{X}$.

Let $\lambda : [0,1] \to [0,1]$ be a strictly increasing piecewise linear function with $\lambda(0) = 0$, $\lambda(1) = 1$, and $\lambda\left(\frac{i}{n}\right) = t_{n-i+1}$. Reparametrize H using λ and then push $Z \times [0,1]$ completely into X by using the map $F : Z \times [0,1] \to X$ defined by $F(z,s) = H(z, \lambda(s))$ for $s \in \left[\frac{1}{n}, 1\right]$ and $F(z,s) = H\left(z, \frac{1}{n}\right)$ for $s \in \left[0, \frac{1}{n}\right]$.

We show that $\mathcal{V} = \{F^{-1}(U) | U \in \mathcal{U}'\}$ is an open cover of $Z \times [0,1]$ with mesh at most ϵ and order at most $n+1$.

Let $(z,s), (z',s') \in F^{-1}(U)$ and set $y = F(z,s), y' = F(z',s')$ and $t = \lambda(s), t' = \lambda(s')$. Choose $j \in \{1, 2, \ldots, n+1\}$ such that $y \in K_j - K_{j-1}$. By the choice of K_i and t_i values from above, $t_{j+1} < t < t_{j-1}$. Thus, $\frac{n-j}{n} < s < \frac{2+n-j}{n}$. Since $y, y' \in U$ and $y \in K_j$, then $U \cap K_j = \emptyset$, so $y' \in K_{j+1}$. Furthermore, $y' \notin K_{j-2}$ because if it were, $U \cap K_{j-2} \neq \emptyset$ and $U \subset K_{j-1}$, a contradiction to the choice of j. Thus, $y' \in K_{j+1} - K_{j-2}$. Similar reasoning as above for t shows that $t_{j+2} < t' < t_{j-2}$ and $\frac{n-1-j}{n} < s' < \frac{n+3-j}{n}$. Thus,

$$|s - s'| < \frac{n+3-j}{n} - \frac{n-j}{n} = \frac{3}{n} < \epsilon$$

Moreover,

$$\begin{aligned}\overline{d}(z,z') &\leq \overline{d}(z,y) + \overline{d}(y,y') + \overline{d}(y',z') \\ &= \overline{d}(z, H(z, \lambda(s))) + \overline{d}(y,y') + \overline{d}(z' H(z', \lambda(s'))) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon\end{aligned}$$

By the above $d_\infty((z,s),(z',s')) < \epsilon$, proving $\mathrm{mesh}_{d_\infty}\mathcal{V} < \epsilon$. Since $\mathcal{V}$ consists of the pre-images of $\mathcal{U}'$ and $\mathrm{order}\mathcal{U}' \leq n+1$, then $\mathrm{order}\mathcal{V} \leq n+1$. Using the definition of dimension in Lemma 4 we have $\dim(Z\times[0,1]) \leq n$ and an application of Lemma 10 finishes the claim. □

Remark 3. We have chosen to follow the traditional definition of $\mathcal{Z}$-sets and require $\overline{X}$ to be an ANR. However, the above proof also applies to more general metric spaces. In particular, we make no use of the ANR properties of $\overline{X}$ or X; if Z is a closed subset of any compact metric space $\overline{X}$ and it is possible to instantly homotope $\overline{X}$ off of Z, then the proof of Theorem 2 will go through as above.

From Theorem 2 we obtain a correct proof of the main assertion of [6, Cor. 1.2], which does not involve groups.

Corollary 11. *If X is a proper CAT(0) space, then* $\mathrm{asdim}\, X \geq \dim \partial X + 1$.

To obtain Theorem 1, we first must show that the notion of controlled $\mathcal{Z}$-compactification applies to a $\mathcal{Z}$-structure $(\overline{X}, Z)$ on a group G. Since $Z \subseteq \overline{X}$ is a $\mathcal{Z}$-set, all that remains to show is that open balls in X become small near the boundary. The cocompact action by isometries combined with the nullity condition will grant that control.

Lemma 12. *Suppose a group G admits a $\mathcal{Z}$-structure, $(\overline{X}, Z)$. Then $\overline{X}$ is a controlled $\mathcal{Z}$-compactification of $X = \overline{X} - Z$.*

Proof. Fix a metric d on X and $\overline{d}$ on $\overline{X}$ and let $\epsilon > 0$ and $R > 0$. By cocompactness, there is a compact set $C \subset X$ such that $X \subset GC$. Choose $r > 0$ and $x_0 \in X$ such that $C \subset B(x_0, r) \subset X$. By the nullity condition, all but finitely many G translates of $\overline{B(x_0, r+R)}$ have diameter less than ϵ in $\overline{X}$ (see Remark 2). Set $\Gamma = \{g \in G | \mathrm{diam}_{\overline{d}} gB(x_0, r+R) \geq \epsilon\}$ and let $K' = \overline{X} - \cup_{g\in G-\Gamma} gB(x_0, r+R)$. Thus, K' is a compact subset of X that satisfies the property that if $gB(x_0, r+R) \cap K' = \emptyset$ for some $g \in G$, then $\mathrm{diam}_{\overline{d}} gB(x_0, r+R) < \epsilon$. Let $K = \overline{N_{2r+R}(K')}$ be the closed $2r+R$ neighborhood of K' in X. We show this is the desired compact set. Thus, let $B(x, R) \subset X$ for some $x \in X$ with $B(x, R) \cap K = \emptyset$. Choose $g \in G$ such that $gx \in C$. Then, $B(x, R) \subset g^{-1}B(x_0, r+R)$ since for any $y \in B(x, R)$,

$$d(y, g^{-1}x_0) \leq d(y, x) + d(x, g^{-1}x_0) < R + r$$

Furthermore, $g^{-1}B(x_0, r+R) \cap K' = \emptyset$. Otherwise, there would be some $z \in g^{-1}B(x_0, r+R) \cap K'$ and $d(x, z) \leq d(x, g^{-1}x_0) + d(g^{-1}x_0, z) < 2r + R$. However, $B(x, R) \cap K = \emptyset$, so, $d(x, K') > 2r + R$. Because $z \in K'$, we obtain the required contradiction.

Thus $\mathrm{diam}_{\overline{d}} g^{-1}B(x_0, r+R) < \epsilon$. $B(x, R)$, being a subset of $g^{-1}B(x_0, r+R)$, will also have diameter smaller than ϵ. □

Proof of Theorem 1. Suppose a group G admits a $\mathcal{Z}$-structure $(\overline{X}, Z)$. By Lemma 12, $\overline{X}$ is a controlled $\mathcal{Z}$-compactification of X. Thus, by Theorem 2, $\mathrm{asdim} X \geq \dim Z + 1$. Since G acts geometrically on X, G is coarsely equivalent to X (see Corollary 0.9 in [4]). Moreover, by [16], asymptotic dimension is a coarse invariant; so $\mathrm{asdim} X = \mathrm{asdim} G$. □

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