GROUP BOUNDARIES FOR SEMIDIRECT PRODUCTS WITH $\mathbb{Z}$

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Abstract. Bestvina's notion of a $\mathcal{Z}$-structure provides a general framework for group boundaries that includes Gromov boundaries of hyperbolic groups and visual boundaries of CAT(0) groups as special cases. A refinement, known as an $E\mathcal{Z}$-structure has proven useful in attacks on the Novikov Conjecture and related problems. Characterizations of groups admitting a $\mathcal{Z}$- or $E\mathcal{Z}$-structure are long-standing open problems. In this paper, we examine groups of the form $G \rtimes_\phi \mathbb{Z}$. For example, we show that, if G is torsion-free and admits a $\mathcal{Z}$-structure, then so does every semidirect product of this type. We prove a similar theorem for $E\mathcal{Z}$-structures, under an additional hypothesis.

As applications, we show that all closed 3-manifold groups admit $\mathcal{Z}$-structures, as do all strongly polycyclic groups and all groups of polynomial growth. In those latter cases our $\mathcal{Z}$-boundaries are always spheres. This allows one to make strong conclusions about the group cohomology and end invariants of those groups. In another direction, we expand upon the notion of an $E\mathcal{Z}$-structure and discuss new applications to the Novikov Conjecture.

Contents

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1. Introduction

Bestvina [Bes96] introduced the notion of a *$\mathcal{Z}$-boundary* for a group as a means to both unify the study of Gromov boundaries of hyperbolic groups with visual boundaries of CAT(0) groups and to provide a framework for assigning boundaries to a wider class of groups. Initially restricted to torsion-free groups, work by Dranishnikov [Dra06] expanded Bestvina's definition to allow for groups with torsion. Roughly speaking, a groups G admits a *$\mathcal{Z}$-structure* if it acts nicely (properly and cocompactly) on a nice space X (an absolute retract or AR) which admits a nice compactification $\overline{X}$ (a $\mathcal{Z}$-set compactification) such that compact subsets of X vanish in $\overline{X}$ as they are pushed toward the *$\mathcal{Z}$-boundary*, $Z := \overline{X} - X$, by elements of G. Work from those papers, as well as [GO07] and [GM19], highlights some of the useful properties of a $\mathcal{Z}$-boundary.

In order to admit a $\mathcal{Z}$-boundary, G must first admit a proper cocompact action on an absolute retract X. This rules out many groups. For example, a torsion free group admits this type of action if and only if it is *type F*. The general class of groups which admit such an action have been defined to be *type F^*_{AR}*. (When X is a CW complex and the action is cellular, it is simply called *type F^**.) A significant open question asks whether all type F or type $\mathrm{F}^*_{\mathrm{AR}}$ groups admit $\mathcal{Z}$-structures. Current progress has focused on special classes of groups. For example, in addition to hyperbolic and CAT(0) groups, all systolic and all generalized Baumslag-Solitar groups have been shown to admit $\mathcal{Z}$-boundaries. In a more general direction, all free and direct products of groups which admit $\mathcal{Z}$-boundaries are known to admit $\mathcal{Z}$-boundaries (references to be provided later). One of the main results in this paper is in that spirit.

Theorem 1.1. *If a torsion-free group G admits a $\mathcal{Z}$-structure with boundary Z, then every semidirect product $G \rtimes_\phi \mathbb{Z}$ admits a $\mathcal{Z}$-structure with boundary equal to the suspension of Z.*

A significant special case of Theorem 1.1 includes all free-by-$\mathbb{Z}$ groups—a well-studied class that is notable because it contains some hyperbolic groups [BF92], some CAT(0) groups, and *many* groups which are neither. Hyperbolic group experts might be surprised that, for this collection of groups, our $\mathcal{Z}$-boundary is *always* a suspended Cantor set—an illustration that even hyperbolic groups need not admit unique $\mathcal{Z}$-boundaries.

With the aid of Theorem 1.1 we obtain a variety of new results.

Theorem 1.2. *Every closed 3-manifold group admits a $\mathcal{Z}$-structure.*

Theorem 1.3. *Every strongly polycyclic and every finitely generated nilpotent group G admits a $\mathcal{Z}$-structure with a k-sphere as boundary, for some $k \geq -1$.*

By combining Theorem 1.3 with Gromov's theorem on groups of polynomial growth [Gro81], the main result of [KMPN09], and a boundary swapping trick, Theorem 1.3 can be pushed further to obtain:

Theorem 1.4. *Every group G of polynomial growth admits a $\mathcal{Z}$-structure with a k-sphere as boundary, for some $k \geq -1$.*

By applying standard properties of $\mathcal{Z}$-structures (see [Bes96]) and some recent extensions (see [GM21]), one may immediately deduce:

Corollary 1.5. *Every group G that is polycyclic or of polynomial growth, has the same $\mathbb{Z}G$-cohomology as $\mathbb{Z}^n$, for some n. In addition, G is semistable and has the same pro-homotopy groups at infinity as $\mathbb{Z}^n$. If G is torsion-free, it is a Poincaré duality group.*

Several of the assertions covered by Corollary 9.8 are known by other methods. Details and references will be provided in Section 9.6.

The reader will notice that the above results include groups with torsion. Indeed, we will prove a more general version of Theorem 1.1 which can be applied whenever there exists a cocompact $\underline{E}G$-space. As a corollary, if G is hyperbolic, CAT(0), or systolic, then $G\rtimes_{\phi}\mathbb{Z}$ admits a $\mathcal{Z}$-structure; and by applying our own Theorem 1.4, $G\rtimes_{\phi}\mathbb{Z}$ admits a $\mathcal{Z}$-structure whenever G is of polynomial growth, . We save the most general statements for Section 6.

Farrell and Lafont [FL05] defined $E\mathcal{Z}$-structures—a refinement of $\mathcal{Z}$-structures which adds a requirement that the G-action on X extends to $\overline{X}$. Their main application was to show that each torsion-free group that admits an $E\mathcal{Z}$-structure satisfies the famous Novikov Conjecture. For that and other reasons, we will expend significant effort proving the existence of $E\mathcal{Z}$-structures whenever possible. The following is a corollary of a more general theorem that will be proved in Section 7.

Theorem 1.6. *Suppose G is a hyperbolic group, a finitely generated abelian group, or a CAT(0) group with the isolated flats property. Then, for any $\phi \in \mathrm{Aut}(G)$, $G\rtimes_{\phi}\mathbb{Z}$ admits an $E\mathcal{Z}$-structure with boundary equal to the suspension of the Gromov or visual boundary of G.*

By [FL05], Theorem 1.6 implies the Novikov Conjecture, for the groups covered there, whenever they are torsion-free. (Other proofs of Novikov are known in these particular cases. See Section 9.1 for details.)

A related approach to the Novikov Conjecture, that can be applied to groups with torsion, has been developed by Rosenthal [Ros04], [Ros06], [Ros12]. In order to apply that work, we develop a refinement of $E\mathcal{Z}$-structures which we call $\underline{E\mathcal{Z}}$-structures. The additional requirement is that, for each finite $H \leq G$, the fixed set X^H is an absolute retract whose closure in $\overline{X}$ is a $\mathcal{Z}$-compactification of X^H. By existing work, to be detailed later, all hyperbolic, CAT(0), and systolic groups support $\underline{E\mathcal{Z}}$-structures. We will prove that, under appropriate hypotheses, an $\underline{E\mathcal{Z}}$-structure on G implies the existence of an $\underline{E\mathcal{Z}}$-structure on $G \rtimes_{\phi} \mathbb{Z}$. Special cases include all of the groups mentioned in Theorem 1.6.

The layout of this paper is as follows. In Section 2 we provide much of the necessary background, including definitions and a variety of notational conventions to be used throughout the paper. In Section 3 we explain the role of mapping tori and mapping telescopes in the study of $G \rtimes_\phi \mathbb{Z}$ (thereby explaining why the groups themselves are sometimes called mapping tori). All of this works best when G is torsion-free, so that is our focus in this section. From there we turn to the proof of Theorem 1.1 which occupies Sections 4 and 5. This is the heart of the paper and also the most technical portion. In Section 6 material from the previous sections is generalized—to the extent possible—to allow for groups with torsion. In Sections 7 and 8 we develop and discuss $E\mathcal{Z}$-structures and $\underline{E\mathcal{Z}}$-structures. From there we turn to applications of our main theorems. In particular, Section 9 contains a detailed discussion of implications to the Novikov Conjecture. From there, we move on to proofs of Theorems 1.2-1.4. We close the paper with a discussion of some open questions. In addition, we include an appendix which looks at an interesting special case of the work presented here—the integral Heisenberg group. That discussion can be viewed as motivation for the more general methods used elsewhere. Before delving into the more abstract constructions and proofs, the reader might benefit from a quick look at this appendix.

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2. Background

2.1. **Semidirect products with $\mathbb{Z}$.** Here we establish some conventions regarding semidirect products with $\mathbb{Z}$. First note that a group Γ is a semidirect product of G with $\mathbb{Z}$ if and only if there exists a short exact sequence

$$1 \to G \to \Gamma \to \mathbb{Z} \to 1$$

In other words (in this special case) a semidirect product is the same as a group extension. Viewed as a semidirect product, Γ is determined by a single element $\phi \in \mathrm{Aut}\,(G)$ which then determines the homomorphism $\Phi : \mathbb{Z} \to \mathrm{Aut}\,(G)$ taking n to ϕ^n. This group is also an HNN extension. It can be defined by the presentation

$$G \rtimes_\phi \mathbb{Z} = \left\langle G, t \mid t^{-1} g t = \phi\,(g) \;\; \forall g \in G \right\rangle$$

Alternatively, if $G = \langle S \mid R \rangle$, then

$$G \rtimes_\phi \mathbb{Z} = \left\langle S, t \mid R,\; t^{-1} s t = \phi\,(s) \;\; \forall s \in S \right\rangle$$

Some authors use an alternative convention that uses relators $tgt^{-1} = \phi(g)$. We prefer the above presentation for geometric reasons that will become apparent as we proceed. Those geometric reasons also explain why some authors ([BF92], [FH99]) refer to this group as a *mapping torus*. We reserve that terminology for the associated topological construction.

2.2. **Absolute retracts, group actions, and ϕ-variant maps.** A locally compact separable metric space X is an *absolute neighborhood retract (ANR)* if, whenever X is embedded as a closed subset of a space Y, some neighborhood of X retracts onto X. A contractible ANR is called an *absolute retract (AR)*. Note that all ARs and ANRs in this paper are assumed to be locally compact, separable, and metrizable.

Definition 2.1. Let G be a group and X a topological space [respectively AR]. We say that X is a *G-space* [respectively *G-AR*] when there is a specified homomorphism $i : G \to \text{Homeo}(X)$. When no confusion can arise, we write $g \cdot x$ to denote $i\,(g)\,(x)$ (omitting the notation i).

Definition 2.2. The action associated to a G-space is *cocompact* if the quotient space $G\backslash X$ is compact, where $x \sim g \cdot x$ for all x, g. The action is *proper* if, for any compact set $K \subset X$, $\{g \in G \mid g \cdot K \cap K \neq \emptyset\}$ is a finite set.

Note that our definition of "proper" matches the more general definition for topological groups whenever G is discrete—a condition that will be satisfied throughout this paper.

Definition 2.3. A G-space is called a *G-complex* if it is a CW-complex and the action is cellular. It is said to be a *rigid* G-complex if the action has the property that if σ is a cell for which $g \cdot \sigma = \sigma$, then σ is point-wise fixed by g.

We are also interested in maps between G-spaces which respect the action, or which are compatible with a 'twisted' version of the action induced by an automorphism.

Definition 2.4. Let X, Y be G-spaces. A map $f : X \to Y$ is *G-equivariant* (or just *equivariant* when the group is understood) if for all $x \in X$, $f(g \cdot x) = g \cdot f(x)$.

More generally, for $\phi \in \text{Aut}\,(G)$, we say f is *ϕ-variant* if

$$f(g \cdot x) = \phi(g) \cdot f(x)$$

for all $g \in G$ and $x \in X$.

Remark 2.5. Clearly a G-equivariant map is just a ϕ-variant map where $\phi = \text{id}_G$. Conversely, a ϕ-variant map can be viewed as a G-equivariant map, where the G-action on Y has been twisted by precomposing with ϕ. For our purposes, ϕ-variance is an extremely useful concept.

2.3. **$\mathcal{Z}$-sets and $\mathcal{Z}$-structures.** A collection $\mathcal{A}$ of subsets of a compact space Y is a *null family* if, for any open cover $\mathcal{U}$ of Y, there exists a finite subcollection $\mathcal{B} \subseteq \mathcal{A}$ such that for all $A \in \mathcal{A} - \mathcal{B}$, there exists $U \in \mathcal{U}$ such that $A \subseteq U$. By the Lebesgue Covering Theorem, if Y is metrizable, $\mathcal{A}$ is a null family if and only if, for any $\varepsilon > 0$, there exists a finite subcollection $\mathcal{B} \subseteq \mathcal{A}$ such that $\text{diam}(A) < \varepsilon$ for all $A \in \mathcal{A} - \mathcal{B}$. This condition is independent of the metric chosen, but different metrics may require different choices of $\mathcal{B}$.

A closed set $Z \subseteq Y$ is a *$\mathcal{Z}$-set* if, there exists a homotopy $\alpha : Y \times [0, 1] \to Y$ such that $\alpha_0 = \text{id}_Y$ and $\alpha_t(Y) \subseteq Y - Z$ for all $t > 0$. In this case, we call α a *$\mathcal{Z}$-set homotopy.* A compactification $\overline{X}$ of a space X is a *$\mathcal{Z}$-compactification* if $Z := \overline{X} - X$

is a $\mathcal{Z}$-set in $\overline{X}$. In that case, we call Z a *$\mathcal{Z}$-boundary* for X (keeping in mind this boundary may not coincide with other boundaries, such as the visual boundary of a CAT(0) or Gromov hyperbolic space).

By an application of Hanner's Theorem [Han51], a $\mathcal{Z}$-compactification of an ANR is always an ANR (see [Tir11] for a detailed discussion); hence, a $\mathcal{Z}$-compactification of an AR is an AR. When X is an AR, we can choose a $\mathcal{Z}$-set homotopy which contracts $\overline{X}$ to a point $x_0 \in X$, keeping x_0 fixed throughout [Tir11, Lemma 1.10]. For a more detailed discussion of ANRs and $\mathcal{Z}$-sets, see [GM19].

Definition 2.6. A *$\mathcal{Z}$-structure* on a group G is a pair of spaces $(\overline{X}, Z)$ satisfying:

(1) $\overline{X}$ is a compact AR,
(2) Z is a $\mathcal{Z}$-set in $\overline{X}$,
(3) $X = \overline{X} - Z$ admits a proper, cocompact action by G, and
(4) (nullity condition) for any compact set $K \subseteq X$, the collection $\{gK \mid g \in G\}$ of subsets of X is a null family in $\overline{X}$. (Informally, compact subsets of X get small when translated toward Z.)

When $(\overline{X}, Z)$ is $\mathcal{Z}$-structure on G, we call Z a *$\mathcal{Z}$-boundary* for G. A $\mathcal{Z}$-structure for which the G-action on X extends to an action on $\overline{X}$ is called an *$E\mathcal{Z}$-structure.*

Fact. *Without loss of generality, we can require that the G-action in item (3) to be geometric with respect to some proper metric on X. See* [GM19, Remark 8].

The question of which groups admit $(E)\mathcal{Z}$-structures is very much open, but a variety of special cases are now understood, including hyperbolic groups [BM91], CAT(0) groups, Baumslag-Solitar groups [GMT19] and [GMS21], systolic groups [OP09], and certain relatively hyperbolic groups [Dah03]. Work of Tirel [Tir11] demonstrates that this class is closed under direct and free products. The latter of these results can also be obtained from Dahmani's theorem.

2.4. **Mapping cylinders, mapping tori, and mapping telescopes.** The following definitions, notation, and background material will play a primary role in our main constructions.

Definition 2.7. Let X be a space and $f : X \to X$ a continuous map.

(1) The *mapping cylinder of f* based on the interval $[a, b]$ is the quotient space $\mathcal{M}_{[a,b]}(f) = X \times [a, b]) \sqcup X/\sim$ where $\sim$ is generated by the rule $(x, b) \sim f(x)$. For each $t \in [a, b)$, the quotient map $q_{[a,b]} : (X \times [a, b]) \sqcup X \to \mathcal{M}_{[a,b]}(f)$ restricts to an embedding of $X \times \{t\} \hookrightarrow \mathcal{M}_{[a,b]}(f)$ whose image will be denoted X_t. The quotient map also restricts to an embedding on the disjoint copy of X; we denote its image in $\mathcal{M}_{[a,b]}(f)$ by X_b and refer to it as the *range end* of $\mathcal{M}_{[a,b]}(f)$. Similarly, X_a is the *domain end* of $\mathcal{M}_{[a,b]}(f)$.
(2) The *mapping torus of f* is the quotient space $\mathrm{Tor}_f(X) = X \times [0, 1]/\sim$ where $\sim$ is generated by the rule $(x, 1) \sim (f(x), 0)$.
(3) The *bi-infinite mapping telescope* corresponding to f is the space

$$\mathrm{Tel}_f(X) = \cdots \cup \mathcal{M}_{[-1,0]}(f) \cup \mathcal{M}_{[0,1]}(f) \cup \mathcal{M}_{[1,2]}(f) \cup \mathcal{M}_{[2,3]}(f) \ldots$$

Implicit in this notation is the identification of the range end of each $\mathcal{M}_{[k-1,k]}(f)$ with the domain end of $\mathcal{M}_{[k,k+1]}(f)$, both denoted by X_k. Using the notation from (1), there is a quotient map $\lambda : \mathrm{Tel}_f(X) \to \mathbb{R}$ such that $\lambda^{-1}(r) = X_r$ for each r.

Remark 2.8. The reader is warned that the "direction" of our mapping cylinders and telescopes is the reverse of what is found in some of the literature, such as [Gui14] and [Gui16].

Definition 2.9. The *suspension* of a space Z is the quotient space $SZ = Z \times [-\infty, \infty] / \sim$ where $(z, \infty) \sim (z', \infty)$ and $(z, -\infty) \sim (z', -\infty)$ for all $z, z' \in X$.

Notation 1. We frequently find ourselves working with a Cartesian product $X \times \mathbb{R}$, a mapping telescope $\mathrm{Tel}_f(X)$, and a related suspension SZ, all at the same time. To aid in distinguishing between points of these spaces, we adopt the following notational conventions.

(1) A point in $X \times \mathbb{R}$ is represented in the usual way, as an ordered pair (x, r) enclosed in ordinary parentheses.
(2) A point in $\mathrm{Tel}_f(X)$ is represented by $\lceil x, r \rceil$ when it is the equivalence class of (x, r) in $\mathcal{M}_{[k,k+1]}(f) \subseteq \mathrm{Tel}_f(X)$ and $k \le r < k+1$. In particular, when a point projects to $k \in \mathbb{Z}$ under $\lambda : \mathrm{Tel}_f(X) \to \mathbb{R}$, it takes its coordinates from the *domain end* of $\mathcal{M}_{[k,k+1]}(f)$. This gives a bijective correspondence between symbols $\{\lceil x, r \rceil \mid x \in X \text{ and } r \in \mathbb{R}\}$ and points of $\mathrm{Tel}_f(X)$. Under this convention, a sequence $\left\{\left\lceil x, k + \frac{1}{i+1} \right\rceil\right\}_{i=1}^{\infty}$ converges to $\lceil x, k \rceil$ while $\left\{\left\lceil x, k + \frac{i}{i+1} \right\rceil\right\}_{i=1}^{\infty}$ converges to $\lceil f(x), k+1 \rceil$.
(3) A point in SZ will be represented by $\langle z, r \rangle$ when it is the equivalence class of (z, r). As such, equivalence classes $Z \times \{\infty\}$ and $Z \times \{-\infty\}$ have non-unique representations which we sometimes abbreviate to $\langle \infty \rangle$ and $\langle -\infty \rangle$. In either case, the delimiters $\langle \ , \ \rangle$ indicates an element of a suspension.

For later use, we compile some basic facts about mapping cylinders, mapping tori, mapping telescopes, and suspensions, tailored to our present needs.

Lemma 2.10. *Let $f : X \to X$ be a proper self map of a locally compact, separable metrizable space. Then*

(1) $\mathcal{M}_{[a,b]}(f)$, $\mathrm{Tor}_f(X)$, *and* $\mathrm{Tel}_f(X)$ *are locally compact, separable, and metrizable,*
(2) *if X is contractible, then $\mathcal{M}_{[a,b]}(f)$ and $\mathrm{Tel}_f(X)$ are contractible,*
(3) *if X is an ANR, then $\mathcal{M}_{[a,b]}(f)$, $\mathrm{Tor}_f(X)$, and $\mathrm{Tel}_f(X)$ are ANRs,*
(4) *if X is an AR, then $\mathcal{M}_{[a,b]}(f)$ and $\mathrm{Tel}_f(X)$ are ARs,*
(5) *if X is a locally finite CW complex and f is a cellular map, then $\mathcal{M}_{[a,b]}(f)$,* $\mathrm{Tor}_f(X)$, *and* $\mathrm{Tel}_f(X)$ *can be endowed with corresponding cell structures making each a locally finite CW complex.*

Proof. Item 1 is an exercise in general topology. In each case, the space in question is endowed with the quotient topology induced by a map $q : Y \to Y/\sim$, where the space Y has all of the desired properties. By using the properness of f one may view the quotient space as the result of an upper semicontinuous decomposition of Y in the sense of [Dav86]. From there, the desired conclusions can be deduced from results found in the first three sections of that book.

Since $\mathcal{M}_{[a,b]}(f)$ strong deformation retracts onto its domain end X_a, the first assertion of Item 2 is clear. By applying that deformation retraction inductively, one sees that any finite subtelescope of $\mathrm{Tel}_f(X)$ is contractible. By thickening these finite telescopes to contractible open subsets of $\mathrm{Tel}_f(X)$, one may apply [AE16] to deduce the contractibility of $\mathrm{Tel}_f(X)$.

Items 3 and 4 follow from [Hu65, p.178]. Item 5 is standard—see, for example, [FP90]. □

3. Mapping tori and telescopes as classifying spaces for $G \rtimes_\phi \mathbb{Z}$

The first step in placing a $\mathcal{Z}$-structure on any group is to find an AR on which that group acts properly and cocompactly. In this paper, we most frequently begin a $\mathcal{Z}$-structure $(\overline{X}, Z)$ on G and look to place one on $G \rtimes_\phi \mathbb{Z}$. As such, our first step is to construct an AR Y on which $G \rtimes_\phi \mathbb{Z}$ acts properly and cocompactly. Due to basic covering space theory, this is accomplished most easily and most intuitively when G is torsion-free. For that reason we deal only with the torsion-free case in this section. We will return to the cases where G has torsion in Section 6.

Throughout this section, prime symbols will be used for maps occurring 'downstairs', while their lifts to universal covers will be denoted without primes.

Beginning with a proper and cocompact G-action on an AR X, the assumption that G is torsion-free ensures that the quotient map $q : X \to G\backslash X$ is a covering projection. Since the action is cocompact and X is contractible, $G\backslash X$ is compact and aspherical. Furthermore, since being an ANR is a local property, $G\backslash X$ is an ANR. We now use some trickery to replace our generic AR X with a locally finite contractible CW complex (and a cellular G-action). This step is not strictly necessary for much of what follows, but it leads to a more intuitive version of a $(G \rtimes_\phi \mathbb{Z})$-space Y. In addition, a CW structure is useful in some applications.

By a famous theorem of West [Wes77], $G\backslash X$ is homotopy equivalent to a finite CW complex K, which is a $K(G, 1)$ complex. The universal cover of K, which we temporarily label as X^*, admits a proper, cocompact, cellular G-action, and by a well-known boundary swapping trick (see [Bes96] or [GM19]), we may attach Z to X^* to obtain an alternative $\mathcal{Z}$-structure $(\overline{X^*}, Z)$ on G. For ease of notation we replace the X with X^* and omit the star—assuming from now on that X is this CW complex.

Without loss of generality, we may assume that the 2-skeleton of K is a presentation 2-complex for $G = \langle S \mid R \rangle$ with vertex v_0. By asphericity, there is a cellular map $f' : (K, v_0) \to (K, v_0)$ such that $f'_* = \phi : G \to G$. Since ϕ is an automorphism, f' is a homotopy equivalence. The mapping torus $\mathrm{Tor}_{f'}(K)$ has fundamental group $G \rtimes_\phi \mathbb{Z}$. In fact, if we give $\mathrm{Tor}_{f'}(K)$ the usual cell structure consisting of K together with a new $(k+1)$-cell for each k-cell of K, then the 2-skeleton of $\mathrm{Tor}(f')$ is a presentation 2-complex for $G \rtimes_\phi \mathbb{Z} = \langle S,\ t \mid R,\ t^{-1}st = \phi(s)\ \forall s \in S \rangle$. The new 1-cell (corresponding to the 0-cell v_0) gives rise to t, and each new 2-cell (one for each $s \in S$) results in a relator $t^{-1}st = \phi(s)$.

It is easy to see that the mapping telescope $\mathrm{Tel}_{f'}(K)$ is an infinite cyclic cover of $\mathrm{Tor}(f')$ with fundamental group corresponding to $G \trianglelefteq G \rtimes_\phi \mathbb{Z}$. From there, the universal cover may be viewed as the mapping telescope $\mathrm{Tel}_f(X)$ where $f : X \to X$ is a lift of f'. More specifically, let $x_0 \in q^{-1}(v_0)$ be a preferred basepoint, where $q : X \to K$ is the covering projection, and choose f taking x_0 to x_0. By lifting the CW structure of $\mathrm{Tor}_{f'}(K)$ to $\mathrm{Tel}_f(X)$, we can realize the Cayley graph and Cayley 2-complex of $G \rtimes_\phi \mathbb{Z}$ as the 1- and 2-skeleta of $\mathrm{Tel}_f(X)$. By Lemma 6.13, $\mathrm{Tel}_f(X)$ is contractible, and since locally finite CW complexes are ANRs, $\mathrm{Tel}_f(X)$ is an AR. This is the space Y we set out to construct.

Before proceeding, we pause to examine the action of $G \rtimes_\phi \mathbb{Z}$ on $Y = \mathrm{Tel}_f(X)$. First we focus on the 1-skeleton, which is $\mathrm{Cay}(G \rtimes_\phi \mathbb{Z}, S \cup \{t\})$. For each integer i, X_i is a copy of X, so its 1-skeleton is a copy of $\mathrm{Cay}(G, S)$. By placing the identity element of $G \rtimes_\phi \mathbb{Z}$ at $x_0 \in X_0$ we identify $X_0^{(1)}$ as the subgraph of $\mathrm{Cay}(G \rtimes_\phi \mathbb{Z}, S \cup \{t\})$ corresponding to $G \trianglelefteq G \rtimes_\phi \mathbb{Z}$. All edges of $\mathrm{Cay}(G \rtimes_\phi \mathbb{Z}, S \cup \{t\})$ not in some $X_i^{(1)}$ are labeled by t and are oriented in the positive $\mathbb{R}$-direction with respect to the projection $\lambda : \mathrm{Tel}_f(X) \to \mathbb{R}$. As a homeomorphism of Y, t maps each X_i onto X_{i+1} via the "identity". As such, each subcomplex $X_i^{(1)}$ corresponds to the left coset t^iG, and each vertex $t^ig \in t^iG$ is taken to its counterpart $t^{i+1}g \in t^iG$. Left multiplication by an element of G is more interesting. For each vertex $g \in X_0^{(1)}$, the outgoing t-edge ends at $gt = t\phi(g)$. So, if $ta \in tG = X_1^{(0)}$, it is the terminal vertex of a t-edge emanating from $\phi^{-1}(a)$. Left-multiplication by g moves that edge to one emanating from $g\phi^{-1}(a)$ and ending at $\phi(g\phi^{-1}(a)) = \phi(g)a$. In other words, from the perspective of $X_1^{(1)}$, left multiplication by g is replaced by left multiplication by $\phi(g)$. More generally, left-multiplication by g shows up in $X_i^{(1)}$ as left-multiplication by $\phi^i(g)$.

As a homeomorphism of Y, $g|_{X_i} = \phi^i(g)$, with mapping cylinder lines being taken to mapping cylinder lines.

For another interesting perspective, note that the left coset $g\langle t\rangle$ of the infinite cyclic subgroup generated by t is the line in the Cayley graph whose vertex set is the bi-infinite sequence $\{\phi^i(g)\}_{i=-\infty}^{\infty}$, and whose edges are all labeled by t.

Let us now consider the geometry of $G \rtimes_\phi \mathbb{Z}$, again focusing on $\mathrm{Cay}(G \rtimes_\phi \mathbb{Z}, S \cup \{t\}) = Y^{(1)}$. If we ignore the t-edges, this Cayley graph is identical to that of the direct product $G \times \mathbb{Z}$; we have a copy $X_i^{(1)}$ of $\mathrm{Cay}(G, S)$ over each integer $i \in \mathbb{R}$. But, unlike the Cayley graph of the direct product, the edges between vertices of $X_i^{(1)}$ and $X_{i+1}^{(1)}$ are not "vertical". Instead the t-edge emanating from a vertex t^ia has terminal point $t^{i+1}\phi(a)$ where a and $\phi(a)$ can be far apart as elements of G. In addition, elements t^ia and t^ib which are far apart in $X_i^{(1)}$ can be close together in $\mathrm{Cay}(G \rtimes_{\overline{\phi}} \mathbb{Z}, S \cup \{t\})$, due to shortcuts made available by the t-edges. In particular, depending on the automorphism ϕ, the geometry of $G \rtimes_\phi \mathbb{Z}$ can be very different from that of $G \times \mathbb{Z}$. See Figure 1.

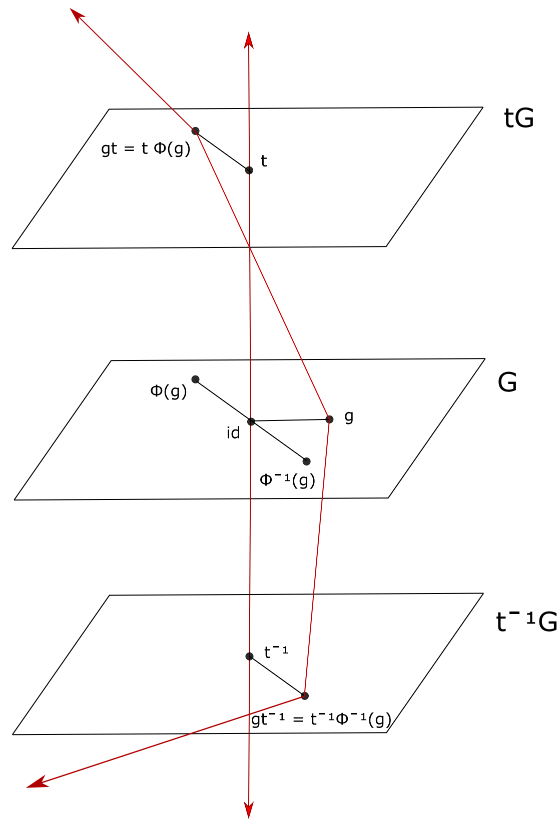

FIGURE 1. A snapshot of the Cayley graph of of a semidirect product with the integers. Red edges are labeled by t, black by elements of G.

We have already noted that Y is contractible, but we can be more precise. We can build a proper homotopy equivalence between Y and $X \times \mathbb{R}$.

Since $f' : (K, v_0) \to (K, v_0)$ is a homotopy equivalence, there exists a cellular map $h' : (K, v_0) \to (K, v_0)$ and cellular homotopies $A', B' : K \times [0, 1] \to K$ such that $A'_0 = \mathrm{id}_K$, $A'_1 = h'f'$, $B'_0 = \mathrm{id}_K$, and $B_1 = f'h'$. Since these maps are proper (all spaces being compact), so are their lifts (see [Geo08, Theorem 10.1.23]). Hence we obtain proper maps $f, h : (X, x_0) \to (X, x_0)$ and proper homotopies $A, B : X \times [0, 1] \to X$ such that $A_0 = \mathrm{id}_X$, $A_1 = hf$, $B_0 = \mathrm{id}_X$, and $B_1 = fh$. Moreover, f is ϕ-variant under the standard covering action on (X, x_0) and h is ϕ^{-1}-variant.

To obtain the desired proper homotopy equivalence $v : Y \to X \times \mathbb{R}$, we will first construct a proper homotopy equivalence $v' : \mathrm{Tel}(f') \to K \times \mathbb{R}$. That map will be lifted to universal covers to obtain v.

The building blocks of v' will be maps $v'_n : \mathcal{M}_{[n,n+1]}(f') \to K \times [n, n+1]$. For the purposes assuring continuity, recall that $\mathcal{M}_{[n,n+1]}(f') = K \times [n, n+1] \sqcup K/\sim$ where $(x, n+1) \sim f'(x)$. We will first describe continuous maps of $K \times [n, n+1] \sqcup K$ into $K \times [n, n+1]$, then observe that they respect $\sim$ thereby inducing continuous maps on $\mathcal{M}_{[n,n+1]}(f')$.

For integers $n \geq 0$, v'_n is induced by the map

$$\begin{array}{lll} (x,r) & \mapsto & \left(h'^n\left(A'_{r-n}(x)\right), r\right) \quad \text{for } (x,r) \in K \times [n,n+1] \\ x & \mapsto & \left(h'^{(n+1)}(x), n+1\right) \quad \text{for } x \text{ in range copy of } K \end{array}$$

Since each $(x, n+1) \in K \times [n, n+1]$ is sent to $(h'^n(h'f'(x)), n+1) = \left(h'^{(n+1)}(f(x)), n+1\right)$, which is also where it sends the point $f(x) \in K$, we get a well-defined continuous map $v'_n : \mathcal{M}_{[n,n+1]}(f') \to K \times [n, n+1]$. For integers $n < 0$, use following similar (but simpler) rule.

$$\begin{array}{lll} (x,r) & \mapsto & (f'^{|n|}(x), r) \quad \text{for } (x,r) \in K \times [n,n+1] \\ x & \mapsto & (f'^{|n|-1}(x) \quad \text{for } x \text{ in range copy of } K \end{array}$$

Notice that, for each integer n, v'_{n-1} and v'_n agree on K_n, so these maps can be pasted together to obtain $v' : \mathrm{Tel}(f') \to K \times \mathbb{R}$. An argument similar to the one in [Gui14] , shows that v' is a proper homotopy equivalence with a proper homotopy inverse $u' : K \times \mathbb{R} \to \mathrm{Tel}(f')$ which takes $K \times \{n\}$ into K_n and $K \times [n, n+1]$ into $\mathcal{M}_{[n,n+1]}(f')$ for each integer n. We say that v' is *level-preserving* and u' is *nearly level-preserving.* (A level-preserving version of u' can be provided if need arises.)

For use later in this paper, we simplify the description of v'. We will make use of the *floor function* for real numbers $\lfloor r \rfloor$, and define $\mathring{r} = r - \lfloor r \rfloor \in [0, 1]$. Using the notation established in Section 2.4 we have:

$$v' \lceil x, r \rceil = \begin{cases} \left(h'^{\lfloor r \rfloor}\left(A'_{\mathring{r}}(x)\right), r\right) & \text{if } r \geq 0 \\ \left(f'^{|\lfloor r \rfloor|}(x), r\right) & \text{if } r < 0 \end{cases} \tag{3.1}$$

By associating the fundamental groups of $\mathrm{Tel}(f')$ and $K \times \mathbb{R}$ with G via the inclusions of $K \times \{0\}$, and noting that v' restricts to the identity on these subspaces, we may view v' as inducing id_G on fundamental groups. As such, the lift to universal covers $v : \mathrm{Tel}(f) \to X \times \mathbb{R}$ is a G-equivariant proper homotopy equivalence, where X is the universal cover of K and $f : X \to X$ is the lift of f'. This map is level-preserving with nearly level-preserving G-equivariant proper homotopy inverse $u : X \times \mathbb{R} \to \mathrm{Tel}(f)$. Letting $h : X \to X$ and $A : X \times [0, 1] \to X$ be the appropriately chosen lifts of f' and A', and adapting the conventions used above, we may specify v by the formula

$$v \lceil x, r \rceil = \begin{cases} \left(h^{\lfloor r \rfloor}\left(A_{\mathring{r}}(x)\right), r\right) & \text{if } r \geq 0 \\ \left(f^{|\lfloor r \rfloor|}(x), r\right) & \text{if } r < 0 \end{cases} \tag{3.2}$$

For future use, let $H' : \mathrm{Tel}(f') \times [0, 1] \to \mathrm{Tel}(f')$ and $J' : K \times \mathbb{R} \to X$ be the near-level preserving proper homotopies $u' \circ v' \overset{H'}{\simeq} \mathrm{id}_{\mathrm{Tel}(f')}$ and $v' \circ u' \overset{J'}{\simeq} \mathrm{id}_K$ promised above, and let H and J be their G-equivariant, near level-preserving lifts $u \circ v \overset{H}{\simeq} \mathrm{id}_{\mathrm{Tel}(f)}$ and $v \circ u \overset{J}{\simeq} \mathrm{id}_X$.

Remark 3.1. A benefit of constructing u, v, H, and J as lifts of maps between $\mathrm{Tel}(f')$ and $K \times \mathbb{R}$ (and related spaces) is that they are G-equivariant. One might ask: Why not start even lower, i.e., begin with homotopy equivalences between $\mathrm{Tor}_{f'}(K)$ and $K \times S^1$, so as to end up with $(G \rtimes_\phi \mathbb{Z})$-equivariant maps? In all but the simplest cases, that is impossible since $\mathrm{Tor}_{f'}(K)$ and $K \times S^1$ have non-isomorphic fundamental groups. The spaces under consideration become homotopy equivalent only after the "t-factor" is unfurled.

For later reference, we provide a summary of the many spaces and maps introduced in this section.

Spaces	
K	a finite $K(G,1)$ complex
$\mathrm{Tor}_{f'}(K)$	a finite $K(G \rtimes_\phi \mathbb{Z})$ complex
$\mathrm{Tel}_{f'}(K)$	an infinite cyclic cover of $\mathrm{Tor}_{f'}(K)$
X	the universal cover of K
$Y = \mathrm{Tel}_f(X)$	universal cover of $\mathrm{Tor}_{f'}(K)$ and $\mathrm{Tel}(f')$
SZ	The suspension of a space Z

Maps	
$\phi : G \to G$	a group isomorphism with inverse $\overleftarrow{\phi}$
$f' : (K, v_0) \to (K, v_0)$	cellular map inducing ϕ on fundamental groups
$h' : (K, v_0) \to (K, v_0)$	a homotopy inverse for f
A' and B'	homotopies: $A' : h' \circ f' \simeq \mathrm{id}_K$; $B' : f' \circ h' \simeq \mathrm{id}_K$
$v' : \mathrm{Tel}_{f'}(K) \to K \times \mathbb{R}$	a level-preserving proper homotopy equivalence
$u' : K \times \mathbb{R} \to \mathrm{Tel}_{f'}(K)$	a nearly level-preserving proper homotopy inverse for v'
H' and J'	homotopies; $H' : u' \circ v' \simeq \mathrm{id}_{\mathrm{Tel}_{f'}(K)}$; $J' : v' \circ u' \simeq \mathrm{id}_{K \times \mathbb{R}}$
$f, h : (X, x_0) \to (X, x_0)$	lifts of f' and h'which are ϕ-variant and ϕ^{-1}-variant, respectively
A and B	lifts of A' and B'; G-equivariant homotopies $A : h \circ f \simeq \mathrm{id}_X$; $B : f \circ h \simeq \mathrm{id}_X$
$v : Y \to X \times \mathbb{R}$	lift of v; a level-preserving G-equivariant proper homotopy equivalence
$u : X \times \mathbb{R} \to Y$	lift of u; a G-equivariant proper homotopy inverse for v
H and J	lifts of H' and J'; G-equivariant homotopies $H : u \circ v \simeq \mathrm{id}_Y$; $J : v \circ u \simeq \mathrm{id}_{X \times \mathbb{R}}$

It is now possible to give a rough outline of the proof of Theorem 1.1.

(1) Using the hypothesis that G admits a $\mathcal{Z}$-structure $(\overline{X}, Z)$ and $\phi : G \to G$ is an isomorphism, obtain a nice (ϕ-variant) continuous quasi-isometry $f : X \to X$ and use this map to construct an AR, $Y = \mathrm{Tel}_f(X)$, on which $G \rtimes_\phi \mathbb{Z}$ acts properly and cocompactly.
(2) Build a carefully controlled proper homotopy equivalence $v : Y \to X \times \mathbb{R}$.
(3) Using the $\mathcal{Z}$-compactifiability of X we may $\mathcal{Z}$-compactify $X \times \mathbb{R}$ by the addition of SZ. This compactification is not unique. Delicate techniques from [Tir11], allow us to choose a compactification for which specific collections of

compact subsets of $X\times\mathbb{R}$ become null families in $\overline{X\times\mathbb{R}}$. In particular we are interested in the v-images of $(G\rtimes_\phi\mathbb{Z})$-translates of compact subsets of Y.

(4) Finally, we use "boundary swapping" techniques developed in [GM19] to pull back the above boundary onto Y. Additional controls must be built into Step 3 to ensure that $\overline{Y}=Y\sqcup SZ$ is an AR and that the nullity condition is satisfied.

When successful with the above, we can also ask whether the $(G\rtimes_\phi\mathbb{Z})$-action on Y can be extended to $\overline{Y}$. That is the topic of Section 7.

4. A controlled $\mathcal{Z}$-compactification of $X\times\mathbb{R}$

Our proof of Theorem 1.1 requires a $\mathcal{Z}$-compactification of the $(G\rtimes_\phi\mathbb{Z})$-space $Y=\mathrm{Tel}_f(X)$ that satisfies the nullity condition of Definition 2.6. The strategy is indirect. We will first $\mathcal{Z}$-compactify $X\times\mathbb{R}$, then use the map $v:Y\to X\times\mathbb{R}$, constructed above, to swap the boundary onto Y using a process like the one described in [GM19]. Unfortunately, our setup does not exactly match the hypotheses found there— v is not a coarse equivalence and the homotopies H and J are not bounded. To compensate, we show how to impose some extreme controls on the $\mathcal{Z}$-compactification of $\overline{X\times\mathbb{R}}$ which can be adjusted to ensure that the boundary swapping procedure succeeds.

Definition 4.1. A *controlled compactification* of a proper metric space (Y,d) is a compactification $\overline{Y}$ satisfying the following property:

For every $R>0$ and every open cover $\mathcal{U}$ of $\overline{Y}$, there is a compact set $C\subset Y$ so that if $A\subset Y\setminus C$ and $\mathrm{diam}_d(A)<R$, then $A\subset U$ for some $U\in\mathcal{U}$.

A *controlled $\mathcal{Z}$-compactification* is one that is simultaneously a controlled compactification and a $\mathcal{Z}$-compactification

Whenever $(\overline{Y},Z)$ is a $\mathcal{Z}$-structure for a group G and d is a metric under which the corresponding G-action on Y is by isometries, $\overline{Y}$ is a controlled $\mathcal{Z}$-compactification of (Y,d) (see Lemma 6.4 in [GM19]). Other examples (not requiring a group action) include the addition of the Gromov boundary to a proper δ-hyperbolic space or the visual boundary to a proper CAT(0) space. For the remainder of this section, we set aside group actions and focus on the following metric/topological goal.

Theorem 4.2. *Let $\overline{X}=X\sqcup Z$ be a controlled $\mathcal{Z}$-compactification of a contractible proper metric space (X,d) and let $\eta:[0,\infty)\to[0,\infty)$ be an arbitrary monotone increasing function with $\lim_{r\to\infty}\eta(r)=\infty$. Then there is a $\mathcal{Z}$-compactification $\overline{X\times\mathbb{R}}$ of $X\times\mathbb{R}$ with boundary SZ satisfying the following control condition:*
(‡) For each open cover $\mathcal{U}$ of $\overline{X\times\mathbb{R}}$, there exists a compact set $Q\times[-N,N]\subseteq X\times\mathbb{R}$, such that every set $B_d[x,\eta(|k|)]\times[k,k+1]$ lying outside $Q\times[-N,N]$ is contained in some $U\in\mathcal{U}$.

Observe that it will suffice to demonstrate this result under the assumption that η is a continuous function. Indeed, we may always replace an arbitrary such function by a continuous function, which is also monotone increasing and is larger than or equal to our original function. One benefit of continuity is that η surjects onto $[\eta(0), \infty)$.

The first order of business is to place an appropriate topology on $X \times \mathbb{R} \sqcup SZ$. The following list of (semi-arbitrary) choices will be used:

(1) Choose metrics $\overline{d}$ for $\overline{X}$ and $\widehat{\sigma}$ for $[-\infty, \infty] = \mathbb{R} \cup \{\pm\infty\}$.
(2) Fix a basepoint $x_0 \in X$ and choose a monotone increasing continuous function $\lambda : [0, \infty) \to [0, \infty)$ with the property that, for every $s > 0$:
  (a) $\overline{d}(x, Z) < 1/s$ for all $x \in X - B_d[x_0, \lambda(s)]$, and
  (b) every ball $B_d[x, s]$ in (X, d) lying outside $B[x_0, \lambda(s)]$ has diameter $\leq 1/s$ in $(\overline{X}, \overline{d})$.
(3) Let $\psi : [0, \infty) \to [0, \infty)$ be a continuous monotone increasing function satisfying:
  (a) $\psi(s) \geq \max\{\eta(s), \lambda(s)\}$, and
  (b) $\psi(s + 1) \geq 3\psi(s)$ for all $s \geq 0$
(4) Define $p : X \to [0, \infty)$ by

$$p(x) = \log(\psi^{-1}(d(x, x_0) + \psi(0)) + 1) \tag{4.1}$$

(5) Finally, we arrive at the *slope function* $\mu : X \times \mathbb{R} \to [-\infty, \infty]$ defined by

$$\mu(x, r) = \begin{cases} \frac{r}{p(x)} & \text{if } p(x) > 0 \\ \infty & \text{if } p(x) = 0 \text{ and } r \geq 0 \\ -\infty & \text{if } p(x) = 0 \text{ and } r < 0 \end{cases} \tag{4.2}$$

Notation 2. In what follows, an undecorated x will indicate a point in X, while $\bar{x}$ will denote a point in Z. We use $\overline{X}$ for a point in $\overline{X}$ when no distinction is intended. Open and closed ball notation, $B_d(x, r)$ and $B_d[x, r]$, will be used primarily for open and closed balls in X.

To differentiate points of $X \times \mathbb{R}$ from those of SZ, we will use (x, r) for the former and $\langle \bar{x}, \mu \rangle$ for the latter (recall our convention that $\mu \in [-\infty, \infty]$.) Equivalence classes of $\langle \bar{x}, -\infty \rangle$ and $\langle \bar{x}, \infty \rangle$ will often (but not always) be abbreviated to $\langle -\infty \rangle$ and $\langle \infty \rangle$, respectively.

Definition 4.3 (The topology on $\overline{X \times \mathbb{R}}$)**.** Let $\overline{X \times \mathbb{R}} = X \times \mathbb{R} \sqcup SZ$ with the topology generated by all open subsets of $X \times \mathbb{R}$ together with sets of the form:

i) For $\langle z, \mu \rangle \in SX$ with $\mu \neq \pm\infty$ and $\varepsilon > 0$,

$$U(\langle z, \mu \rangle, \varepsilon) = \{(x, r) \mid \overline{d}(x, z) < \varepsilon,\ |\mu(x, r) - \mu| < \varepsilon\} \cup \{\langle z', \mu' \rangle \mid \overline{d}(z', z) < \varepsilon,\ |\mu' - \mu| < \varepsilon\}$$

ii) For each $\varepsilon > 0$,

$$U(\langle \infty \rangle, \varepsilon) = \{(x, r) |\ r > \frac{1}{\varepsilon},\ \mu(x, r) > \frac{1}{\varepsilon}\} \cup \{\langle z', \mu' \rangle |\ \mu' > \frac{1}{\varepsilon}\}, \text{ and}$$

$$U(\langle -\infty \rangle, \varepsilon) = \{(x, r) \mid r < \frac{-1}{\varepsilon},\ \mu(x, r) < \frac{-1}{\varepsilon}\} \cup \{\langle z', \mu' \rangle \mid \mu' < \frac{-1}{\varepsilon}\}$$

Proofs of the following two lemmas are nearly identical to their analogs in [Tir11].

Lemma 4.4. $\overline{X \times \mathbb{R}}$ *is a compactification of* $X \times \mathbb{R}$.

Lemma 4.5. *For any open cover $\mathcal{U}$ of $\overline{X \times \mathbb{R}}$ there exists $\delta > 0$ such that for each $\langle z, \mu \rangle \in SZ$, there is an element of $\mathcal{U}$ containing $U(\langle z, \mu \rangle, \delta)$*

Remark 4.6. We allow for the possibility that $\mu = \pm\infty$, i.e. $\langle z, \mu \rangle = \langle \pm\infty \rangle$, in Lemma 4.5; so $U(\langle z, \mu \rangle, \varepsilon)$ can denote a basic open set of type (i) or (ii).

For the rest of this section, fix an open cover $\mathcal{U}$ of $\overline{X \times \mathbb{R}}$, and let δ be the value promised by Lemma 4.5.

Fix a function η as in the statement of Theorem 4.2. Then the following three propositions verify the control condition (‡). The goal is to find a compact set $Q \times [-N, N] \subseteq X \times \mathbb{R}$ such that all sets of the form $B_d\,[x, \eta(|k|)] \times [k, k+1]$ which lie outside $Q \times [-N, N]$ are contained in some basic open set $U(\langle z, \mu \rangle, \delta)$.

Proposition 4.7. *For each compact set of the form $Q = B_d[x_0, \eta(M)]$ and $\delta > 0$, there exists $N_Q \in \mathbb{N}$ such that if $(B_d[x, \eta(|k|)] \times [k, k+1]) \cap (Q \times [-N_Q, N_Q]) = \varnothing$ and $Q \cap B_d[x, \eta(|k|)] \neq \varnothing$, then $B_d[x, \eta(|k|)] \times [k, k+1]) \subseteq U(\langle \pm\infty \rangle, \delta)$.*

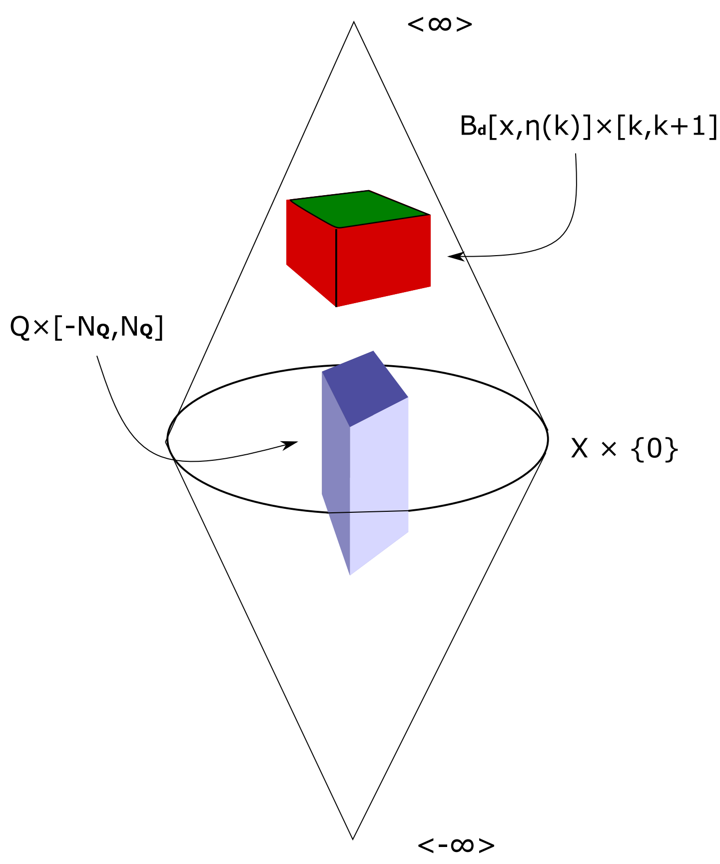

FIGURE 2. A set meeting the criterion of the Proposition 4.7.

Proof. Choose $N_Q \geq M$ so large that $\frac{N_Q}{\log(N_Q+2)} > \frac{1}{\delta}$. Let k be any value such that $Q \cap B_d[x, \eta(k)] \neq \varnothing$ and observe that it must be true that $|k| > N_Q$. Assume first that $k > N_Q$. Then

$$\begin{aligned}\min\{\mu(x',r') \mid (x',r') \in (B(x,\eta(k)) \times [k,k+1])\}&\\ &\geq \frac{\min\{r' \mid r' \in [k,k+1]\}}{\max\{p(x') \mid x' \in B(x,\eta(k))\}}\\ &\geq \frac{k}{\log(\psi^{-1}(\eta(M)+2\eta(k))+1)}\\ &\geq \frac{k}{\log(\psi^{-1}(3\eta(k))+1)}\\ &\geq \frac{k}{\log(\psi^{-1}(3\psi(k))+1)}\\ &\geq \frac{k}{\log(\psi^{-1}(\psi(k+1))+1)} > \frac{1}{\delta}\end{aligned}$$

Thus, $(B(x,\eta(k)) \times [k,k+1]) \subseteq U(\langle\infty\rangle, \delta)$. The case where $k < -N_Q$ is similar, with a conclusion that $(B(x,\eta(k)) \times [k,k+1]) \subseteq U(\langle -\infty\rangle, \delta)$. □

Proposition 4.8. *For each $[-N,N] \subseteq \mathbb{R}$, there exists a compact set $Q_N \subseteq X$ such that if $(B_d[x,\eta(|k|)] \times [k,k+1]) \cap (Q_N \times [-N,N]) = \varnothing$ and $[-N,N] \cap [k,k+1] \neq \varnothing$, then there exists $z \in Z$ such that $B[x,\eta(k)] \times [k,k+1] \subseteq U(\langle z,0\rangle,\delta)$.*

Proof. Observe that it follows from the hypotheses that Q_N must be disjoint from $(B_d[x,\eta(|k|)]$. Thus we may choose Q_N sufficiently large that $\overline{d}(B_d[x,\eta(k)],Z) < \frac{\delta}{2}$ and $\operatorname{diam}_{\overline{d}}(B_d[x,\eta(k)]) < \frac{\delta}{2}$. For example, let $Q_N = B_d[x_0,R]$ where $R \geq \lambda(2/\delta)$.

Suppose now that $(B[x,\eta(k)] \times [k,k+1]) \cap (Q_N \times [-N,N]) = \varnothing$ and $[-N,N] \cap [k,k+1] \neq \varnothing$. Choose $xz \in Z$ for which $\overline{d}(B_d[x,\eta(k)],\overline{x}) < \delta/2$, and assume for the moment that $k \geq 0$. Note that for any $(x',r') \in B(x,\eta(k)) \times [k,k+1]$,

$$\mu(x',r') \leq \frac{N+1}{\log(\psi^{-1}(R+\psi(0))+1)}$$

So, by choosing R sufficiently large, we can ensure that $\mu(x',r') < \delta$. In that case, $B[x,\eta(k)] \times [k,k+1] \subseteq U(\langle z,0\rangle,\delta)$

The case where $k < 0$ is similar. □

Now let $Q' = B_d[x_0,\eta(S)]$ where $S \gg 0$ is so large that, for all $x \in X - Q'$:

- $p(x) > \frac{2}{\delta}$,
- $\overline{d}(x,Z) < \frac{\delta}{2}$, and
- $\log(\frac{\psi^{-1}(3\eta(S))+1}{\psi^{-1}(\eta(S))+1}) < \delta$.

Then choose $N' \gg 0$ so large that:

- $\frac{1}{N'} < \frac{\delta}{2}$,
- $\frac{N'}{\log(N'+2)} > \frac{1}{\delta}$ (so $\frac{|r|}{\log(|r|+2)} > \frac{1}{\delta}$ for all $|r| > N'$), and
- $\widehat{\sigma}(r,\{\pm\infty\}) < \frac{\delta}{2}$ for all $r \in \mathbb{R} - [-N',N']$ (so $\operatorname{diam}_{\widehat{\sigma}}([k,k+1]) < \frac{\delta}{2}$ whenever $[k,k+1] \cap [-N',N'] = \varnothing$).

Given $Q' \subseteq X$ as chosen above, choose $N_{Q'} > 0$ in accordance with Proposition 4.7; and given $N' > 0$ as chosen above, choose $Q_{N'} \subseteq X$ in accordance with Proposition 4.8. Let $N = \max\{N', N_{Q'}\}$ and $Q = Q' \cup Q_{N'}$

Proposition 4.9. *If $(B(x, \eta(k)) \times [k, k+1]) \cap (Q \times [-N, N]) = \varnothing$, then there exists $\langle z, \mu\rangle \in SZ$ such that $B(x, \eta(k)) \times [k, k+1] \subseteq U(\langle z, \mu\rangle, \delta)$.*

Proof. If $B(x, \eta(k)) \cap Q$ or $[k, k+1] \cap [-N, N]$ is nonempty, the conclusion follows from Proposition 4.7 or 4.8, so we assume that $B(x, \eta(k)) \cap Q = \varnothing = [k, k+1] \cap [-N, N]$. For convenience, assume also that $k > 0$.

Let $M \in [0, \infty)$ be such that $d(x, x_0) - \eta(k) = \eta(M)$. Such an M exists since $B(x, \eta(k)) \cap Q = \varnothing$ and we assumed η was continuous and $\eta \to \infty$.

Case 1. *There exists $(x', r') \in B(x, \eta(k)) \times [k, k+1]$ such that $\mu(x', r') \leq \frac{1}{\delta}$.*

By the choice of J, there exists $z \in Z$ such that $\hat{d}(x', \bar{x}) < \frac{\delta}{2}$. By the choice of N (and since $k > 0$), we know that $\widehat{\sigma}(r', \infty) < \frac{\delta}{2}$.

For any other $(x'', r'') \in B(x, \eta(k)) \times [k, k+1]$,

$$\begin{aligned}
|\mu(x'', r'') - \mu(x', r')| &= |\mu(x'', r'') - \mu(x'', r') + \mu(x'', r') - \mu(x', r')| \\
&= \left| \frac{r''}{p(x'')} - \frac{r'}{p(x'')} + \frac{r'}{p(x'')} - \frac{r'}{p(x')} \right| \\
&\leq \frac{1}{p(x'')} |r'' - r'| + \mu(x', r') \frac{|p(x') - p(x'')|}{p(x'')} \\
&< \frac{\delta}{2} + \frac{1}{\delta} \cdot \frac{|p(x') - p(x'')|}{2/\delta}
\end{aligned}$$

So if we can show that $|p(x') - p(x'')| < \delta$, we may conclude that slopes of points in $B(x, \eta(k)) \times [k, k+1]$ differ by no more than δ.

Claim. $\eta(M) > \psi(k)$

Slopes of points in $B(x, \eta(k)) \times [k, k+1]$ are bounded below by

$$\mu_{\min} = \frac{k}{\log(\psi^{-1}(\eta(M) + 2\eta(k)) + 1)}$$

By construction, $\psi(k) \geq \eta(k)$. Suppose that $\psi(k) \geq \eta(M)$. Then,

$$\begin{aligned}
\mu_{\min} &\geq \frac{k}{\log(\psi^{-1}(3\psi(k)) + 1)} \\
&\geq \frac{k}{\log(\psi^{-1}(\psi(k+1)) + 1)} > \frac{1}{\delta}
\end{aligned}$$

contradicting the existence of $(x', r') \in B(x, \eta(k)) \times [k, k+1]$ with $\mu(x', r') \leq \frac{1}{\delta}$. The claim follows.

Now

$$\begin{aligned}|p(x') - p(x'')| &\leq \log(\psi^{-1}(\eta(M) + 2\eta(k)) + 1) - \log(\psi^{-1}(\eta(M)) + 1) \\ &< \log(\frac{\psi^{-1}(3\eta(M)) + 1}{\psi^{-1}(\eta(M)) + 1}) < \delta\end{aligned}$$

Since $\eta(M) > \psi(k)$, we are guaranteed that $\mathrm{diam}_{\overline{d}}(B(x, \eta(k))) < \frac{1}{k} < \frac{\delta}{2}$; so by the triangle inequality $B(x, \eta(k)) \times [k, k+1] \subseteq U(\langle \bar{x}, \mu(x', r')\rangle, \delta)$.

Case 2. *There exists no $(x', r') \in B(x, \eta(k)) \times [k, k+1]$ such that $\mu(x', r') \leq \frac{1}{\delta}$.*

Then $\mu(x', r') > \frac{1}{\delta}$ for all $(x', r') \in B(x, \eta(k)) \times [k, k+1]$. Since all of the slopes are greater than $\frac{1}{\delta}$, the choice of N guarantees that $B(x, \eta(k)) \times [k, k+1] \subseteq U(\langle \infty \rangle, \delta)$. $\square$

To complete the proof of Theorem 4.2, we need only prove the following:

Proposition 4.10. *Given X, $\overline{X}$, and $\overline{X \times \mathbb{R}}$ as defined above, SZ is a $\mathcal{Z}$-set in $\overline{X \times \mathbb{R}}$. If X is an AR, then so are $X \times \mathbb{R}$, $\overline{X}$, and $\overline{X \times \mathbb{R}}$.*

To prove Proposition 4.10, we will construct a contraction $\gamma : \overline{X \times \mathbb{R}} \times [0, 1] \to \overline{X \times \mathbb{R}}$ with $\gamma_0 = \mathrm{id}_{\overline{X \times \mathbb{R}}}$, $\gamma_1(\overline{X \times \mathbb{R}}) = \{x_0\}$, and $\gamma(\overline{X \times \mathbb{R}} \times (0, 1]) \subseteq X \times \mathbb{R}$. From there, the second sentence of the proposition follows from the discussion of $\mathcal{Z}$-compactifications in Section 2.3. Our construction of γ involves relatively minor modifications to the analogous construction in Section 3 of [Tir11]. The first modification is entirely superficial, owing to the fact that she is working with a general product $X \times Y$ and join $Z_X * Z_Y$ while we are working with a special case: $X \times \mathbb{R}$ and SZ. The second modification is due to the more delicate nature of our choice of the function $p : X \to [0, \infty)$, used to define the slope function.

The strategy for contracting $\overline{X \times \mathbb{R}}$ is motivated by the contraction of the visual compactification $\overline{Y}$ of a proper CAT(0) space Y to a basepoint y_0, whereby interior points and boundary points are slid toward y_0 along geodesic segments and geodesic rays, respectively. There it is useful to view—and parameterize—the geodesic segments as "eventually constant" nonproper rays. In this way, one associates to the points of $\overline{Y}$, a continuously varying family of rays $\gamma_y : [0, \infty) \to Y$ beginning at y_0 and ending at $y \in Y$ or else determining a point y in the visual boundary by virtue of being a geodesic ray. The definition of the cone topology allows us to extend each to $\gamma_y : [0, \infty] \to \overline{Y}$ without losing continuity. The contraction is obtained by applying a deformation retraction of $[0, \infty]$ onto $\{0\}$ simultaneously to all (extended) rays.

In our setting, we will identify a continuous family of preferred rays in $X \times \mathbb{R}$, all emanating from $(x_0, 0)$. Some will be eventually constant, ending at a point $(x, r) \in X \times \mathbb{R}$; others will limit to a point $\langle z, \mu \rangle \in SZ$. By choosing these rays in coordination with the topology on $\overline{X \times \mathbb{R}}$, we will be able to mimic the above strategy.

Let $\alpha : \overline{X} \times [0, 1] \to \overline{X}$ be a $\mathcal{Z}$-set homotopy that contracts $\overline{X}$ to $x_0 \in X$ keeping x_0 fixed throughout, and $\beta : [-\infty, \infty] \times [0, 1] \to [-\infty, \infty]$ be a $\mathcal{Z}$-set homotopy that contracts $[-\infty, \infty]$ to 0 keeping 0 fixed throughout.

The following lemma is inspired by [Tir11, Lemmas 3.6 & 3.8].

Lemma 4.11. *There are reparameterizations $\widehat{\alpha}$ and $\widehat{\beta}$ of α and β so that $p(\widehat{\alpha}(z,t)) \in [\frac{1}{t}-1, \frac{1}{t}+2]$ and $\left|\widehat{\beta}(\pm\infty,t)\right| \in [\frac{1}{t}-1, \frac{1}{t}+2]$ for all $t \in (0,1]$ and for all $z \in Z$.*

Proof. Our approach differs slightly from that taken in [Tir11]. Whereas she *chose* her function $p : X \to [0,\infty)$ to have the property:

(††) For some sequence $1 = t_0 > t_1 > \cdots > 0$, $p(\alpha(Z \times [t_i, t_{i-1}])) \subseteq (i-1, i+1]$

we, instead, begin with the function $p : X \to [0,\infty)$ arrived at in defining our slope function, then arrange condition (††) by reparameterizing α. This can be accomplished by using methods similar to those used by Tirel to define her function p.

Once α has been adjusted to satisfy property (††), we can implement the proof of [Tir11, Lemma 3.8] to obtain $\widehat{\alpha}$. Obtaining $\widehat{\beta}$ is much simpler. $\square$

Remark 4.12. Notice that, aside from properness, no specific properties of $p : X \to [0,\infty)$ are used in the above proof. This fact will be useful in Section 8.

In order to turn homotopy tracks into rays, we invert and stretch the "time" interval. Define

$$\begin{array}{rll} \xi : [0,\infty] \to [0,1] & \text{by} & \xi(t) = \begin{cases} \frac{1}{1+t} & \text{if } t \in [0,\infty) \\ 0 & \text{if } t = \infty \end{cases} \\ \alpha' : \overline{X} \times [0,\infty] \to \overline{X} & \text{by} & \alpha'(w,t) = \widehat{\alpha}(w, \xi(t)) \\ \beta' : [-\infty,\infty] \times [0,\infty] \to \overline{\mathbb{R}} & \text{by} & \beta'(r,t) = \widehat{\beta}(r, \xi(t)) \end{array}$$

where we allow w to denote an element of either X or Z. Notice that for any $t \in [0,\infty)$ and $z \in Z$, $p(\alpha'(z,t)) \in (t-1, t+3)$; and similarly, $|\beta'(\pm\infty,t)| \in (t-1,t+3)$.

Define

$$\gamma' : \overline{X \times \mathbb{R}} \times [0,\infty) \to X \times \mathbb{R}$$

by

$$\begin{array}{rcll} ((x,r),t) & \mapsto & (\alpha'(x, \frac{t}{\sqrt{(\mu(x,r))^2+1}}), \beta'(r, \frac{\mu(x,r)\cdot t}{\sqrt{(\mu(x,r))^2+1}})) & \\ (\langle z,\mu\rangle,t) & \mapsto & (\alpha'(z, \frac{t}{\sqrt{\mu^2+1}}), \beta'(\infty, \frac{\mu\cdot t}{\sqrt{\mu^2+1}})) & (\text{if } \mu \geq 0) \\ (\langle z,\mu\rangle,t) & \mapsto & (\alpha'(z, \frac{t}{\sqrt{\mu^2+1}}), \beta'(-\infty, \frac{\mu\cdot t}{\sqrt{\mu^2+1}})) & (\text{if } \mu < 0) \\ (\langle\infty\rangle,t) & \mapsto & (x_0, \beta'(\infty,t)) & \\ (\langle-\infty\rangle,t) & \mapsto & (x_0, \beta'(-\infty,t)) & \end{array}$$

Clearly γ' extends continuously over $X \times \mathbb{R} \times [0,\infty]$ by sending each $((x,r),\infty)$ to (x,r). For later use, let $\gamma'_{(x,r)} = \gamma'|_{(x,r)\times[0,\infty]}$. Note that for each $\langle z,\mu\rangle \in SZ$, the map $\gamma'_{\langle z,\mu\rangle} = \gamma'|_{\langle z,\mu\rangle\times[0,\infty)}$ is a proper ray in $X \times \mathbb{R}$. We wish to observe that $\gamma'_{\langle z,\mu\rangle}(t) \to \langle z,\mu\rangle$ in $\overline{X \times \mathbb{R}}$ as $t \to \infty$. In the special case that $\mu = \infty$, i.e. $\langle z,\mu\rangle = \langle\infty\rangle$, then $\gamma'_{\langle\infty\rangle}(\langle\infty\rangle,t) = (x_0, \beta'(\infty,t))$. Since $p(x_0) = 0$, then $\mu(x_0, \beta'(\infty,t)) = \infty$; moreover

$\beta'(\infty, t) \to \infty$ as $t \to \infty$. So $\gamma'_{\langle\infty\rangle}(t) \to \langle\infty\rangle$ as $t \to \infty$. Similarly for $\mu = -\infty$. For a generic boundary point $\langle z, \mu\rangle$ with $0 \leq \mu < \infty$, we have

$$\begin{aligned}
\mu(\gamma'(\langle z,\mu\rangle, t)) &= \frac{\beta'(\infty, \frac{\mu\cdot t}{\sqrt{\mu^2+1}})}{p(\alpha'(z, \frac{t}{\sqrt{\mu^2+1}}))} \\
&\in (\frac{\frac{\mu\cdot t}{\sqrt{\mu^2+1}} - 2}{\frac{t}{\sqrt{\mu^2+1}} + 3}, \frac{\frac{\mu\cdot t}{\sqrt{\mu^2+1}} + 3}{\frac{t}{\sqrt{\mu^2+1}} - 2}) \\
&= (\frac{\mu\cdot t - 2\sqrt{\mu^2+1}}{t + 3\sqrt{\mu^2+1}}, \frac{\mu \cdot t + 3\sqrt{\mu^2+1}}{t - 2\sqrt{\mu^2+1}})
\end{aligned}$$

which implies that $\mu(\gamma'(\langle z,\mu\rangle, t)) \to \mu$ as $t \to \infty$. In addition, $\alpha'(z, \frac{t}{\sqrt{\mu^2+1}}) \to \alpha(z, 0) = z$ in $\overline{X}$ as $t \to \infty$. Thus $\gamma'(\langle z,\mu\rangle, t) \to \langle z, \mu\rangle$ in $\overline{X \times \mathbb{R}}$, and we define $\gamma'(\langle z,\mu\rangle, \infty) = \langle z,\mu\rangle$. Calculations similar to the above show that, for small ε, rays of the form $\gamma'_{(x',r)}$ and $\gamma'_{\langle z',\mu'\rangle}$ which end in a basic open set $U(\langle z,\mu\rangle, \varepsilon)$, track together in $\overline{X \times \mathbb{R}}$. As such, we obtain a continuous function

$$\gamma' : \overline{X \times \mathbb{R}} \times [0, \infty] \to \overline{X \times \mathbb{R}}$$

Reversing and reparameterizing the interval once more, we get the desired $\mathcal{Z}$-set homotopy $\gamma : \overline{X \times \mathbb{R}} \times [0, 1] \to \overline{X \times \mathbb{R}}$.

5. Proof of Theorem 1.1

We now turn to the proof of Theorem 1.1. The goal is a $\mathcal{Z}$-compactification $\overline{Y} = Y \sqcup SZ$ of the $(G \rtimes_\phi \mathbb{Z})$-space $Y = \mathrm{Tel}_f(X)$ that satisfies Definition 2.6. The approach is to use the map $v : Y \to X \times \mathbb{R}$ to swap the boundary from a controlled $\mathcal{Z}$-compactification $\overline{X \times \mathbb{R}} = (X \times \mathbb{R}) \sqcup SZ$ onto Y. The key is to incorporate some of the geometry of the $(G \rtimes_\phi \mathbb{Z})$-action on Y into the choice of control function $\eta : [0, \infty) \to [0, \infty)$ used in Theorem 4.2.

Choose a $(G \rtimes_\phi \mathbb{Z})$-equivariant metric ρ on Y (see [GM19, §6]). The restriction of ρ to any X_i, $i \in \mathbb{Z}$, is a G-invariant metric on X which we will designate as d. Give $\mathbb{R}$ the usual metric, and let d_1 be the corresponding ℓ_1 metric on $X \times \mathbb{R}$. As in the previous section, let $\overline{d}$ be a metric for $\overline{X}$ and $\widehat{\sigma}$ a metric for $[-\infty, \infty]$.

Let $C_Y \subseteq \mathcal{M}_{[0,1]}(f)$ be a compact set whose $(G \rtimes_\phi \mathbb{Z})$-translates cover Y. For example, let $C_X \subseteq X$ be a finite subcomplex whose G-translates cover X, then let C_Y be the sub-mapping cylinder $\mathcal{M}_{[0,1]}(f|_{C_X})$, where the range is restricted to a finite subcomplex containing $f(C_X)$. We will refer to C_X and C_Y informally as *fundamental domains* for the G- and $(G \rtimes_\phi \mathbb{Z})$-actions on X and Y, respectively.

Definition 5.1 (The control function)**.** Let $\eta : [0,\infty) \to [0,\infty)$ be a function satisfying: For all $k \in \mathbb{N}$

$$\eta(|k|) \geq \max \begin{cases} \operatorname{diam}_{d_1}(v(H(t^{\pm k}C_Y \times [0,1]))) \\ \operatorname{diam}_{d_1}(J(C_X \times [k,k+1]) \times [0,1])) \end{cases}$$

Furthermore, choose η to be monotonic and require that $\lim_{r\to\infty} \eta(r) = \infty$.

Remark 5.2. A priori, $\eta(|k|)$ provides a bound only on the diameters of $h(t^{\pm k}C \times [0,1])$ and $J(v(t^{\pm k}C) \times [0,1])$; however the fact that v, H, and J are G-equivariant and level preserving, means that $\eta(|k|)$ bounds the diameters of all $(G \rtimes_\phi \mathbb{Z})$-translates of those sets which are contained in the $[k,k+1]$-level. Since H_0 and J_0 are identities, $\eta(|k|)$ also bounds the diameters of $t^{\pm k}C$ and $v(t^{\pm k}C)$ and their translates contained in the $[k,k+1]$-level.

Now apply Theorem 4.2 to obtain a $\mathcal{Z}$-compactification $(\overline{X \times \mathbb{R}}, SZ)$ of $X \times \mathbb{R}$ satisfying the condition:

($\ddagger$) *For each open cover* U *of* $\overline{X \times \mathbb{R}}$*, there exists a compact set* $Q \times [-N,N] \subseteq X \times \mathbb{R}$*, such that every set* $B_d[x,\eta(|k|)] \times [k,k+1]$ *lying outside* $Q \times [-N,N]$ *is contained in some* $U \in U$*.*

Recall the map $v : Y \to X \times \mathbb{R}$ defined in Section 3. Let $\overline{Y} = Y \sqcup SZ$ and define $\overline{v} = v \cup \mathrm{id}_{SZ} : \overline{Y} \to \overline{X \times \mathbb{R}}$, i.e.

$$\overline{v}(z) = \begin{cases} v(z) & \text{if } z \in Y \\ z & \text{if } z \in SZ \end{cases}$$

Give $\overline{Y}$ the topology $\mathcal{T}$ generated by the open subsets of Y together with

$$\left\{\overline{v}^{-1}(\mathcal{U}) \mid \mathcal{U} \text{ is open in } \overline{X \times \mathbb{R}}\right\}$$

Remark 5.3. We call $\left(\overline{Y}, \mathcal{T}\right)$ the *pull-back compactification* of Y via the map v. This construction can be applied more broadly whenever one has a proper map $v : Y \to W$ between locally compact separable metric spaces and a compactification $\overline{W} = W \sqcup A$. The result is a compactification $\overline{Y} = Y \sqcup A$ (not always a $\mathcal{Z}$-compactification) of Y and a continuous map $\overline{v} : \overline{Y} \to \overline{W}$ which extends v via the identity over A.

Proposition 5.4. *$\overline{Y}$ is a $\mathcal{Z}$-compactification of Y.*

Our proof will be obtained by applying the following lemma which was based on [Fer00, Prop. 1.6].

Lemma 5.5 (see [GM19, Lemma 3.1])**.** *Let X and Y be separable metric spaces and $f : (X,A) \to (Y,B)$ and $g : (Y,B) \to (X,A)$ be continuous maps with $f(X-A) \subseteq Y-B$, $g(Y-B) \subseteq X-A$, and $g \circ f|_A = \mathrm{id}_A$. Suppose further that there is a homotopy $K : X \times [0,1] \to X$ which is fixed on A and satisfies: $K_0 = \mathrm{id}_X$, $K_1 = g \circ f$, and $K((X-A) \times [0,1]) \subseteq X - A$. If B is a $\mathcal{Z}$-set in Y, then A is a $\mathcal{Z}$-set in X.*

Proof of Proposition 5.4. Recall the maps $u : X \times \mathbb{R} \to Y$ and $H : Y \times [0,1] \to Y$ defined in Section 3, and let $\overline{u} : \overline{X \times \mathbb{R}} \to \overline{Y}$ and $\overline{H} : \overline{Y} \times [0,1] \to \overline{Y}$ be extensions via the identity on SZ. In order to apply Lemma 5.5, it suffices to show that $\overline{u}$ and $\overline{H}$ are continuous.

We will use the following notational convention: Whenever $\overline{V}$ denotes a subset of $\overline{Y}$ [resp., $\overline{X \times \mathbb{R}}$], V will denote $\overline{V} \cap Y$ [resp., $\overline{V} \cap (X \times \mathbb{R})$].

Claim 1. $\overline{u}$ is continuous.

It suffices to verify continuity at points of SZ. Let $\langle z_0, \mu_0 \rangle \in SZ$ and $\overline{v}^{-1}(\overline{V})$ be a basic open neighborhood of $\overline{u}(\langle z_0, \mu_0 \rangle) = \langle z_0, \mu_0 \rangle$ in $\overline{Y}$. The goal is to find a basic open neighborhood $U(\langle z_0, \mu \rangle, \varepsilon_0)$ of $\langle z_0, \mu_0 \rangle$ in $\overline{X \times \mathbb{R}}$ such that $\overline{u}(U(\langle z_0, \mu \rangle, \varepsilon_0)) \subseteq \overline{v}^{-1}(\overline{V})$, i.e., $\overline{v}\left(\overline{u}(U(\langle \overline{x}_0, \mu \rangle, \varepsilon_0))\right) \subseteq \overline{V}$.

To begin, assume that $\mu_0 \neq \pm\infty$ and choose $j \in \mathbb{Z}$ so that $j - 1 < \mu_0 < j + 1$. Let $\overline{W}$ of be an open neighborhood of $\langle \overline{x}_0, \mu_0 \rangle$ such that $\mathrm{cl}(\overline{W}) \subseteq \overline{V}$. Then $\left\{\overline{X \times \mathbb{R}} - \mathrm{cl}(\overline{W}), \overline{V}\right\}$ is an open cover of $\overline{X \times \mathbb{R}}$, so by Lemma 4.5, there exists $\varepsilon' > 0$ such that, if $U(\langle z', \mu' \rangle, \varepsilon') \cap \mathrm{cl}(\overline{W}) \neq \varnothing$, then $U(\langle z', \mu' \rangle, \varepsilon') \subseteq \overline{V}$. Choose $Q \times [-N, N]$ be sufficiently large that $N \geq j + 1$ and every set of the form $B_d[x, \eta(k)] \times [k, k+1]$ which lies outside $Q \times [-N, N]$ is contained in some $U(\langle z, \mu \rangle, \varepsilon')$. Choose $\varepsilon_0 > 0$ so that:

- $U(\langle z_0, \mu \rangle, \varepsilon_0) \subseteq \overline{W}$,
- $\varepsilon_0 \leq \mathrm{dist}(\mu_0, \{j-1, j+1\})$, and
- $d(x, Q) > 2\eta(N)$ for all $(x, r) \in U(\langle z_0, \mu_0 \rangle, \varepsilon_0)$.

Let $(x, r) \in U(\langle z_0, \mu \rangle, \varepsilon_0)$. Then $r \in [j-1, j+1]$. Assume $r \in [j, j+1]$ (the case $r \in [j-1, j]$ is similar). Choose $g \in G$ so that $x \in gC_X$. Then

$$(x, r) \in gC_X \times [j, j+1] \subseteq J(gC_X \times [j, j+1] \times [0,1]) \subseteq B_d[gx_0, \eta(j)] \times [j, j+1]$$

Since $d(x, Q) > 2\eta(N) > 2\eta(j)$, then $B_d[gx_0, \eta(j)] \times [j, j+1]$ lies outside $Q \times [-N, N]$, hence entirely in some basic open set $U(\langle z', \mu' \rangle, \varepsilon')$. This basic open set intersects $\overline{W}$ at (x, r), so by choice of ε', $U(\langle z', \mu' \rangle, \varepsilon') \subseteq \overline{V}$. Furthermore, since $B_d[gx_0, \eta(j)] \times [j, j+1]$ contains $J((x, r) \times [0,1])$, then $U(\langle z', \mu' \rangle, \varepsilon')$ contains $\overline{v} \circ \overline{u}(x)$. This means that $\overline{v}(\overline{u}(x, r)) \in \overline{V}$, as desired. Since $\overline{v} \circ \overline{u}$ is the identity on SZ, it follows that $\overline{v}(\overline{u}(U(\langle z_0, \mu \rangle, \varepsilon_0))) \subseteq \overline{V}$.

A similar, but easier, argument verifies continuity of $\overline{u}$ at $\langle \pm\infty \rangle$.

Claim 2. $\overline{H}$ is continuous.

Let $\langle z_0, \mu_0 \rangle \in SZ$ and $\overline{v}^{-1}(\overline{V})$ be is a basic open neighborhood of $\langle z_0, \mu_0 \rangle$ in $\overline{Y}$. Then $\overline{V}$ is an open neighborhood of $\overline{v}(\langle z_0, \mu_0 \rangle) = \langle z_0, \mu_0 \rangle$ in $\overline{X \times \mathbb{R}}$. Arguing in much the same way as above, we may choose a basic open neighborhood $U(\langle z_0, \mu_0 \rangle, \varepsilon_0) \subseteq \overline{V}$ so small that, if $y \in Y$ and $\overline{v}(y) \in U(\langle z_0, \mu_0 \rangle, \varepsilon_0)$ then $\overline{v}(H(y \times [0,1])) \subseteq \overline{V}$. (This uses the first control condition in Definition 5.1.) As such, $\overline{v}(\overline{H}(\overline{v}^{-1}(U(\langle z_0, \mu_0 \rangle, \varepsilon_0)) \times [0,1])) \subseteq \overline{V}$, so $\overline{H}(\overline{v}^{-1}(U(\langle z_0, \mu_0 \rangle, \varepsilon_0)) \times [0,1]) \subseteq \overline{v}^{-1}(\overline{V})$, implying that $\overline{H}$ is continuous at points of $\langle z_0, \mu_0 \rangle \times [0,1]$. $\square$

We are now ready to complete the main task of this section.

Proof of Theorem 1.1. Thus far we have a geometric action of $G \rtimes_\phi \mathbb{Z}$ on $Y = \mathrm{Tel}_f(X)$, and a $\mathcal{Z}$-compactification $\overline{Y} = Y \sqcup SZ$. We have already noted that Y is a contractible locally finite CW complex—hence an AR—and that a $\mathcal{Z}$-compactification of an AR is an AR. It remains only to show that $(\overline{Y}, SZ)$ satisfies the nullity condition. It suffices to show that the $(G \rtimes_\phi \mathbb{Z})$-translates of the fundamental domain C_Y form a null family in $\overline{Y}$.

Let $\mathcal{V}$ be an open cover of $\overline{Y}$. By passing to an open refinement, we may assume $\mathcal{V}$ consists entirely of basic open sets and thus contains a subcollection $\{\overline{v}^{-1}(U_\alpha)\}_{\alpha \in A}$ which covers SZ. It follows that $\{U_\alpha\}_{\alpha \in A}$ covers SZ in $\overline{X \times \mathbb{R}}$. Choose a single open set $U_0 = B_d(x_0, r) \times (-r, r)$ sufficiently large that $\mathcal{U} = \{U_0\} \cup \{U_\alpha\}_{\alpha \in A}$ covers $\overline{X \times \mathbb{R}}$. Then choose a compact set $Q \times [-N, N] \subseteq X \times \mathbb{R}$ satisfying condition (‡). Let Q' be the closed $\eta(N+1)$-neighborhood of Q in (X, d) and choose $P \subseteq Y$ to be a compact set containing both $v^{-1}(Q' \times [-(N+1), N+1])$ and $v^{-1}(B_d[x_0, r] \times [-r, r])$. Let $a \in G \rtimes_\phi \mathbb{Z}$ such that $aC_Y \cap P = \varnothing$. If we use the standard presentation for semidirect products to express a as gt^k, then $aC_Y \subseteq \mathcal{M}_{[k,k+1]}(f)$ and $v(aC_Y) \subseteq X \times [k, k+1]$. Since $\mathrm{diam}_{d_1}(v(aC_Y)) \leq \eta(|k|)$, then $v(aC_Y) \subseteq B_{d_1}[x, \eta(k)] \times [k, k+1]$ for some $x \in X$. Moreover, since $v(aC_Y) \cap (Q' \times [-(N+1), N+1]) = \varnothing$, then $(B_{d_1}[x, \eta(k)] \times [k, k+1]) \cap (Q \times [-N, N]) = \varnothing$. Therefore $B_{d_1}[x, \eta(k)] \times [k, k+1]$, and hence $v(aC_Y)$, lies in some element of $\mathcal{U}$. That element cannot be U_0 since $v(aC_Y) \cap B_d[x_0, r] \times [-r, r] = \varnothing$, so $v(aC_Y) \subseteq U_\alpha$ for some $\alpha \in A$. Thus $aC_Y \subseteq v^{-1}(U_\alpha) \in \mathcal{V}$. Since the action of $G \rtimes_\phi \mathbb{Z}$ on Y is proper, the nullity condition follows. □

6. $\mathcal{Z}$-structures for $G \rtimes_\phi \mathbb{Z}$ when torsion is permitted

We now turn to the general case where G is permitted to have torsion. As before, we assume a $\mathcal{Z}$-structure $(\overline{X}, Z)$ on G, so there is a proper, cocompact G-action on an AR X, but now the action need not be free. As a result, the quotient map is not a covering projection. This disables the tricks used at the beginning of Section 3 to obtain a finite $K(G,1)$ complex, which was used to build: a finite $K(G \rtimes_\phi \mathbb{Z}, 1)$ complex; a corresponding universal cover; and a variety of maps and homotopies. Without all the benefits of covering space theory, a more hands-on approach is required.

Since the trick that allowed us to pass from ARs and ANRs to the category of CW complexes is no longer available, we will deal directly with A(N)Rs whenever possible. In several places, however, we can get slightly stronger conclusions by assuming the existence of cell structures. Sometimes those stronger conclusions are not needed, but in a few of the more delicate applications they are crucial. For that reason, we include both A(N)R and cellular versions in much of what follows.

In this section we will prove:

Theorem 6.1. *Let G be a group admitting a $\mathcal{Z}$-structure $(\overline{X}, Z)$ and let $\phi \in \mathrm{Aut}\,(G)$. If there exist ϕ-variant and ϕ^{-1}-variant self-maps of X then $G \rtimes_\phi \mathbb{Z}$ admits a $\mathcal{Z}$-structure of the form $(\overline{Y}, SZ)$.*

In the torsion-free case, the existence of ϕ-variant and ϕ^{-1}-variant maps was automatic (by covering space theory). As such Theorem 1.1 can be deduced as a corollary of this theorem. When torsion is present, we are not sure if these maps always exist, but they do exist in a wide variety of important cases to be discussed in this section. Most notably, we will see that Theorem 6.1 can be applied whenever X is an $\underline{E}G$-complex.

6.1. **Constructing** $(G \rtimes_\phi \mathbb{Z})$**-spaces.**

For our purposes, the key to constructing a nice $(G \rtimes_\phi \mathbb{Z})$-space is the existence of a ϕ-variant map from the G-space X to itself.

Theorem 6.2. *Let X be a proper cocompact G-space, $\phi \in \operatorname{Aut}(G)$, and $f : X \to X$ a ϕ-variant map. Then $Y = \operatorname{Tel}_f(X)$ admits a corresponding proper cocompact $(G \rtimes_\phi \mathbb{Z})$-action. Moreover,*

1. *if X is contractible then so is Y,*
2. *if X is an AR then so is Y, and*
3. *if X is a [rigid] G-complex and f is cellular, then Y admits a cell structure under which the $(G \rtimes_\phi \mathbb{Z})$-action is [rigid] cellular.*

Proof. Lemma 2.10 assures that Y is locally compact, separable and metrizable. Using the notation from Definition 2.7, for each $g \in G$, define

$$g \cdot \lceil x, r \rceil = \lceil \phi^{\lfloor r \rfloor}(g) \cdot x, r \rceil \tag{6.1}$$

and let

$$t \cdot \lceil x, r \rceil = \lceil x, r+1 \rceil \tag{6.2}$$

Clearly t determines a self-homeomorphism of Y with inverse $t^{-1} \cdot (x, r) = (x, r-1)$. To see that each $g : Y \to Y$ is continuous, note first that g is continuous on the individual subcylinders $\mathcal{M}_{[k,k+1)}(f)$. Furthermore, for each mapping cylinder segment $\{x\} \times [k, k+1)$ in $\mathcal{M}_{[k,k+1)}(f)$ converging to $(f(x), k+1)$ in $\mathcal{M}_{[k+1,k+2)}(f)$, the g-translate $\left\{\phi^k(g) \cdot x\right\} \times [k, k+1)$ converges to $\left(f\left(\phi^k(g) \cdot x\right), k+1\right)$ in $\mathcal{M}_{[k+1,k+2)}(f)$. By ϕ-variance, $\left(f\left(\phi^k(g) \cdot x\right), k+1\right) = \left(\phi^{k+1}(g) \cdot f(x), k+1\right)$ which is precisely $g \cdot (f(x), k+1)$. As such, g is continuous on Y.

Next note that

$$\begin{aligned} t^{-1}gt \cdot (x, r) &= t^{-1}g \cdot (x, r+1) \\ &= t^{-1} \cdot (\phi^{\lfloor r+1 \rfloor}(g)\, x, r+1) \\ &= (\phi^{\lfloor r+1 \rfloor}(g)\, x, r) \\ &= (\phi^{\lfloor r \rfloor}\phi(g)\, x, r) = \phi(g) \cdot (x, r) \end{aligned}$$

So the semidirect product relators are satisfied, and we have the desired action. We leave it to the reader to check that this action is proper and cocompact.

Assertions 1)-3) follow easily from further application of Lemma 2.10. □

6.2. **Fixed sets and the existence ϕ-variant maps.** We now investigate situations where ϕ-variant (and ϕ^{-1}-variant) maps can be shown to exist. Some commonly used assumptions about fixed point sets will be useful.

Definition 6.3. Let G be a group and $\mathcal{F}_G$ the collection of all finite subgroups of G. A G-space X is called $\mathcal{F}_G$*-contractible* if, for every $H \in \mathcal{F}_G$, the fixed set

$$X^H = \{x \in X \mid hx = x \text{ for all } h \in H\}$$

is contractible. If, in addition, each X^H is an AR the action is called $\mathcal{F}_G^{AR}$*-contractible*

Definition 6.4. A proper, rigid, $\mathcal{F}_G$-contractible G-complex is called an $\underline{E}G$*-complex.* A proper, $\mathcal{F}_G^{AR}$-contractible G-AR is called an $\underline{E}G_{\mathrm{AR}}$*-space*.

Remark 6.5. By definition an $\underline{E}G$-complex X is contractible, and by rigidity each X^H is a subcomplex. Since a cocompact $\underline{E}G$-complex X is necessarily locally finite, it can be viewed as a special case of a cocompact $\underline{E}G_{\mathrm{AR}}$-space. (Recall that we require ARs to be locally compact.)

The following proposition provides some key examples; it will also be useful for some of our later applications.

Proposition 6.6. *If X is a proper cocompact $\mathcal{F}_G$-contractible G-space, $\phi \in \mathrm{Aut}\,(G)$, and $f : X \to X$ is a ϕ-variant map, then $Y = \mathrm{Tel}_f\,(X)$ is a proper cocompact $\mathcal{F}_{G\rtimes_\phi\mathbb{Z}}$-contractible $(G \rtimes_\phi \mathbb{Z})$-space. Moreover, if X is an $\underline{E}G_{AR}$-space then Y an $\underline{E}(G \rtimes_\phi \mathbb{Z})_{AR}$-space and if X is an $\underline{E}G$-complex and f is cellular, then Y is an $\underline{E}(G \rtimes_\phi \mathbb{Z})$-complex.*

Proof. For the initial assertion, let $H \leq G \rtimes_\phi \mathbb{Z}$ be finite. Then $H \leq G \leq G \rtimes_\phi \mathbb{Z}$, and by hypothesis, X^H is contractible, as is $X^{\phi^i(H)}$ for all $i \in \mathbb{Z}$. Furthermore, if $h \cdot x = x$, then $\phi\,(h) \cdot f\,(x) = f\,(x)$; so $f(X^{\phi^i(H)}) \subseteq X^{\phi^{i+1}(H)}$ for all $i \in \mathbb{Z}$. It follows that Y^H is the sub-mapping telescope defined by the following subspaces and restriction maps.

$$\cdots \xrightarrow{f|} X^{\phi^{-2}(H)} \xrightarrow{f|} X^{\phi^{-1}(H)} \xrightarrow{f|} X^H \xrightarrow{f|} X^{\phi(H)} \xrightarrow{f|} X^{\phi^2(H)} \xrightarrow{f|} \cdots$$

By an application of Lemma 2.10, since each $X^{\phi^i(H)}$ is contractible, so is Y^H.

The additional assertions from similar reasoning. □

See [LN03] for examples of groups that do not admit any cocompact $\underline{E}G$-space but which do act cocompactly on a contractible CW-complex.

We now turn to the task of constructing ϕ-variant maps.

Proposition 6.7. *If X is a proper cocompact G-space, X' is an $\mathcal{F}_G$*-contractible *G-space, and $\phi \in \mathrm{Aut}\,(G)$, then there exists a ϕ-variant map $f : X \to X'$. If X and X' are $\underline{E}G$-complexes, then f can be chosen to be cellular.*

We will use the following special case of [FJ93, Theorem A.2].

Theorem 6.8. *Let X be a rigid G-CW complex with finite cell stabilizers, and let X' an $\mathcal{F}_G$-contractible G-space. Then there is a G-equivariant map from X to X', and any two such maps are homotopic through G-maps. If X' is a rigid G-CW complex (hence an $\underline{E}G$-complex), then the maps and homotopies can be chosen to be cellular.*

Remark 6.9. If X and X' are rigid G-CW complexes, Proposition 6.7 follows almost immediately from Theorem 6.8. Let X'' denote X' with the modified G-action $g \cdot x \equiv \phi(g) \cdot x$, then note that a G-equivariant map from X to X'' is a ϕ-variant map from X to X'. In the more general case of Proposition 6.7, more work is needed. Much of our strategy is borrowed from [Ont05].

Proof of Proposition 6.7. Choose a proper metric d on X so that G acts by isometries [GM19, Proposition 6.3]. Since the orbit Gx of any $x \in X$ is discrete, there is a radius r_x such that the closed ball $B_d[x, r_x] \cap (Gx) = \{x\}$. Then, for all $g \in G$, $B(x, \frac{r_x}{2}) \cap gB(x, \frac{r_x}{2}) = \varnothing$ or $gx = x$, with the latter implying that $B(x, \frac{r_x}{2}) = gB(x, \frac{r_x}{2})$. By cocompactness, there is a finite collection $\mathcal{V} = \left\{B_d(x_i, \frac{r_{x_i}}{4})\right\}_{i=1}^{k}$ of balls, with $Gx_i \neq Gx_j$ for $i \neq j$, such that $\mathcal{U} := \{gV |\ g \in G,\ V \in \mathcal{V}\}$ is an open cover of X.

Claim. *Every ball $U \in \mathcal{U}$ intersects only finitely many elements in $\mathcal{U}$.*

If not, then there would be some $B(x, \frac{r_x}{4})$ and $B(y, \frac{r_y}{4})$ along with a sequence $\{g_i\} \subseteq G$ such that infinitely many distinct $g_i B(x, \frac{r_x}{4})$ all have nonempty intersection with $B(y, \frac{r_y}{4})$. It can be assumed that $r_x > r_y$. Thus, if $g_i B(x, \frac{r_x}{4})$ and $g_j B(x, \frac{r_x}{4})$ both intersect $B(y, \frac{r_y}{4})$, then $y \in g_i B(x, \frac{r_x}{2}) \cap g_j B(x, \frac{r_x}{2})$. This contradicts our assumption that $B(x, \frac{r_x}{2}) \cap gB(x, \frac{r_x}{2}) = \varnothing$ or $B(x, \frac{r_x}{2}) = gB(x, \frac{r_x}{2})$. The claim follows.

Let $N(\mathcal{U})$ be the nerve of $\mathcal{U}$ and note that, by the above claim, $N(\mathcal{U})$ is locally finite and finite-dimensional. By construction $N(\mathcal{U})$ admits a proper, cocompact, simplicial G-action; and since $gU \cap U \neq \varnothing$ implies that $gU = U$, this action is rigid. Apply Theorem 6.8 and the trick described in Remark 6.9 to obtain a ϕ-variant map $h : |N(\mathcal{U})| \to X'$, where $|N(\mathcal{U})|$ is the geometric realization of $N(\mathcal{U})$.

Next let $\beta : X \to |N(\mathcal{U})|$ be the barycentric map. In other words, $\beta(x) = \sum_{U \in \mathcal{U}} \lambda_U(x) v_U$, where $\{\lambda_U\}$ is the partition of unity defined by

$$\lambda_{U_0}(x) = d(x, X - U_0)/(\sum_{U \in \mathcal{U}} d(x, X - U))$$

and v_U denotes the vertex in $|N(\mathcal{U})|$ defined by U. By construction, this map is G-equivariant, so $f = h \circ \beta : X \to X'$ is ϕ-variant. □

6.3. **Proof of Theorems 6.11 and 6.1.** We are ready to complete the main task of this section. Roughly speaking, we aim to mimic, to the extent possible, the proof used in the torsion-free case. In addition to ϕ-variant and ϕ^{-1}-variant maps $f : X \to X$ and $h : X \to X$, we will need analogs of the homotopies A and B used in Section 3.

Proposition 6.10. *Let $f : X \to X'$ and $h : X' \to X$ be ϕ-variant and ϕ^{-1}-variant maps, respectively, between proper cocompact G-ARs. Then there exist a pair of bounded homotopies $A : X \times [0, 1] \to X$ and $B : X' \times [0, 1] \to X'$ with $A_0 = \mathrm{id}_X$, $A_1 = hf$, $B_0 = \mathrm{id}_{X'}$ and $B_1 = fh$. As a result, A and B are proper homotopies and f and h are proper homotopy inverses of one another. If X and X' are $\underline{E}G$-complex* *and f and h are cellular maps, then A and B can be chosen so that A_t and B_t are G-equivariant for all t.*

Proof. As usual, choose proper metrics on X and X' so that the actions are geometric. It follows that X and X' are uniformly contractible and have finite macroscopic dimension. By cocompactness, since hf is G-equivariant, it is boundedly close to id_X. This assures that hf is a coarse equivalence, so we may apply [GM19, Cor.5.3] to obtain the bounded homotopy A. The same argument produces B.

When X and X' are $\underline{E}G$-complexes, this proposition and the G-equivariant conclusion follow from Theorem 6.8. □

Proof of Theorem 6.1. By hypothesis, we have ϕ-variant and ϕ^{-1}-variant maps $f : X \to X$ and $h : X \to X$, so we may apply Proposition 6.10 to obtain bounded homotopies $A : X \times [0,1] \to X$ and $B : X \times [0,1] \to X$ with $A_0 = \mathrm{id}_X$, $A_1 = hf$, $B_0 = \mathrm{id}_X$ and $B_1 = fh$. (If X is an $\underline{E}G$-complex and f and h are cellular maps, then A and B can be chosen so that A_t and B_t are G-equivariant for all t.)

Let $Y = \mathrm{Tel}_f(X)$, endowed with the $(G \rtimes_\phi \mathbb{Z})$-action described in Theorem 6.2. Section 3. Then construct a level-preserving proper homotopy equivalence $v : Y \to X \times \mathbb{R}$ by using formula (3.2) directly (as opposed to obtaining v as a lift). In a similar manner, construct a proper homotopy inverse $u : X \times \mathbb{R} \to Y$ and homotopies $H : u \circ v \simeq \mathrm{id}_Y$; $J : v \circ u \simeq \mathrm{id}_{X\times\mathbb{R}}$.

From this point on, the proof of Theorem 1.1 can be used without changes. □

For the purpose of applications, the following variation on Theorem 6.1 is probably the most useful.

Theorem 6.11. *If a group G (possibly with torsion) admits a $\mathcal{Z}$-structure $(\overline{X}, Z)$ where X is an $\underline{E}G_{AR}$-space, then every semidirect product of the form $G \rtimes_\phi \mathbb{Z}$ admits a $\mathcal{Z}$-structure $(\overline{Y}, SZ)$ where Y is an $\underline{E}(G \rtimes_\phi \mathbb{Z})_{AR}$ space. If X is an $\underline{E}G$-complex, then Y can be chosen to be an $\underline{E}(G \rtimes_\phi \mathbb{Z})$-complex.*

Proof. The initial assertion combines Theorem 6.1 and Proposition 6.7. The latter assertion adds in Proposition 6.6. □

Remark 6.12. Theorem 6.11 can be applied whenever G is hyperbolic, CAT(0), or systolic. We save the details of that discussion for Section 8 where even stronger results will be obtained.

6.4. **Some closing comments on the proof of Theorem 6.1.** Unlike the torsion-free case, we did not claim that the map $v : Y \to X \times \mathbb{R}$, used in the proof of Theorem 6.1, is G-equivariant. That is due to the fact that $A : X \times [0,1] \to X$ was not constructed to be G-equivariant—except in cases where X is a CW-complex (see remark below). That is not an issue for the proofs just completed. For later applications, however, it is be useful to note that, since $A|_{X\times\{0,1\}}$ is G-equivariant, v is G-equivariant when restricted to the integer levels of Y. For completeness, we add a quick proof.

Lemma 6.13. *Whenever k is an integer, v has the property that*

$$v(g \cdot \lceil x, k \rceil) = g \cdot (v\lceil x, k \rceil).$$

Proof. First assume $k \geq 0$. Then

$$\begin{aligned} v(g \cdot \lceil x, k \rceil) &= v\lceil \phi^k(g) \cdot x, k \rceil \\ &= (h^k(\phi^k(g) \cdot x), k) \\ &= (\phi^{-k}(\phi^k(g)) \cdot h^k(x), k) \qquad \text{(by } \phi^{-1}\text{-variance of } h) \\ &= (g \cdot h^k(x), k) \\ &= g \cdot v\lceil x, k \rceil. \end{aligned}$$

When $k \leq 0$, the argument is similar. □

Remark 6.14. For later use, we make some additional observations about the above proof in cases where X is an $\underline{E}G$-complex. In that case, f and h can be chosen to be cellular maps by Theorem 6.8. so, as noted in Proposition 6.10, we may choose A and B such that A_t and B_t are G-equivariant for all t. In that case, the maps v and u are G-equivariant, so by another application of Theorem 6.8, we may choose H and J to be G-equivariant as well.

7. $E\mathcal{Z}$-structures

We now turn to the study of $E\mathcal{Z}$-structures. Our primary goal is a proof of the following theorem.

Theorem 7.1. *If a group G admits an $E\mathcal{Z}$-structure $(\overline{X}, Z)$, $\phi \in \mathrm{Aut}(G)$, and there exist ϕ-variant and ϕ^{-1}-variant maps $f : X \to X$ and $g : X \to X$, respectively, which extend continuously to maps $\overline{f} : \overline{X} \to \overline{X}$ and $\overline{g} : \overline{X} \to \overline{X}$, then $G \rtimes_\phi \mathbb{Z}$ admits an $E\mathcal{Z}$-structure with boundary SZ.*

From here it is easy to deduce Theorem 1.6 from the introduction. We save that discussion for the end of this section.

The following pair of elementary lemmas helps to clarify some technical questions. The second offers an alternative hypothesis for our main theorem.

Lemma 7.2. *Let $\overline{X} = X \sqcup Z$ and $\overline{Y} = Y \sqcup Z'$ be controlled compactifications of proper metric spaces X and Y and let $f : X \to Y$ be a proper map which admits a continuous extension $\overline{f} : \overline{X} \to \overline{Y}$. Then $\overline{f}(Z) \subseteq Z'$ and every continuous map $f' : X \to Y$ which is boundedly close to f (as measures in (Y, d_Y)) extends continuously to $\overline{f'} : \overline{X} \to \overline{Y}$ by letting $\overline{f'}\big|_Z = \overline{f}\big|_Z$. Continuous extensions of this type are unique when they exist.*

Lemma 7.3. *Let $\overline{X} = X \sqcup Z$ and $\overline{Y} = Y \sqcup Z'$ be controlled compactifications of proper metric spaces X and Y and let $f : X \to Y$ be a coarse equivalence with coarse inverse $h : Y \to X$. Then the following are equivalent.*

(1) *There exists a continuous extension $\overline{f} : \overline{X} \to \overline{Y}$ of f which takes Z homeomorphically onto Z'.*

(2) *There exists a continuous extensions* $\overline{f} : \overline{X} \to \overline{Y}$ *and* $\overline{h} : \overline{Y} \to \overline{X}$ *of* f *and* g.

Suppose now that G admits an $E\mathcal{Z}$-structure $\left(\overline{X}, Z\right)$ and $\phi \in \mathrm{Aut}(G)$. Assume also the existence of ϕ-variant and ϕ^{-1}-variant maps $f : X \to X$ and $h : X \to X$ which extend continuously to maps $\overline{f} : \overline{X} \to \overline{X}$ and $\overline{h} : \overline{X} \to \overline{X}$. By Lemma 7.3 the restrictions $f_Z : Z \to Z$ and $h_Z : Z \to Z$ are homeomorphisms. As usual, we let $Y = \mathrm{Tel}_f(X)$ with the $(G \rtimes_\phi \mathbb{Z})$-action defined earlier. To prove Theorem 7.1, we will first define a $(G \rtimes_\phi \mathbb{Z})$-action on SZ; then we will show that this action continuously extends the action on Y. The latter of these tasks is surprisingly delicate.

The notational conventions established in Section 2.4 will be especially useful in this section.

7.1. **The** $(G \rtimes_\phi \mathbb{Z})$**-action on** SZ**.** Working with the presentation

$$G \rtimes_\phi \mathbb{Z} = \langle G, t \mid t^{-1}gt = \phi(g) \forall g \in G\} \rangle$$

it suffices to specify a G-action on SZ together with a self-homeomorphism of SZ corresponding to the generator t, and to check that the conjugating relators hold. To begin with, let g represent both an element of g and the corresponding self-homeomorphism of X; let $\overline{g} : \overline{X} \to \overline{X}$ denote the extension of g implied by the $E\mathcal{Z}$-structure; and let $g_Z = \overline{g}|_Z$, a self-homeomorphism of Z.

We choose the G-action on SZ to be the suspension of the G-action on Z. In other words, for each $g \in G$, define $g_{SZ} : SZ \to SZ$ by $g_{SZ} \cdot \langle z, r\rangle = \langle g_Z \cdot z, r\rangle$. Then define $t_{SZ} : SZ \to SZ$ to be the suspension of h_Z, i.e., $t_{SZ} \cdot \langle z, r\rangle = \langle h_Z(z), r\rangle$. Since $h : X \to X$ is ϕ^{-1}-equivariant, $h^{-1}gh = \phi(g)$ as a self-homeomorphism of X, for all $g \in G$. Then, since extensions over $\overline{X}$ are unique when they exist, $h_Z^{-1} \circ g_Z \circ h_Z = \phi(g)_Z$ as self-homeomorphisms of Z. It follows that $t_{SZ}^{-1} \circ g_{SZ} \circ t_{SZ} = \phi(g)_{SZ}$, so the conjugating relators of $G \rtimes_\phi \mathbb{Z}$ are satisfied.

7.2. **The** $(G \rtimes_\phi \mathbb{Z})$**-action on** $\overline{Y}$**.** To save on notation, now let g and t denote the self-homeomorphisms of Y defined in equations (6.1) and (6.2), and let $\overline{g} := g \sqcup g_{SZ}$ and $\overline{t} := t \sqcup t_{SZ}$. Our task is complete if we can show (or arrange) that these maps are continuous on $\overline{Y}$. Recall that $\overline{Y} = Y \sqcup SZ$ was defined by first compactifying $X \times \mathbb{R}$ to $\overline{X \times \mathbb{R}} = (X \times \mathbb{R}) \sqcup SZ$ (in a very precise manner) then applying the pullback compactification (see Remark 5.3) to Y using the map $v : Y \to X \times \mathbb{R}$ described in formula (3.2). To prove continuity, we will use a pair of simple facts. The first is a general property of pullback compactifications; the second follows directly from Definition 4.3.

- A sequence $\{\lceil x_i, r_i \rceil\}$ in Y converges in $\overline{Y}$ to $\langle z, \mu\rangle \in SX$ if and only if $\{v(\lceil x_i, r_i \rceil)\}$ converge to $\langle z, \mu\rangle$ in $\overline{X \times \mathbb{R}}$.
- A sequence $\{(x_i, r_i)\}$ in $X \times \mathbb{R}$ converges in $\overline{X \times \mathbb{R}}$ to $\langle z, \mu\rangle \in SX$ if and only if $\{x_i\}$ converges to z in $\overline{X}$ and the sequence of slopes $\{\mu(x_i, r_i)\}$ converges to μ.

The following lemma allows us to focus on sequences with integral second coordinates which, in turn, allows us to make use of Lemma 6.13.

Lemma 7.4. *Let* $\{\lceil x_i, r_i\rceil\}$ *be a sequence in* $Y = \mathrm{Tel}_f(X)$. *Then* $v(\lceil x_i, r_i\rceil) \to \langle z, \mu\rangle$ *in* $\overline{X \times \mathbb{R}}$ *if and only if* $v(\lceil x_i, \lfloor r_i\rfloor\rceil) \to \langle z, \mu\rangle$.

Proof. For each i, the entire mapping cylinder line containing $\lceil x_i, r_i\rceil$ and $\lceil x_i, \lfloor r_i\rfloor\rceil$ lies in some translate of the fundamental domain defined at the beginning of Section 5. The nullity condition arranged in that section assures that the diameters of the v-images of these fundamental domains, measured in $\overline{X \times \mathbb{R}}$, approach 0 as they are pushed to infinity. The conclusion follows easily. $\square$

7.3. **An important special case.** Verifying continuity of $\overline{g}$ and $\overline{t}$ is a delicate matter. In fact, without an adjustment to the earlier construction, continuity could fail. The adjustment involves the choice of slope function defined in Section 4. To make the new choice as intuitive as possible, we start with a key special case: We assume that the original proper metric space (X, d), on which G is acting geometrically, is *quasi-geodesic* space. This condition holds in nearly all commonly studied cases (it is built-in when G is hyperbolic or CAT(0) and can be arranged whenever G is torsion-free), but it is traditionally not required in the definition of a $\mathcal{Z}$-structure. After handling the special case, we will return to address the generic case.

The usefulness of the quasi-geodesic hypothesis is that, by the Švarc-Milnor Lemma, it ensures that the ϕ- and ϕ^{-1}-variant maps $f : X \to X$ and $h : X \to X$ are quasi-isometries. As such, we may choose a single pair of constants $K \geq 1$ and $\varepsilon \geq 0$ such that

$$\frac{1}{K} d(x, y) - \varepsilon \leq d(f(x), f(y)) \leq K d(x, y) + \varepsilon$$

and

$$\frac{1}{K} d(x, y) - \varepsilon \leq d(h(x), h(y)) \leq K d(x, y) + \varepsilon$$

for all $x, y \in X$.

Now let us return to the continuous, monotone increasing function $\psi : [0, \infty) \to [0, \infty)$ described in item (3) of Section 4. The two properties assigned to ψ were the following:

a) $\psi(s) \geq \max\{\eta(s), \lambda(s)\}$, and
b) $\psi(s+1) \geq 3\psi(s)$ for all $s \geq 0$

For the purposes of this section, we introduce a third requirement that can easily be added to the above.

c) ψ is smooth with $\psi'(s) \geq 1$ for all $s \geq 0$.

Now define $\Psi : [0, \infty) \to [0, \infty)$ by $\Psi(s) = e^{\psi(s)}$ and notice that Ψ also satisfies conditions a)-c). As such, we can go back to Section 4 and replace ψ with Ψ. That will change the slope function (hence, the way SZ is glued to $X \times \mathbb{R}$ to obtain $\overline{X \times \mathbb{R}}$) but the proofs that follow remain valid. Given that $\Psi^{-1}(s) = \psi^{-1}(\log s)$, the new slope formula takes the form

$$\begin{aligned}\mu(x,r) &= \frac{r}{\log\left(\Psi^{-1}\left(d\left(x,x_0\right)+\Psi\left(0\right)+1\right)\right)} \\ &= \frac{r}{\log(\psi^{-1}(\log\left(d\left(x,x_0\right)+\Psi\left(0\right)+1\right)))}\end{aligned}$$

Most of our continuity arguments hinge on calculations of limits. The following lemma will aid in several calculations.

Lemma 7.5. *Let $A_1, A_2, B_1, B_2 \in \mathbb{R}$ with $A_1, A_2 > 0$ and let $\theta : [0,\infty) \to [0,\infty)$ be a smooth monotone increasing function such that $\theta(x) \to \infty$ and $\theta'(x) \leq 1$ for all sufficiently large x. Then*

$$\lim_{x\to\infty} \frac{\theta(\log\left(A_1 x + B_1\right)))}{\theta(\log\left(A_2 x + B_2\right)))} = 1$$

Proof. First note that if $\sigma : [0,\infty) \to \mathbb{R}$ is a smooth monotone increasing function with $\sigma'(x) \leq 1$ for all sufficiently large x, then

$$\sigma(x) - |c| \leq \sigma(x+c) \leq \sigma(x) + |c|$$

for sufficiently large x. This fact will be applied to both θ and the log function.

In particular, since

$$\log\left(A_i x + B_i\right) = \log\left(A_i(x + \frac{B_i}{A_i})\right) = \log\left(x + \frac{B_i}{A_i}\right) + \log A_i$$

then

$$\log(x) + \log A_i - \left|\frac{B_i}{A_i}\right| \leq \log\left(A_i x + B_i\right) \leq \log(x) + \log A_i + \left|\frac{B_i}{A_i}\right|$$

Letting $d_i := \max\left\{\left|\log A_i - \left|\frac{B_i}{A_i}\right|\right|, \left|\log A_i + \left|\frac{B_i}{A_i}\right|\right|\right\}$, we can conclude that

$$\theta(\log(x)) - d_i \leq \theta(\log\left(A_i x + B_i\right)) \leq \theta(\log(x)) + d_i$$

Applying this inequality multiple times yields

$$\frac{\theta(\log(x)) - d_1}{\theta(\log(x)) + d_2} \leq \frac{\theta(\log\left(A_1 x + B_1\right)))}{\theta(\log\left(A_2 x + B_2\right)))} \leq \frac{\theta(\log(x)) + d_1}{\theta(\log(x)) - d_2}$$

for sufficiently large x. Our main assertion follows easily. □

We are now ready to proceed with the proof.

Claim 1. *For each $g \in G$, the function $\overline{g} : \overline{Y} \to \overline{Y}$, defined above, is continuous.*

Proof. Since $\overline{g}|_Y = g$ is continuous, it suffices to check continuity at points of SZ, and since $\overline{g}|_{SZ} = g_{SZ}$ is continuous, it suffices consider the effect of $\overline{g}$ on sequences $\{\lceil x_i, r_i\rceil\}$ in Y which converge in $\overline{Y}$ to a point $\langle z, \mu\rangle \in SX$. Specifically, we need to show that

$$\{\lceil x_i, r_i\rceil\} \to \langle z, \mu\rangle \implies \{\overline{g} \cdot \lceil x_i, r_i\rceil\} \to \langle g_Z \cdot z, \mu\rangle$$

By Lemma 7.4, we may replace each r_i with $\lfloor r_i \rfloor$. To simplify notation, we simply assume that each r_i is an integer.

Case 1. $0 < \mu \leq \infty$

Then r_i is eventually non-negative, so we can assume $r_i \geq 0$ for all i. Applying formula (3.2) and the fact that $A_0 = \mathrm{id}_X$, we have $v\left(\lceil x_i, r_i \rceil\right) = \left(h^{r_i}\left(x_i\right), r_i\right)$. By the above bullet points, $h^{r_i}\left(x_i\right) \to z$ in $\overline{X}$ and $\mu\left(h^{r_i}\left(x_i\right), r_i\right) \to \mu$. If we apply $\overline{g}$ to $\{\lceil x_i, r_i \rceil\}$ and $\langle z, \mu \rangle$, we get $\{\overline{g} \cdot \lceil x_i, r_i \rceil\} = \{\lceil \phi^{r_i}\left(g\right) \cdot x_i, r_i \rceil\}$ and $\overline{g} \cdot \langle z, \mu \rangle = \langle g_Z \cdot z, \mu \rangle$. By the first bullet point, it remains to check that $v\left(\lceil \phi^{r_i}\left(g\right) \cdot x_i, r_i \rceil\right) \to \langle g_Z \cdot z, \mu \rangle$ in $\overline{X \times \mathbb{R}}$. Now

$$\begin{aligned} v\left(\lceil \phi^{r_i}\left(g\right) \cdot x_i, r_i \rceil\right) &= \left(h^{r_i}\left(\phi^{r_i}\left(g\right) \cdot x_i\right), r_i\right) \\ &= \left(\phi^{-r_i}(\phi^{r_i}(g)) \cdot h^{r_i}(x), r_i\right) \qquad \text{(by } \phi^{-1}\text{-variance of } h\text{)} \\ &= \left(g \cdot h^{r_i}\left(x_i\right), r_i\right) \end{aligned}$$

Since $h^{r_i}\left(x_i\right) \to z$ and $\overline{g} : \overline{X} \to \overline{X}$ is continuous, $g \cdot h^{r_i}\left(x_i\right) \to \overline{g} \cdot z = g_Z \cdot z$ in $\overline{X}$, as desired.

It remains to show that $\mu\left(g \cdot h^{r_i}\left(x_i\right), r_i\right) \to \mu$. We already know that $\mu\left(h^{r_i}\left(x_i\right), r_i\right) \to \mu$, so by comparing these two sequences and applying formula (7.3), it suffices to show that

$$\frac{\log(\psi^{-1}(\log(d(h^{r_i}\left(x_i\right), x_0) + \Psi(0)) + 1)))}{\log(\psi^{-1}(\log(d(g \cdot h^{r_i}\left(x_i\right), x_0) + \Psi(0)) + 1)))} \to 1 \tag{7.1}$$

Since g is an isometry of X, the triangle inequality assures us that

$$d\left(h^{r_i}\left(x_i\right), x_0\right) - d\left(x_0, g \cdot x_0\right) \leq d(g \cdot h^{r_i}\left(x_i\right), x_0) \leq d\left(h^{r_i}\left(x_i\right), x_0\right) + d\left(x_0, g \cdot x_0\right)$$

This allows us to squeeze limit 7.1 between a pair of limits of the type addressed in Lemma 7.5. In both cases $\theta = \log \circ \psi^{-1}$, $A_1 = A_2 = 1$, and $B_1 = \Psi(0)) + 1$. For the lower limit, let $B_2 = \Psi(0)) + 1 + d\left(x_0, g \cdot x_0\right)$ and for the upper limit, let $B_2 = \Psi(0)) + 1 - d\left(x_0, g \cdot x_0\right)$.

Case 2. $-\infty \leq \mu < 0$

Then r_i is eventually negative, so we can assume $r_i < 0$ for all i, so formula (3.2) yields $v\left(\lceil x_i, r_i \rceil\right) = \left(f^{|r_i|}\left(x_i\right), r_i\right)$. By the earlier bullet points, $f^{|r_i|}\left(x_i\right) \to z$ in $\overline{X}$ and $\mu\left(f^{|r_i|}\left(x_i\right), r_i\right) \to \mu$. As in Case 1, $\{\overline{g} \cdot \lceil x_i, r_i \rceil\} = \{\lceil \phi^{r_i}\left(g\right) \cdot x_i, r_i \rceil\}$ and $\overline{g} \cdot \langle z, \mu \rangle = \langle g_Z \cdot z, \mu \rangle$, so it remains to check that $v\left(\lceil \phi^{r_i}\left(g\right) \cdot x_i, r_i \rceil\right) \to \langle \overline{g} \cdot z, \mu \rangle$ in $\overline{X \times \mathbb{R}}$. In this case,

$$\begin{aligned} v\left(\lceil \phi^{r_i}\left(g\right) \cdot x_i, r_i \rceil\right) &= \left(f^{|r_i|}\left(\phi^{-|r_i|}\left(g\right) \cdot x_i\right), r_i\right) \\ &= \left(\phi^{|r_i|}(\phi^{-|r_i|}(g)) \cdot f^{|r_i|}(x), r_i\right) \qquad \text{(by } \phi\text{-variance of } g\text{)} \\ &= \left(g \cdot f^{|r_i|}\left(x_i\right), r_i\right) \end{aligned}$$

The rest of the proof follows the reasoning used in Case 1, with$f^{|r_i|}$ replacing h^{r_i}.

Case 3. $\mu = 0$

Split the sequence $\{\lceil x_i, r_i \rceil\}$ into a pair of subsequences, one with all $r_i \geq 0$ and the other with $r_i < 0$. Then apply the arguments used in Cases 1 and 2 to the subsequence individually. $\square$

Claim 2. *The function $\bar{t} : \overline{Y} \to \overline{Y}$, defined above, is continuous.*

Proof. Following the same strategy used above, we will show that

$$\{\lceil x_i, r_i \rceil\} \to \langle z, \mu \rangle \implies \{\bar{t} \cdot \lceil x_i, r_i \rceil\} \to \langle h_Z(z), \mu \rangle$$

As before, we may assume each r_i is an integer.

Case 1. $0 < \mu \leq \infty$

Then r_i is eventually non-negative, so we can assume $r_i \geq 0$ for all i. Then $v(\lceil x_i, r_i \rceil) = (h^{r_i}(x_i), r_i)$ so $h^{r_i}(x_i) \to z$ in $\overline{X}$ and $\mu(h^{r_i}(x_i), r_i) \to \mu$. Applying $\bar{t}$ to $\{\lceil x_i, r_i \rceil\}$ and $\langle z, \mu \rangle$, we get $\{\lceil x_i, r_i + 1 \rceil\}$ and $\langle h_Z(z), \mu \rangle$, respectively. By the first bullet point, it remains to show that $v(\lceil x_i, r_i + 1 \rceil) \to \langle h_Z(z), \mu \rangle$ in $\overline{X \times \mathbb{R}}$. By formula (3.2), we must show that $(h^{r_i+1}(x_i), r_i + 1) \to \langle h_Z(z), \mu \rangle$ in $\overline{X \times \mathbb{R}}$.

Since $h^{r_i}(x_i) \to z$ and $\overline{h} : \overline{X} \to \overline{X}$ is continuous, $h^{r_i+1}(x_i) \to h_Z(z)$ in $\overline{X}$, as desired. It remains to show that $\mu(h^{r_i+1}(x_i), r_i + 1) \to \mu$. We already know that $\mu(h^{r_i}(x_i), r_i) \to \mu$, so by comparing these two sequences and applying formula (7.3), it suffices to show that

$$\frac{\log(\psi^{-1}(\log(d(h^{r_i}(x_i), x_0) + \Psi(0)) + 1)))}{\log(\psi^{-1}(\log(d(h^{r_i+1}(x_i), x_0) + \Psi(0)) + 1)))} \to 1 \tag{7.2}$$

By applying (7.3) we have

$$\frac{1}{K} d(h^{r_i}(x_i), x_0) - \varepsilon \leq d(h^{r_i+1}(x_i), x_0) \leq K d(h^{r_i}(x_i), x_0) + \varepsilon$$

which allows us squeeze limit (7.2) between a pair of limits like those addressed in Lemma 7.5. For both limits $\theta = \log \circ \psi^{-1}$, $A_1 = 1$, and $B_1 = \Psi(0)) + 1$. For the lower limit $A_2 = K$ and $B_2 = \Psi(0)) + 1 + \varepsilon$, while for the upper limit $A_2 = 1/K$ and $B_2 = \Psi(0)) + 1 - \varepsilon$.

Case 2. $0 < \mu \leq \infty$

Then r_i is eventually negative so we assume $r_i < 0$ for all i. Now $v(\lceil x_i, r_i \rceil) = \left(f^{|r_i|}(x_i), r_i\right)$ and $v(\lceil x_i, r_i + 1 \rceil) = \left(f^{|r_i+1|}(x_i), r_i + 1\right)$, so we rely on the continuity of $\overline{f} : \overline{X} \to \overline{X}$ and inequalities (7.3) instead of the analogs for h. Everything else follows as in Case 1.

Case 3. $\mu = 0$

As we did earlier, split $\{\lceil x_i, r_i \rceil\}$ into subsequences to which the arguments of Cases 1 and 2 can be applied. $\square$

Remark 7.6. In proving Claim 1, we showed that if a sequence $\{(x_i, r_i)\}$ converges to $\langle z, \mu \rangle$ in $X \times \mathbb{R}$ and $g \in G$, then $\{(g \cdot x_i, r_i)\} \to \langle g_Z \cdot z, \mu \rangle$. A useful, and not entirely obvious, corollary is that, when we have an $E\mathcal{Z}$-structure $\left(\overline{X}, Z\right)$ on G, the product G-action on $X \times \mathbb{R}$ extends to our compactification $\overline{X \times \mathbb{R}}$ by suspending the given G-action on Z.

7.4. **Spaces that are not quasi-geodesic.** It is possible to meet the hypotheses of Theorem 7.1 with respect to a space X such that the ϕ-variant map is not a quasi-isometry. Indeed, when (X, d) is not quasi-geodesic, we can only conclude that ϕ-variant maps are coarse equivalences. See [BDM07] for a discussion of this topic.

Definition 7.7. Let X, W be metric spaces. A map $\phi : X \to W$ is a *coarse embedding* if there exist increasing *control functions* $\rho_-, \rho_+ : [0, \infty) \to [0, \infty)$ with $\lim_{r\to\infty} \rho_-(r) = \infty$ such that for all $x, x' \in X$

$$\rho_-(d_X(x, x')) \leq d_W(\phi(x), \phi(x')) \leq \rho_+(d_X(x, x')).$$

A coarse embedding is called a *coarse equivalence* if it is quasi-onto, i.e., there exists $C > 0$ such that for all $w \in W$, $d_W(w, \phi(X)) < C$.

Example 1. As a simple illustration, consider $(\mathbb{R}, d')$ where $d'(r, s) = \log(1 + |x - y|)$. The usual $\mathbb{Z}$-action is still by isometries, but the standard orbit map $\lambda : \mathbb{Z} \to \mathbb{R}$ is not a quasi-isometry. For a more extreme example, let $d'' = \log(1 + d')$, etc.

Faced with more general control functions—imagine ρ_+ being super-exponential and ρ_- growing slower than an iterated logarithm—the adjustment made to the function ψ in the earlier argument (where we replaced ψ with e^{ψ}) may not suffice. In this more general context, we will make an adjustment that depends on the *growth rates* of control functions ρ_- and ρ_+. Without loss of generality, we assume those functions are continuous. As another simplification, we may make the following substitution.

Lemma 7.8. *Let X, Y be spaces which are coarsely equivalent with control functions ρ_-, ρ_+. Then there exists a function $\rho : [0, \infty) \to [0, \infty)$ such that X, Y are also coarsely equivalent with respect to functions ρ^{-1}, ρ.*

Proof. Let $\rho(x) := \min\big(\rho_-(x), \rho_+^{-1}(x)\big)$. □

Definition 7.9. Let $\phi : [0, \infty) \to [0, \infty)$ be a continuous, increasing function such that $\phi(x) \leq \frac{x}{2}$ and define $\phi^* : [0, \infty) \to [0, \infty)$ by thee following rule:

$$\phi^*(x) = \begin{cases} 1 & \text{if } x \leq 1 \\ 1 + \phi^*(\phi(x)) & \text{else.} \end{cases}$$

One may view the output $\phi^*(x)$ to be "one more than the number of times one needs to apply ϕ to x to achieve a value less than 1". This is well-defined and finite by the assumption that $\phi(x) \leq \frac{x}{2}$. The construction of the star function is inspired by the *iterated logarithm*, which is traditionally denoted as $\log^*$. The $\log^*$ function (not to be confused with probability's "Law of the Iterated Logarithm") has roots in complexity theory and logic; for an example see [PSV06]. Clearly this function is not continuous, but rather looks like a floor function which steps one greater at intervals of length $\phi(k)$, $k \in \mathbb{N}$.

The following is a direct consequence of the definition of ϕ^*.

Lemma 7.10. *Let $\phi : [0, \infty) \to [0, \infty)$ be a continuous, increasing function such that $\phi(x) \leq \frac{x}{2}$. Then for all $z \in [0, \infty)$, the following hold:*

- $\phi^*(\phi(z)) = \phi^*(z) - 1$
- $\phi^*(\phi^{-1}(z)) = \phi^*(z) + 1$.

We now prove the generic (non-quasigeodesic) case.

Proof of Theorem 7.1. Most of the necessary work has been done in the preceding subsections, with the sole exception of an analog of Lemma 7.5. Thus, we need to, for an arbitrary pair of control functions ρ_-, ρ_+, develop an a function ψ satisfying analogous limit laws to the above. Before doing this, we invoke Lemma 7.8 to use control functions of the form ρ^{-1}, ρ, assuming without loss of generality $\rho^{-1} < \rho$. Furthermore, we can also assume that $\rho(x) > 3x$. Let $(\rho^{-1})^*$ be defined as above with respect to ρ^{-1}.

To tackle the limits, we first show the following.

Claim. For all $A, B \geq 0$

$$\lim_{x\to\infty} \frac{(\rho^{-1})^*(x+A)}{(\rho^{-1})^*\left(\rho_-(x)+B\right)} = 1 \quad (\dagger)$$

Proof

$$\frac{(\rho^{-1})^*(x+A)}{(\rho^{-1})^*\left(\rho_-(x)+B\right)} \geq \frac{(\rho^{-1})^*(x+A)}{(\rho^{-1})^*\left(\rho_-(x)\right)+B} = \frac{(\rho^{-1})^*(x+A)}{(\rho^{-1})^*(x)+B-1} \geq \frac{(\rho^{-1})^*(x)}{(\rho^{-1})^*(x)+B-1}$$

The last ratio clearly approaches 1. □

Observe the use of Lemma 7.10. Verification of the following inequality for all A, B proceeds mutatis mutandis:

$$\lim_{x\to\infty} \frac{(\rho^{-1})^*(x+A)}{(\rho^{-1})^*\left(\rho_+(x)+B\right)} = 1 \quad (\ddagger)$$

With these limits in hand, the function $(\rho^{-1})^*$ almost serves our purpose. Earlier arguments require ψ (and thus ψ^{-1}) to be continuous and bijective. To accomplish that take ψ^{-1} to be the function which linearly connects the points

$$(0,0), (1, (\rho^{-1})^*(1)), (2, (\rho^{-1})^*(2)), \ldots$$

Observe that ψ^{-1} is continuous and bijective, and also satisfies $|\psi^{-1}(x) - (\rho^{-1})^*(x)| \leq 1$. This closeness of ψ^{-1} to $(\rho^{-1})^*$ guarantees that the inequalities $(\dagger), (\ddagger)$ are satisfied by ψ^{-1} as well. We note that the original condition on ψ given by $\psi(s+1) \geq 3\psi(s)$ is satisfied by the assumption that $\rho \geq 3x$, as in this case we know that $\rho^{-1} \leq \frac{x}{3}$, so $(\rho^{-1})^*(x) \leq (\frac{x}{3}^*) = \log_3(x)$. This means that ψ^{-1} is bounded above by $\log_3(x)$, ensuring that ψ grows at least as fast as 3^x. Requirement c), from the previous section, can arranged by carefully rounding the corners on the piecewise-linear graph. □

7.5. **Proof of Theorem 1.6.** Theorem 1.6 simply identifies some interesting special cases where the continuous maps in the hypothesis of Theorem 7.1 always exist. In the case of hyperbolic groups, this follows from [Gro87]. For $G = \mathbb{Z}^n$, one can begin with the standard Euclidean $E\mathcal{Z}$-structure $\left(\overline{\mathbb{R}^n}, S^{n-1}\right)$ and note that every element of $\mathrm{Aut}\left(\mathbb{Z}^n\right)$ can be realized by a linear map which has a natural extension to the sphere at infinity. For abelian groups with torsion, one can simply let the torsion elements act trivially on $\mathbb{R}^n$ and repeat the previous construction. For CAT(0) groups with the isolated flats property, the existence of these maps is an application of [HK05]. The appendix of that paper includes additional cases that can be added to this collection.

8. $\underline{E\mathcal{Z}}$-structures

In Section 6 we saw that, when working on groups that have torsion, it can be useful to use classifying spaces ($\underline{E}G_{AR}$-spaces and $\underline{E}G$-complexes) which allow fixed points but place restrictions on the fixed-point sets. In that spirit, we introduce analogous definitions for $\mathcal{Z}$- and $E\mathcal{Z}$-structures. Motivation is contained in work by Rosenthal, which allows us to use these structures to make conclusions about the Novikov Conjecture. Specific applications of that type will be addressed in Section 9.1.

Definition 8.1. An $\underline{E\mathcal{Z}}$*-structure* on G is an $E\mathcal{Z}$-structure $\left(\overline{X}, Z\right)$ with the following additional properties:

(1) X is an $\underline{E}G_{\mathrm{AR}}$-space, and
(2) for each $H \in \mathcal{F}_G$, $\overline{X}^H$ is a $\mathcal{Z}$-compactification of X^H.

Notice that, when this definition is satisfied, then

a) $\overline{X}^H$, i.e., the subset of $\overline{X}$ fixed by H, is equal to the closure of X^H in $\overline{X}$,
b) $\overline{X}^H \cap Z = Z^H$, and
c) $\left(\overline{X}^H, Z^H\right)$ is an $E\mathcal{Z}$-structure for $N_G(H)$ (the normalizer of H in G) and also for $N_G(H)/H$.

If, in addition to the above, X is an $\underline{E}G$-complex, we call $\left(\overline{X}, Z\right)$ a *cellular* $\underline{E\mathcal{Z}}$*-structure*.

For a variation on the above definition, we can relax equivariance by requiring only that $\left(\overline{X}, Z\right)$ be a $\mathcal{Z}$-structure, while keeping conditions 1 and 2 in place. We call this a $\underline{\mathcal{Z}}$*-structure* on G. Under this definition, observations a)-c) remain valid, except that $\left(\overline{X}^H, Z^H\right)$ is only a $\mathcal{Z}$-structure for $N_G(H)$ and $N_G(H)/H$ rather than an $E\mathcal{Z}$-structure. A *cellular* $\underline{\mathcal{Z}}$*-structure* is defined in the obvious way.

For the purposes of this paper, we are especially interested in $\underline{E\mathcal{Z}}$-structures. That will be the focus of the remainder of this section.

Example 2. Every hyperbolic group G admits a cellular $\underline{E\mathcal{Z}}$-structure $\left(\overline{X}, \partial G\right)$, where X is an appropriately chosen Rips complex for G and ∂G is the Gromov boundary. This is precisely the content of [RS05].

Example 3. Every systolic group G admits a cellular $\underline{E\mathcal{Z}}$-structure $\left(\overline{X}, \partial X\right)$, where X is the implied systolic simplicial complex acted upon by G, and ∂X is the systolic boundary as defined in [OP09]. This observation follows from the main theorem of that paper along with their Theorem 14.1, Claim 14.2, and its proof.

Example 4. Every CAT(0) group G admits an $\underline{E\mathcal{Z}}$-structure $\left(\overline{X}, \partial_\infty X\right)$, where X is the implied proper CAT(0) space acted upon by geometrically by G, and $\partial_\infty X$ is its visual boundary. It is well documented that $\left(\overline{X}, \partial_\infty X\right)$ is an $E\mathcal{Z}$-structure for G. Conditions 1) and 2) hold because the fixed set of every finite subgroup is nonempty and convex in X. See [BH99, Cor.II.2.8].

Our primary applications of $\underline{E\mathcal{Z}}$-structures (see Section 9.1) requires that they be cellular. For that reason, the following refinement of Example 4 will be useful.

Theorem 8.2. *Every CAT(0) group G admits a cellular $\underline{E\mathcal{Z}}$-structure.*

Proof. Let $\left(\overline{X}, \partial_\infty X\right)$ be the $E\mathcal{Z}$-structure implied by the definition of CAT(0) group, then apply [Ont05, Prop.A] to obtain a rigid $\underline{E}G$-simplicial complex K and a G-equivariant map $f : K \to X$. If we give K the path-length metric, then f is a quasi-isometry, so we may use the map f and the $E\mathcal{Z}$-boundary swapping theorem from [GM19] To obtain an $E\mathcal{Z}$-structure of the form $\left(\overline{K}, \partial_\infty X\right)$ and a continuous extension $\overline{f} : \overline{K} \to \overline{X}$ which is the identity on $\partial_\infty X$.

Since f is G-equivariant, it maps K^H into X^H for every $H \in \mathcal{F}_G$. Moreover, since $N_G(H)$ acts properly and cocompactly on both K^H and X^H, $f|_{K^H} : K^H \to X^H$ is a quasi-isometry. We know that $\overline{X}^H = X^H \sqcup \partial_\infty(X^H)$, so the boundary swap between K and X restricts to a boundary swap between K^H and X^H. In particular, $\overline{K}^H = K^H \sqcup \partial_\infty(X^H)$ is a $\mathcal{Z}$-compactification. Therefore $\left(\overline{K}, \partial_\infty X\right)$ is an $\underline{E\mathcal{Z}}$-structure. $\square$

We now state and prove our main theorem about $\underline{E\mathcal{Z}}$-structures on groups of the form $G \rtimes_\phi \mathbb{Z}$. Due to the delicate nature of the argument, we begin with the assumption of a *cellular* $\underline{E\mathcal{Z}}$-structure on G. That allows us to choose G-equivariant maps and homotopies in a number of places—a property will be used in the proof. It also leads to the conclusion of a *cellular* $\underline{E\mathcal{Z}}$-structure on $G \rtimes_\phi \mathbb{Z}$ —a property that is required for our main applications. It seems likely that a more generic version of this theorem is true, but the argument would be even more delicate.

Theorem 8.3. *Suppose G admits a cellular $\underline{E\mathcal{Z}}$-structure $\left(\overline{X}, Z\right)$, $\phi \in \mathrm{Aut}(G)$, and the corresponding ϕ-variant map(s) $f : X \to X$ extends to a continuous map $\overline{f} : \overline{X} \to \overline{X}$ which is a homeomorphism on Z. Then $G \rtimes_\phi \mathbb{Z}$ admits a cellular $\underline{E\mathcal{Z}}$-structure with boundary equal to SZ.*

Proof. Propositions 6.7 and 6.2 guarantee the existence of a cellular ϕ-variant map f and a corresponding rigid $(G \rtimes_\phi \mathbb{Z})$-complex $Y = \mathrm{Tel}_f(X)$. Proposition 6.6 assures that Y is an $\underline{E}(G \rtimes_\phi \mathbb{Z})$-complex. The assumption of the existence of $\overline{f} : \overline{X} \to \overline{X}$ implies a corresponding $E\mathcal{Z}$-structure $\left(\overline{Y}, SZ\right)$ for $G \rtimes_\phi \mathbb{Z}$, as proved in Theorem 7.1. It remains only to verify condition 2) of Definition 8.1.

Let $H \in \mathcal{F}_{G \rtimes_\phi \mathbb{Z}}$ and recall from our discussion in Section 6.2 that $H \leq G$, and Y^H is the sub-mapping telescope defined by the following subspaces and restriction maps.

$$\cdots \xrightarrow{f|} X^{\phi^{-2}(H)} \xrightarrow{f|} X^{\phi^{-1}(H)} \xrightarrow{f|} X^H \xrightarrow{f|} X^{\phi(H)} \xrightarrow{f|} X^{\phi^2(H)} \xrightarrow{f|} \cdots$$

Clearly $(X \times \mathbb{R})^H = X^H \times \mathbb{R}$, and since $v : Y \to X \times \mathbb{R}$ and $u : X \times \mathbb{R} \to Y$ are G-equivariant (see Remark 6.14), $v\left(Y^H\right) \subseteq X^H \times \mathbb{R}$ and $u(X \times \mathbb{R}) \subseteq Y^H$. By our hypothesis, $\overline{X}^H$ is a $\mathcal{Z}$-compactification $X^H \sqcup Z^H$ of X^H where $Z^H \subseteq Z$. By the definition of the topology on $\overline{X \times \mathbb{R}}$, the closure of $X^H \times \mathbb{R}$ in $\overline{X \times \mathbb{R}}$ is precisely $\left(X^H \times \mathbb{R}\right) \sqcup S(Z^H)$ topologized according to the same rules (i.e., the restriction of the same slope function used to topologize $\overline{X \times \mathbb{R}}$) where $S(Z^H)$ is also SZ^H. From here, the same argument used in proving 4.10 (See, in particular, Lemma 4.11 and the remark that follows it.) shows that $\overline{X \times \mathbb{R}}^H = \left(X^H \times \mathbb{R}\right) \sqcup S(Z^H)$ is a $\mathcal{Z}$-compactification of $(X \times \mathbb{R})^H$. Recalling that the $\mathcal{Z}$-compactification $\overline{Y} = Y \sqcup SZ$ was obtained as the pull-back of the compactification $\overline{X \times \mathbb{R}} = (X \times \mathbb{R}) \sqcup SZ$ via the map $v : Y \to X \times \mathbb{R}$, it is clear that $\overline{Y}^H$ is the pull-back compactification of $\overline{X \times \mathbb{R}}^H = \left(X^H \times \mathbb{R}\right) \sqcup S(Z^H)$ via $v|_{Y^H} : Y^H \to X^H \times \mathbb{R}$. Now the same boundary swapping proof used in Proposition 5.4 applies to show that $\overline{Y}^H = Y^H \sqcup S(Z^H)$ is a $\mathcal{Z}$-compactification. Here one should note that, by G-equivariance (see Remark 6.14 again), the homotopy $\overline{H}$ used in the earlier argument restricts to an appropriate homotopy between self-maps of $\overline{Y}^H$. □

Corollary 8.4. *Let G be a hyperbolic group and $\phi \in \mathrm{Aut}\,(G)$. Then $G \rtimes_\phi \mathbb{Z}$ admits a cellular $\underline{E\mathcal{Z}}$-structure with boundary the suspension of the Gromov boundary of G.*

Proof. Begin with the cellular $\underline{E\mathcal{Z}}$-structure $\left(\overline{X}, \partial G\right)$ on G discussed in Example 2. A ϕ-variant map $f : X \to X$ exists by Theorem 6.7. Since f is a quasi-isometry, the well-known theory of hyperbolic spaces ensures a continuous extension $\overline{f} : \overline{X} \to \overline{X}$ which takes ∂G homeomorphically onto ∂G. □

Corollary 8.5. *Let G be a CAT(0) or systolic group with corresponding $\underline{E\mathcal{Z}}$-structure $(\overline{X}, \partial_\infty X)$, $\phi \in \mathrm{Aut}\,(G)$, and $f : X \to X$ a corresponding ϕ-variant map. If there exists a continuous extension $\overline{f} : \overline{X} \to \overline{X}$ which is a homeomorphism on $\partial_\infty X$, then $G \rtimes_\phi \mathbb{Z}$ admits a cellular $\underline{E\mathcal{Z}}$-structure with boundary the suspension of $\partial_\infty X$.*

Proof. For systolic G, the $\underline{E\mathcal{Z}}$-structure $(\overline{X}, \partial_\infty X)$ is automatically cellular, so the conclusion is immediate. For CAT(0) G, we must first swap the $\underline{E\mathcal{Z}}$-structure $(\overline{X}, \partial_\infty X)$ for the cellular version $\left(\overline{K}, \partial_\infty X\right)$ promised in Theorem 8.2. □

9. Applications of the main theorems

In this section we look at some concrete applications of the main results of this paper. We begin with a look at $E\mathcal{Z}$-structures and $\underline{E\mathcal{Z}}$-structures and their relationship to the Novikov Conjecture. We point out situations where our methods can add to the collection of groups for which the Novikov Conjecture is known to be true, and provide new proofs that other groups belong to that collection.

Next we examine polycyclic groups (a class which contains all finitely generated nilpotent groups) from the perspective of group boundaries. After that, we show that fundamental groups of *all* closed 3-manifolds admit $\mathcal{Z}$-structures. For these latter two applications, we also discuss $E\mathcal{Z}$-structures and $\underline{E\mathcal{Z}}$-structures.

9.1. **Applications of $\underline{E\mathcal{Z}}$-structures to the Novikov Conjecture.** Notice that for torsion-free groups there is no difference between a $E\mathcal{Z}$-structure and an $\underline{E\mathcal{Z}}$-structure. Work by Carllson and Pedersen [CP95] and Farrell and Lafont [FL05] showed that the existence of an $E\mathcal{Z}$-structure on a torsion-free group Γ implies the Novikov Conjecture for Γ. In fact, Farrell and Lafont's motivation for defining an $E\mathcal{Z}$-structure was precisely that application. Rosenthal [Ros04], [Ros06], [Ros12] expanded upon that work to provide a similar approach to the Novikov Conjecture for groups with torsion. His conditions motivated our definition of an $\underline{E\mathcal{Z}}$-structure. When the $\underline{E\mathcal{Z}}$-structure is cellular, all of Rosenthal's conditions are satisfied, so we have:

Theorem 9.1 (after Rosenthal)**.** *If a group G admits a cellular $\underline{E\mathcal{Z}}$-structure, then the Baum-Connes map, $KK_i^G\left(C_0\left(\underline{E}G\right);\mathbb{C}\right) \to K_i\left(C_r^*G\right)$, is split injective. In particular, the Novikov Conjecture holds for G.*

For example, this theorem, combined with work discussed in the previous section, implies the Novikov Conjecture for all hyperbolic, CAT(0), and systolic groups, including those with torsion. (In many cases, other proofs are known.)

For the purposes of this paper, we are interested in groups of the form $G \rtimes_\phi \mathbb{Z}$. Work presented above yields the following.

Theorem 9.2. *Let G be a hyperbolic group and $\phi \in \operatorname{Aut}(G)$. Then the Novikov Conjecture holds for $G \rtimes_\phi \mathbb{Z}$.*

Theorem 9.3. *Let G be a CAT(0) or systolic group with corresponding $E\mathcal{Z}$-structure $(\overline{X}, \partial_\infty X)$, $\phi \in \operatorname{Aut}(G)$, and $f : X \to X$ a ϕ-variant map. If f extends continuously to a map $\overline{f} : \overline{X} \to \overline{X}$ which is a homeomorphism on $\partial_\infty X$, then the Novikov Conjecture holds for $G \rtimes_\phi \mathbb{Z}$.*

Remark 9.4. Our assertion about the Novikov Conjecture in Theorem 9.2 is not new. An existing proof goes as follows: groups with finite asymptotic dimension satisfy the Novikov Conjecture; hyperbolic groups have finite asymptotic dimension; and extensions of groups with finite asymptotic dimension by groups with finite asymptotic dimension have finite asymptotic dimension. See [BD01].

It is an open question whether CAT(0) or systolic groups have finite asymptotic dimension. As such, to the best of our knowledge, the assertions about the Novikov Conjecture in Theorem 9.3 are new. Notice that the hypothesis about the existence of $\overline{f} : \overline{X} \to \overline{X}$ is not vacuous. See [BR96] for a relevant example.

9.2. **Polycyclic groups, nilpotent groups, and groups of polynomial growth.** A group G is *polycyclic* if it admits a subnormal series

$$G = G_k \rhd G_{n-1} \rhd \cdots \rhd G_0 = \{1\}$$

for which each quotient group G_{i+1}/G_i is cyclic. If it admits such a series for which each quotient is infinite cyclic, then G is called *strongly polycyclic* (or sometimes *poly-$\mathbb{Z}$*). The *Hirsch length* of polycyclic G is the number of infinite cyclic factors in its subnormal series. It is a standard fact that Hirsch length is an invariant of G.

The following comes from inducting on the Hirsch length, applying Theorem 1.1 at each step.

Theorem 9.5. *Every strongly polycyclic group G admits a $\mathcal{Z}$-structure. If the Hirsch length of G is n, the $\mathcal{Z}$-structure $(\overline{X}, Z)$ can be chosen so that $Z = S^{n-1}$.*

A group G is *nilpotent* if there exists a finite sequence of normal subgroups $G = G_k \rhd G_{n-1} \rhd \cdots \rhd G_0 = \{1\}$ such that $[G, G_{i+1}]$ is contained in G_i (where brackets indicate the commutator). Observe that a finitely generated nilpotent group is polycyclic.

Theorem 9.6. *Every finitely generated nilpotent group admits a $\mathcal{Z}$-structure with spherical boundary.*

Proof. Let Γ be a finitely generated nilpotent group. The set of all torsion elements forms a finite, characteristic subgroup $T(\Gamma) \lhd \Gamma$, which gives us a short exact sequence

$$1 \to T(\Gamma) \to \Gamma \to \Gamma/T(\Gamma) \to 1$$

where $\Gamma/T(\Gamma)$ is a torsion-free, nilpotent group [Seg83, Cor. 1.10]. Recall then that a *torsion-free* nilpotent group is strongly polycyclic group (see [Rob96, 5.2.20]). Therefore $\Gamma/T(\Gamma)$ admits a $\mathcal{Z}$ structure $(\overline{X}, \mathbb{S}^k)$ by Theorem 9.5. The proper cocompact action of $\Gamma/T(\Gamma)$ on X can be extended to a cocompact Γ-action using the homomorphism $\Gamma \to \Gamma/T(\Gamma)$. Since the kernel is finite, this action is also proper, so $(\overline{X}, \mathbb{S}^k)$ is a $\mathcal{Z}$-structure for Γ. $\square$

Already, the above results expand greatly on the class of groups known to admit a $\mathcal{Z}$-structures. That is because non-elementary nilpotent groups are never hyperbolic (they have polynomial growth, [Wol68]) and seldom CAT(0) (nilpotent groups which are CAT(0) are virtually abelian). See [BH99, Theorem 7.8, p. 249]. By invoking some powerful theorems, we can obtain more.

Theorem 9.7. *Every group of polynomial growth admits a $\mathcal{Z}$-structure with spherical boundary.*

Proof. Suppose G has polynomial growth. Then, by [Gro81], G contains a finite index nilpotent subgroup H. Since G is finitely generated, H is finitely generated, so by Theorem 9.6, H admits a $\mathcal{Z}$-structure of the form $\left(\overline{X}, S^k\right)$ for some k.

Since G contains a finite index nilpotent group, it contains a finite index normal nilpotent group, assuring us that G is elementary amenable. Since finitely generated nilpotent groups are polycyclic, G is also polycyclic-by-finite, so it satisfies condition (x) of [KMPN09, Th.1.1]. As such, G satisfies condition (i) of that theorem, meaning that there exists a cocompact $\underline{E}G$-complex Y. From here, we may apply the Generalized Boundary Swapping Theorem [GM19, Cor.7.2] to obtain a $\mathcal{Z}$-structure for G of the form $\left(\overline{Y}, S^k\right)$. $\square$

Corollary 9.8. *For every group G that is strongly polycyclic or of polynomial growth, there exists an integer $k \geq -1$ such that $H^*(G; \mathbb{Z}G) \cong H^{*-1}\left(\mathbb{S}^k; \mathbb{Z}\right)$. In addition, G has the same homotopy at infinity as $\mathbb{Z}^{k+1}$, in particular, G is semistable and* pro-$\pi_i(G)$ *is stably isomorphic to $\pi_i\left(\mathbb{S}^k\right)$ for all i.*

Proof. These observations follow easily from standard applications of $\mathcal{Z}$-structures. See [Bes96] for torsion-free cases and [Dra06], [GM19] and [GM21] for extensions to groups with torsion. Earlier proofs of some of the group cohomology assertions can be found at [Bro82, p.213]. For analogous conclusions regarding semistability and pro-$\pi_i(G)$ for strongly polycyclic groups one can inductively apply [Geo08, Prop.17.3.1]. □

Remark 9.9. Unlike the class of groups with polynomial growth, a group G quasi-isometric to a strongly polycyclic group is not known to be strongly polycyclic or to admit a cocompact $\underline{E}G$-complex. As such, we cannot mimic the above strategies to endow G with an $\mathcal{Z}$-structure. We can, however, place a *coarse $\mathcal{Z}$-structure* on G which allows for many of the same applications as a genuine $\mathcal{Z}$-structure. See [GM21] for details.

By combining Theorem 6.11 with 9.7 and its proof, we get a little more.

Theorem 9.10. *Every group of the form $G \rtimes_\phi \mathbb{Z}$, where G is of polynomial growth, admits a $\mathcal{Z}$-structure with spherical boundary.*

Existence of $E\mathcal{Z}$-structures and $\underline{E\mathcal{Z}}$-structures for the groups discussed in this section is an interesting open question. For a strongly polycyclic group, each step in the inductive proof of Theorem 9.5 involves some $\phi_i \in \mathrm{Aut}(G_i)$ whose realization as ϕ_i-variant map $f : X_i \to X_i$ (implicit in that proof) would need to be extended over $\overline{X}_i$ in order to move from an $E\mathcal{Z}$-structure on G_i to an $E\mathcal{Z}$-structure on G_{i+1}. By using work found in [Wei21], one sees that this is always possible when the Hirsch length is ≤ 3. For higher Hirsch length, the corresponding sequence of progressively more complicated groups provides an interesting test case for Question 1.

As for a finitely generated nilpotent group Γ, the $\mathcal{Z}$-structure described above depends entirely on the $\mathcal{Z}$-structure $(\overline{X}, \mathbb{S}^n)$ on $\Gamma/T(\Gamma)$, where we simply allow $T(\Gamma)$ to act trivially on X. This means that whenever $\Gamma/T(\Gamma)$ admits an $E\mathcal{Z}$-structure, so does Γ. Moreover, if $H \leq \Gamma$ is finite, then $H \leq T(\Gamma)$, so $X^H = X$ and $\overline{X}^H = \overline{X}$. Therefore, we have an $\underline{E\mathcal{Z}}$-structure.

9.3. **$\mathcal{Z}$-structures on 3-manifold groups.** We now prove Theorem 1.2 from the introduction—that the fundamental groups of every closed 3-manifold admits a $\mathcal{Z}$-structure. Our proof brings together a tremendous amount established knowledge from 3-manifold topology, beginning with the classical theory and extending through Perelman's proof of the Geometrization Conjecture. It also uses tools from geometric group theory, such as theorems by Dahmani and Tirel regarding boundaries of free products, as well as boundary swapping techniques introduced by Bestvina and expanded upon in [GM19]. As for new ingredients, Theorem 1.1 plays a decisive role in handling fundamental groups of manifolds with *Nil* and *Sol* geometries. For a general discussion of 3-manifold groups, see [AFW15].

A closed 3-manifold is *prime* if it admits no nontrivial connected sum decomposition; it is *irreducible* if every tamely embedded 2-sphere bounds a 3-ball; it is $\mathbb{P}^2$-*irreducible* if it is irreducible and contains no 2-sided projective planes. Note that every orientable irreducible 3-manifold is $\mathbb{P}^2$-irreducible.

Theorem 9.11. *Every closed 3-manifold group admits a $\mathcal{Z}$-structure.*

Proof. Let M^3 be a closed connected 3-manifold and $G = \pi_1(M^3)$. By Kneser's existence theorem for prime decompositions [Kne29] (see also [Hem76, Th.3.15]) we may express M^3 and a finite connected sum $P_1^3 \# P_2^3 \# \cdots \# P_n^3$ of closed prime 3-manifolds and G as a free product $G_1 * G_2 * \cdots * G_n$, where $G_i = \pi_1(P_i^3)$. By [Dah03] or [Tir11], it suffices to show that each G_i admits a $\mathcal{Z}$-structure. That will be accomplished by examining the possible structures for the individual P_i^3.

Case 1. *P_i^3 is not irreducible.*

Then P_i^3 is either the trivial 2-sphere bundle over S^1 or the twisted (nonorientable) 2-sphere bundle $S^2 \widetilde{\times} S^1$. In either case, $G_i \cong \mathbb{Z}$ which admits the $\mathcal{Z}$-structure $\left(\overline{\mathbb{R}}, \{\pm\infty\}\right)$.

Case 2. *P_i^3 is $\mathbb{P}^2$-irreducible.*

By the Geometrization Conjecture, this class of 3-manifolds can be divided into three disjoint subclasses: geometric manifolds; non-geometric mixed manifolds; and non-geometric graph manifolds. We will discuss each of these in some detail.

SUBCASE 2A). *P_i^3 is a geometric manifold.*

Of the eight 3-dimensional geometries allowed by the Geometrization Theorem, seven remain possible (the geometry $S^2 \times \mathbb{R}$ having been taken care of in Case 1).

If P_i^3 admits the geometry of S^3 then G_i is finite, and we can obtain a $\mathcal{Z}$-structure, with empty boundary, by letting G_i act on a one-point space.

The geometries $\mathbb{E}^3$, $\mathbb{H}^3$, $\mathbb{H}^2 \times \mathbb{R}$ are all CAT(0), so if P_i^3 admits one of these geometries, then G_i admits a $\mathcal{Z}$-structure with the corresponding visual 2-sphere at infinity serving as its $\mathcal{Z}$-boundary.

The geometry $\widetilde{SL_2}(\mathbb{R})$ is not CAT(0), but by [Ger92] (see also [Rie01] for a short elegant proof) it is quasi-isometric to $\mathbb{H}^2 \times \mathbb{R}$. So by the boundary swapping trick described in [GM19], $\widetilde{SL_2}(\mathbb{R})$ admits a controlled $\mathcal{Z}$-compactification. Since G_i acts geometrically on $\widetilde{SL_2}(\mathbb{R})$, this is a $\mathcal{Z}$-structure for G_i.

The remaining geometries Nil and Sol are Lie groups homeomorphic to $\mathbb{R}^3$ which contain cocompact lattices of the form $\mathbb{Z}^2 \rtimes_{\phi_1} \mathbb{Z}$ and $\mathbb{Z}^2 \rtimes_{\phi_2} \mathbb{Z}$. (For example, let ϕ_1 and ϕ_2 be induced by matrices $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, respectively.) For simplicity, let us begin with Nil. Since $\mathbb{R}^2$ with the Euclidean metric and the visual $\mathbb{S}^1$ at infinity provide a $\mathcal{Z}$-structure for $\mathbb{Z}^2$, Theorem 1.1 allows us to build a $\mathcal{Z}$-structures $\left(\overline{Y}, \mathbb{S}^2\right)$ for the lattice $\mathbb{Z}^2 \rtimes_{\phi_1} \mathbb{Z}$. (With a little effort, we can arrange that Y is a geodesic space homeomorphic to $\mathbb{R}^3$, but that is not essential.) By the Švarc-Milnor Lemma, Nil is quasi-isometric to Y, so we can again use the boundary swapping trick to obtain a controlled $\mathcal{Z}$-compactification $\left(\overline{Nil}, \mathbb{S}^2\right)$ which serves as a $\mathcal{Z}$-structure for all cocompact lattices in Nil. A similar argument yields a controlled $\mathcal{Z}$-compactification $\left(\overline{Sol}, \mathbb{S}^2\right)$ which handles the case of Sol.

SUBCASE 2B). *P_i^3 is a non-geometric mixed manifold.*

Recall that a *mixed manifold* is one whose prime JSJ-decomposition includes at least one hyperbolic block. Leeb proved that Haken mixed manifolds admit nonpositively curved Riemannian metrics [Lee95]. Since P_i^3 is prime and non-geometric, it contains at least one JSJ-torus, and is therefore Haken. So Leeb's theorem tells us that G_i is a CAT(0) group with a 2-sphere boundary.

SUBCASE 2C). *P_i^3 is a non-geometric graph manifold.*

A *graph manifold* is one whose prime JSJ decomposition contains no hyperbolic blocks. Reasoning as above, P_i^3 is Haken, so by Kapovich-Leeb [KL98], there exists a nonpositively curved 3-manifold N^3 and a bi-Lipschitz homeomorphism $h : \widetilde{P_i^3} \to \widetilde{N^3}$. Another boundary swap places the visual boundary of $\widetilde{N^3}$ (necessarily a topological 2-sphere) onto $\widetilde{P_i^3}$ thereby giving G_i a $\mathcal{Z}$-structure.

Case 3. *P_i^3 is irreducible but not $\mathbb{P}^2$-irreducible.*

Let $p : Q_i^3 \to P_i^3$ be the orientable double covering. Then $\mathbb{Z}_2$ acts on Q_i^3 by covering transformations, and by [MSY82] or [Dun85], Q_i^3 contains a $\mathbb{Z}_2$-equivariant collection of pairwise disjoint essential 2-spheres $\{\Sigma_k\}_{k=1}^n$ which generates $\pi_2(Q_i^3)$ as a $\mathbb{Z}\pi_1$-module.

CLAIM. No Σ_k separates Q_i^3.

The claim breaks into two cases, depending on whether $p(\Sigma_k)$ is a 2-sphere or a projective plane.

First assume that $p(\Sigma_k) = \Pi$ is a projective plane. Since P_i^3 is prime and nonorientable, Π is 2-sided (see [Hei73, Lemma 2]). By an Euler characteristic argument, a single projective plane cannot be the boundary of a compact 3-manifold, so Π cannot separate P_i^3. Let γ be a path in $P_i^3 - \Pi$ which begins at a point on one side of a product neighborhood of Π and ends at a point on the other side. A lift of γ will lie in $Q_i^3 - \Sigma_k$ and connect points on opposite sides of a collar neighborhood of Σ_k, therefore Σ_k does not separate Q_i^3.

Next suppose that $p(\Sigma_k)$ is a 2-sphere Σ, and assume Σ_k separates Q_i^3. Write $Q_i^3 = A \cup B$ where A and B are connected codimension 0 submanifolds of Q_i^3 intersecting in a common boundary Σ_k. Without loss of generality, assume B contains the $\mathbb{Z}_2$ translate of Σ_k. Notice that $p|_A : A \to p(A)$ is a covering map. (Just check the definition locally.) Since points of Σ have only one preimage in A then $p|_A : A \to p(A)$ is a homeomorphism. It follows that Σ separates P_i^3 so, by irreducibility, Σ bounds a 3-ball in P_i^3. That 3-ball can be lifted to a 3-ball in P_i^3 bounded by Σ_k, contradicting the inessentiality of Σ_k. The claim follows.

Since no Σ_k separates it, Q_i^3 is a prime manifold. By its definition, Q_i^3 is orientable and contains at least one essential 2-sphere. It follows that $Q_i^3 \approx S^1 \times S^2$. The only nonorientable manifold double covered by $S^1 \times S^2$ is $\mathbb{P}^2 \times S^1$. It follows that $G_i \cong \mathbb{Z} \times \mathbb{Z}_2$, a group which admits a geometric action on $\mathbb{R}$ and a corresponding $\mathcal{Z}$-structure $\left(\overline{\mathbb{R}}, \{\pm\infty\}\right)$ □

Remark 9.12. By the same strategy applied above (but using the equivariant versions of [Dah03] or [Tir11]), one can prove that a given closed 3-manifold group $G =$

$\pi_1(M^3)$ admits an $E\mathcal{Z}$-structure by showing that the fundamental group $G_i = \pi_1(P_i^3)$ of each of its prime factors does. Revisiting the above proof, we see that in many cases, the work has already been done. Cases 1 and 3 involve the groups $\mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z}_2$, both of which admit $E\mathcal{Z}$-structures. Of the many groups G_i that arise in Subcase 2a), the $\mathcal{Z}$-structures associated to fundamental groups of manifolds with geometries of S^3, $\mathbb{E}^3$, $\mathbb{H}^3$, and $\mathbb{H}^2 \times \mathbb{R}$ are immediately $E\mathcal{Z}$-structure. As for the remaining geometries $\widetilde{SL_2(\mathbb{R})}$, Nil, and Sol, many of the Nil and Sol groups (those isomorphic to a semidirect product of the form $\mathbb{Z}^2 \rtimes_\phi \mathbb{Z}$) have been shown, in this paper, to admit $E\mathcal{Z}$-structures. Unfortunately, there may be cocompact lattices in Nil and Sol that do not fall into this category. We are not yet sure if they admit $E\mathcal{Z}$-structures. For similar reasons, the existence of $E\mathcal{Z}$-structures for groups arising in Subcase 2c)—fundamental groups of non-geometric graph manifolds—is still an open question. On the other hand, all groups arising in Subcase 2b)—fundamental groups of non-geometric mixed manifolds—admit $E\mathcal{Z}$-structures by virtue of being CAT(0).

Without formulating a detailed statement, we can say that *many* (even most) closed 3-manifold groups admit $E\mathcal{Z}$-structures. For the reader interested in a specific 3-manifold or collection of 3-manifolds, the above discussion provides a roadmap for checking whether an $E\mathcal{Z}$-structure is known to exist.

10. Further Questions

The work presented in this paper raises a number of questions. We close by highlighting two of them.

Question 1. *Given a $\mathcal{Z}$-structure $\left(\overline{X}, Z\right)$ on a group G and $\phi \in \mathrm{Aut}\,(G)$, when does there exist a ϕ-variant maps $f : X \to X$ which can be continuously extended to a map $\overline{f} : \overline{X} \to \overline{X}$. Given a ϕ for which the answer is no, is there a different $\mathcal{Z}$-structure for which the answer is yes?*

When $\left(\overline{X}, Z\right)$ is a canonical $\mathcal{Z}$-structure on a hyperbolic group, the answer is "always". The same holds for finitely generated abelian groups. Work by Hruska, Kleiner and Hindawi [HK05] identifies other interesting classes of CAT(0) groups (e.g., those with the isolated flats property) for which the answer is "always". But for general CAT(0) groups, such as $F_2 \times \mathbb{Z}^n$, there are difficulties. See [BR96]. What more can be said about about these groups? What about strongly polycyclic and finitely generated nilpotent groups? Systolic groups? Baumslag-Solitar groups?

In a different direction, we ask for generalizations of our main theorems.

Question 2. *Given a short exact sequence of groups*

$$1 \to N \to G \to Q \to 1$$

where N and Q admit $\mathcal{Z}$- or $E\mathcal{Z}$- or $\underline{E\mathcal{Z}}$-structures, what can be said about G? In particular, when does G admit an analogous structure? Does it help to assume that G is a semidirect product?

Special cases of this question are of interest. For example, what if N and Q are hyperbolic? CAT(0)? free? As a starting point, one might look at [Gui14] where it is shown that, whenever N and Q are nontrivial and of type F, then G admits a *weak $\mathcal{Z}$-structure*, meaning that all conditions for a $\mathcal{Z}$-structure are satisfied, except for the nullity condition.

Appendix A. Example: A $\mathcal{Z}$-structure for the Discrete Heisenberg Group

In this appendix we take a quick look at the main construction, within the narrow context of a well-known group realizable as an infinite cyclic extension of $\mathbb{Z}^2$. In particular, we analyze the issues involved in placing a $\mathcal{Z}$-structure on the discrete Heisenberg group $H_3(\mathbb{Z})$. The reader who might otherwise be put off by the abstractions found in the body of this paper should consider first reading this appendix as a warm-up exercise (while using Section 2, as needed, for definitions and notation).

Begin with $\mathbb{Z}^2$, its standard geometric action on $\mathbb{R}^2$, and the well-known $\mathcal{Z}$-structure obtained by adding the a circle at infinity. Let $\phi : \mathbb{Z}^2 \to \mathbb{Z}^2$ be the automorphism induced by $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and construct the semidirect product

$$\mathbb{Z}^2 \rtimes_\phi \mathbb{Z} = \left\langle x, y, t \mid xy = yx, t^{-1}xt = x, t^{-1}yt = xy \right\rangle$$

This is one realization of the discrete Heisenberg group $H_3(\mathbb{Z})$. The torus T^2 is a $K(\mathbb{Z}^2, 1)$ and the standard Dehn twist homeomorphism $f : T^2 \to T^2$ induces ϕ on fundamental groups, so $\mathrm{Tor}_f(T^2)$ has fundamental group $\mathbb{Z}^2 \rtimes_\phi \mathbb{Z}$. The infinite cyclic cover of $\mathrm{Tor}_f(T^2)$ is the mapping telescope $\mathrm{Tel}_f(T^2)$. Since f is a homeomorphism, each $\mathcal{M}_{[n,n+1]}(f)$ is homeomorphic to $T^2 \times [n, n+1]$ and $\mathrm{Tel}_f(T^2) \approx T^2 \times \mathbb{R}$. The universal cover is homeomorphic to $\mathbb{R}^2 \times \mathbb{R}$, but for the purposes of geometry should also be viewed as $\mathrm{Tel}_f(\mathbb{R}^2)$, where $f : \mathbb{R}^2 \to \mathbb{R}^2$ (the lift of f) is the linear isomorphism defined by matrix A.

Associate each $\mathcal{M}_{[n,n+1]}(f)$ with $\mathbb{R}^2 \times [n, n+1]$ by sending the domain and range copies of $\mathbb{R}^2$ to $\mathbb{R}^2 \times n$ and $\mathbb{R}^2 \times (n + 1)$, respectively, via identity maps, and each interval $x \times [n, n + 1]$ (as in the definition of mapping cylinder) linearly to the line segment from (x, n) to $(f(x), n + 1)$. A little linear algebra shows that no two of these segments intersect, so we have a homeomorphism. Under this realization of $\mathrm{Tel}_f(\mathbb{R}^2)$, $\mathrm{Cay}(\mathbb{Z}^2 \rtimes_\phi \mathbb{Z})$ (for generating set $\{x, y, t\}$) is made up of the standard 1×1 grids in each hyperplane $\mathbb{R}^2 \times n$ together with the line segments connecting each $((i, j), n)$ to $(f(i, j), n + 1)$. Here the vertex $((i, j), n)$ represents the group element $t^n x^i y^j = \phi^n(x^i y^j)t^n$, and the action of t is by translation of $\mathbb{R}^2 \times \mathbb{R}$ along the z-axis.

It is tempting to use the Euclidean metric on $\mathbb{R}^2 \times \mathbb{R}$ and its corresponding visual boundary in an attempt at a $\mathcal{Z}$-structure for $\mathbb{Z}^2 \rtimes_\phi \mathbb{Z}$. Although this gives a $\mathcal{Z}$-compactification of $\mathrm{Tel}_f(\mathbb{R}^2)$, the nullity condition fails badly. For example, y^n-translations of the segment connecting $(0, 0, 0)$ and $(0, 0, 1)$ grow linearly with n. Those segments look small when viewed from the origin since they lie within $\mathbb{R}^2 \times [0, 1]$; however, it we now translate them by powers of t, they cast large shadows in 2-sphere at infinity.

It is possible to "straighten" the $\mathbb{Z}^2$-portion of the above $\mathbb{Z}^2 \rtimes_\phi \mathbb{Z}$ action on $\mathbb{R}^2 \times \mathbb{R}$ by conjugating with a homeomorphism $v : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2 \times \mathbb{R}$ that sends the cosets of $\langle t \rangle \leq \mathbb{Z}^2 \rtimes_\phi \mathbb{Z}$ to vertical lines. (That is roughly what the map v in the body of this paper does.) This solves the problem uncovered in the previous paragraph. But now, if we translate the unit square $[0,1] \times [0,1] \times 0$ upward or downward using t^n, the effect is to apply larger and larger (positive and negative) powers of f to the 2-dimensional hyperplanes. Since the diameters of the resulting parallelograms grow linearly with n, their shadows in the 2-sphere at infinity are large.

At this point, we have a pair of (non-geometric) proper cocompact $(\mathbb{Z}^2 \rtimes_\phi \mathbb{Z})$-action on $\mathbb{R}^2 \times \mathbb{R}$ for which the natural $\mathcal{Z}$-compactification fails the nullity condition. To solve this problem, we look for an alternative way to attach a 2-sphere at infinity. Determining the right gluing is the delicate task at the heart of Theorem 1.1.

Remark A.1. Due to the simplicity of this example, we were able to realize $\mathrm{Tel}_f(\mathbb{R}^2)$ topologically as $\mathbb{R}^2 \times \mathbb{R}$. In general, however, the relevant mapping cylinders and telescopes will only be *(proper) homotopy equivalent* to, and not *homeomorphic to*, products. Our special case does generalize to semidirect products of the form $\mathbb{Z}^n \rtimes_\phi \mathbb{Z}$. But for a more typical situation, consider free-by-$\mathbb{Z}$ groups, illustrated in Figure 3.

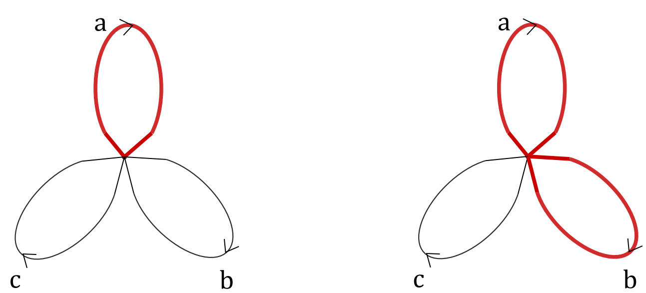

FIGURE 3. A free group automorphism need not be induced by a homemorphism of a $K(F_3, 1)$.

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