ALL BOUNDARIES OF CROKE-KLEINER'S GROUP ARE CE EQUIVALENT

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ABSTRACT. In 2000, Croke and Kleiner showed that a CAT(0) group G can admit more than one boundary. This contrasted with the situation for δ-hyperbolic groups, where it was well-known that each such group admitted a unique boundary—in a very stong sense. Prior to Croke and Kleiner's discovery, it had been observed by Geoghegan and Bestvina that a weaker sort of uniqueness does hold for boundaries of torsion free CAT(0) groups; in particular, any two such boundaries always have the same shape. Hence, the boundary really does carry significant information about the group itself. In an attempt to strengthen the correspondence between group and boundary, Bestvina asked whether boundaries of CAT(0) groups are unique up to cell-like equivalence. For the types of space that arise as boundaries of CAT(0) groups, this is a notion that is weaker than topological equivalence and stronger than shape equivalence.

In an earlier paper, we introduced a construction which assigns to every pair of boundaries of a CAT(0) group, an intermediate compactum admitting a group action which comes equipped with equivariant maps onto the boundaries. We showed that in some simple product examples, these maps are cell-like. Here we extend our theory to prove that all boundaries of the Croke-Kleiner group are cell-like equivalent. Indeed, our proof extends to a wider class of CAT(0) groups known as Croke-Kleiner admissible groups.

1. Introduction

One striking difference between the category of negatively curved groups and that of nonpositively curved groups occurs at their ends; whereas a δ-hyperbolic group admits a topologically unique boundary, a CAT(0) group can admit uncountably many distinct boundaries [CK00, Wil05, Moo08, Moo10]. On its surface, that observation might lead one to believe that a boundary for a CAT(0) group is not a useful object, but that is not the case. Many properties remain constant across the spectrum of boundaries of a given CAT(0) group, and thus may be viewed as properties of the group itself. One substantial such property, which implies many others, is the *shape* of the boundary. That observation was made indirectly by Geoghegan [Geo86] and, specifically for CAT(0) groups, by Bestvina [Bes96]. The upshot is that all boundaries of a given CAT(0) group are topologically similar in a manner made precise by shape theory—a classical branch of geometric topology developed specifically for dealing with spaces with the sort of bad local properties that frequently occur in boundaries of groups. Looking for an even stronger correlation between CAT(0) groups and their boundaries, Bestvina posed the following:

Bestvina's Cell-like Equivalence Question. *For a given CAT(0) group G, are all boundaries cell-like equivalent?*

Precise formulations of the notion of 'shape equivalence' and 'cell-like equivalence' can be found in [GM11] along with examples illustrating the contrast between the concepts.

Roughly speaking, two finite dimensional compacta X and Y are declared to be *shape equivalent* if whenever they are embedded in some high-dimensional Euclidean space, 'typical neighborhoods' of one are homotopy equivalent to typical neighborhoods of the other. So, for instance, the topologist's sine curve is shape equivalent to a single point, since 'typical neighborhoods' when embedded in $\mathbb{R}^2$ are disks. This is formalized by writing X and Y as the limits of inverse sequences of polyhedral neighborhoods and constructing a ladder diagram which commutes up to homotopy (after possibly passing to subsequences).

A compactum X is called *cell-like* if it is shape equivalent to a point. An equivalent definition for finite dimensional compacta is to say that X is cell-like if whenever it is embedded in a high-dimensional Euclidean

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space, it contracts in every neighborhood. So contractible sets are cell-like. In fact, all of the cell-like sets considered in this paper are contractible. Therefore the reader unfamiliar with the term "cell-like space" may replace it with "contractible space" for the purposes of understanding our results.

A *cell-like map* is a continuous surjection $X \to Y$ such that the preimage of every point is cell-like. A pair of compacta X and Y are declared to be cell-like equivalent if there exists a third compactum Z and a pair of cell-like maps $X \xleftarrow{f_1} Z \xrightarrow{f_2} Y$. To obtain an equivalence relation we permit several intermediate spaces: X and Y are declared to be cell-like equivalent if there exists a diagram of compacta and cell-like maps of the form:

$$\begin{array}{ccccccccccccccc} & & Z_1 & & & & Z_3 & & & & & & Z_{2n+1} & & \\ & \swarrow & & \searrow & & \swarrow & & \searrow & & \swarrow & \cdots & \searrow & & \swarrow & & \searrow \\ X & & & & Z_2 & & & & Z_4 & & & & Z_{2n} & & & & Y \end{array} \tag{1}$$

In this setup we write $X \overset{\mathrm{CE}}{\sim} Y$. Clearly, cell-like equivalence is weaker than topological equivalence; moreover, if we require that all spaces involved be finite-dimensional, then cell-like equivalence is stronger than shape equivalence [She72]. Since boundaries of CAT(0) groups are always finite dimensional, this is the case for us [Swe99].

In addition to lying between the notions of topological equivalence and shape equivalence, cell-like equivalence has the advantage of allowing for an easily understood equivariant variation. Compacta X and Y, each equipped with a G-action, are declared to be 'G-equivariantly cell-like equivalent' if there exists a diagram of type (1) for which each of the Z_i also admits a G-action, and the cell-like maps are equivariant. Bestvina has indicated an interest in the following:

Bestvina's Equivariant Cell-like Equivalence Question. *For a given CAT(0) group G, are all boundaries G-equivariantly cell-like equivalent?*

In our previous paper we proposed a general strategy for obtaining an affirmative solution to the equivariant version of Bestvina's question [GM11]. That strategy is straighforward; it is described at the end of this section. We also presented some specific cases where our strategy works. For the groups in that paper, all boundaries were already known to be equivariantly homeomorphic [BR96]. However our results provide something more for those groups; namely, our maps arise naturally as the extensions of equivariant quasi-isometries.

The purpose of this sequel is to extend our work to a more interesting class of groups, namely Croke-Kleiner admissible (or CKA) groups. Described in Section 3, this class includes Croke and Kleiner's original group and the non-rigid examples of [Moo08]. To our knowledge, we provide the first example where it is proven for a group G with multiple boundaries that all of its boundaries are cell-like equivalent.

1.1. CAT(0) groups and their boundaries.

A geodesic metric space X is called a *CAT(0) space* if each of its triangles is at least as thin as the corresponding comparison triangle in the Euclidean plane. A group G is called a *CAT(0) group* if it acts *geometrically* (properly and cocompactly via isometries) on a proper CAT(0) space. A metric d on a CAT(0) space X satisfies a property called *convexity of metric*, which says that given any pair of geodesics α and β parameterized to have constant speed over $[0,1]$, the function $t \mapsto d(\alpha(t), \beta(t))$ is a convex function.

If X is locally compact, then it can be compactified by the addition of its *visual boundary* ∂X which may be defined as the space of all equivalence classes of geodesic rays in X, where a pair of rays $\alpha, \beta : [0,\infty) \to X$ are equivalent if they are asymptotic, i.e., if $\{d(\alpha(t), \beta(t)) \mid t \in [0,\infty)\}$ is bounded above. When G acts geometrically on X we call ∂X *a boundary* for G. Clearly, the action of G on X induces an action by G on ∂X. We put the *cone topology* on ∂X by declaring two geodesic rays to be close in ∂X if they track together for a long time before they diverge.

∂X is seen to be a compactification of X in the following way. Fix a basepoint $x_0 \in X$, and identify X with the space of geodesic line segments emanating from x_0 by identifying the point x with the geodesic $[x_0, x]$. If a sequence of points (x_n) remains unbounded, then the geodesics $\gamma_n = [x_0, x_n]$ get longer and longer. Since X was assumed to be locally compact, convexity of the metric and the Arzela-Ascoli theorem guarantee that, after possibly passing to a subsequence, (γ_n) has a limit γ which is a geodesic ray. Formally, if the γ_n are parameterized to have constant speed, then $\gamma_n \to \gamma$ uniformly on compact subsets of $[0,\infty)$. For more details on this construction (and other properties of CAT(0) spaces) the reader may wish to consult [BH99].

Nonuniqueness of the boundary of a CAT(0) group G is possible since G can act on more than one CAT(0) space. The first example of a group acting on multiple CAT(0) spaces whose boundaries are not homeomorphic was given by Croke and Kleiner in [CK00]. When the action by G is free, covering space techniques and other topological tools allowed Bestvina [Bes96] to show that all boundaries of G are shape equivalent. Later, Ontaneda [Ont05] extended that obsevation to include *all* CAT(0) groups. In those cases where all CAT(0) boundaries of a given G are homeomorphic we say that G is *rigid*. Clearly Bestvina's Cell-like Equivalence Question has a positive answer for all such groups. A positive answer has also been given for groups which split as products with infinite factors [Moo09].

1.2. **Quasi-Isometric Embeddings.** When a group G acts nicely on multiple spaces, a key relationship between those spaces is captured by the notion of 'quasi-isometry'. A function $f:(X,d)\to(X',d')$ between metric spaces is called a *quasi-isometric embedding (QIE)* if there exist positive constants λ and ε such that for all $x,y\in X$

$$\frac{1}{\lambda}d\left(x,y\right)-\varepsilon\leq d'\left(f\left(x\right),f\left(y\right)\right)\leq\lambda d\left(x,y\right)+\varepsilon.$$

If, in addition, X' is contained in some tubular neighborhood of the image of f, then we call f a *quasi-isometry* and declare X and X' to be *quasi-isometric*.

By choosing a finite generating set and endowing it with the corresponding word metric, any finitely generated group can be viewed as a metric space. It follows from the Švarc-Milnor Lemma that, up to quasi-isometry, this metric space is independent of the choice of generating set; in fact if X is any length space on which G acts geometrically, then for any base point $x_0\in X$ the *orbit map* $G\to X$ given by $g\mapsto gx_0$ is a quasi-isometry [Šva55, Mil68].

Given a subset A of a CAT(0) space X, define the *limset* of A to be the collection of all limit points of A lying in ∂X. In other words, $\operatorname{limset}A=\overline{A}-X$ where the closure is taken in $\overline{X}$. Clearly any such limset is a closed subset of ∂X. If G acts on a proper CAT(0) space properly discontinuously by isometries, then we denote by $\operatorname{limset}(X,G)$ the limset of the image of G under the orbit map. This provides a compactification $G\cup\operatorname{limset}(X,G)$ for G. This is easily seen to be independent of basepoint, since the Hausdorff distance between any pair of G-orbits is finite. Note that if this action is cocompact then $\operatorname{limset}(X,G)=\partial X$.

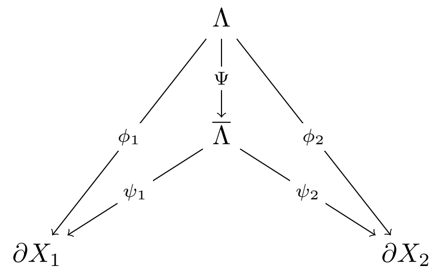
If G acts properly discontinuously on two proper CAT(0) spaces X and Y, then we may compare the two compactifications $\Lambda=\operatorname{limset}(X,G)$ and $\Lambda'=\operatorname{limset}(Y,G)$. If the identity map on G extends continuously to a map $G\cup\Lambda\to G\cup\Lambda'$, then the restriction $\Lambda\to\Lambda'$ is called a *limset map*. The existence of such a map is very strong. It means that whenever an unbounded sequence of group elements converges in one compactification, it also converges in the other. Two limsets are considered *equivalent* if there is a limset map between them which is a homeomorphism.

We call G *strongly rigid* if whenever G acts geometrically on proper CAT(0) spaces X and Y, the boundaries ∂X an ∂Y are equivalent in the above sense. Examples of such groups include free abelian groups, δ-hyperbolic CAT(0) groups (or *negatively curved* groups), and others [KL97, HK05]. Clearly Bestvina's Equivariant Cell-like Equivalence Question has a positive answer for all strongly rigid groups. The question also has a positive answer for certain products [BR96, Rua99], although these are not strongly rigid in the sense of this paper. CKA groups are never strongly rigid, as Croke and Kleiner proved in [CK02].

If G acts properly discontinuously by isometries on X, then so does any subgroup $H\leq G$. If H has infinite index, then it does not act geometrically, since cocompactness has been lost. Moreover, even when H is finitely generated, it is not always the case that $H\hookrightarrow G$ (or equivalently $h\mapsto hx_0$) is a QIE. An object of special interest to us will be $\operatorname{limset}H$ for certain subgroups H of CAT(0) groups.

1.3. **The standard strategy and our Main Conjecture.** Suppose G acts geometrically on a pair of proper CAT(0) spaces X_1 and X_2. Then the l_2-metric $d=\sqrt{d_1^2+d_2^2}$ makes $X_1\times X_2$ a proper CAT(0) space on which $G\times G$ acts geometrically via the product action. It is a standard fact that $\partial\left(X_1\times X_2\right)$ is homeomorphic to the topological join of the original boundaries [BH99, Example II.8.11(6)]. To see this, first choose a base point $(x_1,x_2)\in X_1\times X_2$ and define slopes of segments and rays in $X_1\times X_2$ based at (x_1,x_2) in the obvious way. A ray α may be projected into X_1 and X_2 to obtain a pair of rays α_1 and α_2 —except in those cases where the slope is 0 or ∞ which produce an α_i that is constant. Assign to each α

Figure 1. The Schmear and the Furstenberg Limit Set



three coordinates: α_1, α_2, and the slope of α. Keeping in mind the exceptional cases where α has slope 0 or ∞, we get a correspondence between $\partial(X_1 \times X_2)$ and the quotient space

$$\partial X_1 * \partial X_2 = \partial X_1 \times \partial X_2 \times [0,\infty] / \sim$$

where $(\alpha_1, \alpha_2, 0) \sim (\alpha_1, \alpha_2', 0)$ for all $\alpha_2, \alpha_2' \in \partial X_2$ and $(\alpha_1, \alpha_2, \infty) \sim (\alpha_1', \alpha_2, \infty)$ for all $\alpha_1, \alpha_1' \in \partial X_1$. This join contains a *preferred copy* of ∂X_1 (all points with slope 0) and a *preferred copy* of ∂X_2 (all points with slope ∞) which may be identified with the boundaries of convex subspaces $X_1 \times \{x_2\}$ and $\{x_1\} \times X_2$.

Now consider the diagonal subgroup $G^\Delta = \{(g,g) \mid g \in G\}$ of $G \times G$. Clearly, G^Δ is isomorphic to G and acts on $X_1 \times X_2$ properly by isometries. For $g \in G$, we will denote $g^\Delta = (g,g)$. In [GM11, Section 4.1], we make the following observations:

(i) The map $g \longmapsto g^\Delta(x_1, x_2)$ is a QIE of G into $X_1 \times X_2$, and
(ii) $\operatorname{limset} G^\Delta$ is a closed subset of $\partial X_1 * \partial X_2$ that misses the preferred copies of ∂X_1 and ∂X_2.

We refer to $\Lambda = \operatorname{limset} G^\Delta$ as a *schmear* of ∂X_1 and ∂X_2. Item (i) above is used in proving (ii) and offers hope that Λ resembles a *boundary* for G. Item (ii) allows us to restrict the projections of $\partial X_1 \times \partial X_2 \times (0,\infty)$ onto ∂X_1 and ∂X_2 to obtain a pair of G-equivariant *schmear maps* $\phi_1 : \Lambda \to \partial X_1$ and $\phi_2 : \Lambda \to \partial X_2$.

Since Λ lives in the join and misses ∂X_1 and ∂X_2, we may think of it as living in the product $\partial X_1 \times \partial X_2 \times (0,\infty)$. Here the schmear maps are just the coordinate projection maps onto ∂X_1 and ∂X_2. Let $\overline{\Lambda}$ denote the image of the coordinate projection map $\Lambda \to \partial X_1 \times \partial X_2$, and $\overline{\phi_i} : \overline{\Lambda} \to \partial X_i$ also be coordinate projections, as in Figure 1. Following the language of Link [Lin10], we refer to $\overline{\Lambda}$ as the *Furstenberg limit set* (or *F-set*) of G^Δ. We will also refer to it as the F-set for the actions of G on X_1 and X_2, the maps $\overline{\phi_i}$ as *the associated F-maps*, and point preimages of these F-maps as *F-fibers*. These maps are automatically continuous, equivariant, and surjective.

Our standard strategy is summed up by the following:

Main Conjecture. *Suppose G acts geometrically on a pair of CAT(0) spaces X_1 and X_2. Then both F-maps are cell-like; hence* ∂X_1 and ∂X_2 *are G-equivariantly cell-like equivalent.*

In fact, we hope for something stronger, namely that the schmear maps themselves are cell-like. In the case where G contains a pair of independent rank-one elements (which includes the groups studied here), the two conjectures are equivalent by [Lin10, Theorems B and C]. The advantage to the schmear is that it can be realized as the limit set of an actual group action. When we pass to the F-set, this action is lost, although there is still a natural action of G on $\partial X_1 \times \partial X_2$.

1.4. **The main results.** The main result of [GM11] is the following. We have stated it for a class of actions slightly more general than geometric actions, although in this paper the actions will all be geometric.

Theorem 1 (G-M,2011)**.** *Assume an infinite group G acts properly discontinuously by isometries on CAT(0) spaces X_1 and X_2 such that $G \to X_1$ and $G \to X_2$ are QIEs. Then there exists an action of G by isometries on a third CAT(0) space X such that $G \to X$ is a QIE and there are natural limset maps* $\operatorname{limset} G \to \partial X_i$. *If the action of G on both X_i is by semi-simple isometries, then so is the action on X.*

As an application, we used this to prove

Theorem 2 (G-M,2011)**.** *Whenever $G = \mathbb{F}_m \times \mathbb{Z}^d$ acts geometrically on two proper CAT(0) spaces, the corresponding schmear fibers are topological cells. In particular, the schmear maps are cell-like.*

Note that since those groups have higher rank, the work of [Lin10] does not apply to them. The main theorem of this paper is

Theorem 3 (Main Theorem)**.** *Let G be a Croke-Kleiner admissible group acting geometrically on two proper CAT(0) spaces X_1 and X_2 and $\overline{\Lambda}$ denote the F-set of the pair. Then the corresponding F-fibers are contractible. In particular, the F-maps $\overline{\Lambda} \to \partial X_i$ are cell-like.*

To illustrate some of the subtlety, the reader should keep in mind that a geodesic ray may be quasi-isometrically embedded in $\mathbb{E}^2$ in such a way that its limit set is the entire circle boundary, which is certainly not cell-like! In fact, Staley [Sta11] has shown that for the same class of groups considered in Theorem 2, when the dimension of the boundary is bigger than 1, there are geometric actions on CAT(0) spaces X and Y for which the images of geodesic rays under equivariant quasi-isometries $X \to Y$ have exotic limit sets at infinity. The lesson learned from our theorem seems to be that "taking the whole schmear" has a tendency to paint over oddities in the local behavior of limsets.

As mentioned above, when we combine our theorem with the results of [Lin10], we get

Corollary 4. *Schmear fibers for Croke-Kleiner admissible groups are cell-like.*

In any case, all boundaries of the Croke-Kleiner group are now seen to be equivariantly cell-like equivalent.

Along the way, we prove a much weaker result about schmear maps for general CAT(0) groups.

Theorem 5 (Schmear Fibers are Connected)**.** *Let G be a CAT(0) group acting geometrically on two proper CAT(0) spaces. Then the corresponding schmear fibers are connected.*

1.5. **Relationship to the Tits Boundary.** The Tits metric induces another common topology on a CAT(0) boundary. This gives it a beautiful geometric structure where geodesics in the boundary correspond to the presence of "flatness" in the space. It is known, for instance, that the dimension of the Tits boundary is exactly one less than the dimension of the largest copy of Euclidean space which can be embedded [Kle99].

All Tits boundaries of CKA groups are obviously homeomorphic, so Bestvina's question does not appear interesting on that level. Nonetheless we find it curious that in the examples of this paper (and those of our previous paper), F-fibers turn out also to be contractible when given the Tits topology. This leads us to wonder if sequences converging to a common boundary point under one group action and which "fan out" in another, are only allowed to do so in "directions of flatness".

2. Schmear Fibers are Connected

We begin by proving Theorem 5. Recall that if C is a metric compactum, then the Gromov-Hausdorff metric on the space C' of subcompacta turns C' into a metric space. It is an exercise to prove that a Gromov-Hausdorff limit of connected compacta is connected.

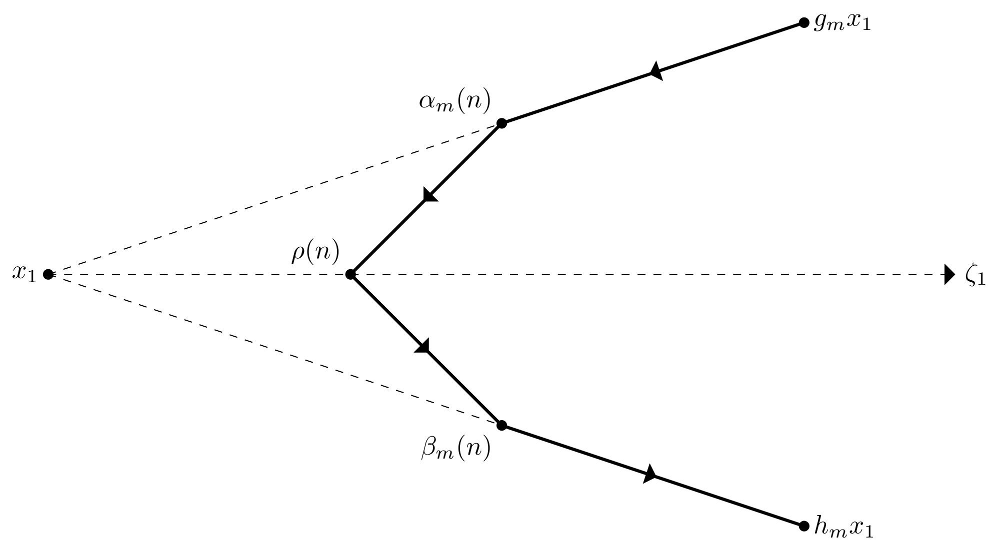
Proof of Theorem 5. Denote $X = X_1 \times X_2$ and choose basepoints $x_i \in X_i$ and $x = (x_1, x_2) \in X$. It will be easier to see the proof if G is torsion free. Then

$$\pi_1(X_1/G) = \pi_1(X_2/G) = G.$$

Choose a homotopy equivalence $\overline{f} : X_1/G \to X_2/G$ which sends the image of x_1 to the image of x_2. This can be lifted to an equivariant homotopy equivalence $f : X_1 \to X_2$ which sends x_1 to x_2. Since this restricts to the orbit map $Gx_1 \to Gx_2$, it is a quasi-isometry. Let $\widetilde{f} : X_1 \to X_1 \times X_2$ denote the graph of f, which is an equivariant (λ, ϵ)-quasi-isometric proper homeomorphic embedding for some $\lambda \geq 1$ and $\epsilon \geq 0$. The image $\widetilde{X}$ contains Gx as a quasi-dense subset, and hence has the same limset. ϕ_i is the extension of coordinate projection $\widetilde{X} \to X_i$.

Consider the following claim:

Claim. *Let $(g_n), (h_n) \subset G$ be two sequences such that $g_n x_1, h_n x_1 \to \zeta_1 \in \partial X_1$. Then there exists a sequence of paths $\widetilde{\gamma}_n$ in $\widetilde{X}$ joining $g_n x$ to $h_n x$ such that their images γ_n in X_1 converge as a Gromov-Hausdorff limit in $X_1 \cup \partial X_1$ to the point ζ_1.*



Suppose this claim holds. Then choose $\nu, \nu' \in \phi_1^{-1}(\zeta_1)$ and sequences $(g_n), (h_n) \subset G$ such that $g_n x \to \nu$ and $h_n x \to \nu'$ and $(g_n x_1)$ and $(h_n x_1)$ both converge to ζ_1. Let $\widetilde{\gamma}_n$ be the paths prescribed by the claim. By passing to a subsequence, we may assume this sequence converges in the Gromov-Hausdorff limit to $K \subset \widetilde{X} \cup \Lambda$. Since $\widetilde{f}$ is a quasi-isometric embedding, the sequence of paths $\widetilde{\gamma}_n$ eventually leave every compact set, which guarantees that $K \subset \Lambda$. Certainly K contains ν and ν'. Furthermore, every point of K can be written as the limit of a sequence of points $(\widetilde{y}_n)$ where $\widetilde{y}_n \in \widetilde{\gamma}_n$. By construction, the image of this sequence in X_1 gives a sequence converging to ζ_1. It follows that $\phi_1(K) = \{\zeta_1\}$. Therefore for every pair of points in $\phi_1^{-1}(\zeta_1)$, we have found a connected subset of $\phi_1^{-1}(\zeta_1)$ containing both. Therefore $\phi_1^{-1}(\zeta_1)$ is connected.

We now prove the claim. Consider the geodesics $\alpha_m = [x_1, g_m x_1]$, $\beta_m = [x_1, h_m x_1]$ both parameterized to have unit speed and let ρ be the ray based at x_1 going out to ζ_1. Choose a basis $\{U_n\}$ of ζ_1 in $X_1 \cup \partial X_1$. Given any $n \geq 0$, there is an $N_n \geq 0$ such that whenever $m \geq N_n$, $d(\alpha_m(n), \rho(n))$ and $d(\beta_m(n), \rho(n))$ are both less than 1. For $N_n \leq m \leq N_{n+1}$, we choose γ_m to be the path from $g_m x_1$ which follows α_m back down to $\alpha_m(n)$, jumps over to $\beta_m(n)$ by a path of length ≤ 2, and heads back up β_m, ending at $h_m x_1$. Convexity of the metric guarantees that $\gamma_n \subset U_n$. It is easy to verify that $\widetilde{\gamma}_n = \widetilde{f}(\gamma_n)$ satisfies the claim.

In closing, we observe that by replacing the paths $\widetilde{\gamma}_n$ with k-chains in Gx, we get an argument which does not require the group to be torsion-free. $\square$

3. Croke-Kleiner Admissible Groups

Recall that a geodesic space is called δ-hyperbolic if given any triangle (possibly with ideal vertices) then each side lies in the δ-tubular neighborhood of the union of the other two sides.

In the language of [CK02], a graph of groups $\mathcal{G}$ is called *admissible* if it satisfies all of the following:

(1) $\mathcal{G}$ is a finite graph with at least one edge.

(2) Each vertex group $\overline{G_v}$ has center $Z(\overline{G_v}) \cong \mathbb{Z}$, $\overline{H_v} = \overline{G_v}/Z(\overline{G_v})$ is nonelementary hyperbolic, and every edge subgroup $\overline{G_e}$ is isomorphic to $\mathbb{Z}^2$.

(3) Let e_1 and e_2 be distinct directed edges entering a vertex v, and for $i = 1, 2$ let $K_i \subset \overline{G_v}$ be the image of the edge homomorphism $\overline{G_{e_i}} \to \overline{G_v}$. Then for every $g \in \overline{G_v}$, gK_1g^{-1} is not commensurable with K_2, and for every $g \in \overline{G_v} \setminus K_i$, gK_ig^{-1} is not commensurable with K_i.

(4) For every edge group $\overline{G_e}$, if $\alpha_i : \overline{G_e} \to \overline{G_{v_i}}$ are the edge monomorphisms, then the subgroup generated by $\alpha_1^{-1}(Z(\overline{G_{v_1}}))$ and $\alpha_2^{-1}(Z(\overline{G_{v_2}}))$ have finite index in $\overline{G_e} \cong \mathbb{Z}^2$.

The fundamental group of such a graph of groups will be called *Croke-Kleiner Admissible (CKA)*.

The conditions listed above are satisfied by Croke and Kleiner's original example of a non-rigid CAT(0) group, and the non-rigid knot groups discussed in [Moo08]. They also include many other examples, for which the rigidity question is not known. Some of these may even have locally connected boundaries. However, as a Corollary to their main theorem, CKA groups are never strongly rigid.

3.1. **Decompositions of CKA Spaces.** Let G be a CKA group acting geometrically on a proper CAT(0) space X. As shown in [CK02, Section 3.2], X admits a decomposition corresponding to the decomposition of G as a graph of groups. Let T be the Bass-Serre tree for the underlying graph of groups. Given a simplex σ of T, its stabilizing subgroup is denoted by by G_σ (these are isomorphic copies of the groups $\overline{G_\sigma}$ coming from the graph of groups.) For every vertex $v \in T$, $Z(G_v)$ is infinite cyclic and $H_v = G_v/Z(G_v)$ is a nonelementary hyperbolic group.

Recall that the *minset* of an isometry i of a CAT(0) space X is the set of points $x \in X$ such that $d(x, ix)$ is minimal. For a group Γ of isometries, the minset of Γ is the intersection of the minsets its elements. This is a closed conved subspace of X. For vertices v of T, let Y_v denote the minset in X of $Z(G_v)$. The family $\{Y_v\}$ is clearly periodic. Choose also a periodic family $\{Y_e\}$ of G_e-invariant 2-flats (e ranging over the edges of T). Both familes $\{Y_e\}$ and $\{Y_v\}$ are locally finite by [CK02, Lemma 3.10].

The following Lemma summarizes the results of [CK02, Section 3.2] which are relevant here.

Lemma 3.1. *There is a periodic family of closed, convex subspaces $\{X_\sigma\}_{\sigma\in T}$ and a $K > 0$ satisfying the following properties:*

(1) *Both familes $\{X_e\}$ and $\{X_v\}$ cover X.*

(2) *For every simplex σ of T, X_σ is G_σ-invariant with compact quotient.*

(3) *For every simplex σ of T, $Y_\sigma \subset X_\sigma \subset N_K(Y_\sigma)$.*

(4) *For every vertex v of T, Y_v splits as $\overline{Y_v} \times \mathbb{R}$ where $Z(G_v)$ acts only in the $\mathbb{R}$-coordinate and H_v projects to a cocompact action of $\overline{Y_v}$. In particular, $\overline{Y_v}$ is δ-hyperbolic.*

(5) *Whenever an edge e separates a pair of vertices u and v of T, any path α from a point of X_u to a point of X_v must pass through X_e. In fact, the point at which α leaves X_u is a point interior to X_e.*

The spaces X_σ are called *vertex* or *edge spaces* depending on whether σ is a vertex or an edge.

3.2. **Boundaries of CKA Groups.** A large part of a boundary of a CKA group is just the union of boundaries of vertex spaces. If v is a vertex of T, then parts (3) and (4) of Lemma 3.1 tells us that ∂X_v decomposes as the suspension of $\partial\overline{Y_v}$. The suspension points are the endpoints of the $\mathbb{R}$-factor. We will refer to the suspenion points as *poles* and the suspension arcs as *longitudes*.

Points of $\bigcup_{v\in T^0} \partial X_v$ are called *rational*, and points in the complement are called *irrational*. Denote the former set by RX and the latter by IX. IX is very easy to understand – components are either singletons or intervals [CK02, Proposition 7.3].

This next result tells us that a geodesic ray limits out to an irrational point iff it eventually stays far away from every vertex space (thus it is safe to refer to geodesic rays themselves as irrational and rational). As discussed in [CK02], there is a G-equivariant coarse Lipschitz map $\rho : X \to T^0$, and the vertex and edge spaces may be chosen so that the following holds [CK02, Lemmas 3.19 and 3.22].

Lemma 3.2. *Let γ be a geodesic ray going out to $\zeta \in \partial X$. Then exactly one of the following is true:*

(1) *$\rho \circ \gamma$ is unbounded and its image lies in a uniform tubular neighborhood of a unique geodesic ray, τ, in T starting at $\rho(\gamma(0))$. The geodesic γ intersects X_e for all but finitely many edges e of τ. In this case, $\zeta \in IX$. Furthermore, whenever γ' is an asymptotic ray $\rho \circ \gamma'$ is also in a tubular neighborhood of the same τ.*

(2) *$\rho \circ \gamma$ is bounded, and γ eventually stays inside X_v for some vertex v. In this case, there is a subcomplex $T_\gamma \subset T$ defined by the property that for each simplex σ of T, σ is in T_γ if and only if γ is asymptotic to a ray in X_σ. The possibilities for T_γ are:*
   (a) *a single vertex v (in which case $\zeta \in \partial X_v$ is not in the boundary of any edge space).*

   (b) *an edge e (in which case $\zeta \in \partial X_e$).*

   (c) *the closed star at a vertex v (in which case ζ is one of the suspension points of ∂X_v).*

Part (1) says that if γ and γ' are asymptotic irrational geodesic rays, then their image in T completely determines which vertex spaces and edge spaces they pass through. Specifically, if $e_1, e_2, ...$ are the sequence of edges in the ray τ, then both γ and γ' must pass through X_{e_i} for all but finitely many i. Even further, if we put this together with Lemma **??**, we see that if v_i is the vertex shared by e_i and e_{i+1}, then γ must pass through X_v, and when it leaves, it does so at a point of $X_{e_{i+1}}$.

4. Notes on δ-Hyperbolic Spaces

There are many statements about δ-hyperbolic spaces of the form "For every $C > 0$, there is a constant R depending only on δ and C such that whenever x and y are points satisfying property $P(C)$, then $d(x, y) < R$". Thus R is a measure of coarse closeness. Since often times C also was a measure of coarse closeness, these constants can pile up quickly and make it difficult to keep track of precisely. Really all that is necessary is keeping track of the order of statements and imagining that there is one universal constant large to satisfy everything. In this paper we will simply use the word *near* or *close* or say that *the distance is bounded* in a statement to mean that a distance is bounded by a constant depending only on δ, and any other constant which may arise in the statement. When α and β are geodesics for which $\alpha(t)$ and $\beta(t)$ are close for every t, we say the geodesics *track eachother.*

Here is a well-known fact.

Lemma 4.1 (Bounded Tracking Property)**.** *Let α and β be a pair of geodesics in a hyperbolic space parameterized to have constant speed over* $[0, 1]$*. α and β track eachother.*

When there is a (λ, ϵ)-quasi-isometry or quasi-isometric embedding of another space into the hyperbolic space, then we will assume that "near" also takes into account the unlisted constants. Recall that *quasi-geodesic* is a QIE of an interval. These behave well in hyperbolic spaces [BH99, Theorem III.H.1.7]:

Lemma 4.2 (Stability of Quasi-Geodesics)**.** *A quasi-geodesic in a hyperbolic space remains close to a geodesic joining its endpoints.*

A point p in a hyperbolic space is called a *center* for a triangle $\triangle xyz$ (possibly with ideal vertices) if it is close to each of the three sides. It is well-known the set of centers for a triangle is bounded. Since the proof of this fact is in the flavor of later proofs, we include it here.

Whenever $\triangle xyz$ is a triangle in a hyperbolic space (without ideal vertices), there are points $\overline{x} \in [y, z]$, $\overline{y} \in [x, z]$, and $\overline{z} \in [x, y]$ such that $d(x, \overline{y}) = d(x, \overline{z})$, $d(y, \overline{x}) = d(y, \overline{z})$, and $d(z, \overline{x}) = d(z, \overline{y})$. These points are called the *internal points* of the triangle. Internal points close to eachother and the geodesics joining them to the vertices of the triangle track together [BH99, Proposition III.H.1.17].

Lemma 4.3. *If p and q are a pair of points close to all three sides of a common triangle, then p and q are close.*

Proof. Suppose the triangle in question has no ideal vertices. Choose unit speed parameterizations α, β, and γ for the three sides of the triangle so that $\alpha(0)$, $\beta(0)$, and $\gamma(0)$ are the internal points, and $\alpha(a) = \beta(-a)$, $\beta(b) = \gamma(-b)$, and $\gamma(c) = \alpha(-c)$ for positive numbers a, b, and c. Let $\alpha(r)$, $\beta(s)$, and $\gamma(t)$ be the points on the respective sides of the triangle which are closest to p. Then two of the numbers r, s, and t must have the same sign. Without loss of generality, assume r and s are both positive. Since $\alpha(r)$ is close to $\beta(-r)$ and $\beta(s)$ and $\alpha(r)$ are both close p, it follows that $\beta(-r)$ is close to $\beta(s)$. Since $\beta(0)$ is between these two, it must be close to $\beta(s)$ and hence to p. Similarly, q is also close to $\beta(0)$. So p and q are close.

Now what if the triangle has some ideal vertices? Suppose x is an ideal vertex of $\triangle xyz$. Let α and β be the sides $[x, y]$ and $[x, z]$ parameterized to have unit speed and so that $\alpha(t)$ and $\beta(t)$ go out to x as $t \to \infty$. Find points $\alpha(a)$ and $\beta(b)$ on these sides close to p and $\alpha(a')$ and $\beta(b')$ close to q. Take T larger than a, b, a', and b'. By Bounded Tracking, p and q are also close to (any geodesics) $[\alpha(T), z]$ and $[\alpha(T), y]$. So can replace x with $\alpha(T)$, thereby removing an ideal vertex. Continue until all ideal vertices have been removed. □

Lemma 4.4. *Let α and β be a pair of lines in a hyperbolic space and a pair of points and p on α and q on β such that p is closest to β and q is closest to α. If x is close to α and y is close to β, then any geodesic $[x, y]$ passes near both p and q.*

Proof. By Bounded Tracking, we may assume x and y lie on α and β respectively. Look at the triangle $\triangle yqp$ where $[y, q]$ is chosen as a subsegment of β. Since the internal point on $[y, q]$ is no closer to the internal point on $[q, p]$ than q, q is close to this internal point. Then the internal point on $[y, p]$is also close to q. Thus we have shown that $[y, p]$ passes near q. Similarly, $[x, q]$ passes near p. **Something missing – what if $p = q$?** □

5. F-Fibers of CKA Groups

Assume now that G acts geometrically on another CAT(0) space X' which also has a decomposition into vertex and edge spaces. When we wish to refer to aspects of the decomposition of X', we will use primes.

Let $\overline{\Lambda} \subset \partial X \times \partial X'$ denote the F-set of the pair X and X', $\phi : \overline{\Lambda} \to \partial X$ and $\phi' : \overline{\Lambda} \to \partial X'$ be the F-maps. Since these are just coordinate projection maps, for any $\zeta \in \partial X$, ϕ' restricts to a homeomorphic embedding of the F-fiber $\phi^{-1}(\zeta) \subset \{\zeta\} \times \partial X'$. Therefore to prove that this F-fiber is contractible, it suffices to prove that its image, $\Lambda'(\zeta)$ in $\partial X'$ is contractible.

Remark 5.1. Since Y_v is quasi-dense in X_v, whenever a geodesic γ in X passes through X_v, there is a geodesic β in Y_v which tracks with γ as long as it passes through X_v. Saying that its projection passes near a point $x \in \overline{Y_v}$ is the same as saying that β (hence γ) passes near the line $\{x\} \times \mathbb{R}$.

Let v be a vertex and e an edge in its star with w the other endpoint of e. Property (4) above tells us that $G_e/Z(G_v)$ is a vitually cyclic subgroup of H_v. Let $L = L(v, e) \subset \overline{Y_v}$ be an axis for this subgroup, so that the 2-flat $Y(v, e) = L(v, e) \times \mathbb{R}$ is G_e-invariant.

Let $\mathcal{L}_v$ denote the collection of such lines in $\overline{Y_v}$; this collection is locally finite. Let $x_v \in Y_v$ be chosen basepoints for every vertex v of T and that y_v is the $\overline{Y_v}$-coordinate of x_v.

If $(x_n) \subset Y_v$ is a sequence of points converging to a point $\zeta \in \partial Y_v$ not a pole of ∂Y_v, then it is easy to check which longitude ζ lies in by looking at the image y_n of the sequence under coordinate projection $Y_v \to \overline{Y_v}$. By identifying $\overline{Y_v}$ with the subspace $\overline{Y_v} \times \{0\}$ of Y_v, y_n must converge to a point in the same longitude as x_n. Conversely, if y_n converges to a point in a longitude l of ∂Y_v, then ζ lies in the closure of l (either it lies in l or it is a pole of ∂Y_v).

Remark 5.2. Since H_v is negatively curved, f_v extends to a homeomorphism $\partial f_v : \partial \overline{Y_v} \to \partial \overline{Y'_V}$. As proven in [BR96], this extends to a G_v-equivariant homeomorphism $\partial X_v \to \partial X'_v$ taking poles to poles and longitudes to longitudes (in fact, this is an isometry in the Tits metric). If l is a longitude of ∂X_v, we will refer to its image under this homeomorphism as *the corresponding longitude of $\partial X'_v$.*

5.1. Types of Sequences.

Assume that $\zeta \in RX$, v is a vertex of T, and $(g_n) \subset G$ is a sequence such that $g_n x_v \to \zeta$. We may assume, after possibly passing to a subsequence, that (g_n) has one of the following types. In each case, $[v, g_n v]$ denotes the geodesic edge path in T from v to $g_n v$.

- (Type A) $g_n v = v$ for all n.
- (Type B) $d_T(v, g_n v) \geq 1$ but no pair of $[v, g_n v]$ shares the same first edge.
- (Type C) $g_n v = w$ for some w in the link of v.
- (Type D) $d_T(v, g_n v) \geq 2$ and all $[v, g_n v]$ share the first edge, but no pair shares a second.

- (Type E) $d_T(v, g_n v) \geq 2$ and all $[v, g_n v]$ share the same first two edges.

Lemma 5.3 (Type E)**.** *Suppose (g_n) has Type E and $\zeta \in \partial X_v$. Then ζ is a pole of ∂X_w for some w in the link of v.*

Proof. Denote the geodesic ray emanating from x_v and going out to ζ by γ. Let e_1 and e_2 denote the first two edges shared by all geodesics $[v, g_n v]$, w denote the vertex shared by e_1 and e_2. Let γ_n be the geodesic $[x_v, g_n x_v]$; then $\gamma_n \to \gamma$. In $\overline{Y_w}$, let $\overline{p} \in L_1 = L(w, e_1)$ be a closest point to $L(w, e_2)$. By Lemma 4.4 and Remark 5.1, every γ_n passes near a point $x_n \in \{\overline{p}\} \times \mathbb{R}$; If (x_n) is bounded, then γ leaves X_v, and $\zeta \notin \partial X_v$. So (x_n) must be unbounded, and ζ must be a pole of ∂X_w. □

Lemma 5.4 (Types A and B)**.** *If (g_n) has Type A or B, then $\zeta \in \partial X_v$.*

Proof. Assume $\zeta \notin \partial X_v$. Let γ_n denote the geodesic $[x_v, g_n x_v]$. By hypothesis, γ_n converges to some geodesic ray γ emanating from x_v and going out to ζ. Since we assumed that $\zeta \notin \partial X_v$, γ leaves X_v at some point z interior to X_e for an edge e in the star of v. After leaving X_v, γ must immediately pass through the interior of X_w where w is the other endpoint e. Since $\gamma_n \to \gamma$ and γ passes through the interior of $X_w \setminus X_v$, so does γ_n (when n is large). If the sequence has Type A, then $\{g_n x_v\}$ is contained in X_v and $\zeta \in \partial X_v$. If it has Type B, then convexity of X_v guarantees that w lies between v and $g_n v$ (which means that all $[v, g_n v]$ share the same first edge). Either way, we are in trouble. □

5.2. **F-Fibers of Rational Non-Poles.** The purpose of this section is to prove

Proposition 5.5. *Suppose $\zeta \in \partial X_v$ is not a pole. Then $\Lambda(\zeta)$ is a subset of an arc. To be precise*

(1) *If $\zeta \notin \partial X_w$ for any vertex w in the link of v, then $\Lambda(\zeta)$ is contained in the closure of the longitude of $\partial X'_v$ corresponding to the longitude containing ζ.*

(2) *If $\zeta \in \partial X_w$ for some vertex w in the link of v, then it lies in an intersection of two longitudes – one from ∂X_v and one from ∂X_w. Then $\Lambda(\zeta)$ lies in the union of the corresponding longitudes of $\partial X'_v$ and $\partial X'_w$.*

Since $\Lambda(\zeta)$ is connected, the intermediate value theorem guarantees that it is an arc. So

Corollary 5.6. *If $\zeta \in RX_v$ is not a pole, then $\Lambda(\zeta)$ is contractible.*

We will assume for the remainder of this subsection that $(g_n) \subset G$ is a sequence such that $g_n x_v \to \zeta \in \partial X$ and $g_n x'_v \to \zeta' \in \partial X'$. Let $f_v : \overline{Y_v} \to \overline{Y'_v}$ be an H_v-equivariant quasi-isometry.

Lemma 5.7. *Let v be a vertex of T and e_1 and e_2 be two edges in the star of v. Let $\overline{p} \in L_2 = L(v, e_2)$ be a closest point to $L_1 = L(v, e_1)$ and $\overline{p'} \in L'_2 = L'(v, e_2)$ be a closest point to $L'_1 = L'(v, e_1)$. Then $f_v(\overline{p})$ is close to $\overline{p}'$.*

Proof. We know that L_1 and L'_1 are axes for the same group element $h_1 \in H_w$. Similarly, L_2 and L'_2 are axes for the same group element h_2. Let A be a geodesic line in Y_v joining the points h_1^∞ and h_2^∞, B be a geodesic line in Y_v joining the points h_1^∞ and $h_2^{-\infty}$, and A' and B' be chosen analogously in Y'_v. By Stability of Quasi-Geodesics, $f_v(A)$ tracks with A, $f_v(B)$ tracks with B, and $f_v(L_2)$ tracks with L'_2. In particular, $f_v(\overline{p})$ is close to A', B', and L'_2. Now apply Lemma 4.3. □

Lemma 5.8 (Containment in Vertex Space Boundary)**.** *If $\zeta' \in \partial X'_v$ and ζ' is not a pole of $\partial X'_w$ for any w in the link of v, then $\zeta \in \partial X_v$.*

Proof. Assume $\zeta \notin \partial X_v$. Types A, B and E are impossible by Lemmas 5.3 and 5.4.

Suppose the sequence has Type D. Let e be the edge in the link of v shared by all $[v, g_n v]$, w be its second endpoint, and e_n denote the second edge in $[v, g_n v]$. Denote $L = L(w, e)$ and $L_n = L(w, e_n)$, and let $\overline{q_n} \in L_n$ be a closest point to L. Denote by γ_n the geodesic $[x_v, g_n x_v]$. Lemma 4.4 and Remark 5.1 guarantees that γ_n passes near $\overline{q_n} \times \mathbb{R}$. If the set $\{\overline{q_n}\}$ is bounded, then by local finiteness of $\mathcal{L}$, the lines L_n cannot be all distinct. So $\{\overline{q_n}\}$ is unbounded, and by passing to a subsequence, we may assume that $\overline{q_n} \to \nu \in \partial \overline{Y_w}$. Note that $\nu \notin \partial L$, since otherwise we would have $\zeta \in \partial X_e \subset \partial X_v$.

Now let γ'_n denote the geodesic $[x'_v, g_n x'_v]$. Get analogous lines L' and L'_n in $\overline{Y'_w}$ and $\overline{q'_n} \in L'_n$. Again, each γ_n passes near $\{\overline{q_n}'\} \times \mathbb{R}$ where $\overline{q_n}' \in L'_n$ is a closest point to L'. By Lemma 5.7, the sequence $\{\overline{q_n}'\}$ is unbounded and converges to $\nu' = \partial f_w(\nu)$. Since L and L' are axes of the same group element and f_w is equivariant, $\nu \notin \partial L'$. So $\zeta' \notin \partial X'_v$ for a contradiction.

Finally, suppose the sequence has type C. Let γ and γ' denote the geodesic rays in X and X' based at x_v and x'_v going out to ζ and ζ' respectively and $L = L(w, e)$. Since ζ' is not a pole of $\partial X'_w$, it is contained in a longitude, say determined by the point $\overline{\zeta} \in \partial\overline{Y_w}$.

Now, ζ is also the limit of the sequence $a_n = g_n g_1^{-1} \subset G_{g_1 v}$, and $\overline{\zeta}$ is the limit of the sequence $b_n = a_n Z(G_{g_1 v}) \subset H_{g_1 v}$. The hypothesis that $\zeta \notin \partial X_v$ means that $\overline{\zeta} \notin \partial L$. As in the previous case, strong rigidity of H_w guarantees that in $\overline{Y'_w}$, the sequence (b_n) cannot converge to either boundary point of $L'(w, e)$. In X'_w, this means that the sequence (a_n) does not converge to a point of ∂X_e other than a pole of $\partial X'_w$. That is, either ζ' is a pole of $\partial X'_w$ or $\zeta' \notin \partial X'_v$ which is a contradiction. □

Proof of Proposition 5.5. By Lemma 5.8, we know that $\zeta' \in \partial X'_v$. The sequence cannot be of Type E by Lemma 5.3. If the sequence has Type A, then write $h_n = g_n Z(G_v) \in H_v$. Since $h_n y_v$ converges to a point of l_v, Remark 5.2 guarantees that $h_n y'_v$ will converge to a point of l'_v and so ζ' must lie in the closure of l'_v. If the sequence has Type B, then denote by e_n the first edge in the geodesic edge path $[v, g_n v]$, $L_n = L(v, e_n)$, and $L'_n = L'(v, e_n)$. Choose $p_n \in L_n$ to be a point closest to y_v and $p'_n \in L'_n$ to be a point closest to y'_v. Since the $\{e_n\}$ are all distinct, the sequences (p_n) and (p'_n) remain unbounded and converge to points $\nu \in \partial\overline{Y_v}$ and $\nu' \in \partial\overline{Y_v}$. Since the geodesics $[x_v, g_n x_v]$ and $[x'_v, g_n x'_v]$ pass near the lines $\{p_n\} \times \mathbb{R}$ and $\{p'_n\} \times \mathbb{R}$, $\nu \in l_v$, and, by Remark 5.2, ν' lies in the closure of l'_v.

Finally suppose the sequence has Type C or D. Let e denote the common edge, w the other endpoint of e, and e_n the second edge in the geodesic edge path $[v, g_n v]$. Denote also $L = L(w, e)$. If (g_n) has Type C, choose for all n a point $\overline{p_n} \in L$ closest to $g_n g_1^{-1} Z(G_w) y_w$. If (g_n) has Type D, then let e_n denote the second edge of $[v, g_n v]$ and choose $\overline{p_n} \in L$ to be a closest point to $L(v, e_n)$. Either way, $[x_v, g_n x_v]$ passes near the line $\{p_n\} \times \mathbb{R}$ in Y_v. If (p_n) remains bounded, then $g_n x_v$ converges to a point of $\partial X_w \setminus \partial X_e$, which is a contradiction, since we assumed that $\zeta \in \partial X_e$. So (p_n) converges to an endpoint ν of L. Let l_w denote the longitude of ∂X_w containing ν. Remark 5.2 guarantees that $(f_w(p_n))$ converges to the endpoint of L' corresponding to ν. Since the geodesics $[x'_v, g_n x'_v]$ all pass near the lines $\{f_w(p_n)\} \times \mathbb{R}$ (lines in Y_w), it follows that ζ' is in the closure of l'_w, as desired. □

5.3. **F-Fibers of Poles.** Here we deal with the pole case. We will show

Proposition 5.9. *Let $\zeta \in \partial X_v$ be a pole. Then $\Lambda(\zeta)$ is homeomorphic to a cone. To be precise, $\Lambda(\zeta)$ contains exactly one pole ζ' of $\partial X'_v$, and whenever $\nu' \in \Lambda(\zeta)$ is another point and η' is between ζ' and ν' (on the longitude containing ν'), then $\eta' \in \Lambda(\zeta)$.*

This leads to

Corollary 5.10. *If $\zeta \in \partial X_v$ is a pole, then $\Lambda(\zeta)$ is contractible.*

The following technical lemma is an exercise in real analysis.

Lemma 5.11. *Let $b > a$ and $\big\{Q(n, m) \big| n, m$ are nonnegative integers$\big\} \subset [a, b]$ such that all of the following hold:*

(1) $\lim_{n\to\infty} Q(n, 0) = a$.

(2) *For fixed $n \geq 0$, $\lim_{m\to\infty} Q(n, m) = b$.*

(3) *For all $\epsilon > 0$, there is an $N \geq 0$ large enough so that whenever $n \geq N$, $Q(n, m+1) < Q(n, m) + \epsilon$.*

Then given any $q \in [a, b]$, there are increasing sequences n_k and m_k such that

$$\lim_{k\to\infty} Q(n_k, m_k) = q.$$

Recall that the *Alexandrov angle* between a pair of geodesics (either segments or rays) α and β emanating from a common basepoint x_0 is defined as the limit as $t \to 0$ of corresponding angles in comparison triangles

$\overline{\triangle}x\alpha(t)\beta(t)$ in Euclidean space. If the other endpoints of α are y and z, then this angle is denoted by $\angle_x(y,z)$.

Things to mention:

- Continuity with fixed basepoint
- CAT(0) angles smaller than flat angles
- Triangle inequality

Lemma 5.12. *Let $\epsilon > 0$ be given. Then there exists a $D \geq 0$ such that whenever w is a vertex next to v such that $d(y_v, L(v,e)) \geq D$ and $z \in X$ such that $[x_v, z]$ passes through X_w, then*

$$\angle_{x_v}(c^{-\infty}, cz) \leq \angle_{x_v}(c^{-\infty}, z) + \epsilon.$$

Proof. Let $y_e \in L(v,e)$ be the closest point to y_v. By Lemma 4.4, both geodesics $[x_v, z]$ and $[x_v, cz]$ pass near the line $L = \{y_e\} \times \mathbb{R}$. By convexity of metric, they must remain close to the flat strip $[y_v, y_e] \times \mathbb{R}$ before this time. Let $x_e \in L$ be a point near $[x_v, z]$ and α be the geodesic ray emanating from cx_e going out to $c^{-\infty}$. Now, $[x_v, cz]$ either passes near $[cx_v, cx_e]$ or it passes near α. If the former holds, then it also passes near $[cx_v, cz]$ before this passes near cx_e and remains close afterwards. Hence $[x_v, cz]$ also passes near cx_e. Either way, there is a constant $R \geq \delta$ depending only on δ such that $[x_v, cz]$ passes within a distance of R from α. Now let $D \geq 0$ be large enough so that whenever $\triangle abc$ is a triangle in Euclidean space such that $d(a,b)$ and $d(a,c)$ both exceed D and $d(b,c) \leq R$, then $\angle_a(b,c) < \epsilon/3$ and such that $\arctan(\tau/D) < \epsilon/3$ where τ is the minimal translation length of c ($\tau = d(x_e, cx_e)$). Then $\angle x_v(x_e, cx_e) < \epsilon/3$ and the conclusion follows from the triangle inequality for Alexandrov angles. □

Lemma 5.13. *Suppose $(g_n) \subset G$ is a sequence such that $g_n x_v \to c^\infty$ in ∂X. Then $g_n x_v$ cannot converge to $c^{-\infty}$ in $\partial X'$.*

Proof. Given any vertex w in the link of v, choose a point $a_w \in L(v,w)$ and let M_w be the line joining the poles of ∂X_w which contains the point $(a_w, 0) \in Y_v$. Choose also $a'_w \in L'(v,w)$ such that a'_w is close to $f_v(a_w)$, and let M'_w be the corresponding line in Y'_v. Since $g_n \to c^\infty$, $k_n \to \infty$ and since $g_n \to c^{-\infty}$, $k_n \to -\infty$. This is a contradiction. □

Proof of Proposition 5.9. We will assume that $\zeta = c^\infty$ in ∂X_v. Certainly the corresponding pole $\zeta' = c^\infty \in \partial X'_v$ is also in $\Lambda(\zeta)$, since c_n converges to c^∞ in both ∂X_v and $\partial X'_v$. Let $(g_n) \subset G$ be a sequence of group elements such that $g_n x_v \to \zeta$ and $g_n x_v \to \nu' \in \partial X'$. By Lemma **??**, $\nu' \in \partial X'_v$, and by the previous Lemma, ν' is not a pole of $\partial X'_v$. So it must lie in a longitude l'. Consider what Lemma 5.11 says about $Q(n,m) = \angle_{x'_v}(c^{-\infty}, c^m g_n x'_v)$ with $a = \angle_{x_v}(c^{-\infty}, \zeta)$ and $b = \pi$. Item (1) is satisfied by continuity of Alexandrov angles with fixed basepoint, (2) is satisfied because $c^m x \to c^\infty$ regardless of which $x \in X$ is chosen, and (3) is Lemma 5.12. Therefore, for any $\theta \in [a, \pi]$, there is a sequence $h_k = c^{m_k} g_{n_k}$ such that $\angle_{x'_v}(c^{-\infty}, h_k x'_v) \to \theta$.

Next we verify that $h_k x'_v$ converges to a point on l'. There is a sequence $(y_k) \subset Y_v$ converging to a point on l' and such that every geodesic $[x'_v, g_{n_k} x'_v]$ passes near the line $y_k \times \mathbb{R}$. But these lines are axes of c, which means that $[x'_v, h_k x'_v]$ also pass near them. Therefore all limit points of $\{h_k x'_v\}$ also lie in l'. But of course, since the angles converge, there is only one.

Finally, we check that $h_k x_v \to \zeta$. Take y_k to be the image of y'_k under the quasi-isometry $Y'_v \to Y_v$. Again, each geodesic $[x_v, h_k x_v]$ passes near the line $L_k = \{y_k\} \times \mathbb{R}$. Since $\{y_k\}$ is unbounded, we may apply Lemma 5.12, to show that $\angle_{x_v}(c^\infty, h_k x_v) \to 0$. Since the lines L_k have all their limit points in ∂X_v, $h_k x_v$ has no choice but to converge to ζ. □

5.4. **F-Fibers of Irrational Points.** The summary of the previous two subsections is that F-fibers of rational points are contractible. It remains to prove that F-fibers of irrational points are contractible. This turns out to be very easy.

Proposition 5.14. *Let $\zeta \in IX$. Then $\Lambda(\zeta)$ is a subset of an arc. To be precise, $\Lambda(\zeta)$ is contained in the set of points of $\partial X'$ with the same infinite itinerary as ζ (shown in* [CK02] *to be a Tits arc).*

Proof. It follows from 5.8 that $\Lambda(\zeta) \subset IX$. Since components of IX in both topologies are arcs and $\Lambda(\zeta)$ is connected, the proposition is clear. $\square$

The main Theorem follows from Propositions 5.5, 5.9, and 5.14.
Say more?

6. Notes for the Authors on Hyperbolic Spaces

Throughout this section, X will be a δ_0-hyperbolic space. That is, whenever $\triangle xyz$ is a triangle in X (possibly with some ideal vertices), then each side is contained in the δ_0-tubular neighborhood of the other two sides.

Pairs of geodesics in X satisfy the following *bounded tracking property.* Whenever α and β are geodesics (parameterized arbitrarily over $[0,1]$) such that $d(\alpha(0),\beta(0)) \leq C$ and $d(\alpha(1),\beta(1)) \leq C$, then the Hausdorff distance between α and β is bounded by $C + 2\delta_0$.

Recall that whenever $\triangle xyz$ is a triangle in a geodesic space X, there are points $\overline{x} \in [y,z]$, $\overline{y} \in [x,z]$, and $\overline{z} \in [x,y]$ such that $d(x,\overline{y}) = d(x,\overline{z})$, $d(y,\overline{x}) = d(y,\overline{z})$, and $d(z,\overline{x}) = d(z,\overline{y})$. These points are called the *internal points* of the triangle. If X is δ_0-hyperbolic, then there is a δ_1 (depending only on δ_0) such that given any triangle, the distance between any pair of internal points is bounded above by δ_1 [BH99, Proposition III.H.1.17]. Furthermore, if $c(t)$ and $c'(t)$ are unit speed parameterizations of the sides $[x,y]$ and $[x,z]$ such that $c(0) = c'(0) = x$ and $c(T)$ and $c'(T)$ are the internal points on these sides, then $d(c(t),c'(t)) \leq \delta_1$.

Lemma 6.1. *Let $C \geq 0$ be given, $\triangle xyz$ be a triangle, and $p \in X$ such that all three sides of the triangle pass within a distance of C from p. Then p is within a distance of $3C + 2\delta_1$ from each of the internal points.*

Proof. Choose unit speed parameterizations α, β, and γ for the three sides of the triangle so that $\alpha(0)$, $\beta(0)$, and $\gamma(0)$ are the internal points, and $\alpha(a) = \beta(-a)$, $\beta(b) = \gamma(-b)$, and $\gamma(c) = \alpha(-c)$ for positive numbers a, b, and c. Let $\alpha(r)$, $\beta(s)$, and $\gamma(t)$ be the points on the respective sides of the triangle which are closest to p. Then two of the numbers r, s, and t must have the same sign. Without loss of generality, assume r and s are both positive. Since $d(\alpha(r),\beta(-r)) \leq \delta_1$ and $d(\beta(s),\alpha(r)) \leq 2C$, $d(\beta(-r),\beta(s)) \leq 2C + \delta_1$. Since $\beta(0)$ lies between these two, $d(\beta(0),\beta(s)) \leq 2C + \delta_1$. The conclusion is now clear. $\square$

Lemma 6.2. *Let $\triangle xyz$ be a triangle (possibly with some ideal vertices) in a δ_0-hyperbolic space, $C \geq 0$, and $p, q \in X$ be a pair of points both of which lie within a distance of C from all three sides of the triangle. Then there is a constant R depending only on δ_0 and C such that $d(p,q) \leq R$.*

Proof. Suppose x is an ideal vertex. Denote the sides of the triangle by $A = [x,y]$, $B = [x,z]$, and $C = [y,z]$. Let $a \in A$ and $b \in B$ be closest points to p and $a' \in A$ and $b' \in B$ be closest points to q. Choose $\overline{x} \in A$ and $\overline{y} \in B$ such that $d(\overline{x},\overline{y}) \leq \delta_0$ and a and a' both lie in the same component of $A \setminus \{\overline{x}\}$ and b and b' both lie in the same component of $B \setminus \{\overline{y}\}$. Then $[\overline{x},z]$ passes within $3\delta_0$ of both b and b'. Therefore both p and q lie within a distance of $C + 3\delta_0$ of all three sides of a triangle $\triangle\overline{x}yz$ which has one fewer ideal vertex. Continue until all ideal vertices are gone and apply the previous lemma. $\square$

Here is another application of internal points. If $\triangle xyz$ is a triangle in X such that y is a point on the side $[x,y]$ closest to z, then y lies in the δ_1-neighborhood of the internal point on the side $[y,z]$. Therefore $[x,z]$ passes through the $2\delta_1$-neighborhood of y. Extending this idea we can get

Lemma 6.3. *Let X be a δ_0-hyperbolic geodesic space, $C \geq 0$, A and B be a pair of lines and $p \in A$ and $q \in B$ be a pair of closest points. If x lies within a distance of C from A and y lies within a distance of C from B, then any geodesic $[x,y]$ passes within a distance of $C + 6\delta_1 + 6\delta_0$ from both p and q.*

Proof. Let $\overline{x} \in A$ and $\overline{y} \in B$ be points within C of x and y, and a be a point on $[\overline{x},p]$ closest to $\overline{y}$. Then a geodesic $[p,\overline{y}]$ passes within $2\delta_1$ of both q and a. Using bounded tracking, q is a distance of $4\delta_1 + 2\delta_0$ from $[a,\overline{y}]$. Similarly, since $[\overline{x},\overline{y}]$ also passes within $2\delta_1$ of a, it passes within $6\delta_1 + 4\delta_0$ of q. The conclusion now follows from bounded tracking. $\square$

References

[Bes96] Mladen Bestvina. Local homology properties of boundaries of groups. *Michigan Math. J.*, 43(1):123–139, 1996.

[BH99] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[BR96] Philip L. Bowers and Kim Ruane. Boundaries of nonpositively curved groups of the form $G \times \mathbb{Z}^n$. *Glasgow Math. J.*, 38(2):177–189, 1996.

[CK00] Christopher B. Croke and Bruce Kleiner. Spaces with nonpositive curvature and their ideal boundaries. *Topology*, 39(3):549–556, 2000.

[CK02] Christopher B. Croke and Bruce Kleiner. The geodesic flow of a nonpositively curved graph manifold. *Geom. Funct. Anal.*, 12(3):479–545, 2002.

[Geo86] Ross Geoghegan. The shape of a group—connections between shape theory and the homology of groups. In *Geometric and algebraic topology*, volume 18 of *Banach Center Publ.*, pages 271–280. PWN, Warsaw, 1986.

[GM11] Craig Guilbault and Christopher Mooney. Cell-like equivalences and boundaries of CAT(0) groups. *Geom. Dedicata*, 2011.

[HK05] G. Christopher Hruska and Bruce Kleiner. Hadamard spaces with isolated flats. *Geom. Topol.*, 9:1501–1538 (electronic), 2005. With an appendix by the authors and Mohamad Hindawi.

[KL97] B. Kleiner and B. Leeb. Rigidity of quasi-isometries for symmetric spaces and euclidean buildings. *Inst. Hautes tudes Sci. Publ. Math.*, (86):115–197, 1997.

[Kle99] Bruce Kleiner. The local structure of length spaces with curvature bounded above. *Math. Z.*, 231(3):409–456, 1999.

[Lin10] Gabriele Link. Asymptotic geometry in products of Hadamard spaces with rank one isometries. *Geom. Topol.*, 14(2):1063–1094, 2010.

[Mil68] J. Milnor. A note on curvature and fundamental group. *J. Differential Geometry*, 2:1–7, 1968.

[Moo08] Christopher Mooney. Examples of non-rigid CAT(0) groups from the category of knot groups. *Algebr. Geom. Topol.*, 8(3):1666–1689, 2008.

[Moo09] Christopher Mooney. All CAT(0) boundaries of a group of the form $H \times K$ are CE equivalent. *Fund. Math.*, 203(2):97–106, 2009.

[Moo10] Christopher Mooney. Generalizing the Croke-Kleiner construction. *Topology Appl.*, 157(7):1168–1181, 2010.

[Ont05] Pedro Ontaneda. Cocompact CAT(0) spaces are almost geodesically complete. *Topology*, 44(1):47–62, 2005.

[Rua99] Kim E. Ruane. Boundaries of CAT(0) groups of the form $\Gamma = G \times H$. *Topology Appl.*, 92(2):131–151, 1999.

[She72] Richard B. Sher. Realizing cell-like maps in Euclidean space. *General Topology and Appl.*, 2:75–89, 1972.

[Sta11] Dan Staley. Erratic boundary images of CAT(0) geodesics under G-equivariant maps. to appear in Geom. Dedic., September 2011.

[Šva55] A. S. Švarc. A volume invariant of coverings. *Dokl. Akad. Nauk SSSR (N.S.)*, 105:32–34, 1955.

[Swe99] Eric L. Swenson. A cut point theorem for CAT(0) groups. *J. Differential Geom.*, 53(2):327–358, 1999.

[Wil05] Julia M. Wilson. A CAT(0) group with uncountably many distinct boundaries. *J. Group Theory*, 8(2):229–238, 2005.