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Topology and Geometric Group Theory

Ohio State University, Columbus, USA, 2010–2011

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Preface

During the academic year 2010–2011, the Ohio State University Mathematics Department hosted a special year on geometric group theory. Over the course of the year, four-week-long workshops, two weekend conferences, and a week-long conference were held, each emphasizing a different aspect of topology and/or geometric group theory. Overall, approximately 80 international experts passed through Columbus over the course of the year, and the talks covered a large swath of the current research in geometric group theory. This volume contains contributions from the workshop on "Topology and geometric group theory," held in May 2011.

One of the basic questions in manifold topology is the Borel Conjecture, which asks whether the fundamental group of a closed aspherical manifold determines the manifold up to homeomorphism. The foundational work on this problem was carried out in the late 1980s by Farrell and Jones, who reformulated the problem in terms of the K-theoretic and L-theoretic Farrell–Jones Isomorphism Conjectures (FJIC). In the mid-2000s, Bartels, Lück, and Reich were able to vastly extend the techniques of Farrell and Jones. Notably, they were able to establish the FJICs (and hence the Borel Conjecture) for manifolds whose fundamental groups were Gromov hyperbolic. Lück reported on this progress at the 2006 ICM in Madrid. At the Ohio State University workshop, **Arthur Bartels** gave a series of lectures explaining their joint work on the FJICs. The write-up of these lectures provides a gentle introduction to this important topic, with an emphasis on the techniques of proof.

Staying on the theme of the Farrell–Jones Isomorphism Conjectures, **Daniel Juan-Pineda** and **Jorge Sánchez Saldaña** contributed an article in which both the K- and L-theoretic FJIC are verified for the braid groups on surfaces. These are the fundamental groups of configuration spaces of finite tuples of points, moving on the surface. Braid groups have been long studied, both by algebraic topologists, and by geometric group theorists.

A major theme in geometric group theory is the study of the behavior "at infinity" of a space (or group). This is a subject that has been studied by geometric topologists since the 1960s. Indeed, an important aspect of the study of open manifolds is the topology of their ends. The lectures by **Craig Guilbault** present the state of the art on these topics. These lectures were subsequently expanded into a graduate course, offered in Fall 2011 at the University of Wisconsin (Milwaukee).

An important class of examples in geometric group theory is given by CAT(0) cubical complexes and groups acting geometrically on them. Interest in these has grown in recent years, due in large part to their importance in 3-manifold theory (e.g., their use in Agol and Wise's resolution of Thurston's virtual Haken conjecture). A number of foundational results on CAT(0) cubical spaces were obtained in Michah Sageev's thesis. In his contributed article **Daniel Farley** gives a new proof of one of Sageev's key results: any hyperplane in a CAT(0) cubical complex embeds and separates the complex into two convex sets.

One of the powers of geometric group theory lies in its ability to produce, through geometric or topological means, groups with surprising algebraic properties. One such example was Burger and Mozes' construction of finitely presented, torsion-free simple groups, which were obtained as uniform lattices inside the automorphism group of a product of two trees (a CAT(0) cubical complex!). The article by **Pierre-Emmanuel Caprace** and **Bertrand Rémy** introduces a geometric argument to show that some nonuniform lattices inside the automorphism group of a product of trees are also simple.

An important link between algebra and topology is provided by the cohomology functors. Our final contribution, by **Peter Kropholler**, contributes to our understanding of the functorial properties of group cohomology. He considers, for a fixed group G, the set of integers n for which the group cohomology functor $H^n(G,-)$ commutes with certain colimits of coefficient modules. For a large class of groups, he shows this set of integers is always either finite or cofinite.

We hope these proceedings provide a glimpse of the breadth of mathematics covered during the workshop. The editors would also like to take this opportunity to thank all the participants at the workshop for a truly enjoyable event.

Columbus, OH, USA
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James Fowler
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Chapter 1
On Proofs of the Farrell–Jones Conjecture

Arthur Bartels

Abstract These notes contain an introduction to proofs of Farrell–Jones Conjecture for some groups and are based on talks given in Ohio, Oxford, Berlin, Shanghai, Münster and Oberwolfach in 2011 and 2012.

Keywords K-theory · L-theory · Controlled topology · Controlled algebra · Geodesic flow · CAT(0)-Geometry

Introduction

Let R be a ring and G be a group. The Farrell–Jones Conjecture [25] is concerned with the K- and L-theory of the group ring $R[G]$. Roughly it says that the K- and L-theory of $R[G]$ is determined by the K- and L-theory of the rings $R[V]$ where V varies over the family of virtually cyclic subgroups of G and group homology. The conjecture is related to a number of other conjectures in geometric topology and K-theory, most prominently the Borel Conjecture. Detailed discussions of applications and the formulation of this conjecture (and related conjectures) can be found in [10, 32–35].

These notes are aimed at the reader who is already convinced that the Farrell–Jones Conjecture is a worthwhile conjecture and is interested in recent proofs [3, 6, 9] of instances of this Conjecture. In these notes I discuss aspects or special cases of these proofs that I think are important and illustrating. The discussion is based on talks given over the last two years. It will be much more informal than the actual proofs in the cited papers, but I tried to provide more details than I usually do in talks. I took the liberty to express opinion in some remarks; the reader is encouraged to disagree with me. The cited results all build on the seminal work of Farrell and Jones surrounding their conjecture, in particular, their introduction of the geodesic flow as a tool in K- and L-theory [23]. Nevertheless, I will not assume that the reader is already familiar with the methods developed by Farrell and Jones.

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A brief summary of these notes is as follows. Section 1.1 contains a brief discussion of the statement of the conjecture. The reader is certainly encouraged to consult [10, 32–35] for much more details, motivation and background. Section 1.2 contains a short introduction to geometric modules that is sufficient for these notes. Three axiomatic results, labeled Theorems A, B and C, about the Farrell–Jones Conjecture are formulated in Sect. 1.3. Checking for a group G the assumptions of these results is never easy. Nevertheless, the reader is encouraged to find further applications of them. In Sect. 1.4 an outline of the proof of Theorem A is given. Section 1.5 describes the role of flows in proofs of the Farrell–Jones Conjecture. It also contains a discussion of the flow space for CAT(0)-groups. Finally, in Sect. 1.6 an application of Theorem C to some groups of the form $\mathbb{Z}^n \rtimes \mathbb{Z}$ is discussed.

1.1 Statement of the Farrell–Jones Conjecture

Classifying Spaces for Families

Let G be a group. A family of subgroups of G is a non-empty collection $\mathcal{F}$ of subgroups of G that is closed under conjugation and taking subgroups. Examples are the family Fin of finite subgroups, the family Cyc of cyclic subgroups, the family of virtually cyclic subgroups VCyc, the family Ab of abelian subgroups, the family $\{1\}$ consisting of only the trivial subgroup and the family All of all subgroups. If $\mathcal{F}$ is a family, then the collection $V\mathcal{F}$ of all $V \subseteq G$ which contain a member of $\mathcal{F}$ as a finite index subgroup is also a family. All these examples are closed under abstract isomorphism, but this is not part of the definition. If G acts on a set X then $\{H \leq G \mid X^H \neq \emptyset\}$ is a family of subgroups.

Definition 1.1.1 A G-CW-complex E is called a classifying space for the family $\mathcal{F}$, if E^H is non-empty and contractible for all $H \in \mathcal{F}$ and empty otherwise.

Such a G-CW-complex always exists and is unique up to G-equivariant homotopy equivalence. We often say such a space E is a model for $E_{\mathcal{F}}G$; less precisely we simply write $E = E_{\mathcal{F}}G$ for such a space.

Example 1.1.2 Let $\mathcal{F}$ be a family of subgroups. Consider the G-set $S := \coprod_{F \in \mathcal{F}} G/F$. The full simplicial complex $\Delta(S)$ spanned by S (i.e., the simplicial complex that contains a simplex for every non-empty finite subset of S) carries a simplicial G-action. The isotropy groups of vertices of $\Delta(S)$ are all members of $\mathcal{F}$, but for an arbitrary point of $\Delta(S)$ the isotropy group will only contain a member of $\mathcal{F}$ as a finite index subgroup. The first barycentric subdivision of $\Delta(S)$ is a G-CW-complex and it is not hard to see that it is a model for $E_{V\mathcal{F}}G$.

This construction works for any G-set S such that $\mathcal{F} = \{H \leq G \mid S^H \neq \emptyset\}$.

More information about classifying spaces for families can be found in [31].

Statement of the Conjecture

The original formulation of the Farrell–Jones Conjecture [25] used homology with coefficients in stratified and twisted Ω-spectra. We will use the elegant formulation of the conjecture developed by Davis and Lück [21]. Given a ring R and a group G Davis–Lück construct a homology theory for G-spaces

$$X \mapsto H_*^G(X; \mathbf{K}_R)$$

with the property that $H_*^G(G/H; \mathbf{K}_R) = K_*(R[H])$.

Definition 1.1.3 Let $\mathcal{F}$ be a family of subgroups of G. The projection $E_{\mathcal{F}}G \twoheadrightarrow G/G$ to the one-point G-space G/G induces the $\mathcal{F}$-assembly map

$$\alpha_{\mathcal{F}} \colon H_*^G(E_{\mathcal{F}}G; \mathbf{K}_R) \to H_*^G(G/G; \mathbf{K}_R) = K_*(R[G]).$$

Conjecture 1.1.4 (Farrell–Jones Conjecture) *For all groups G and all rings R the assembly map α_{VCyc} is an isomorphism.*

Remark 1.1.5 Farrell–Jones really only conjectured this for $R = \mathbb{Z}$. Moreover, they wrote (in 1993) that they regard this and related conjectures *only as estimates which best fit the known data at this time*. It still fits all known data today.

For arbitrary rings the conjecture was formulated in [2]. The proofs discussed in this article all work for arbitrary rings and it seems unlikely that the conjecture holds for $R = \mathbb{Z}$ and all groups, but not for arbitrary rings.

Remark 1.1.6 Let $\mathcal{F}$ be a family of subgroups of G. If R is a ring such that $K_*R[F] = 0$ for all $F \in \mathcal{F}$, then $H_*^G(E_{\mathcal{F}}G; \mathbf{K}_R) = 0$.

In particular, the Farrell–Jones Conjecture predicts the following: if R is a ring such that $K_*(R[V]) = 0$ for all $V \in \mathrm{VCyc}$ then $K_*(R[G]) = 0$ for all groups G.

Transitivity Principle

The family in the Farrell–Jones Conjecture is fixed to be the family of virtually cyclic groups. Nevertheless, it is beneficial to keep the family flexible, because of the following transitivity principle [25, A. 10].

Proposition 1.1.7 *Let $\mathcal{F} \subseteq \mathcal{H}$ be families of subgroups of G. Write $\mathcal{F} \cap H$ for the family of subgroups of H that belong to $\mathcal{F}$. Assume that*

(a) $\alpha_{\mathcal{H}} \colon H_*^G(E_{\mathcal{H}}G; \mathbf{K}_R) \to K_*(R[G])$ is an isomorphism,
(b) $\alpha_{\mathcal{F}\cap H} \colon H_*^H(E_{\mathcal{F}\cap H}H; \mathbf{K}_R) \to K_*(R[H])$ is an isomorphism for all $H \in \mathcal{H}$.

Then $\alpha_{\mathcal{F}} \colon H_*^G(E_{\mathcal{F}}G; \mathbf{K}_R) \to K_*(R[G])$ is an isomorphism.

Remark 1.1.8 The following illustrates the transitivity principle.

Assume that R is a ring such that $K_*(R[F]) = 0$ for all $F \in \mathcal{F}$. Assume moreover that the assumptions of Proposition 1.1.7 are satisfied. Combining Remark 1.1.6 with (b) we conclude $K_*(R[H]) = 0$ for all $H \in \mathcal{H}$. Then combining Remark 1.1.6 with (a) it follows that $K_*(R[G]) = 0$.

Remark 1.1.9 The transitivity principle can be used to prove the Farrell–Jones Conjecture for certain classes by induction. For example the proof of the Farrell–Jones Conjecture for $\mathrm{GL_n}(\mathbb{Z})$ uses an induction on n [11]. Of course the hard part is still to prove in the induction step that $\alpha_{\mathscr{F}_{n-1}}$ is an isomorphism for $\mathrm{GL_n}(\mathbb{Z})$ where the family $\mathscr{F}_{n-1}$ contains only groups that can be build from $\mathrm{GL_{n-1}}(\mathbb{Z})$ and poly-cyclic groups. The induction step uses Theorem B from Sect. 1.3. See also Remark 1.5.18.

More General Coefficients

Farrell and Jones also introduced a generalization of their conjecture now called the fibered Farrell–Jones Conjecture. This version of the conjecture is often not harder to prove than the original conjecture. Its advantage is that it has better inheritance properties. An alternative to the fibered conjecture is to allow more general coefficients where the group can act on the ring. As K-theory only depends on the category of finitely generated projective modules and not on the ring itself, it is natural to also replace the ring by an additive category. We briefly recall this generalization from [13].

Let $\mathscr{A}$ be an additive category with a G-action. There is a construction of an additive category $\mathscr{A}[G]$ that generalizes the twisted group ring for actions of G on a ring R. (In the notation of [13, Definition 2.1] this category is denoted as $\mathscr{A} *_G G/G$; $\mathscr{A}[G]$ is a more descriptive name for it.) There is also a homology theory $H^G_*(-;\mathbf{K}_{\mathscr{A}})$ for G-spaces such that $H^G_*(G/H;\mathbf{K}_{\mathscr{A}}) = K_*(\mathscr{A}[H])$. Therefore there are assembly maps

$$\alpha_{\mathscr{F}}\colon H^G_*(E_{\mathscr{F}}G;\mathbf{K}_{\mathscr{A}}) \to H^G_*(G/G;\mathbf{K}_{\mathscr{A}}) = K_*(\mathscr{A}[G]).$$

Conjecture 1.1.10 (Farrell–Jones Conjecture with coefficients) *For all groups G and all additive categories $\mathscr{A}$ with G-action the assembly map α_{VCyc} is an isomorphism.*

An advantage of this version of the conjecture is the following inheritance property.

Proposition 1.1.11 *Let $N \rightarrowtail G \twoheadrightarrow Q$ be an extension of groups. Assume that Q and all preimages of virtually cyclic subgroups under $G \twoheadrightarrow Q$ satisfies the Farrell–Jones Conjecture with coefficients 1.1.10. Then G satisfies the Farrell–Jones Conjecture with coefficients 1.1.10.*

Remark 1.1.12 Proposition 1.1.11 can be used to prove the Farrell–Jones Conjecture with coefficients for virtually nilpotent groups using the conjecture for virtually abelian groups, compare [10, Theorem 3.2].

It can also be used to reduce the conjecture for virtually poly-cyclic groups to irreducible special affine groups [3, Sect. 3]. The latter class consists of certain groups G for which there is an exact sequence $\Delta \to G \to D$, where D is infinite cyclic or the infinite dihedral group and Δ is a crystallographic group.

Remark 1.1.13 For additive categories with G-action the consequence from Remark 1.1.6 becomes an equivalent formulation of the conjecture: A group G satisfies the Farrell–Jones Conjecture with coefficients 1.1.10 if and only if for additive categories $\mathscr{B}$ with G-action we have

$$K_*(\mathscr{B}[V]) = 0 \text{ for all } V \in \mathrm{VCyc} \implies \mathrm{K}_*(\mathscr{B}[\mathrm{G}]) = 0.$$

(This follows from [9, Proposition 3.8] because the obstruction category $\mathscr{O}^G(E_{\mathscr{F}}G; \mathscr{A})$ is equivalent to $\mathscr{B}[G]$ for some $\mathscr{B}$ with $K_*(\mathscr{B}[F]) = 0$ for all $F \in \mathscr{F}$.)

In particular, surjectivity implies bijectivity for the Farrell–Jones Conjecture with coefficients.

Remark 1.1.14 The Farrell–Jones Conjecture 1.1.4 should be viewed as a conjecture about finitely generated groups. If it holds for all finitely generated subgroups of a group G, then it holds for G. The reason for this is that the conjecture is stable under directed unions of groups [27, Theorem 7.1].

With coefficients the situation is even better. This version of the conjecture is stable under directed colimits of groups [4, Corollary 0.8]. Consequently the Farrell–Jones Conjecture with coefficients holds for all groups if and only if it holds for all *finitely presented* groups, compare [1, Corollary 4.7]. It is therefore a conjecture about finitely presented groups.

Despite the usefulness of this more general version of the conjecture I will mostly ignore it in this paper to keep the notation a little simpler.

L-Theory

There is a version of the Farrell–Jones Conjecture for L-Theory. For some applications this is very important. For example the Borel Conjecture asserting the rigidity of closed aspherical topological manifolds follows in dimensions ≥ 5 via surgery theory from the Farrell–Jones Conjecture in K- and L-theory. The L-theory version of the conjecture is very similar to the K-theory version. Everything said so far about the K-theory version also holds for the L-theory version.

For some time proofs of the L-theoretic Farrell–Jones conjecture have been considerably harder than their K-theoretic analoga. Geometric transfer arguments used in L-theory are considerably more involved than their counterparts in K-theory. A change that came with considering arbitrary rings as coefficients in [2], is that transfers became more algebraic. It turned out [6] that this more algebraic point of view allowed for much easier L-theory transfers. (In essence, because the world of chain complexes with Poincaré duality is much more flexible than the world of manifolds.) This is elaborated at the end of Sect. 1.4.

I think that it is fair to say, that, as far as proofs are concerned, there is as at the moment no significant difference between the K-theoretic and the L-theoretic Farrell–Jones Conjecture. For this reason L-theory is not discussed in much detail in these notes.

1.2 Controlled Topology

The Thin h-Cobordism Theorem

An *h-cobordism* W is a compact manifold whose boundary is a disjoint union $\partial W = \partial_0 W \amalg \partial_1 W$ of closed manifolds such that the inclusions $\partial_0 W \to W$ and $\partial_1 W \to W$ are homotopy equivalences. If $M = \partial_0 W$, then we say W is an h-cobordism over M. If W is homeomorphic to $M \times [0, 1]$, then W is called trivial.

Definition 1.2.1 Let M be a closed manifold with a metric d. Let $\varepsilon \geq 0$.

An h-cobordism W over M is said to be ε-controlled over M if there exists a retraction $p\colon W \to M$ for the inclusion $M \to W$ and a homotopy $H\colon \mathrm{id_W} \to \mathrm{p}$ such that for all $x \in W$ the track

$$\{p(H(t, x)) \mid t \in [0, 1]\} \subseteq M$$

has diameter at most ε.

Remark 1.2.2 Clearly, the trivial h-cobordism is 0-controlled. Thus it is natural to think of being ε-controlled for small ε as being close to the trivial h-cobordism.

The following theorem is due to Quinn [39, Theorem 2.7]. See [18, 19, 28] for closely related results by Chapman and Ferry.

Theorem 1.2.3 (Thin ***h***-cobordism theorem) *Assume* $\dim M \geq 5$. *Fix a metric d on M (generating the topology of M).*

Then there is $\varepsilon > 0$ such that all ε-controlled h-cobordisms over M are trivial.

Remark 1.2.4 Farrell–Jones used the thin h-cobordism Theorem 1.2.3 and generalizations thereof to study $K_*(\mathbb{Z}[G])$, $* \leq 1$. For example in [23] they used the geodesic flow of a negatively curved manifold M to show that any element in $\mathrm{Wh}(\pi_1 M)$ could be realized by an h-cobordism that in turn had to be trivial by an application of (a generalization of) the thin h-cobordism theorem. Thus $\mathrm{Wh}(\pi_1 M) = 0$. In later papers they replaced the thin h-cobordism theorem by controlled surgery theory and controlled pseudoisotopy theory.

The later proofs of the Farrell–Jones Conjecture that we discuss here do not depend on the thin h-cobordism theorem, controlled surgery theory or controlled pseudoisotopy theory, but on a more algebraic control theory that we discuss in the next subsection.

An Algebraic Analog of the Thin h-Cobordism Theorem

Geometric groups (later also called geometric modules) were introduced by Connell-Hollingsworth [20]. The theory was developed much further by, among others, Quinn and Pedersen and is sometimes referred to as controlled algebra. A very pleasant introduction to this theory is given in [37].

Let R be a ring and G be a group.

Definition 1.2.5 Let X be a free G-space and $p\colon X \to Z$ be a G-map to a metric space with an isometric G-action.

(a) A *geometric $R[G]$-module over X* is a collection $(M_x)_{x\in X}$ of finitely generated free R-modules such that the following two conditions are satisfied.

- $M_x = M_{gx}$ for all $x \in X$, $g \in G$.
- $\{x \in X \mid M_x \neq 0\} = G \cdot S_0$ for some finite subset S_0 of X.

(b) Let M and N be geometric $R[G]$-modules over X. Let $f\colon \bigoplus_{x\in X} M_x \to \bigoplus_{x\in X} N_x$ be an $R[G]$-linear map (for the obvious $R[G]$-module structures). Write $f_{x'',x'}$ for the composition

$$M_{x'} \rightarrowtail \bigoplus_{x\in X} M_x \xrightarrow{f} \bigoplus_{x\in X} N_x \twoheadrightarrow N_{x''}.$$

The *support* of f is defined as $\operatorname{supp} f := \{(x'', x') \mid f_{x'',x'} \neq 0\} \subseteq X \times X$. Let $\varepsilon \geq 0$. Then f is said to be *ε-controlled over Z* if

$$d_Z(p(x''), p(x')) \leq \varepsilon \quad \text{for all} \quad (x'', x') \in \operatorname{supp} f.$$

(c) Let M be a geometric $R[G]$-module over X. Let $f\colon \bigoplus_{x\in X} M_x \to \bigoplus_{x\in X} M_x$ be an $R[G]$-automorphism. Then f is said to be an *ε-automorphism over Z* if both f and f^{-1} are ε-controlled over Z.

Remark 1.2.6 Geometric $R[G]$-modules over X are finitely generated free $R[G]$-modules with an additional structure, namely an G-equivariant decomposition into R-modules indexed by points in X. This additional structure is not used to change the notion of morphisms which are still $R[G]$-linear maps. But this structure provides an additional point of view for $R[G]$-linear maps: the set of morphisms between two geometric $R[G]$-modules now carries a filtration by control.

A good (and very simple) analog is the following. Consider finitely generated free R-modules. An additional structure one might be interested in are bases for such modules. This additional information allows us to view R-linear maps between them as matrices.

Controlled algebra is really not much more than working with (infinite) matrices whose index set is a (metric) space. Nevertheless this theory is very useful and flexible.

It is a central theme in controlled topology that sufficiently controlled obstructions (for example Whitehead torsion) are trivial. Another related theme is that assembly maps can be constructed as *forget-control* maps. In this paper we will use a variation of this theme for K_1 of group rings over arbitrary rings. Before we can state it we briefly fix some conventions for simplicial complexes.

Convention 1.2.1 *Let $\mathcal{F}$ be a family of subgroups of G. By a simplicial $(G, \mathcal{F})$-complex we shall mean a simplicial complex E with a simplicial G-action whose isotropy groups $G_x = \{g \in G \mid g \cdot x = x\}$ belong to $\mathcal{F}$ for all $x \in E$.*

Convention 1.2.2 *We will always use the l^1-metric on simplicial complexes. Let $Z^{(0)}$ be the vertex set of the simplicial complex Z. Then every element $z \in Z$ can be uniquely written as $z = \sum_{v \in Z^{(0)}} z_v \cdot v$ where $z_v \in [0,1]$, all but finitely many z_v are zero and $\sum_{v \in Z^{(0)}} z_v = 1$. The l^1-metric on Z is given by*

$$d^1_Z(z, z') = \sum_{v \in V} |z_v - z'_v|.$$

Remark 1.2.7 If E is a simplicial complex with a simplicial G-action such that the isotropy groups G_v belong to $\mathcal{F}$ for all vertices $v \in E^{(0)}$ of E, then E is a simplicial $(G, V\mathcal{F})$-complex, where $V\mathcal{F}$ consists of all subgroups H of G that admit a subgroup of finite index that belongs to $\mathcal{F}$.

Theorem 1.2.8 (Algebraic thin ***h***-cobordism theorem) *Given a natural number N, there is $\varepsilon_N > 0$ such that the following holds: Let*

(a) Z be a simplicial $(G, \mathcal{F})$-complex of dimension at most N,
(b) $p\colon X \to Z$ be a G-map, where X is a free G-space,
(c) M be a geometric $R[G]$-module over X,
(d) $f\colon M \to M$ be an ε_N-automorphism over Z (with respect to the l^1-metric on Z).

Then the K_1-class $[f]$ of f belongs to the image of the assembly map

$$\alpha_{\mathcal{F}}\colon H^G_1(E_{\mathcal{F}}G; \mathbf{K}_R) \to K_1(R[G]).$$

Remark 1.2.9 I called Theorem 1.2.8 the algebraic thin h-cobordism theorem here, because it can be used to prove the thin h-cobordism theorem. Very roughly, this works as follows. Let W be an ε-thin h-cobordism over M. Let $G = \pi_1 M = \pi_1 W$. The Whitehead torsion of W can be constructed using the singular chain complexes of the universal covers $\widetilde{W}$ and $\widetilde{M}$. This realizes the Whitehead torsion $\tau_W \in \mathrm{Wh}(G)$ of W by an $\widetilde{\varepsilon}$-automorphism f_W over $\widetilde{M}$, i.e. $[f_W]$ maps to τ_W under $K_1(\mathbb{Z}[G]) \to \mathrm{Wh}(G)$. Moreover, $\widetilde{\varepsilon}$ can be explicitly bounded in terms of ε, such that $\widetilde{\varepsilon} \to 0$ as $\varepsilon \to 0$. Because $\widetilde{M}$ is a free $G = \pi_1 M$-space it follows from Theorem 1.2.8 that $[f_W]$ belongs to the image of the assembly map $\alpha\colon H^G_1(EG, \mathbf{K}_{\mathbb{Z}}) \to K_1(\mathbb{Z}[G])$. But $\mathrm{Wh}(G)$ is the cokernel of α and therefore $\tau_W = 0$. This reduces the thin h-cobordism theorem to the s-cobordism theorem.

I believe that—at least in spirit—this outline is very close to Quinn's proof in [39].

Remark 1.2.10 If $f\colon M \to M'$ is ε-controlled over Z and $f'\colon M' \to M''$ is ε'-controlled over Z, then their composition $f' \circ f$ is $\varepsilon + \varepsilon'$-controlled. In particular, there is no category whose objects are geometric modules and whose morphisms are ε-controlled for fixed (small) ε. However, there are very elegant substitutes for this ill-defined category. These are built by considering a variant of the theory over an open cone over Z and taking a quotient category. In this quotient category every morphisms has for every $\varepsilon > 0$ an ε-controlled representative. Pedersen–Weibel [38] used this to construct homology of a space E with coefficients in the K-theory spectrum as the K-theory of an additive category. Similar constructions can be used to describe the assembly maps as forget-control maps [2, 17]. This also leads to a category (called the obstruction category in [9]), whose K-theory describes the fiber of the assembly map. A minor drawback of these constructions is that they usually involve a dimension shift.

A very simple version of such a construction is discussed at the end of this section. See in particular Theorem 1.2.18.

Remark 1.2.11 It is not hard to deduce Theorem 1.2.8 from [6, Theorem 5.3]. The latter result is a corresponding result for the obstruction category mentioned in Remark 1.2.10. In fact this result about the obstruction is stronger and can be used to prove that the assembly map is an isomorphism and not just surjective, see [6, Theorem 5.2]. I have elected to state the weaker Theorem 1.2.8 because it is much easier to state, but still grasps the heart of the matter. On the other hand, I think it is not at all easier to prove Theorem 1.2.8 than to prove the corresponding statement for the obstruction category. (The result in [6] deals with chain complexes, but this is not an essential difference.)

Remark 1.2.12 Results like Theorem 1.2.8 are very useful to prove the Farrell–Jones Conjecture. But it is not clear to me, that it really provides the best possible description of the image of the assembly map. For $g \in G$ we know that $[g]$ lies in the image of the assembly map. But its most natural representative (namely the isomorphism of $R[G]$ given by right multiplication by g) is not ε-controlled for small ε.

It may be beneficial to find other, maybe more algebraic and less geometric, characterizations of the image of the assembly map. But I do not know how to approach this.

Remark 1.2.13 The use of the l^1-metric in Theorem 1.2.8 is of no particular importance. In order for ε_N to only depend on N and not on Z, one has to commit to some canonical metric.

Remark 1.2.14 If $\mathcal{F}$ is closed under finite index supergroups, i.e., if $\mathcal{F} = V\mathcal{F}$ then there is no loss of generality in assuming that Z is the N-skeleton of the model for $E_{\mathcal{F}}G$ discussed in Example 1.1.2. This holds because there is always a G-map $Z^{(0)} \to S := \coprod_{F \in \mathcal{F}} G/F$ and this map extends to a simplicial map $Z \to \Delta(S)^{(N)}$. Barycentric subdivision only changes the metric on the N-skeleton in a controlled (depending on N) way.

Remark 1.2.15 There is also a converse to Theorem 1.2.8. If $a \in K_1(R[G])$ lies in the image of the assembly map $\alpha_{\mathcal{F}}$ then there is some N such that it can for any $\varepsilon > 0$ be realized by an ε-automorphism over an N-dimensional simplicial complex Z with a simplicial G-action all whose isotropy groups belong to $\mathcal{F}$. The simplicial complex can be taken to be the N-skeleton of a simplicial complex model for $E_{\mathcal{F}}G$.

This is a consequence of the description of the assembly map as a forget-control map as for example in [2, Corollary 6.3].

Remark 1.2.16 It is not hard to extend the theory of geometric $R[G]$-modules from rings to additive categories. In this case one considers collections $(A_x)_{x \in X}$ where each

A_x is an object from $\mathscr{A}$. In fact [6, Theorem 5.3], which implies Theorem 1.2.8, is formulated using additive categories as coefficients.

Remark 1.2.17 Results for K_1 often imply results for K_i, $i \leq 0$, using suspension rings. For a ring R, there is a suspension ring ΣR with the property that $K_i(R) = K_{i+1}(\Sigma R)$ [44]. This construction can be arranged to be compatible with group rings: $\Sigma(R[G]) = (\Sigma R)[G]$. A consequence of this is that for a fixed group G the surjectivity of $\alpha_{\mathscr{F}} \colon H_1^G(E_{\mathscr{F}}G; \mathbf{K}_R) \to K_1(R[G])$ for all rings R implies the surjectivity of $\alpha_{\mathscr{F}}$ for all $i \leq 1$, compare [2, Corollary 7.3].

Because of this trick there is no need for a version of Theorem 1.2.8 for K_i, $i \leq 0$.

Higher K-Theory

We end this section by a brief discussion of a version of Theorem 1.2.8 for higher K-theory. Because there is no good concrete description of elements in higher K-theory it will use slightly more abstract language.

Let $p_n \colon X_n \to Z_n$ be a sequence of G-maps where each X_n is a free G-space and each Z_n is a simplicial $(G, \mathscr{F})$-complex of dimension N. Define a category $\mathscr{C}$ as follows. Objects of $\mathscr{C}$ are sequences $(M_n)_{n \in \mathbb{N}}$ where for each n, M_n is a geometric $R[G]$-module over X_n. A morphism $(M_n)_{n \in \mathbb{N}} \to (N_n)_{n \in \mathbb{N}}$ in $\mathscr{C}$ is given by a sequence $(f_n)_{n \in \mathbb{N}}$ of $R[G]$-linear maps $f_n \colon \bigoplus_{x \in X_n} (M_n)_x \to \bigoplus_{x \in X_n} (N_n)_x$ satisfying the following condition: there is $\alpha > 0$ such that for each n, f_n is $\frac{\alpha}{n}$-controlled over Z_n. For each $k \in \mathbb{N}$,

$$(M_n)_{n \in \mathbb{N}} \mapsto \bigoplus_{x \in X_k} (M_k)_x$$

defines a functor π_k from $\mathscr{C}$ to the category of finitely generated free $R[G]$-modules. The following is a variation of [14, Corollary 4.3]. It can be proven using [9, Theorem 7.2].

Theorem 1.2.18 *Let $a \in K_*(R[G])$. Suppose that there is $A \in K_*(\mathscr{C})$ such that for all k*

$$(\pi_k)_*(A) = a.$$

Then a belongs to the image of $\alpha_{\mathscr{F}} \colon H_*^G(E_{\mathscr{F}}G; \mathbf{K}_R) \to K_*(R[G])$.

1.3 Conditions that Imply the Farrell–Jones Conjecture

In [6, 9] the Farrell–Jones Conjecture is proven for hyperbolic and CAT(0)-groups. Both papers take a somewhat axiomatic point of view. They both contain careful (and somewhat lengthy) descriptions of conditions on groups that imply the Farrell–Jones conjecture. The conditions in the two papers are closely related to each other. A group satisfying them is said to be *transfer reducible* over a given family of subgroups in [6]. Further variants of these conditions are introduced in [11, 45]. The existence of all these different versions of these conditions seem to me to suggest that we have not found the ideal formulation of them yet. The notion of transfer reducible groups (and all its variations) can be viewed as an axiomatization of the work of Farrell–Jones using the geodesic flow that began with [23]. Somewhat different conditions—related to work of Farrell–Hsiang [22]—are discussed in [5].

Transfer Reducible Groups—Strict Version

Let R be a ring and G be a group.

Definition 1.3.1 An *N-transfer space* X is a compact contractible metric space such that the following holds.

For any $\delta > 0$ there exists a simplicial complex K of dimension at most N and continuous maps and homotopies $i: X \to K$, $p: K \to X$, and $H: p \circ i \to \mathrm{id}_X$ such that for any $x \in X$ the diameter of $\{H(t, x) \mid t \in [0, 1]\}$ is at most δ.

Example 1.3.2 Let T be a locally finite simplicial tree. The compactification $\overline{T}$ of T by equivalence classes of geodesic rays is a 1-transfer space.

Theorem A *Suppose that G is finitely generated by S. Let $\mathcal{F}$ be a family of subgroups of G. Assume that there is $N \in \mathbb{N}$ such that for any $\varepsilon > 0$ there are*

(a) an N-transfer space X equipped with a G-action,
(b) a simplicial $(G, \mathcal{F})$-complex E of dimension at most N,
(c) a map $f: X \to E$ that is G-equivariant up to ε: $d^1(f(s \cdot x), s \cdot f(x)) \leq \varepsilon$ for all $s \in S$, $x \in X$.

Then $\alpha_{\mathcal{F}}: H_*^G(E_{\mathcal{F}}G; \mathbf{K}_R) \to K_*(RG)$ is an isomorphism.

Remark 1.3.3 It follows from [8] that Theorem A (with $\mathcal{F}$ the family of virtually cyclic subgroups VCyc) applies to hyperbolic groups.

Example 1.3.4 Let G be a group and K be a finite contractible simplicial complex with a simplicial G-action. Then for the family $\mathcal{F} := \mathcal{F}_K$ the assembly map $\alpha_{\mathcal{F}}: H_*^G(E_{\mathcal{F}}G; \mathbf{K}_R) \to K_*(RG)$ is an isomorphism. This follows from Theorem A by setting $N := \dim K$ and $X := K$, $E := K$, $f := \mathrm{id}_K$ (for all $\varepsilon > 0$). Since K is finite, the group of simplicial automorphisms of K is also finite. It follow that for all $x \in K$ the isotropy group G_x has finite index in G.

The assumptions of Theorem A should be viewed as a weakening of this example. The properties of K are reflected in requirements on X or on E and the existence of the map f yields a strong relationship between X and E.

Remark 1.3.5 Rufus Willet and Guoliang Yu pointed out that the assumption of Theorem A implies that the group G has finite asymptotic dimension, provided there is a uniform bound on the asymptotic dimension of groups in $\mathcal{F}$. The latter assumptions is of course satisfied for the family of virtually cyclic groups VCyc.

Remark 1.3.6 Martin Bridson pointed out that the assumptions of Theorem A are formally very similar to the concept of amenability for actions on compact spaces. The main difference is that in the latter context E is replaced by the (infinite dimensional) space of probability measures on G.

Remark 1.3.7 Theorem A is a minor reformulation of [9, Theorem 1.1]. In this reference instead of the existence of f the existence of certain covers $\mathcal{U}$ of $G \times X$ are postulated. But the first step in the proof is to use a partition of unity to construct a G-map from $G \times X$ to the nerve $|\mathcal{U}|$ of $\mathcal{U}$. Under the assumptions formulated in Theorem A this map is simply $(g, x) \mapsto g \cdot f(g^{-1}x)$.

Avoiding the open covers makes the theorem easier to state, but there is no real mathematical difference.

Remark 1.3.8 The proof of Theorem A in [9] really shows a little bit more: there is M (depending on N) such that the restriction of $\alpha_{\mathcal{F}}$ to $H_*^G(E_{\mathcal{F}}G^{(M)}; \mathbf{K}_R)$ is surjective. For arbitrary groups and rings with non-trivial K-theory in infinitely many negative degrees there will be no such M. It is reasonable to expect that groups satisfying the assumptions of Theorem A will also admit a finite dimensional model for the space $E_{\mathcal{F}}G$.

Remark 1.3.9 Let E be a simplicial complex of dimension N. Let g be a simplicial automorphism of E. Let $x = \sum_{v \in E^{(0)}} x_v \cdot v$ be a point of E. Let $\operatorname{supp} x := \{v \in E^{(0)} \mid x_v \neq 0\}$. It is a disjoint union of the sets

$$\begin{aligned} P_x &:= \{v \in \operatorname{supp} x \mid \forall n \in \mathbb{N} \ : \ g^n \in \operatorname{supp} x\}, \\ D_x &:= \{v \in \operatorname{supp} x \mid \exists n \in \mathbb{N} \ : \ g^n \notin \operatorname{supp} x\}. \end{aligned}$$

Observe that for $v \in D_x$, we have $d^1(x, gx) \geq x_v$. Assume now that $d^1(x, gx) < \frac{1}{N+1}$. As $\sum_v x_v = 1$ there is a vertex v with $v \geq \frac{1}{N+1}$. Such a vertex v belongs then to P_x and it follows that $\{g^n v \mid n \in \mathbb{N}\}$ is finite and spans a simplex of E whose barycenter is fixed by g.

Assume now that $f \colon X \to E$ is as in assumption (c) of Theorem A. If G_x is the isotropy group for $x \in X$ (and if G_x is finitely generated by S_x say) then if ε is sufficiently small it follows that $d^1(f(x), gf(x)) < \frac{1}{N+1}$. The previous observation implies then $G_x \in \mathcal{F}$.

On the other hand one can apply the Lefschetz fixed point theorem to the simplicial dominations to X and finds for fixed $g \in G$ and each $\varepsilon > 0$ a point $x_\varepsilon \in X$ such that $d(gx_\varepsilon, x_\varepsilon) \leq \varepsilon$. The compactness of X implies that there is a fixed point in X for each element of G. Altogether, it follows that $\mathcal{F}$ will necessarily contain the family of cyclic subgroups.

Remark 1.3.10 Frank Quinn has shown that one can replace the family of virtually cyclic groups in the Farrell–Jones Conjecture by the family of (possibly infinite) hyper-elementary groups [40].

It is an interesting question whether one can (maybe using Smith theory) build on the argument from Remark 1.3.9 to conclude that in order for the assumptions of Theorem A to be satisfied it is necessary for $\mathcal{F}$ to contain this family of (possibly infinite) hyper-elementary groups.

Remark 1.3.11 One can ask for which N-transfer spaces X with a G-action it is possible to find for all $\varepsilon > 0$ a map $f \colon X \to E$ as in assumptions (b) and (c).

Remark 1.3.9 shows that a necessary condition is $G_x \in \mathcal{F}$ for all $x \in X$, but it is not clear to me that this condition is not sufficient.

In light of the observation of Willet and Yu from Remark 1.3.5 a related question is whether there is a group G of infinite asymptotic dimension for which there is an N-transfer space with a G-action such that the asymptotic dimension of G_x, $x \in X$ is uniformly bounded.

Remark 1.3.12 The reader is encouraged to try to check that finitely generated free groups satisfy the assumptions of Theorem A with respect to the family of (virtually) cyclic subgroups. In this case one can use the compactification $\bar{T}$ of the usual tree by equivalence classes of geodesic rays as the transfer space. I am keen to see a proof of this that is easier than the one coming out of [8] and avoids flow spaces. Maybe a clever application of Zorn's Lemma could be useful here.

I am not completely sure whether or not it is possible to write down the maps $f : \bar{T} \to E$ in assumption (c) explicitly for finitely generated free groups.

Transfer Reducible Groups—Homotopy Version

Let R be a ring.

Definition 1.3.13 Let $G = \langle\, S \mid R \,\rangle$ be a finitely presented group. A homotopy action of G on a space X is given by

- for all $s \in S \cup S^{-1}$ maps $\varphi_s : X \to X$,
- for all $r = s_1 \cdot s_2 \cdots s_l \in R$ homotopies $H_r : \varphi_{s_1} \circ \varphi_{s_2} \circ \cdots \circ \varphi_{s_l} \to \mathrm{id}_X$

Theorem B *Suppose that $G = \langle\, S \mid R \,\rangle$ is a finitely presented group. Let $\mathcal{F}$ be a family of subgroups of G. Assume that there is $N \in \mathbb{N}$ such that for any $\varepsilon > 0$ there are*

(a) an N-transfer space X equipped with a homotopy G-action (φ, H),
(b) a simplicial $(G, \mathcal{F})$-complex E of dimension at most N,
(c) a map $f : X \to E$ that is G-equivariant up to ε: for all $x \in X$, $s \in S \cup S^{-1}$, $r \in R$

- *$d^1(f(\varphi_s(x)), s \cdot f(x)) \leq \varepsilon$,*
- *$\{H_r(t, x) \mid t \in [0, 1]\}$ has diameter at most ε.*

Then $\alpha_{\mathcal{F}} : H_i^G(E_{\mathcal{F}}G; \mathbf{K}_R) \to K_i(RG)$ is an isomorphism for $i \leq 0$ and surjective for $i = 1$.

Remark 1.3.14 It follows from [7] that Theorem B applies to CAT(0)-groups (where $\mathcal{F}$ is the family of virtually cyclic groups). We will sketch the proof of this fact in Sect. 1.5.

Wegner introduced the notion of a strong homotopy action and proved a version of Theorem B where the conclusion is that $\alpha_{\mathcal{F}}$ is an isomorphism in all degrees [45]. This result also applies to CAT(0)-groups.

Remark 1.3.15 Theorem B is a reformulation of [6, Theorem 1.1] (just as in Remark 1.3.7).

The assumptions of Theorem A feel much cleaner than the assumptions of Theorem B. It would be very interesting if one could show, maybe using some kind of limit that promotes a (strong) homotopy action to an actual action, such that the latter (or Wegner's variation of them) do imply the former.

In light of Remark 1.3.5 this would imply in particular that CAT(0)-groups have finite asymptotic dimension and is therefore probably a difficult (or impossible) task.

Remark 1.3.16 I do not know whether semi-direct products of the form $\mathbb{Z}^n \rtimes \mathbb{Z}$ satisfy the assumptions of Theorem B, for example if $\mathcal{F}$ is the family of abelian groups. On the other hand the Farrell–Jones Conjecture is known to hold for such groups and more general for virtually poly-cyclic groups [3].

Remark 1.3.17 Remark 1.3.8 also applies to Theorem B.

Remark 1.3.18 There is an L-theory version of Theorem B, see [6, Theorem 1.1(ii)]. There, the conclusion is that the assembly map $\alpha_{\mathcal{F}_2}$ is an isomorphism in L-theory where $\mathcal{F}_2$ is the family of subgroups that contain a member of $\mathcal{F}$ as a subgroup of index at most 2. Of course $\mathrm{VCyc} = \mathrm{VCyc}_2$. There is no restriction on the degree i in this L-theoretic version and so this also provides an L-theory version of Theorem A.

Farrell–Hsiang Groups

Definition 1.3.19 A finite group H is said to be *hyper-elementary* if there exists a short exact sequence

$$C \rightarrowtail H \twoheadrightarrow P$$

where C is a cyclic group and the order of P is a prime power.

Quinn generalized this definition to infinite groups by allowing the cyclic group to be infinite [40].

Hyper-elementary groups play a special role in K-theory because of the following result of Swan [43]. For a group G we denote by $\mathrm{Sw}(G)$ the Swan group of G. It can be defined as K_0 of the exact category of $\mathbb{Z}[G]$-modules that are finitely generated free as $\mathbb{Z}$-modules. This group encodes information about transfer maps in algebraic K-theory.

Theorem 1.3.20 (Swan) *For a finite group F the induction maps combine to a surjective map*

$$\bigoplus_{H \in \mathcal{H}(F)} \mathrm{Sw}(H) \twoheadrightarrow \mathrm{Sw}(F),$$

where $\mathcal{H}(F)$ denotes the family of hyper-elementary subgroups of F.

Let R be a ring and G be a group.

Theorem C *Suppose that G is finitely generated by S. Assume that there is $N \in \mathbb{N}$ such that for any $\varepsilon > 0$ there are*

(a) a group homomorphism $\pi : G \to F$ where F is finite,
(b) a simplicial $(G, \mathcal{F})$-complex E of dimension at most N
(c) a map $f : \coprod_{H \in \mathcal{H}(F)} G/\pi^{-1}(H) \to E$ that is G-equivariant up to ε: $d^1(f(sx), s \cdot f(x)) \leq \varepsilon$ for all $s \in S$, $x \in \coprod_{H \in \mathcal{H}(F)} G/\pi^{-1}(H)$.

Then $\alpha_{\mathcal{F}} : H_*^G(E_{\mathcal{F}}G; \mathbf{K}_R) \to K_*(RG)$ is an isomorphism.

Remark 1.3.21 Theorem C is proven in [5] building on work of Farrell–Hsiang [22]. The main difference to Theorems A and B is that the transfer space X is replaced by the discrete space $\coprod_{H \in \mathcal{H}(F)} G/\pi^{-1}(H)$. It is Swan's Theorem 1.3.20 that replaces the contractibility of X.

I have no conceptual understanding of Swan's theorem. For this reason Theorem C is to me not as conceptually satisfying as Theorem A. Moreover, I expect that a version of Theorem C for Waldhausen's A-theory will need a larger family than the family of hyper-elementary subgroups.

Remark 1.3.22 Groups satisfying the assumption of Theorem C are called *Farrell–Hsiang groups with respect to $\mathcal{F}$* in [5].

Remark 1.3.23 Theorem C can be used to prove the Farrell–Jones Conjecture for virtually poly-cyclic groups [3, Sects. 3 and 4]. We will discuss some semi-direct products of the form $\mathbb{Z}^n \rtimes \mathbb{Z}$ in Sect. 1.6.

Remark 1.3.24 Remark 1.3.8 also applies to Theorem C.

Remark 1.3.25 Theorem C holds without change in L-theory as well [5].

Remark 1.3.26 It would be good to find a natural common weakening of the assumptions in Theorems A, B and C that still implies the Farrell–Jones Conjecture. Ideally such a formulation should have similar inheritance properties as the Farrell–Jones Conjecture, see Propositions 1.1.7 and 1.1.11.

Injectivity

It is interesting to note that injectivity of the assembly map $\alpha_{\{1\}}$ or α_{Fin} is known for seemingly much bigger classes of groups, than the class of groups known to satisfy the Farrell–Jones Conjecture. Rational injectivity of the L-theoretic assembly map $\alpha_{\{1\}}$ is of particular interest, as it implies Novikov's conjecture on the homotopy invariance of higher signatures. Yu [46] proved the Novikov conjecture for all groups admitting a uniform embedding into a Hilbert-space. This class of groups contains all groups of finite asymptotic dimension. Integral injectivity of the assembly map $\alpha_{\{1\}}$ for K- and L-theory is known for all groups that admit a finite CW-complex as a model for BG and are of finite decomposition complexity [30, 41]. The latter property is a generalization of finite asymptotic dimension. Rational injectivity of the K-theoretic assembly map $\alpha_{\{1\}}$ for the ring $\mathbb{Z}$ is proven by Bökstedt–Hsiang–Madsen [15] for all groups G satisfying the following homological finiteness condition: for all n the rational group-homology $H_*(G; \mathbb{Q})$ is finite dimensional.

1.4 On the Proof of Theorem A

Using the results from controlled topology discussed in Sect. 1.2 we will outline a proof of the surjectivity of

$$\alpha_{\mathscr{F}}\colon H_1^G(E_{\mathscr{F}}G;\mathbf{K}_R) \to K_1(R[G])$$

under the assumptions of Theorem A.

Step 1: Preparations

Let G be a finitely generated group and $\mathscr{F}$ be a family of subgroups of G. Let $N \in \mathbb{N}$ be the number appearing in Theorem A. Let $a \in K_1(R[G])$. Then $a = [\psi]$ where $\psi\colon R[G]^n \to R[G]^n$ is an $R[G]$-right linear automorphism. We write $R[G]^n = \mathbb{Z}[G] \otimes_{\mathbb{Z}} R^n$. There is a finite subset $T \subseteq G$ and there are R-linear maps $\psi_g\colon R^n \to R^n$, ${\psi^{-1}}_g\colon R^n \to R^n$, $g \in T$ such that

$$\psi(h \otimes v) = \sum_{g \in T} hg^{-1} \otimes \psi_g(v) \quad \text{and} \quad \psi^{-1}(h \otimes v) = \sum_{g \in T} hg^{-1} \otimes {\psi^{-1}}_g(v).$$

Because of Theorem 1.2.8 it suffices to find

- a G-space Y,
- a $(G,\mathscr{F})$-complex E of dimension at most N,
- a G-map $Y \to E$,
- a geometric $R[G]$-module M over Y,
- an ε_N-automorphism over E, $\varphi\colon M \to M$,

such that $[\varphi] = a \in K_1(R[G])$. Here ε_N is the number depending on N, whose existence is asserted in Theorem 1.2.8.

Let L be a (large) number. We will later specify L; it will only depend on N. From the assumption of Theorem A we easily deduce that there are

(a) an N-transfer space X equipped with a G-action,
(b) a simplicial $(G,\mathscr{F})$-complex E of dimension at most N,
(c) a map $f\colon X \to E$ such that $d^1(f(g \cdot x), g \cdot f(x)) \leq \varepsilon_N/2$ for all $x \in X$ and all $g \in G$ that can be written as $g = g_1 \dots g_L$ with $g_1, \dots, g_L \in T$.

By compactness of X there is $\delta_0 > 0$ such that $d^1(f(x), f(x')) \leq \varepsilon_N/2$ for all $x, x' \in X$ with $d(x,x') \leq L\delta_0$. We will use $Y := G \times X$ with the G-action defined by $g \cdot (h,x) := (gh, x)$. We will also use the G-map $G \times X \to E$, $(g,x) \mapsto gf(x)$. The action of G on X will be used later.

Step 2: A Chain Complex Over X

To simplify the discussion let us assume that for X the maps i and p appearing in Definition 1.3.1 can be arranged to be δ-homotopy equivalences. This means that in addition to H there is also a homotopy $H'\colon i \circ p \to \mathrm{id}_K$ such that for any $y \in K$ the diameter of $\{H'(t,y) \mid t \in [0,1]\}$ with respect to the l^1-metric on K is at most δ.

Let C_* be the simplicial chain complex of the l-fold simplicial subdivision of K. Using $p\colon K \to X$ we can view C_* as a chain complex of geometric $\mathbb{Z}$-modules over X. If we choose l sufficiently large, then we can arrange that the boundary maps ∂^{C_*} of C_* are δ_0-controlled over X. Moreover, using the action of G on X and a δ-homotopy equivalence between K and X (for $0 < \delta \ll \delta_0$) and enlarging l we can produce chain maps $\varphi_g\colon C_* \to C_*$, $g \in G$, chain homotopies $H_{g,h}\colon \varphi_g \circ \varphi_h \to \varphi_{gh}$ satisfying the following control conditions

- if $g \in T$ and $(x', x) \in \operatorname{supp}\varphi_g$ then $d(x', gx) \le \delta_0$ (recall that we view C_* as a chain complex over X),
- if $g, h \in T$ and $(x', x) \in \operatorname{supp} H_{g,h}$ then $d(x', ghx) \le \delta_0$.

Remark 1.4.1 If we drop the additional assumption on X (i.e., if we no longer assume the existence of the homotopy H'), then it is only possible to construct the chain complex C_* in the idempotent completion of geometric $\mathbb{Z}$-modules over X. This is a technical but—I think—not very important point.

Remark 1.4.2 A construction very similar to this step 2 is carried out in great detail in [6, Sect. 8].

Step 3: Transfer to a Chain Homotopy Equivalence

Recall our automorphism ψ of $R[G]^n = \mathbb{Z}[G]\otimes_{\mathbb{Z}} R^n$. We will now replace the R-module R^n by the R-chain complex $C_*\otimes_{\mathbb{Z}} R^n$ to obtain the chain complex $D_* := \mathbb{Z}[G]\otimes_{\mathbb{Z}} C_*\otimes_{\mathbb{Z}} R^n$. As C_* is a chain complex of geometric $\mathbb{Z}$-modules over X, D_* is naturally a geometric $R[G]$-module over $G\times X$. Here $(D_*)_{h,x} = \{h\otimes w\otimes v \mid v \in R^n, w \in (C_*)_x\}$ for $h \in G$, $x \in X$. Recall that we use the left action defined by $g \cdot (h, x) = (gh, x)$ on $G\times X$. We can now use the data from Step 2 to transfer ψ to a chain homotopy equivalence $\Psi = \sum_{g\in T} g\otimes\varphi_g\otimes\psi_g\colon D_* \to D_*$. Similarly, there is a chain homotopy inverse Ψ' for Ψ and there are chain homotopies $\mathscr{H}\colon \Psi \circ \Psi' \to \mathrm{id}_{D_*}$ and $\mathscr{H}'\colon \Psi' \circ \Psi \to \mathrm{id}_{D_*}$. In more explicit formulas these are defined by

$$\begin{aligned}
\Psi(h\otimes w\otimes v) &= \sum_{g\in T} hg^{-1}\otimes\varphi_g(w)\otimes\psi_g(v),\\
\Psi'(h\otimes w\otimes v) &= \sum_{g\in T} hg^{-1}\otimes\varphi_g(w)\otimes{\psi^{-1}}_g(v),\\
\mathscr{H}(h\otimes w\otimes v) &= \sum_{g,g'\in T} h(gg')^{-1}\otimes H_{g,g'}(w)\otimes\psi_g \circ {\psi^{-1}}_{g'}(v),\\
\mathscr{H}'(h\otimes w\otimes v) &= \sum_{g,g'\in T} h(gg')^{-1}\otimes H_{g,g'}(w)\otimes{\psi^{-1}}_g \circ \psi_{g'}(v),
\end{aligned}$$

for $h \in G$, $w \in C_*$, $v \in R^n$.

Digression on Torsion

Let S be a ring. If Φ is a self-homotopy equivalence of a bounded chain complex D_* of finitely generated free S-modules then its *self-torsion* $\tau(\Phi) \in K_1(S)$ is the K-theory class of an automorphism $\tilde{\tau}(\Phi)$ of $\bigoplus_{n\in\mathbb{Z}} D_n$. There is an explicit formula for $\tilde{\tau}(\Phi)$ that involves the boundary map of D_*, Φ, a chain homotopy inverse Φ' for Φ and chain homotopies $\Phi \circ \Phi' \to \mathrm{id}_{\mathrm{D}_*}$, $\Phi' \circ \Phi \to \mathrm{id}_{\mathrm{D}_*}$. The ingredients for such a formula can be found for example in [2, Sect. 12.1]. A key property is that given a commutative diagram

$$\begin{tikzcd} D_* \arrow[r, "\Phi_1"] \arrow[d, "q"] & D_* \arrow[d, "q"] \\ D'_* \arrow[r, "\Phi_2"] & D'_* \end{tikzcd}$$

where Φ_1, Φ_2 and q are chain homotopy equivalences one has $\tau(\Phi_1) = \tau(\Phi_2) \in K_1(S)$.

Remark 1.4.3 It is possible to formulate Theorem 1.2.8 directly for self-chain homotopy equivalences of chain complexes of geometric modules of bounded dimension. Then the discussion of torsion can be avoided here. This is the point of view taken in [6, Theorem 5.3].

Step 4: $\tau(\Psi) = a$

Because X is contractible, the augmentation map $C_* \to \mathbb{Z}$ induces a homotopy equivalence

$$q\colon D_* = \mathbb{Z}[G]\otimes_{\mathbb{Z}} C_* \otimes_{\mathbb{Z}} R^n \to \mathbb{Z}[G]\otimes_{\mathbb{Z}}\mathbb{Z}\otimes_{\mathbb{Z}} R^n = \mathbb{Z}[G]\otimes_{\mathbb{Z}} R^n.$$

Moreover, $q \circ \Psi = \psi \circ q$. It follows that

$$a = [\psi] = \tau(\psi) = \tau(\Psi) = [\tilde{\tau}(\Psi)]$$

Step 5: control of $\tilde{\tau}(\Psi)$

In order to understand the support of $\tilde{\tau}(\Psi)$ we first need to understand the support of its building blocks. If $((h', x'), (h, x)) \in (G \times X)^2$ belongs to the support of ∂^{D_*}, then $h' = h$ and $d(x', x) \le \delta_0$. If $((h', x'), (h, x))$ belongs to the support of Ψ or of its homotopy inverse Ψ', then there is $g \in T$ such that $h' = hg^{-1}$ and $d(x', gx) \le \delta_0$. If $((h', x'), (h, x))$ belongs to the support of the chain homotopy $\mathscr{H}$ or $\mathscr{H}'$ then there are $g, g' \in T$ such that $h' = h(gg')^{-1}$ and $d(x', gg'x) \le \delta_0$. From the explicit formula for $\tilde{\tau}(\Psi)$ one can then read off that there is a number K, depending only on the dimension of D_* (which is in our case bounded by N), such that the support of $\tilde{\tau}(\Psi)$ satisfies the following condition: if $((h', x'), (h, x)) \in \operatorname{supp} \tilde{\tau}(\Psi)$ then there are $g_1, \ldots, g_K \in T$ such that

$$h' = h(g_1 \ldots g_K)^{-1} \qquad \text{and} \quad d(x', g_1 \ldots g_K x) \le K\delta_0.$$

Note that we specified K in this step; note also that K does only depend on N.

Remark 1.4.4 The actual value of K is of course not important. It is not very large; for example $K := 10N$ works—I think.

Step 6: Applying f

Using the map $f\colon X \to E$ we define the G-map $F\colon G\times X \to E$ by $F(h,x) := hf(x)$. Combining the estimates from the end of step 2 and the analysis of $\operatorname{supp}(\tilde{\tau}(\Psi))$ it is not hard to see that $\tilde{\tau}(\Psi)$ is an ε_N-automorphism over E (with respect to F).

This finishes the discussion of the surjectivity of $\alpha_{\mathcal{F}}\colon H_1^G(E_{\mathcal{F}}G;\mathbf{K}_R) \to K_1(R[G])$ under the assumptions of Theorem A. Surjectivity of this map under the assumptions of Theorem B follows from a very similar argument; mostly step 2 is slightly more complicated. For Theorem C the transfer can no longer be constructed using a chain complex associated to a space; instead Swan's Theorem 1.3.20 is used to construct a transfer. Otherwise the proof is again very similar.

L-Theory Transfer

The proof of the L-theory version of Theorems A and B follows the same outline. Now elements in L-theory are given by quadratic forms. The analog of chain homotopy self-equivalences in L-theory are ultra-quadratic Poincaré complexes [42]. These are chain complex versions of quadratic forms. The main difference is that to construct a transfer it is no longer sufficient to have just the chain complex C, in addition we need a symmetric structure on this chain complex. Moreover, this symmetric structure needs to be controlled (just as the boundary map ∂ is controlled). While there may be no such symmetric structure on C, there is a symmetric structure on the product of C with its dual $D := C\otimes C^{-*}$. This symmetric structure is given (up to signs) by $\langle a\otimes\alpha, b\otimes\beta\rangle = \alpha(b)\beta(a)$ and turns out to be suitably controlled. This is the only significant change from the proof in K-theory to the proof in L-theory.

Transfer for Higher K-Theory

We end this section with a very informal discussion of one aspect of the proof of Theorem A for higher K-theory. Again, we focus on surjectivity. In this case we use Theorem 1.2.18 in place of Theorem 1.2.8. Thus we need to produce an element in $K_*(\mathcal{C})$. Recall that objects of $\mathcal{C}$ are sequences $(M_\nu)_{\nu\in\mathbb{N}}$ of geometric $R[G]$-modules and that morphisms are sequences of $R[G]$-linear maps that become more controlled with $\nu \to \infty$. The general idea is to apply the transfer from Step 3 to each ν to produce a functor from $R[G]$-modules to $\mathcal{C}$. The problem is, however, that the construction from Step 3 is not functorial. The reason for this in turn is that the group G only acts up to homotopy on the chain complex C_*. The remedy for this failure is to use the singular chain complex of $C^s_*ing(X)$ in place of C_*. It is no longer finite, but it is homotopy finite, which is finite enough. For the control consideration from Step 5 it was important, that the boundary map of C_* is δ_0-controlled. This is no longer true for $C^s ing_*(X)$. One might be tempted to use the subcomplex $C^{sing,\delta_0}(X)$ spanned by singular simplices in X of diameter $\leq \delta_0$. However, the action of G on X is not isometric and therefore there is no G-action on $C^{sing,\delta_0}(X)$. Finally, the solution is to use $C^s_*ing(X)$ together with its filtration by the subcomplexes $(C^{sing,\delta}_*(X))_{\delta>0}$.

Using this idea it is possible to construct a transfer functor from the category of $R[G]$-modules to a category $\widetilde{ch}_{hfd}\mathscr{C}$. The latter is a formal enlargement of the Waldhausen category $ch_{hfd}\mathscr{C}$ of homotopy finitely dominated chain complexes over the category $\mathscr{C}$ [12, Appendix]. Both the higher K-theory of $ch_{hfd}\mathscr{C}$ and of $\widetilde{ch}_{hfd}\mathscr{C}$ coincide with the higher K-theory of $\mathscr{C}$. Similar constructions are used in [9, 45].

1.5 Flow Spaces

Convention 1.5.1 *A* CAT(0)-group *is a group that admits a cocompact, proper and isometric action on a finite dimensional* CAT(0)*-space.*

The goal of this section is to outline the proof of the fact [7] that CAT(0)-groups satisfy the assumptions of Theorem B. Note that CAT(0)-groups are finitely presentable [16, Theorem III.Γ.1.1(1), p. 439].

Proposition 1.5.1 *Let G be a* CAT(0)*-group. Exhibit G as a finitely presented group $\langle S \mid R\rangle$. Then there is $N \in \mathbb{N}$ such that for any $\varepsilon > 0$ there are*

(a) an N-transfer space X equipped with a homotopy G-action (φ, H),
(b) a simplicial (G, VCyc)-complex E of dimension at most N,
(c) a map $f\colon X \to E$ that is G-equivariant up to ε: for all $x \in X$, $s \in S \cup S^{-1}$, $r \in R$

  - $d^1(f(\varphi_s(x)), s \cdot f(x)) \le \varepsilon$,
  - $\{H_r(t, x) \mid t \in [0, 1]\}$ *has diameter at most* ε.

An $(\boldsymbol{\alpha}, \boldsymbol{\varepsilon})$*-Version of the Assumptions of Theorem* B

Let G be a group.

Definition 1.5.2 An N-flow space FS for G is a metric space with a continuous flow $\phi\colon FS \times \mathbb{R} \to FS$ and an isometric proper action of G such that

(a) the flow is G-equivariant: $\phi_t(gx) = g\phi_t(x)$ for all $x \in X$, $t \in \mathbb{R}$ and $g \in G$;
(b) $FS \setminus \{x \mid \phi_t(x) = x \text{ for all } t \in \mathbb{R}\}$ is locally connected and has covering dimension at most N.

Notation 1.5.1 *Let $\alpha, \varepsilon \ge 0$. For $x, y \in FS$ we write*

$$d_{FS}^{fol}(x, y) \le (\alpha, \varepsilon)$$

if there is $t \in [-\alpha, \alpha]$ such that $d(\phi_t(x), y) \le \varepsilon$.

Of course, ε will usually be a small number while α will often be much larger.

Proposition 1.5.3 *Let G be a* CAT(0)*-group. Exhibit G as a finitely presented group $\langle S \mid R\rangle$. Then there exists $N \in \mathbb{N}$ and a cocompact N-flow space for G and $\alpha > 0$ such that for all $\varepsilon > 0$ there are*

(a) an N-transfer space X equipped with a homotopy G-action (φ, H),
(b) a map $f : X \to FS$ that is G-equivariant up to (α, ε): for all $x \in FS$, $s \in S \cup S^{-1}$, $r \in R$, $t \in [0, 1]$
   - $d^{fol}_{FS}(f(\varphi_s(x)), s \cdot f(x)) \leq (\alpha, \varepsilon)$,
   - $d^{fol}_{FS}(f(H_r(t, x)), f(x)) \leq (\alpha, \varepsilon)$.

The proof of Proposition 1.5.3 will be discussed in a later subsection. The key ingredient that allows to deduce Proposition 1.5.1 from Proposition 1.5.3 are the long and thin covers for flow spaces from [8], that in turn generalize the long and thin cell structures of Farrell–Jones [23, Sect. 7].

Definition 1.5.4 Let $R > 0$. A collection $\mathcal{U}$ of open subsets of FS is said to be an *R-long cover* of $A \subseteq FS$ if for all $x \in A$ there is $U \in \mathcal{U}$ such that

$$\phi_{[-R,R]}(x) := \{\phi_t(x) \mid t \in [-R, R]\} \subseteq U.$$

Notation 1.5.2 (Periodic orbits) *Let $\gamma > 0$. Write $FS_{\leq\gamma}$ for the subset of FS consisting of all points x for which there are $0 < \tau \leq \gamma$ and $g \in G$ such that $\phi_\tau(x) = gx$.*

Theorem 1.5.5 (Existence of long thin covers) *Let FS be a cocompact N-flow space for G. Then there is $\tilde{N}$ such that for all $R > 0$ there exists $\gamma > 0$ and a collection $\mathcal{U}$ of open subsets of FS such that*

(a) $\dim \mathcal{U} \leq \tilde{N}$*: any point of FS is contained in at most $\tilde{N} + 1$ members of $\mathcal{U}$,*
(b) $\mathcal{U}$ is an R-long cover of $FS \setminus FS_{\leq\gamma}$,
(c) $\mathcal{U}$ is G-invariant: for $g \in G$, $U \in \mathcal{U}$ we have $g(U) \in \mathcal{U}$,
(d) $\mathcal{U}$ has finite isotropy: for all $U \in \mathcal{U}$ the group $G_U := \{g \in G \mid g(U) = U\}$ is finite.

Example 1.5.6 Let $G := \mathbb{Z}$. Consider $FS := \mathbb{R}$ with the usual $\mathbb{Z}$-action and the flow defined by $\phi_t(x) := x + t$. If $\mathcal{U}_R$ is an R-long $\mathbb{Z}$-invariant cover of $\mathbb{R}$ of finite isotropy then the dimension of $\mathcal{U}_R$ grows linearly with R.

Theorem 1.5.5 states that this is the only obstruction to the existence of uniformly finite dimensional arbitrary long G-invariant covers of FS of finite isotropy.

Remark 1.5.7 Theorem 1.5.5 is more or less [8, Theorem 1.4], see also [7, Theorem 5.6]. The proof depends only on fairly elementary constructions, but is nevertheless very long. (It would be nice to simplify this proof—but I do not know where to begin.)

In these references in addition an upper bound for the order of finite subgroups of G is assumed. This assumption is removed in recent (and as of yet unpublished) work of Adam Mole and Henrik Rüping.

Remark 1.5.8 For the flow spaces, that have been relevant for the Farrell–Jones conjecture so far, it is possible to extend the cover $\mathcal{U}$ from $FS \setminus FS_{\leq\gamma}$ to all of FS. The only price one has to pay for this extension is that in assertion (d) one has to allow virtually cyclic groups instead of only finite groups. Note that with this change Example 1.5.6 is no longer problematic; we can simply set $\mathcal{U}_R := \{\mathbb{R}\}$.

It is really at this point where the family of virtually cyclic subgroups plays a special role and appears in proofs of the Farrell–Jones Conjecture.

Remark 1.5.9 In the case of CAT(0) groups the extension of the cover from $FS \setminus FS_\gamma$ to FS is really the most technical part of the arguments in [7].

It would be more satisfying to have a result that provides this extension (after allowing virtually cyclic groups) for general cocompact flow spaces.

Remark 1.5.10 One may think of Theorem 1.5.5 as a (as of now quite difficult!) parametrized version of the very easy fact that $\mathbb{Z}$ has finite asymptotic dimension.

***Sketch of Proof for Proposition* 1.5.1 *using Proposition* 1.5.3**

The idea is easy. We produce a map $F\colon FS \to E$ that is suitably contracting along the flow lines of ϕ. Then we can compose $f\colon X \to FS$ from Proposition 1.5.3 with F to produce the required map $F \circ f\colon X \to E$.

Let G be a CAT(0)-group. Let $\varepsilon > 0$ be given. Let FS be the cocompact N-flow space for G from Proposition 1.5.3. As discussed in Remark 1.5.8 there is $\tilde{N}$ such that for all $R > 0$ there exists a collection $\mathcal{U}$ of open subsets of FS such that

(a) $\dim \mathcal{U} \leq \tilde{N}$,
(b) $\mathcal{U}$ is an R-long cover of FS,
(c) $\mathcal{U}$ is G-invariant,
(d) $\mathcal{U}$ has virtually cyclic isotropy: for all $U \in \mathcal{U}$ the group $G_U := \{g \in G \mid g(U) = U\}$ is virtually cyclic.

Let now $E := |\mathcal{U}|$ be the nerve of the cover $\mathcal{U}$. The vertex set of this simplicial complex is $\mathcal{U}$ and we have $|\mathcal{U}| = \{\sum_{U \in \mathcal{U}} t_U U \mid t_U \in [0,1], \sum_{U \in \mathcal{U}} t_U = 1 \text{ and } \bigcap_{t_U \neq 0} U \neq \emptyset\}$. Note that $|\mathcal{U}|$ is a simplicial (G, VCyc)-complex. To construct the desired map $F\colon FS \to E$ we first change the metric on FS. For (large) $\Lambda > 0$ we can define a metric that blows up the metric transversal to the flow ϕ, and corresponds to time along flow lines. More precisely,

$$d_\Lambda(x, y) := \inf \Big\{ \sum_{i=1}^{n} \alpha_i + \Lambda\varepsilon_i \;|\exists x = x_0, x_1, \ldots, x_n \text{ such that } d_{FS}^{fol}(x_{i-1}, x_i) \leq (\alpha_i, \varepsilon_i) \text{ for } i = 1, \ldots, n \Big\}$$

For $U \in \mathcal{U}$, $x \in FS$ let $a_U(x) := d_\Lambda(x, FS \setminus U)$ and define $F\colon FS \to |\mathcal{U}|$ by

$$F(x) := \sum_{U \in \mathcal{U}} \frac{a_U(x)}{\sum_{V \in \mathcal{U}} a_V(x)} U.$$

As $\mathcal{U}$ is G-invariant, F is G-equivariant. If $R > 0$ is sufficiently large (depending only on ε), then there are $\Lambda > 0$ and $\delta > 0$ (depending on everything at this point) such that

$$d_{FS}^{fol}(x, x') \leq (\alpha, \delta) \implies d^1(F(x), F(x')) \leq \varepsilon.$$

(More details for similar calculations can be found in [9, Sect. 4.3, Proposition 5.3].) Thus we can compose with F and conclude that Proposition 1.5.3 implies Proposition 1.5.1.

The Flow Space for a CAT(0)*-Space*

This subsection contains an introduction to the flow space for CAT(0)-groups from [7]. Let Z be a CAT(0)-space.

Definition 1.5.11 A *generalized geodesic* in Z is a continuous map $c\colon \mathbb{R} \to Z$ for which there exists an interval (c_-, c_+) such that $c|_{(c_-,c_+)}$ is a geodesic and $c|_{(-\infty,c_-)}$ and $c|_{(c_+,+\infty)}$ are constant. (Here $c_- = -\infty$ and/or $c_+ = +\infty$ are allowed.)

Definition 1.5.12 The *flow space for* Z is the space $FS(Z)$ of all generalized geodesics $c\colon \mathbb{R} \to Z$. It is equipped with the metric

$$d_{FS}(c, c') := \int_{\mathbb{R}} \frac{d(c(t), c'(t))}{2e^{|t|}} dt$$

and the flow

$$\phi_\tau(c)(t) := c(t + \tau).$$

Remark 1.5.13 The fixed point space for the flow $FS(Z)^{\mathbb{R}} := \{c \mid \phi_t(c) = c$ for all $t\}$ is via $c \mapsto c(0)$ canonically isometric to Z.

The flow space $FS(Z)$ is somewhat singular around $Z = FS(Z)^{\mathbb{R}}$. For example there are well defined maps $c \mapsto c(\pm\infty)$ from $FS(Z)$ to the bordification [16, Chap. II.8] $\bar{Z}$ of Z, but these maps fail to be continuous at Z.

Remark 1.5.14 The metric $d_{FS}(c, c')$ cares most about $d(c(t), c'(t))$ for t close to 0. For example if $c(0) = c'(0)$ then $d_{FS}(c, c')$ is bounded by $\int_0^\infty \frac{t}{e^t} dt$. For this reason one can think of $c(0)$ as *marking* the generalized geodesic c. If $c(0)$ is different from both $c(c_-)$ and $c(c_+)$ (equivalently if $c_- < 0 < c_+$) then the triple $(c(c_-), c(0), c(c_+))$ uniquely determines c.

Remark 1.5.15 An isometric action of G on Z induces an isometric action on $FS(Z)$ via $(g \cdot c)(t) := g \cdot c(t)$. If the action of G on Z is in addition cocompact, proper and Z has dimension at most N, then $FS(Z)$ is a cocompact $3N + 2$-flow space for G in the sense of Definition 1.5.2, see [7, Sect. 2].

For cocompactness it is important that we allowed $c_- = -\infty$ and $c_+ = +\infty$ in the definition of generalized geodesics.

Remark 1.5.16 For hyperbolic groups there is a similar flow space constructed by Mineyev [36]. This space is an essential ingredient for the proof that hyperbolic groups satisfy the assumptions of Theorem A. Mineyev's construction motivated the flow space for CAT(0) groups.

However, for hyperbolic groups the construction is really much more difficult. A priori, there is really no local geometry associated to a hyperbolic group, hyperbolicity is just a condition on the large scale geometry and Mineyev extracts local information from this in the construction of his flow space. In contrast, for a CAT(0)-group the corresponding CAT(0)-space provides local and global geometry right from the definition.

***Sketch of Proof for Proposition* 1.5.3**

Let Z be a finite dimensional CAT(0)-space with an isometric, cocompact, proper action of the group G. Let $G = \langle S \mid R' \rangle$ be a finite presentation of G. Pick a base point $x_0 \in Z$. For $R > 0$ let $B_R(x_0)$ be the closed ball in Z of radius R around x_0. This will be our transfer space. Let $\rho_R \colon Z \to B_R(x_0)$ be the closest point projection. For $x, x' \in Z$, $t \in [0,1]$ we write $t \mapsto (1-t) \cdot x + t \cdot x'$ for the straight line from x to x' parametrized by constant speed $d(x, x')$. For $g, h \in G$, $t \in [0,1]$, $x \in B_R(x_0)$ let

$$\begin{aligned} \varphi_g^R(x) &:= \rho_R(g \cdot x), \\ H_{g,h}^R(t,x) &:= \rho_R((1-t) \cdot g\varphi_h^R(x) + t \cdot ghx). \end{aligned}$$

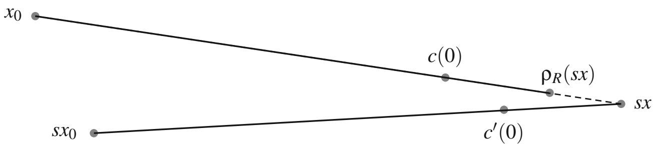
Then $H_{g,h}^R$ is a homotopy $\varphi_g^R \circ \varphi_h^R \to \varphi_{gh}^R$. This data also specifies a homotopy action (φ^R, H^R) on $B_R(x_0)$. We will use the map $\iota_R \colon B_R(x_0) \to FS(Z)$ where $\iota_R(x)$ is the unique generalized geodesic c in Z with $c_- = 0$, $c_+ = d(x, x_0)$, $c(c_-) = c(0) = x_0$ and $c(c_+) = x$, i.e., the generalized geodesic from x_0 to x that starts at time 0 at x_0. For $T \geq 0$ let $f^{T,R} := \phi_T \circ \iota_R \colon B_R(x_0) \to FS(x_0)$. Proposition 1.5.3 follows from the next Lemma; this will conclude the sketch of proof for Proposition 1.5.3.

Lemma 1.5.17 *Let* $\alpha := \max_{s \in S} d(x_0, sx_0)$. *For any* $\varepsilon > 0$ *there are* $T, R > 0$ *such that for all* $x \in FS$, $s \in S \cup S^{-1}$, $r \in R'$, $t \in [0,1]$ *we have*

- $d_{FS}^{fol}(f^{T,R}(\varphi_s^R(x)), s \cdot f^{T,R}(x)) \leq (\alpha, \varepsilon)$,
- $d_{FS}^{fol}(f^{T,R}(H_r^R(t,x)), f^{T,R}(x)) \leq (\alpha, \varepsilon)$.

Proof (Sketch of proof) We will only discuss the first inequality; the second inequality involves essentially no additional difficulties.

Let us first visualize the generalized geodesics $c := f^{T,R}(\varphi_s^R(x))$ and $c' := s \cdot f^{T,R}(x)$. The generalized geodesic c starts at $c(c_-) = x_0$ and ends at $c(c_+) = \varphi_s^R(x)$. If $d(x_0, sx) \leq R$, then the endpoint $\varphi_s^R(x)$ coincides with sx; otherwise we can prolong c (as a geodesic) until it hits sx. If $T \leq d(x_0, \varphi_s^R(x)$ then $c(0)$ is the unique point on the image of c of distance T from x_0, otherwise $c(0) = c(c_+) = \varphi_s^R(x) = \rho_R(sx)$. The generalized geodesic c' starts at $c'(c'_-) = sx_0$ and ends at $c'(c'_-) = sx$. If $T \leq d(sx_0, sx)$, then $c'(0)$ is the unique point on the image of c' of distance T from sx_0, otherwise $c'(0) = c'(c'_+) = sx$. We draw this as



There are two basic cases to consider.

Case I: $d(sx, x_0)$ is small.
Then $\rho_R(sx) = sx$, and both c and c' converge to the constant geodesic at sx with $T \to \infty$. Consequently $d_{FS}(c, c')$ is small for large T.

Case II: $d(sx, x_0)$ is large.
Then we may have $\rho_R(sx) \neq sx$. Note that $d(\rho_R(sx), sx) \leq d(x_0, sx_0) \leq \alpha$. Let $t := d(c(0), sx) - d(c'(0), sx) \in [-\alpha, \alpha]$. Using the CAT(0)-condition one can then check that $d_{FS}(\phi_t(c), c')$ will be small provided that T, $R - T$, $\frac{R}{R-T}$ are large.

A more careful analysis of the two cases shows that it is possible to pick R and T (depending only on ε) such that for any x one of the two cases applies and therefore $d^{fol}_{FS}(c, c') \leq (\alpha, \varepsilon)$.

Remark 1.5.18 The assumption that the action of G on the CAT(0)-space Z is cocompact is important for the proof of Proposition 1.5.1, because it implies that the action of G on the flow space $FS(Z)$ is also cocompact. This in turn is important for the construction of R-long covers: Theorem 1.5.5 otherwise only allows the construction of R-long covers for a cocompact subspace of the flow space.

Nevertheless, there are situations where it is possible to construct R-long covers for flow spaces that are not cocompact. For example $\mathrm{GL_n}(\mathbb{Z})$ acts properly and isometrically but not cocompactly on a CAT(0) space. But it is possible to construct R-long covers for the corresponding flow space [11]. This uses as an additional input a construction of Grayson [29] and enforces a larger family of isotropy groups for the cover. This is the family $\mathcal{F}_{n-1}$ mentioned in Remark 1.1.9.

There are very general results of Farrell–Jones [26] without a cocompactness assumption, but I have no good understanding of these methods.

1.6 $\mathbb{Z}^n \rtimes \mathbb{Z}$ as a Farrell–Hsiang Group

For $A \in \mathrm{GL_n}(\mathbb{Z})$ let $\mathbb{Z}^n \rtimes_A \mathbb{Z}$ be the corresponding semi-direct product. We fix a generator $t \in \mathbb{Z}$. Then for $v \in \mathbb{Z}^n$ we have $t \cdot vt^{-1} = Av$ in $\mathbb{Z}^n \rtimes_A \mathbb{Z}$. The goal of this section is to outline a proof of the following fact from [3]. Recall that Ab denotes the family of abelian subgroups. In the case of $\mathbb{Z}^n \rtimes_A \mathbb{Z}$ these are all finitely generated free abelian.

Proposition 1.6.1 *Suppose that no eigenvalue of A over $\mathbb{C}$ is a root of unity. Then the group $\mathbb{Z}^n \rtimes_A \mathbb{Z}$ is a Farrell–Hsiang group with respect to the family* Ab *of abelian groups, i.e., there is N such that for any $\varepsilon > 0$ there are*

(a) a group homomorphism $\pi : \mathbb{Z}^n \rtimes_A \mathbb{Z} \to F$ *where* F *is finite,*
(b) a simplicial $(\mathbb{Z}^n \rtimes_A \mathbb{Z}, \mathrm{Ab})$*-complex* E *of dimension at most* N
(c) a map $f\colon \coprod_{H \in \mathscr{H}(F)} \mathbb{Z}^n \rtimes_A \mathbb{Z}/\pi^{-1}(H) \to E$ *that is* $\mathbb{Z}^n \rtimes_A \mathbb{Z}$*-equivariant up to* ε*:* $d^1(f(sx), s \cdot f(x)) \leq \varepsilon$ *for all* $s \in S$*,* $x \in \coprod_{H \in \mathscr{H}(F)} G/\pi^{-1}(H)$.

Here S *is any fixed generating set for* G*.*

Remark 1.6.2 The Farrell–Jones Conjecture holds for abelian groups. Thus using Theorem C and the transitivity principle 1.1.7 we deduce from Proposition 1.6.1 that the Farrell–Jones Conjecture holds for the group $\mathbb{Z}^n \rtimes_A \mathbb{Z}$ from Proposition 1.6.1.

Finite Quotients of $\mathbb{Z}^n \rtimes_A \mathbb{Z}$*.*

We write $\mathbb{Z}/s$ for the quotient ring (and underlying cyclic group) $\mathbb{Z}/s\mathbb{Z}$. Let A_s denote the image of A in $\mathrm{GL_n}(\mathbb{Z}/\mathrm{s})$. Choose $r, s \in \mathbb{N}$ such that the order $|A_s|$ of A_s in $\mathrm{GL_n}(\mathbb{Z}/\mathrm{s})$ divides r. Then we can form $(\mathbb{Z}/s)^n \rtimes_{A_s} \mathbb{Z}/r$ and there is canonical surjective group homomorphism

$$\pi : \mathbb{Z}^n \rtimes_A \mathbb{Z} \twoheadrightarrow (\mathbb{Z}/s)^n \rtimes_{A_s} \mathbb{Z}/r.$$

Hyper-Elementary Subgroups of $(\mathbb{Z}/s)^n \rtimes_{A_s} \mathbb{Z}/r$.

Lemma 1.6.3 *Let* $s = p_1 \cdot p_2$ *be the product of two primes. Let* $r := s \cdot |A_s|$*. If* H *is a hyper-elementary subgroup of* $(\mathbb{Z}/s)^n \rtimes_{A_s} \mathbb{Z}/r$ *then there is* $q \in \{p_1, p_2\}$ *such that*

(a) $\pi^{-1}(H) \cap \mathbb{Z}^n \subseteq (q\mathbb{Z})^n$ *or*
(b) the image of $\pi^{-1}(H)$ *under* $\mathbb{Z}^n \rtimes_{A_s} \mathbb{Z} \twoheadrightarrow \mathbb{Z}$ *is contained in* $q\mathbb{Z}$*.*

To prove Lemma 1.6.3 we recall [3, Lemma 3.20].

Lemma 1.6.4 *Let* s *be any natural number. Let* $r := s \cdot |A_s|$*. Let* C *be a cyclic subgroup of* $\mathbb{Z}/s^n \rtimes_{A_s} \mathbb{Z}/r$ *that has nontrivial intersection with* $(\mathbb{Z}/s)^n$*.*

Then there is a prime power q^N *(*$N \geq 1$*) such that*

- q^N *divides* $r = r's$*,*
- q^N *does not divide the order of the image of* C *in* $\mathbb{Z}/r$*,*
- q *divides the order of* $C \cap (\mathbb{Z}/s)^n$*.*

Proof (Proof of Lemma 1.6.3) Let $H \subseteq (\mathbb{Z}/s)^n \rtimes_{A_s} \mathbb{Z}/r$ be hyper-elementary. There is a short exact sequence $C \rightarrowtail H \twoheadrightarrow P$ with P a p-group and C a cyclic group. The cyclic group C can always be arranged to be of order prime to p.

$$\begin{tikzcd}
(\mathbb{Z}/s)^n \arrow[r, tail] & (\mathbb{Z}/s)^n \rtimes_{A_s} \mathbb{Z}/r \arrow[r, two heads, "\mathrm{pr}"] & \mathbb{Z}/r \\
H \cap (\mathbb{Z}/s)^n \arrow[r, tail] \arrow[u, tail] & H \arrow[r, two heads, "\mathrm{pr}"] \arrow[u, tail] & \mathrm{pr}(H) \arrow[u, tail] \\
C \cap (\mathbb{Z}/s)^n \arrow[r, tail] \arrow[u, tail] & C \arrow[r, two heads, "\mathrm{pr}"] \arrow[u, tail] & \mathrm{pr}(C) \arrow[u, tail]
\end{tikzcd}$$

There are two cases.

Case I: $C \cap (\mathbb{Z}/s)^n$ is trivial.
Then $H \cap (\mathbb{Z}/s)^n$ is a p-group. Let q be the prime from $\{p_1, p_2\}$ that is different from p. Then (a) will hold.
Case II: $C \cap (\mathbb{Z}/s)^n$ is nontrivial.
Then there is a prime q as in Lemma 1.6.4. As q divides $|C \cap (\mathbb{Z}/s)^n|$ we have $q \in \{p_1, p_2\}$ and $q \neq p$. It follows that q divides $[\mathbb{Z}/r : \mathrm{pr}(H)]$. This implies (b).

Contracting Maps

Example 1.6.5 Consider the standard action of $\mathbb{Z}^n$ on $\mathbb{R}^n$. Let $\bar{H} := (l\mathbb{Z})^n \subseteq \mathbb{Z}^n$ and $\varphi\colon \bar{H} \to \mathbb{Z}^n$ be the isomorphism $v \mapsto \frac{v}{l}$. Let $\mathrm{res}_\varphi\, \mathbb{R}^n$ be the $\bar{H}$-space obtained by restricting the action of $\mathbb{Z}^n$ on $\mathbb{R}^n$ with φ_l. Then $x \mapsto \frac{x}{l}$ defines an $\bar{H}$-map $F\colon \mathbb{Z}^n \to \mathrm{res}_\varphi\, \mathbb{R}^n$. This map is contracting. In fact by increasing l we can make F as contracting as we like, while we can keep the metric on $\mathbb{R}^n$ fixed.

A variant of this simple construction will be used to produce maps as in (c) of Proposition 1.6.1. This will finish the discussion of the proof of Proposition 1.6.1.

Proposition 1.6.6 *Let $S \subseteq \mathbb{Z}^n \rtimes_A \mathbb{Z}$ be finite. For any $\varepsilon > 0$ there is l_0 such that for all $l \geq l_0$ the following holds.*

Let $\bar{H} := \mathbb{Z}^n \rtimes_A (l\mathbb{Z}) \subseteq \mathbb{Z}^n \rtimes_A \mathbb{Z}$. Then there is a simplicial $(\mathbb{Z}^n \rtimes_A \mathbb{Z}, \mathrm{Ab})$-complex E of dimension 1 and an $\bar{H}$-equivariant map

$$F\colon \mathbb{Z}^n \rtimes_A \mathbb{Z} \to E$$

such that $d^1(F(g), F(h)) \leq \varepsilon$ whenever $g^{-1}h \in S$.

Proof We apply the construction of Example 1.6.5 to the quotient $\mathbb{Z}$ of $\mathbb{Z}^n \rtimes_A \mathbb{Z}$.

Let $E := \mathbb{R}$. We use the standard way of making $E = \mathbb{R}$ a simplicial complex in which $\mathbb{Z} \subseteq \mathbb{R}$ is the set of vertices. Let $\bar{H}$ act on E via $(vt^k) \cdot \xi := \frac{k}{l}\xi$; this is a simplicial action. Finally define $F\colon \mathbb{Z}^n \rtimes_A \mathbb{Z} \to E$ by $F(vt^k) := \frac{k}{l}$. It is very easy to check that F has the required properties for sufficiently large l.

Proposition 1.6.7 *Let $S \subseteq \mathbb{Z}^n \rtimes_A \mathbb{Z}$ be finite. There is $N \in \mathbb{N}$ depending only on n such that for any $\varepsilon > 0$ there is l_0 such that for all $l \geq l_0$ the following holds.*

Let $\bar{H} := (l\mathbb{Z})^n \rtimes_A \mathbb{Z} \subseteq \mathbb{Z}^n \rtimes_A \mathbb{Z}$. Then there is a simplicial $(\mathbb{Z}^n \rtimes_A \mathbb{Z}, \mathrm{Cyc})$-complex E of dimension at most N and an $\bar{H}$-equivariant map

$$F\colon \mathbb{Z}^n \rtimes_A \mathbb{Z} \to E$$

such that $d^1(F(g), F(h)) \leq \varepsilon$ whenever $g^{-1}h \in S$.

Proof (Sketch of proof) As in the proof of Proposition 1.6.6 we start with the construction from Example 1.6.5, now applied to the subgroup $\mathbb{Z}^n \subseteq \mathbb{Z}^n \rtimes_A \mathbb{Z}$. However, unlike the quotient $\mathbb{Z}$, there is no homomorphism from $\mathbb{Z}^n \rtimes_A \mathbb{Z}$ to the subgroup and it will be more difficult to finish the proof.

Let $\mathbb{Z}^n \rtimes_A \mathbb{Z}$ act on $\mathbb{R}^n \times \mathbb{R}$ via $vt^k \cdot (x, \xi) := (v + A^k(x), k + \xi)$. Let $\varphi\colon \bar{H} \to \mathbb{Z}^n \rtimes_A \mathbb{Z}$ be the isomorphism $vt^k \mapsto \frac{v}{l}t^k$. The map $F_0\colon \mathbb{Z}^n \rtimes_A \mathbb{Z} \to \operatorname{res}_\varphi \mathbb{R}^n \times \mathbb{R}$, $(vt^k) \mapsto (v/l, k)$ is $\bar{H}$-equivariant and contracting in the $\mathbb{Z}^n$-direction, but not in the $\mathbb{Z}$-direction. In order to produce a map that is also contracting in the $\mathbb{Z}$-direction we use the flow methods from Sect. 1.5.

There is $\mathbb{Z}^n \rtimes_A \mathbb{Z}$-equivariant flow on $\mathbb{R}^n \times \mathbb{R}$ defined by $\phi_\tau(x, \xi) = (x, \tau + \xi)$. It is possible to produce a simplicial $(\mathbb{Z}^n \rtimes_A \mathbb{Z}, \mathrm{Cyc})$-complex E of uniformly bounded dimension (depending only on n) and $\mathbb{Z}^n \rtimes_A \mathbb{Z}$-equivariant map $F_1\colon \mathbb{R}^n \times \mathbb{R} \to E$ that is contracting in the flow direction (but expanding in the transversal $\mathbb{R}^n$-direction). To do so one uses Theorem 1.5.5; E will be the nerve of a suitable long cover of $\mathbb{R}^n \times \mathbb{R}$.

The fact that F_1 is expanding in the $\mathbb{R}^n$-direction can be countered by the contracting property of F_0. All together, the composition $F_1 \circ F_0\colon \mathbb{Z}^n \rtimes_A \mathbb{Z} \to \operatorname{res}_\varphi E$ has the desired properties.

Remark 1.6.8 As many other things, the idea of using a flow space in the proof of Proposition 1.6.7 originated in the work of Farrell and Jones [24]. I found this trick very surprising when I first learned about it.

Lemma 1.6.9 *Let $\bar{H}$ be a subgroup of $\mathbb{Z}^n \rtimes_A \mathbb{Z}$ and $l, k \in \mathbb{N}$ such that*

(a) $\bar{H} \cap \mathbb{Z}^n \subseteq l\mathbb{Z}$,
(b) $\bar{H}$ maps to $k\mathbb{Z}$ under the projection $\mathbb{Z}^n \rtimes_A \mathbb{Z} \to \mathbb{Z}$,
(c) the index $[\mathbb{Z}^n : (\mathrm{id} - \mathrm{A}^\mathrm{k})\mathbb{Z}^\mathrm{n}]$ is finite and $l \equiv 1$ modulo $[\mathbb{Z}^n : (\mathrm{id} - \mathrm{A}^\mathrm{k})\mathbb{Z}^\mathrm{n}]$.

Then $\bar{H}$ is subconjugated to $(l\mathbb{Z})^n \rtimes_A \mathbb{Z}$.

Proof Consider the image $\bar{H}_l$ of $\bar{H}$ under $\mathbb{Z}^n \rtimes_A \mathbb{Z} \to (\mathbb{Z}/l)^n \rtimes_A \mathbb{Z}$. Then (a) implies that the restriction of the projection $(\mathbb{Z}/l)^n \rtimes_A \mathbb{Z} \to \mathbb{Z}$ to $\bar{H}_l$ is injective. In particular $\bar{H}_l$ is cyclic. By (b) there is $v \in \mathbb{Z}^n$ such that $vt^k \in \mathbb{Z}^n \rtimes_A \mathbb{Z}$ maps to a generator of $\bar{H}_l$. Assumption (c) implies that there is $w \in \mathbb{Z}^n$ such that $v \equiv (\mathrm{id} - \mathrm{A}^\mathrm{k})\mathrm{w}$ modulo $(l\mathbb{Z})^n$. A calculation shows that w conjugates $\bar{H}$ to a subgroup of $(l\mathbb{Z})^n \rtimes_A \mathbb{Z}$.

Proof (Proof of Proposition 1.6.1) Let L be a large number. Since A has no roots of unity as eigenvalues, the index $i_k := [\mathbb{Z}^n : (\mathrm{id} - \mathrm{A}^\mathrm{k})\mathbb{Z}^\mathrm{n}]$ is finite for all k. Let $K := i_1 \cdot i_2 \cdot \dots \cdot i_L$. By a theorem of Dirichlet there are infinitely many primes congruent to 1 modulo K. Let $s = p_1 \cdot p_2$ be the product of two such primes, both $\geq L$, and set $r := s \cdot |A_s|$.

We use the group homomorphism $\pi\colon \mathbb{Z}^n \rtimes_A \mathbb{Z} \twoheadrightarrow (\mathbb{Z}/s)^n \rtimes_{A_s} \mathbb{Z}/r$. Because of Lemma 1.6.3 we find for any hyper-elementary subgroup H of $(\mathbb{Z}/s)^n \rtimes_{A_s} \mathbb{Z}/r$ an $q \in \{p_1, p_2\}$ such that $\pi^{-1}(H) \subseteq \mathbb{Z}^n \rtimes_A (q\mathbb{Z})$ or $\pi^{-1}(H) \cap \mathbb{Z}^n \subseteq (q\mathbb{Z})^n$. In the first case we set $l := q$. In the second case we have either $\pi^{-1}(H) \subseteq \mathbb{Z}^n \rtimes_A (l\mathbb{Z})$ for some $l > L$ or we can apply Lemma 1.6.9 to deduce that (up to conjugation) $\pi^{-1}(H) \subseteq (q\mathbb{Z})^n \rtimes_A \mathbb{Z}$ and we again set $l := q$.

Therefore it suffices to find simplicial $(\mathbb{Z}^n \rtimes_A \mathbb{Z}, \mathrm{Ab})$-complexes E_1, E_2 whose dimension is bounded by a number depending only on n (and not on l) and maps

$$f_1\colon\ \mathbb{Z}^n \rtimes_A \mathbb{Z}/(l\mathbb{Z})^n \rtimes_A \mathbb{Z} \to E_1$$
$$f_2\colon\ \mathbb{Z}^n \rtimes_A \mathbb{Z}/\mathbb{Z}^n \rtimes_A (l\mathbb{Z}) \to E_2$$

that are G-equivariant up to ε. If $f\colon \mathbb{Z}^n \rtimes_A \mathbb{Z} \to E$ is the map from Proposition 1.6.7, then we can set $E_1 := (\mathbb{Z}^n \rtimes_A \mathbb{Z}) \times_{(l\mathbb{Z}^n)\rtimes_A \mathbb{Z}} E$ and define f_1 by $f(vt^k) := \big((vt^k), f((vt^k)^{-1})\big)$. Similarly, we can produce f_2 using Proposition 1.6.6.

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Chapter 2
The *K* and *L* Theoretic Farrell-Jones Isomorphism Conjecture for Braid Groups

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Abstract We prove the K and L theoretic versions of the Fibered Isomorphism Conjecture of F.T. Farrell and L.E. Jones for braid groups on a surface.

Keywords Braid groups · K and L-theory

2.1 Introduction

Aravinda, Farrell, and Roushon in [1] showed that the Whitehead group of the classical pure braid groups vanishes. Later on, in [10], Farrell and Roushon extended this result to the full braid groups. Computations of the Whitehead group of braid groups for the sphere and the projective space were performed by D. Juan-Pineda and S. Millán in [12–14]. In all cases the key ingredient was to prove the Farrell-Jones isomorphism conjecture for the *Pseudoisotopy* functor, this allows computations for lower algebraic K theory groups. In this note, we use results by Bartels, Lück and Reich [6] for word hyperbolic groups and Bartels, Lück and Wegner [5, 18] for CAT(0) groups to prove that the algebraic K and L theory versions of the conjecture holds for braid groups on surfaces.

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2.2 The Farrell-Jones Conjecture and Its Fibered Version

In this work we are interested in the formulations of the Farrell-Jones Isomorphism Conjecture, in both, its fibered and nonfibered versions. Let $\mathbb{K}$ be the Davis-Lück algebraic K theory functor, see [8, Sect. 2] and let $\mathbb{L}$ be the Davis-Lück $\mathscr{L}^{-\infty}$ theory functor, see [8, Sect. 2] and [9, Sect. 1.3] for the definition of $\mathscr{L}^{-\infty}$-theory (or L theory for short). The validity of this conjecture allows us, in principle, to compute the algebraic K or L theory groups of the group ring of a given group G from (a) the corresponding K or L theory groups of the group rings of virtually cyclic subgroups of G and (b) homological information. More precisely, let $\mathscr{L}_*()$ be any of the functors $\mathbb{K}$ or $\mathbb{L}$.

Conjecture 2.2.1 (Farrell-Jones Isomorphism Conjecture, IC) *Let G be a discrete group. Then for, all $n, \in \mathbb{Z}$ the assembly map*

$$A_{Vcyc} : H_n^G(\underline{\underline{E}}G; \mathscr{L}_*()) \to H_n^G(pt; \mathscr{L}_*()) \tag{2.1}$$

induced by the projection $\underline{\underline{E}}G \to pt$ is an isomorphism, where the groups $H_*^G(--;\mathscr{L}_*())$ build up a suitable equivariant homology theory with local coefficients in the functor $\mathscr{L}_*()$, and $\underline{\underline{E}}G$ is a model for the classifying space for actions with isotropy in the family of virtually cyclic subgroups of G.

A generalization of the Farrell-Jones conjecture is what is known as the *Fibered Isomorphism Conjecture* (FIC). This generalization has better hereditary properties (see [4, 6]).

Definition 2.2.2 Given a homomorphism of groups $\varphi : K \to G$ and a family of subgroups $\mathscr{F}$ of G closed under conjugation and finite intersections, we define the induced family

$$\varphi^*\mathscr{F} = \{H \leq K | \varphi(H) \in \mathscr{F}\}.$$

If K is a subgroup of G and φ is the inclusion we denote $\varphi^*\mathscr{F} = \mathscr{F} \cap K$.

In this note, we will use FIC to refer to the Fibered Isomorphism Conjecture with respect to the family $Vcyc$ (of virtually cyclic subgroups), for either K- or L- theory.

Definition 2.2.3 Let G be a group and let $\mathscr{F}$ be a family of subgroups of G. We say that the pair $(G, \mathscr{F})$ satisfies the **Fibered Farrell-Jones Isomorphism Conjecture (FIC)** if for all group homomorphisms $\varphi : K \to G$ the pair $(K, \varphi^*\mathscr{F})$ satisfies that

$$A_{\varphi^*\mathscr{F}} : H_n^K(E_{\varphi^*\mathscr{F}}K; \mathscr{L}_*()) \to H_n^K(pt; \mathscr{L}_*())$$

is an isomorphism for all $n \in \mathbb{Z}$.

One of the most interesting hereditary properties of FIC is the so called *Transitivity Principle* [6, Theorem 2.4]:

Theorem 2.2.4 *Let G be a group and let $\mathscr{F} \subset \mathscr{G}$ be two families of subgroups of G. Assume that $N \in \mathbb{Z} \cup \{\infty\}$. Suppose that for every element $H \in \mathscr{G}$ the group H satisfies FIC for the family $\mathscr{F} \cap H$ for all $n \leq N$. Then $(G, \mathscr{G})$ satisfies FIC for all $n \leq N$ if and only if $(G, \mathscr{F})$ satisfies FIC for all $n \leq N$.*

Let G be a group, we denote $Vcyc(G)$ for the family or virtually cyclic subgroups of G. The next two theorems are fundamental for later sections, see [6, lemma 2.8].

Theorem 2.2.5 *Let $f : G \to Q$ be a surjective homomorphism of groups. Suppose that Q, satisfies FIC and for all $f^{-1}(H)$ with $H \in Vcyc(Q)$, $(f^{-1}(H), Vcyc(f^{-1}(H)))$ FIC is true. Then G, satisfies FIC.*

Theorem 2.2.6 *If G satisfies FIC then every subgroup of G satisfies FIC as well.*

2.3 Pure Braid Groups on Aspherical Surfaces

Definition 2.3.1 [1, Definition 1.1] We say that a group G is strongly poly-free if there exists a filtration $1 = G_0 \subset G_1 \subset \cdots \subset G_n = G$ such that the following conditions hold:

(a) G_i is normal in G for each i.
(b) G_{i+1}/G_i is a finitely generated free group.
(c) For each $g \in G$ there exists a compact surface F and a diffeomorphism $f : F \to F$ such that the action by conjugation of g in G_{i+1}/G_i can be geometrically realized, i.e., the following diagram commutes:

$$\begin{tikzcd} \pi_1(F) \arrow[r, "f_\#"] \arrow[d, "\varphi"] & \pi_1(F) \\ G_{i+1}/G_i \arrow[r, "C_g"] & G_{i+1}/G_i \arrow[u, "\varphi^{-1}"'] \end{tikzcd}$$

where φ is an suitable isomorphism.

In this situation we say that G has rank lower or equal than n. Now we enunciate some theorems that will be useful later.

Theorem 2.3.2 *Every word hyperbolic group satisfies FIC. In particular, every finitely generated free group satisfies FIC.*

Proof See [6, Theorem 1.1] for K-theory and [5, Theorem B] for L-theory.

Theorem 2.3.3 *Every $CAT(0)$ group satisfies FIC.*

Proof See [18] for K-theory and [5, Theorem B] for L-theory.

Remark 2.3.4 Both Theorems 2.3.2 and 2.3.3 were proven for a more general version of the Farrell-Jones Isomorphism conjecture stated here, namely they were proven for generalized homology theories with coefficients in any additive category, these versions imply the one given here and also Theorems 2.2.4, 2.2.5 and 2.2.6, see [7].

Theorem 2.3.5 *Let M be a simply connected complete Riemannian manifold whose sectional curvatures are all nonpositive and let G be a group. Assume that G acts by isometries on M, properly discontinuously and cocompactly, then G is a $CAT(0)$ group. In particular, G satisfies FIC.*

Proof This is the basic example of a $CAT(0)$ group. See [3].

Lemma 2.3.6 *Let F be a finitely generated free group and $f : F \to F$ be an automorphism that can be geometrically realized, in the sense of Definition 2.3.1(c), then $F \rtimes \mathbb{Z}$, with the action of $\mathbb{Z}$ in F given by f, satisfies FIC.*

Proof It is proven in [1, Lemma 1.7] that under our hypotheses $F \rtimes \mathbb{Z}$ is isomorphic to the fundamental group of a compact Riemannian 3-manifold M that supports in its interior a complete Riemannian metric with nonpositive sectional curvature everywhere and the metric is a cylinder near the boundary, in fact, near the boundary is diffeomorphic to finitely many components of the form $\mathbb{R} \times N$ where N is either a torus or a Klein bottle. Let DM be the double of M, by the description of the boundary, we can endow the closed manifold DM with a Riemannian metric with nonpositive curvature everywhere as in [2, Proposition 6]. As DM is now compact and complete, it follows that the fundamental group $\pi_1(DM)$ is CAT(0), hence it satisfies FIC. Moreover, $F \rtimes \mathbb{Z} \cong \pi_1(M)$ injects into $\pi_1(DM)$ our result follows by Theorem 2.2.6 and Theorem 2.3.5.

Our main theorem is now the following.

Theorem 2.3.7 *Every strongly poly-free group G satisfies FIC.*

Proof Let us proceed by induction on the rank of G. The induction base, when G has rank ≤ 1, is true as in this case G is a finitely generated free group, thus the assertion follows from Theorem 2.3.2.

Assume that strongly poly-free groups of rank $\leq n$ satisfy FIC and let G be a group of rank $\leq n+1$ with $n > 0$. We apply Theorem 2.2.5 to the surjective homomorphism $q : G \to G/G_n$. Observe that G/G_n is a finitely generated free group, hence it satisfies FIC. Let $C \subset G/G_n$ be a virtually cyclic (and hence cyclic) subgroup, not excluding the possibility of C being G/G_n. We have the following cases:

(a) If $C = \{1\}$ then $q^{-1}(C) = G_n$, which is strongly poly-free of rank $\leq n$, hence it satisfies FIC.
(b) Assume now that C is an infinite cyclic subgroup of G/G_n. Let

$$f : q^{-1}(C) \to \frac{q^{-1}(C)}{G_{n-1}}$$

be the natural projection and observe that

$$\frac{q^{-1}(C)}{G_{n-1}} \cong \frac{G_n \rtimes C}{G_{n-1}} \cong \frac{G_n}{G_{n-1}} \rtimes C.$$

Moreover, the group $\frac{G_n}{G_{n-1}} \rtimes C$ satisfies the hypotheses of Lemma 2.3.6 by the condition (c) for SPF groups, thus $\frac{q^{-1}(C)}{G_{n-1}}$ satisfies FIC and we apply Theorem 2.2.5 to the homomorphism $f : q^{-1}(C) \to \frac{q^{-1}(C)}{G_{n-1}}$. Let $V \subseteq \frac{q^{-1}(C)}{G_{n-1}}$ be a cyclic subgroup, again we have the following cases:

a. $V = 1$ it follows that $f^{-1}(V) = G_{n-1}$ which is an SPF group of rank $\leq n-1$, and it does satisfy FIC by induction.
b. V is an infinite cyclic subgroup. By the definition of V it fits in a filtration

$$1 = G_0 \subset G_1 \subset \cdots \subset G_{n-1} \subset f^{-1}(V),$$

which gives that $f^{-1}(V)$ is an SPF group of rank $\leq n$, hence it satisfies FIC by induction.

It follows that $q^{-1}(C)$ satisfies FIC and therefore G satisfies FIC.

We now recall the definition of the pure braid groups on a surface.

Definition 2.3.8 Let S be a surface with boundary (possibly empty) and $P_k = \{y_1, \ldots, y_k\} \subset S$ be a finite subset of interior points. Define the configuration space to be $M_n^k(S) = \{(x_1, x_2, \ldots, x_n) | x_i \in S - P_k,\ x_i \neq x_j \text{ for } i \neq j\}$. The Pure Braid group on S with n-strings $B_n(S)$ is by definition $\pi_1(M_n^0(S))$.

Lemma 2.3.9 *Let S be a surface with boundary. For $n > r \geq 1$ and $k \geq 0$, the projection on the first r coordinates $M_n^k(S^0) \to M_r^k(S^0)$ is a fibration with fiber $M_{m-r}^{k+r}(S)$, where $S^0 = S - \partial S$.*

Proof See [15, Lemma 1.27].

Lemma 2.3.10 *Suppose that $S = \mathbb{C}$ or that S is a compact surface with nonempty boundary. Then for all $m \geq 0$, $n \geq 1$ the manifold $M_n^m(S)$ is aspherical.*

Proof Consider the fibration $M_n^m(S^0) \to M_1^m(S^0) = S^0 - P_m$ with fiber $M_{n-1}^{m+1}(S^0)$ given by previous lemma. The exact sequence associated to this fibration is as follows

$$\cdots \to \pi_{i+1}(S - P_m) \to \pi_i(M_{n-1}^{m+1}(S)) \to \pi_i(M_n^m(S)) \to \pi_i(S - P_m) \to \cdots.$$

Since $S - P_m$ is aspherical, because the boundary is nonempty, $\pi_i(S - P_m) = 0$ for all $i \geq 2$. Hence for all $i \geq 2$, $\pi_i(M_{n-1}^{m+1}(S)) \cong \pi_i(M_n^m(S))$. An inductive argument shows that for all $i \geq 2$ we have

$$\pi_i(M_n^m(S)) \cong \pi_i(M_1^{m+n-1}(S)) \cong \pi_i(S - P_{m+n-1}) = 0.$$

Theorem 2.3.11 *Suppose that $S = \mathbb{C}$ or that S is a compact surface with nonempty boundary different from $\mathbb{S}^2$ or $\mathbb{R}P^2$. Then the pure braid group $B_n(S)$ is strongly poly-free of rank $\leq n$ for all $n \geq 1$.*

Proof See [1, Theorem 3.1].

Recall that the braid groups on $\mathbb{C}$ or on a compact surface other than $\mathbb{S}^2$ or $\mathbb{R}P^2$ are torsion free. The above Theorem and Theorem 2.3.7 now gives:

Theorem 2.3.12 *Suppose that $S = \mathbb{C}$ or S is a compact surface other than $\mathbb{S}^2$ or $\mathbb{R}P^2$. Then the pure braid groups $B_n(S)$ satisfy FIC for all $n \geq 1$.*

2.4 Full Braid Groups on Aspherical Surfaces

The main goal of this section is to prove that any extension of a finite group by an SPF group satisfies FIC. In order to prove this we shall need some results.

Definition 2.4.1 Let G and H be groups, with H finite. We define de wreath product $G \wr H$ to be the semidirect product $G^{|H|} \rtimes H$, where $G^{|H|}$ is the group of $|H|$-tuples of elements in G indexed by elements in H, H acts on $G^{|H|}$ by permuting the coordinates as the action of H on H by right translation.

Wreath products have been widely studied, the following properties are well known.

Lemma 2.4.2 *Let $1 \to G \to \Gamma \to H \to 1$ be a group extension with H finite. Then there are injective homomorphisms $\delta : \Gamma \to G \wr H$ and $\theta : G \to G^{|H|}$ which together with $id : H \to H$ define a map to the group extension $1 \to G^{|H|} \to G \wr H \to H \to 1$.*

Proof See [10, Algebraic Lemma].

The following lemmas contain standard facts about wreath products.

Lemma 2.4.3 *Let A, B, S and H be groups with S and H finite.*

(a) If A is a subgroup of B, then $A \wr H$ is a subgroup of $B \wr H$.
(b) $A^{|H|} \wr S$ is a subgroup of $A \wr (H \times S)$.

Lemma 2.4.4 ([1, Fact 2.4]) *Let G be a group and H a finite subgroup of $Aut(G)$. We define $G^{|H|} \rtimes \mathbb{Z}$, where the generator of $\mathbb{Z}$ acts on the left factor via $f = \bigoplus_{h\in H} h \in Aut(G^{|H|})$, and $G \rtimes_h \mathbb{Z}$ for each $h \in H$ in the obvious way. Then $G^{|H|} \rtimes \mathbb{Z}$ is a subgroup of $\prod_{h\in H} G \rtimes_h \mathbb{Z}$.*

Theorem 2.4.5 *If G is a $CAT(0)$ group and H is a finite group, then $G \wr H$ is a $CAT(0)$ group, and hence satisfies FIC.*

Proof Let G act properly, isometrically and cocompactly on the $CAT(0)$ space X, then $G \wr H$ acts properly, isometrically and cocompactly on $X^{|H|}$ with $G^{|H|}$ acting coordinate wise on $X^{|H|}$ and H by permuting the coordinates.

Theorem 2.4.6 *Let G be an SPF group, and let H be a finite group. Then $G \wr H$ satisfies FIC.*

Proof Let us proceed by induction on the rank n of G. When $n = 0$ it follows that $G = 1$ and hence $G \wr H$ is finite, thus is hyperbolic and it satisfies FIC by Theorem 2.3.2.

Now assume G has rank $\leq n$, where $n > 1$, and consider the filtration $1 = G_0 \subset G_1 \subset \cdots \subset G_n = G$ given in the definition of a strongly poly-free group.

Note that $G_1^{|H|}$ is a normal subgroup of $G \wr H$ and hence, we have the group extension

$$1 \to G_1^{|H|} \to G \wr H \to (G/G_1) \wr H \to 1.$$

Let $p : G \wr H \to (G/G_1) \wr H$ denote the above epimorphism. We will apply Theorem 2.2.5, note that this is possible because G/G_1 is an SPF of rank $\leq n - 1$. Hence $(G/G_1) \wr H$ satisfies FIC by induction hypothesis.

Next, let $S \subset (G/G_1) \wr H$ be a virtually cyclic subgroup. We have to prove that $p^{-1}(S)$ satisfies FIC. There are two cases to consider.

Case 1: S is finite. We have an exact sequence $1 \to G_1^{|H|} \to p^{-1}(S) \to S \to 1$. Using Lemmas 2.4.2 and 2.4.3, we get

$$p^{-1}(S) \subset G_1^{|H|} \wr S \subset G_1 \wr (H \times S),$$

Now by Theorems 2.4.5 and 2.2.6, $p^{-1}(S)$ satisfies FIC.

Case 2: S is infinite. S contains a normal subgroup T of finite index such that T is infinite cyclic and $T \subset (G/G_1)^{|H|}$. In fact, we assume $T = S \cap (G/G_1)^{|H|}$. We have the exact sequence $1 \to p^{-1}(T) \to p^{-1}(S) \to S/T \to 1$. By Lemma 2.4.2 we get $p^{-1}(S) \subset T_1 \wr (S/T)$, where $T_1 = p^{-1}(T)$. By Theorem 2.2.6 it suffices to show that $T_1 \wr (S/T)$ satisfies FIC.

Fix $t = (t_h)_{h \in H} \in G^{|H|}$, which maps to a generator of T. Note that each t_h acts geometrically on G_1. Then, by Lemma 2.4.4

$$T_1 = G_1^{|H|} \rtimes_t \mathbb{Z} \subset \prod_{h \in H} (G_1 \rtimes_{t_h} \mathbb{Z}),$$

now using Definition 2.3.1, Lemma 2.3.6, and the fact that the finite product of $CAT(0)$-groups is a $CAT(0)$-group we conclude that $\prod_{h \in H} (G_1 \rtimes_{t_h} \mathbb{Z})$ is a $CAT(0)$-group. Hence by Theorems 2.4.5 and 2.2.6, $T_1 \wr (S/T)$ satisfies FIC. Thus, $p^{-1}(S)$ also satisfies FIC. Thus $G \wr H$ satisfies FIC.

Corollary 2.4.7 *Every extension Γ of a finite group by an SPF satisfies FIC.*

Proof It is immediate from the previous Theorem and Lemma 2.4.2 and Theorem 2.2.6.

Theorem 2.4.8 *Suppose that $M = \mathbb{C}$ or M is a compact surface other than $\mathbb{S}^2$ or $\mathbb{R}P^2$. Let Γ be an extension of a finite group H by $B_n(M)$ for some $n \geq 1$. Then Γ satisfies FIC.*

Proof If M has nonempty boundary or $M = \mathbb{C}$ then $B_n(M)$ is SPF by Theorem 2.3.11, so by the previous Corollary the assertion is true. From now on, we assume that M has empty boundary. By Lemma 2.4.2 and Theorem 2.2.6 it suffices to prove that $B_n(M) \wr H$ satisfies FIC. Considering the fiber bundle projection $p : M_n^0(M) \to M_1^0(M) = M$ with fiber $M_{n-1}^1(M) = M_{n-1}^0(M - pt)$ we have the following short exact sequence

$$1 \to A = B_{n-1}(M - pt) \to B_n(M) \xrightarrow{p} \pi_1(M) \to 1.$$

Consider the exact sequence

$$1 \to A^{|H|} \to B_n(M) \wr H \to \pi_1(M) \wr H \to 1.$$

Let $p : B_n(M) \wr H \to \pi_1(M) \wr H$ be the surjective homomorphism in the above sequence. We proceed to apply Theorem 2.2.5 to p. Note that $\pi_1(M) \wr H$ satisfies FIC by Theorem 2.4.5. Let S be a virtually cyclic subgroup of $\pi_1(M) \wr H$. We claim that $p^{-1}(S)$ contains a SPF of finite index. Let $T = S \cap \pi_1(M)^{|H|}$. Then $p^{-1}(T)$ is of finite index in $p^{-1}(S)$. Now, as A is SPF, it follows that $A^{|H|}$ is also SPF by considering the filtration

$$1 = G_0 \subset \cdots G_n = A \subset A \times G_1 \subset \cdots A \times A \subset \cdots \subset A^{|H|-1} \times G_{n-1} \subset A^{|H|}$$

where $1 = G_0 \subset G_1 \subset \cdots G_n = A$ is an SPF structure on A. On the other hand, we have an exact sequence $1 \to A^{|H|} \to p^{-1}(T) \to T \to 1$. Now, from the monodromy action on the pure braid group of M it can be checked that $p^{-1}(T)$ is SPF. The proof now follows from the previous Corollary.

Definition 2.4.9 We recall from Definition 2.3.8 that

$$M_n^0(M) = \{(x_1, x_2, \ldots, x_n) | x_i \in S,\ x_i \neq x_j \text{ if } i \neq j\}.$$

The symmetric group S_n acts on M_n^0. We define the Full Braid Group $FB_n(M)$ on a surface M to be $\pi_1(M_n^0/S_n)$.

It is not difficult to see that we have an exact sequence

$$1 \to B_n(M) \to FB_n(M) \to S_n \to 1,$$

hence by our previous Theorem we have the following:

Theorem 2.4.10 *Suppose that $M = \mathbb{C}$ or M is a compact surface other than $\mathbb{S}^2$ or $\mathbb{R}P^2$. Then the full braid group $FB_n(M)$ satisfies FIC.*

2.5 Braid Groups on $\mathbb{S}^2$ and $\mathbb{R}P^2$

Theorem 2.5.1 *The pure braid groups $B_n(\mathbb{S}^2)$ satisfy FIC for all $n > 0$.*

Proof We have that $B_1(\mathbb{S}^2) = B_2(\mathbb{S}^2) = 0$, and $B_3(\mathbb{S}^2) \cong \mathbb{Z}_2$ (see [11]), so they satisfy FIC because they are finite. For $n > 3$, we consider the fiber bundle $M_n^0(\mathbb{S}^2) \to M_3^0(\mathbb{S}^2)$ with fiber $M_{n-3}^3(\mathbb{S}^2) \cong M_{n-3}^2(\mathbb{C})$. Applying the long exact sequence of the corresponding fibration and the fact that $\pi_2(M_3^0(\mathbb{S}^2))$ is trivial we get

$$1 \to \pi_1(M_{n-3}^2(\mathbb{C})) \to \pi_1(M_n^0(\mathbb{S}^2)) \to \pi_1(M_3^0(\mathbb{S}^2)) \to 1.$$

Note that $\pi_1(M_{n-3}^2(\mathbb{C}))$ is SPF because it is part of the filtration of $B_{n-3}(\mathbb{C})$, hence by Corollary 2.4.7 we have that $\pi_1(M_n^0(\mathbb{S}^2)) = B_n(\mathbb{S}^2)$ satisfies FIC.

Lemma 2.5.2 *Let $1 \to K \to G \to Q \to 1$ be an extension of groups. Suppose that K is virtually cyclic and Q satisfies FIC. Then G satisfies FIC.*

Proof See [6] Lemma 3.4.

Theorem 2.5.3 *The full braid groups $FB_n(\mathbb{S}^2)$ satisfy FIC for all $n > 0$.*

Proof In [16], Theorem 24 it is proven that $FB_n(\mathbb{S}^2)$ fits in an exact sequence

$$1 \to \Gamma_n \to FB_n(\mathbb{S}^2))/\mathbb{Z}_2 \to S_n \to 1$$

where Γ_n is SPF. Hence $FB_n(\mathbb{S}^2)/\mathbb{Z}_2$ satisfies FIC by Corollary 2.4.7. Now consider the exact sequence

$$1 \to \mathbb{Z}_2 \to FB_n(\mathbb{S}^2) \to FB_n(\mathbb{S}^2)/\mathbb{Z}_2 \to 1$$

applying the previous Lemma, we conclude that $FB_n(\mathbb{S}^2)$ satisfies FIC.

Theorem 2.5.4 *The pure braid groups $B_n(\mathbb{R}P^2)$ satisfy FIC for all $n > 0$.*

Proof In [17] it is proven that $B_1(\mathbb{R}P^2) = \mathbb{Z}_2$, $B_2(\mathbb{R}P^2) \cong Q_8$ and $B_3(\mathbb{R}P^2) \cong F_2 \rtimes Q_8$, where Q_8 is the quaternion group with eight elements and F_2 is the free group on two generators. Hence $B_1(\mathbb{R}P^2)$ and $B_2(\mathbb{R}P^2)$ satisfy FIC because they are finite, while $F_2 \rtimes Q_8$ does as it is hyperbolic. In [16, Theorem 27] it is proven that $B_n(\mathbb{R}P^2)$ fits in an exact sequence

$$1 \to \Lambda_n \to B_n(\mathbb{R}P^2) \to Q_8 \to 1$$

where Λ_n is SPF, for all $n > 3$. Hence $B_n(\mathbb{R}P^2)$ satisfies FIC by Corollary 2.4.7.

Theorem 2.5.5 *The full braid groups $FB_n(\mathbb{R}P^2)$ satisfy FIC for all $n > 0$.*

Proof In [17] it is proven that $FB_1(\mathbb{R}P^2) = \mathbb{Z}_2$ and $FB_2(\mathbb{R}P^2)$ is isomorphic to a dicyclic group of order 16. Hence $FB_1(\mathbb{R}P^2)$ and $FB_2(\mathbb{R}P^2)$ satisfy FIC because they are finite. In [16] Theorem 29 it is proven that $FB_n(\mathbb{R}P^2)$ fits in an exact sequence

$$1 \to S_n \to FB_n(\mathbb{R}P^2) \to FB_n(\mathbb{R}P^2)/S \to 1$$

where S_n is a normal SPF subgroup of $FB_n(\mathbb{R}P^2)$ with finite index, for all $n > 2$. Hence by Corollary 2.4.7 we conclude that $FB_n(\mathbb{R}P^2)$ satisfies FIC for all $n > 2$.

Recall that if the groups G and H satisfy FIC then $F \times H$ also satisfies FIC. Therefore we have the following

Theorem 2.5.6 *Let G be a braid group of a surface in any number of strands then $G \times \mathbb{Z}^n$ satisfies FIC for all $n \geq 1$.*

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Chapter 3
Ends, Shapes, and Boundaries in Manifold Topology and Geometric Group Theory

Craig R. Guilbault

Abstract This survey/expository article covers a variety of topics related to the "topology at infinity" of noncompact manifolds and complexes. In manifold topology and geometric group theory, the most important noncompact spaces are often contractible, so distinguishing one from another requires techniques beyond the standard tools of algebraic topology. One approach uses end invariants, such as the number of ends or the fundamental group at infinity. Another approach seeks nice compactifications, then analyzes the boundaries. A thread connecting the two approaches is shape theory. In these notes we provide a careful development of several topics: homotopy and homology properties and invariants for ends of spaces, proper maps and homotopy equivalences, tameness conditions, shapes of ends, and various types of $\mathcal{Z}$-compactifications and $\mathcal{Z}$-boundaries. Classical and current research from both manifold topology and geometric group theory provide the context. Along the way, several open problems are encountered. Our primary goal is a casual but coherent introduction that is accessible to graduate students and also of interest to active mathematicians whose research might benefit from knowledge of these topics.

Keywords End · Shape · Boundary · Manifold · Group · Fundamental group at infinity · Tame · Open collar · Pseudo-collar · Z-set · Z-boundary · Z-structure

Preface

In [87], a paper that plays a role in these notes, Siebenmann mused that his work was initiated at a time "when 'respectable' geometric topology was necessarily compact". That attitude has long since faded; today's topological landscape is filled with research in which noncompact spaces are the primary objects. Even so, past

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traditions have impacted today's topologists, many of whom developed their mathematical tastes when noncompactness was viewed more as a nuisance than an area for exploration. For that and other reasons, many useful ideas and techniques have been slow to enter the mainstream. One goal of this set of notes is to provide quick and intuitive access to some of those results and methods by weaving them together with more commonly used approaches, well-known examples, and current research. In this way, we attempt to present a coherent "theory of ends" that will be useful to mathematicians with a variety of interests.

Numerous topics included here are fundamental to manifold topology and geometric group theory: Whitehead and Davis manifolds, Stallings' characterization of Euclidean spaces, Siebenmann's Thesis, Chapman and Siebenmann's $\mathscr{Z}$-compactification Theorem, the Freudenthal-Hopf-Stallings Theorem on ends of groups, and applications of the Gromov boundary to group theory—to name just a few. We hope these notes give the reader a better appreciation for some of that work. Many other results and ideas presented here are relatively new or still under development: generalizations of Siebenmann's thesis, Bestvina's $\mathscr{Z}$-structures on groups, use of $\mathscr{Z}$-boundaries in manifold topology, and applications of boundaries to non-hyperbolic groups, are among those discussed. There is much room for additional work on these topics; the natural path of our discussion will bring us near to a number of interesting open problems.

The style of these notes is to provide a lot of motivating examples. Key definitions are presented in a rigorous manner—often preceded by a non-rigorous, but (hopefully) intuitive introduction. Proofs or sketches of proofs are included for many of the fundamental results, while many others are left as exercises. We have not let issues of mathematical rigor prevent the discussion of important or interesting work. If a theorem or example is relevant, we try to include it, even when the proof is too long or deep for these pages. When possible, an outline or key portions of an argument are provided—with implied encouragement for the reader to dig deeper.

These notes originated in a series of four one-hour lectures given at the workshop on *Geometrical Methods in High-dimensional Topology*, hosted by Ohio State University in the Spring of 2011. Notes from those talks were expanded into a one-semester topics course at the University of Wisconsin-Milwaukee in the fall of that year. The author expresses his appreciation to workshop organizers Jean-François Lafont and Ian Leary for the opportunity to speak, and acknowledges all fellow participants in the OSU workshop and the UWM graduate students in the follow-up course; their feedback and encouragement were invaluable. Special thanks go to Greg Friedman and the anonymous referee who read the initial version of this document, pointed out numerous errors, and made many useful suggestions for improving both the mathematics and the presentation. Finally, thanks to my son Phillip Guilbault who created most of the figures in this document.

3.1 Introduction

A fundamental concept in the study of noncompact spaces is the "number of ends". For example, the real line has two ends, the plane has one end, and the uniformly trivalent tree $\mathbb{T}_3$ has infinitely many ends. Counting ends has proven remarkably useful, but certainly there is more—after all, there is a qualitative difference between the single end of the ray $[0, \infty)$ and that of $\mathbb{R}^2$. This provides an idea: If, in the topological tradition of counting things, one can (somehow) use the π_0- or H_0-functors to measure the number of ends, then maybe the π_1- and H_1-functors (or, for that matter π_k and H_k), can be used in a similar manner to measure other properties of those ends. Turning that idea into actual mathematics—the "end invariants" of a space—then using those invariants to solve real problems, is one focus of the early portions of these notes.

Another approach to confronting noncompact spaces is to compactify.[1] The 1-point compactification of $\mathbb{R}^1$ is a circle and the 1-point compactification of $\mathbb{R}^2$ a 2-sphere. A "better" compactification of $\mathbb{R}^1$ adds one point to each end, to obtain a closed interval—a space that resembles the line far more than does the circle. This is a special case of "end-point compactification", whereby a single point is added to each end of a space. Under that procedure, an entire Cantor set is added to $\mathbb{T}_3$, resulting in a compact, but still tree-like object. Unfortunately, the end-point compactification of $\mathbb{R}^2$ again yields a 2-sphere. From the point of view of preserving fundamental properties, a far better compactification of $\mathbb{R}^2$ adds an entire circle at infinity. This is a prototypical "$\mathcal{Z}$-compactification", with the circle as the "$\mathcal{Z}$ -boundary". (The end-point compactifications of $\mathbb{R}^1$ and $\mathbb{T}_3$ are also $\mathcal{Z}$-compactifications.) The topic of $\mathcal{Z}$-compactification and $\mathcal{Z}$-boundaries is a central theme in the latter half of these notes.

Shape theory is an area of topology developed for studying compact spaces with bad local properties, so it may seem odd that "shapes" is one of three topics mentioned in the title of an article devoted to *noncompact* spaces with *nice* local properties. This is not a mistake! As it turns out, the tools of shape theory are easily adaptable to the study of ends—and the connection is not just a similarity in approaches. Frequently, the shape of an appropriately chosen compactum precisely captures the illusive "topology at the end of a space". In addition, shape theory plays a clarifying role by connecting end invariants hinted at in paragraph one of this introduction to the $\mathcal{Z}$-boundaries mentioned in paragraph two. To those who know just enough about shape theory to judge it too messy and set-theoretical for use in manifold topology or geometric group theory (a belief briefly shared by this author), patience is encouraged. At the level of generality required for our purposes, shape theory is actually quite elegant and geometric. In fact, very little set-theoretic topology is involved—instead spaces with bad properties are quickly replaced by simplicial and CW complexes, where techniques are clean and intuitive. A working knowledge of shape theory is one subgoal of these notes.

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[1]Despite our affinity for noncompact spaces, we are not opposed to the practice of compactification, provided it is done in a (geometrically) sensitive manner.

3.1.1 Conventions and Notation

Throughout this article, all spaces are separable metric. A *compactum* is a compact space. We often restrict attention to *absolute neighborhood retracts* (or *ANRs*)—a particularly nice class of spaces, whose most notable property is local contractibility. In these notes, ANRs are required to be locally compact. Notable examples of ANRs are: manifolds, locally finite polyhedra, locally finite CW complexes, proper CAT(0) spaces,[2] and Hilbert cube manifolds. Due to their unavoidable importance, a short appendix with precise definitions and fundamental results about ANRs has been included. Readers anxious get started can safely begin, by viewing "ANR" as a common label for the examples just mentioned. An *absolute retract* (or AR) is a contractible ANR, while an *ENR* [resp., *ER*] is a finite-dimensional ANR [resp., AR].

The unmodified term *manifold* means "finite-dimensional manifold". A manifold is *closed* if it is compact and has no boundary and *open* if it is noncompact with no boundary; if neither is specified, boundary is permitted. For convenience, all manifolds are assumed to be piecewise-linear (PL); in other words, they may be viewed as simplicial complexes in which all links are PL homeomorphic to spheres of appropriate dimensions. A primary application of PL topology will be the casual use of general position and regular neighborhoods. A good source for that material is [83]. Nearly all that we do can be accomplished for smooth or topological manifolds as well; readers with expertise in those categories will have little trouble making the necessary adjustments.

Hilbert cube manifolds are entirely different objects. The *Hilbert cube* is the countably infinite product $\mathcal{Q} = \prod_{i=1}^{\infty} [-1, 1]$, endowed with the product topology. A space X is a *Hilbert cube manifold* if each $x \in X$ has a neighborhood homeomorphic to $\mathcal{Q}$. Like ANRs, Hilbert cube manifolds play an unavoidably key role in portions of these notes. For that reason, we have included a short and simple appendix on Hilbert cube manifolds.

Symbols will be used as follows: $\approx$ denotes homeomorphism, while $\simeq$ indicates homotopic maps or homotopy equivalent spaces; $\cong$ indicates isomorphism. When M^n is a manifold, n indicates its dimension and ∂M^n its manifold boundary. When A is a subspace of X, $\mathrm{Bd}_X A$ (or when no confusion can arise, $\mathrm{Bd}X$) denotes the set-theoretic boundary of A. The symbols $\overline{A}$ and $\mathrm{cl}_X A$ (or just $\mathrm{cl}A$) denote the closure of A in X, while $\mathrm{int}_X A$ (or just $\mathrm{int}A$) denotes the interior. The symbol $\widetilde{X}$ always denotes the universal cover of X. Arrows denote (continuous) maps or homomorphisms, with $\hookrightarrow$, $\rightarrowtail$, and $\twoheadrightarrow$ indicating inclusion, injection and surjection, respectively.

[2] A *proper* metric space is one in which every closed metric ball is compact.

3.2 Motivating Examples: Contractible Open Manifolds

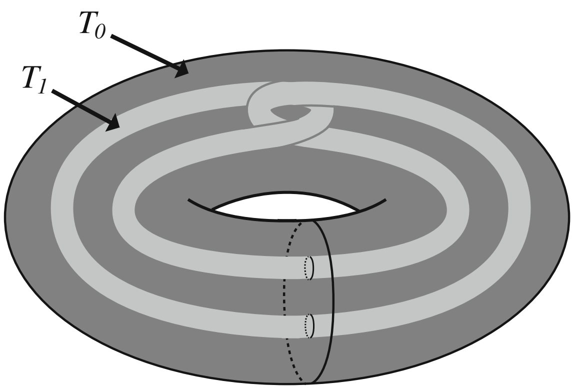
Let us assume that space-time is a large boundaryless 4-dimensional manifold. Recent evidence suggests that this manifold is noncompact (an "open universe"). By running time backward to the Big Bang, we might reasonably conclude that space-time is "just" a contractible open manifold.[3] Compared to the possibilities presented by a closed universe ($\mathbb{S}^4$, $\mathbb{S}^2 \times \mathbb{S}^2$, $\mathbb{R}P^4$, $\mathbb{C}P^2$, the E_8 manifold, …?), the idea of a contractible open universe seems rather disappointing, especially to a topologist primed for the ultimate example on which to employ his/her tools. But there is a mistake in this thinking—an implicit assumption that a contractible open manifold is topologically uninteresting (no doubt just a blob, homeomorphic to an open ball). In this section we take a quick look at the surprisingly rich world of contractible open manifolds.

3.2.1 Classic Examples of Exotic Contractible Open Manifolds

For $n = 1$ or 2, it is classical that every contractible open n-manifold is topologically equivalent to $\mathbb{R}^n$; but when $n \geq 3$, things become interesting. J.H.C Whitehead was among the first to underestimate contractible open manifolds. In an attempt to prove the Poincaré Conjecture, he briefly claimed that, in dimension 3, each is homeomorphic to $\mathbb{R}^3$. In [98] he corrected that error by constructing the now famous *Whitehead contractible* 3*-manifold*—an object surprisingly easy to describe.

Example 3.2.1 (*Whitehead's contractible open 3-manifold*) Let $\mathscr{W}^3 = \mathbb{S}^3 - T_\infty$, where T_∞ is the compact set (the *Whitehead continuum*) obtained by intersecting a nested sequence $T_0 \supseteq T_1 \supseteq T_2 \supseteq \cdots$ of solid tori, where each T_{i+1} is embedded

Fig. 3.1 Constructing the Whitehead manifold



[3] No expertise in cosmology is being claimed by the author. This description of space-time is intended only to motivate discussion.

in T_i in the same way that T_1 is embedded in T_0. See Fig. 3.1. Standard tools of algebraic topology show that $\mathscr{W}^3$ is contractible. For example, first show that $\mathscr{W}^3$ is simply connected (this takes some thought), then show that it is acyclic with respect to $\mathbb{Z}$-homology.

The most interesting question about $\mathscr{W}^3$ is: *Why is it not homeomorphic to* $\mathbb{R}^3$*?* Standard algebraic invariants are of little use, since $\mathscr{W}^3$ has the homotopy type of a point. But a variation on the fundamental group—the "fundamental group at infinity"—does the trick. Before developing that notion precisely, we describe a few more examples of *exotic* contractible open manifolds, i.e., contractible open manifolds not homeomorphic to a Euclidean space.

It turns out that exotic examples are rather common; moreover, they play important roles in both manifold topology and geometric group theory. But for now, let us just think of them as possible universes.

In dimension ≤ 2 there are no exotic contractible open manifolds, but in dimension 3, McMillan [65] constructed uncountably many. In some sense, his examples are all variations on the Whitehead manifold. Rather than examining those examples, let us move to higher dimensions, where new possibilities emerge.

For $n \geq 4$, there exist *compact* contractible n-manifolds not homeomorphic to the standard n-ball $\mathbb{B}^n$. We call these *exotic compact contractible manifolds*. Taking interiors provides a treasure trove of easy-to-understand exotic contractible open manifolds. We provide a simple construction for some of those objects.

Recall that a group is *perfect* if its abelianization is the trivial group. A famous example, the *binary icosahedral group*, is given by the presentation $\left\langle s, t \mid (st)^2 = s^3 = t^5 \right\rangle$.

Example 3.2.2 (*Newman contractible manifolds*) Let G be a perfect group admitting a finite presentation with an equal number of generators and relators. The corresponding presentation 2-complex, K_G has the homology of a point. Embed K_G in $\mathbb{S}^n$ $(n \geq 5)$ and let N be a regular neighborhood of K_G. By general position, loops and disks may be pushed off K_G, so inclusion induces an isomorphism $\pi_1(\partial N) \cong \pi_1(N) \cong G$. By standard algebraic topology arguments ∂N has the $\mathbb{Z}$-homology of an $(n-1)$-sphere and $C^n = \mathbb{S}^n - \mathrm{int} N$ has the homology of a point. A second general position argument shows that C^n is simply connected, and thus contractible—but C^n is clearly not a ball. A compact contractible manifold constructed in this manner is called a *Newman compact contractible manifold* and its interior an *open Newman manifold*.

Exercise 3.2.3 Verify the assertions made in the above example. Be prepared to use numerous tools from a first course in algebraic topology: duality, universal coefficients, the Hurewicz theorem and a theorem of Whitehead (to name a few).

The Newman construction can also be applied to acyclic 3-complexes. From that observation, one can show that every finitely presented *superperfect group* G (that is, $H_i(G; \mathbb{Z}) = 0$ for $i = 1, 2$) can be realized as $\pi_1(\partial C^n)$ for some compact contractible n-manifold $(n \geq 7)$. A related result [42, 59] asserts that *every* $(n-1)$-manifold with the homology of $\mathbb{S}^{n-1}$ bounds a compact contractible n-manifold. For an elementary construction of 4-dimensional examples, see [63].

Exercise 3.2.4 By applying the various Poincaré Conjectures, show that a compact contractible n-manifold is topologically an n-ball if and only if its boundary is simply connected. (An additional nontrivial tool, the Generalized Schönflies Theorem, may also be helpful.)

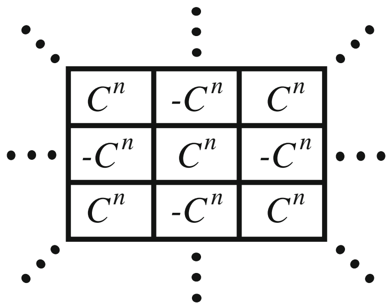
A place where open manifolds arise naturally, even in the study of closed manifolds, is as covering spaces. A place where contractible open manifolds arise naturally is as universal covers of *aspherical* manifolds.[4] Until 1982, the following was a major open problem:

> *Does an exotic contractible open manifold ever cover a closed manifold? Equivalently: Can the universal cover of a closed aspherical manifold fail to be homeomorphic to* $\mathbb{R}^n$?

In dimension 3 this problem remained open until Perelman's solution to the Geometrization Conjecture. It is now known that the universal cover of a closed aspherical 3-manifold is always homeomorphic to $\mathbb{R}^3$. In all higher dimensions, a remarkable construction by Davis [27] produced aspherical n-manifolds with exotic universal covers.

Example 3.2.5 (*Davis' exotic universal covering spaces*) The construction begins with an exotic (piecewise-linear) compact contractible oriented manifold C^n. Davis' key insight was that a certain Coxeter group Γ determined by a triangulation of ∂C^n provides precise instructions for assembling infinitely many copies of C^n into a contractible open n-manifold $\mathscr{D}^n$ with enough symmetry to admit a proper cocompact[5] action by Γ. Figure 3.2 provides a schematic, of $\mathscr{D}^n$, where $-C^n$ denotes a copy of C^n with reversed orientation. Intuitively, $\mathscr{D}^n$ is obtained by repeatedly reflecting copies of C^n across $(n-1)$-balls in ∂C^n. The reflections explain the reversed orientations on half of the copies. By Selberg's Lemma, there is a finite index torsion-free $\Gamma' \leq \Gamma$. By properness, the action of Γ' on $\mathscr{D}^n$ is *free* (no $\gamma \in \Gamma'$ has a fixed point), so the quotient map $\mathscr{D}^n \to \Gamma' \backslash \mathscr{D}^n$ is a covering projection with image a closed aspherical manifold.

Fig. 3.2 A Davis manifold



[4]A connected space X is *aspherical* if $\pi_k(X) = 0$ for all $k \geq 2$.

[5]An action by Γ on X is *proper* if, for each compact $K \subseteq X$ at most finitely many Γ-translates of K intersect K. The action is *cocompact* if there exists a compact C such that $\Gamma C = X$.

Later in these notes, when we prove that $\mathscr{D}^n \not\approx \mathbb{R}^n$, an observation by Ancel and Siebenmann will come in handy. By discarding all of the beautiful symmetry inherent in the Davis construction, their observation provides a remarkably simple topological picture of $\mathscr{D}^n$. Toward understanding that picture, let P^n and Q^n be oriented manifolds with connected boundaries, and let B, B' be $(n-1)$-balls in ∂P^n and ∂Q^n, respectively. A *boundary connected sum* $P^n \overset{\partial}{\#} Q^n$ is obtained by identifying B with B' via an orientation reversing homeomorphism. (By using an orientation reversing gluing map, we may give $P^n \overset{\partial}{\#} Q^n$ an orientation that agrees with both original orientations.)

Theorem 3.2.6 ([3]) *A Davis manifold $\mathscr{D}^n$ constructed from copies of an oriented compact contractible manifold C^n is homeomorphic to the interior of an infinite boundary connected sum:*

$$C_0^n \overset{\partial}{\#} \left(-C_1^n\right) \overset{\partial}{\#} \left(C_2^n\right) \overset{\partial}{\#} \left(-C_3^n\right) \overset{\partial}{\#} \cdots$$

where each C_{2i}^n is a copy of C^n and each $-C_{2i+1}^n$ is a copy of $-C^n$.

Remark 3.2.7 The reader is warned that an infinite boundary connected sum is not topologically well-defined. For example, one could arrange that the result be 2-ended instead of 1-ended. See Fig. 3.3. Remarkably, the *interior* of such a sum is well-defined. The proof of that fact is relatively straight-forward; it contains the essence of Theorem 3.2.6.

Exercise 3.2.8 Sketch a proof that the 1-ended and 2-ended versions of $C_0^n \overset{\partial}{\#} \left(-C_1^n\right) \overset{\partial}{\#} \left(C_2^n\right) \overset{\partial}{\#} \left(-C_3^n\right) \overset{\partial}{\#} \ldots$, indicated by Fig. 3.3 have homeomorphic interiors.

Example 3.2.9 (*Asymmetric Davis manifolds*) To create a larger collection of exotic contractible open n-manifolds (without concern for whether they are universal covers), the infinite boundary connect sum construction can be applied to a collection $\left\{C_j^n\right\}_{j=0}^{\infty}$ of non-homeomorphic compact contractible n-manifolds. Here orientations are less relevant, so mention is omitted. Since there are infinitely many distinct compact contractible n-manifolds, this strategy produces uncountably many examples, which we refer to informally as *asymmetric Davis manifolds*. Distinguishing

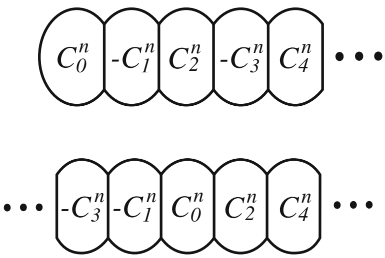


Fig. 3.3 1- and 2-ended boundary connected sums

one from another will be a good test for our soon-to-be-developed tools. Recent applications of these objects can be found in [7] and in the dissertation of P. Sparks.

Exercise 3.2.10 Show that the interior of an infinite boundary connected sum of compact contractible n-manifolds is contractible.

A natural question is motivated by the above discussion:

Among the contractible open manifolds described above, which can or cannot be universal covers of closed n-manifolds?

We will return to this question in Sect. 3.5.1. For now we settle for a fun observation by McMillan and Thickstun [66].

Theorem 3.2.11 *For each $n \geq 3$, there exist exotic contractible open n-manifolds that are not universal covers of any closed n-manifold.*

Proof There are uncountably many exotic open n-manifolds and, by [22], only countably many closed n-manifolds.

3.2.2 Fundamental Groups at Infinity for the Classic Examples

With an ample supply of examples to work with, we begin defining an algebraic invariant useful for distinguishing one contractible open manifold from another. Technical issues will arise, but to keep focus on the big picture, we delay confronting those until later. Once completed, the new invariant will be more widely applicable, but for now we concentrate on contractible open manifolds.

Let W^n be a contractible open manifold with $n \geq 2$. Express W^n as $\cup_{i=0}^{\infty} K_i$ where each K_i is a connected codimension 0 submanifold and $K_i \subseteq \operatorname{int} K_{i+1}$ for each i. With some additional care, arrange that each K_i has connected complement. (Here one uses the fact that W^n is contractible and $n \geq 2$. See Exercise 3.3.3.) The corresponding *neighborhoods of infinity* are the sets $U_i = \overline{W^n - K_i}$.

For each i, let $p_i \in U_i$ and consider the inverse sequence of groups:

$$\pi_1(U_0, p_0) \xleftarrow{\lambda_1} \pi_1(U_1, p_1) \xleftarrow{\lambda_2} \pi_1(U_2, p_2) \xleftarrow{\lambda_3} \cdots \tag{3.1}$$

We would *like* to think of the λ_i as being induced by inclusion, but since $\cap_{i=0}^{\infty} U_i = \varnothing$, a single choice of base point is impossible. Instead, for each i choose a path α_i in U_i connecting p_i to p_{i+1}; then declare λ_i to be the composition

$$\pi_1(U_{i-1}, p_{i-1}) \xleftarrow{\widehat{\alpha}_{i-1}} \pi_1(U_{i-1}, p_i) \leftarrow \pi_1(U_i, p_i)$$

where the first map is induced by inclusion and $\widehat{\alpha}_{i-1}$ is the "change of base point isomorphism". By assembling the α_i end-to-end, we can define a map $r : [0, \infty) \to X$, called the *base ray*. The entire inverse sequence (3.1) is taken as a representation of the *fundamental group at infinity (based at r)* of W^n. Those who prefer a single group can take an inverse limit (defined in Sect. 3.4.1) to obtain the *Čech fundamental group at infinity* (based at r). Unfortunately, that inverse limit typically contains far less information than the inverse sequence itself—more on that later.

Two primary technical issues are already evident:

- *well-definedness:* most obviously, the groups found in (3.1) depend upon the chosen neighborhoods of infinity, and
- *dependence upon base ray:* the "bonding homomorphisms" in (3.1) depend upon the base ray.

We will return to these issues soon; for now we forge ahead and apply the basic idea to some examples.

Example 3.2.13 (*Fundamental group at infinity for* $\mathbb{R}^n$) Express $\mathbb{R}^n$ as $\cup_{i=0}^{\infty} i\mathbb{B}^n$ where $i\mathbb{B}^n$ is the closed ball of radius i. Then, $U_0 = \mathbb{R}^n$ and for $i > 0$, $U_i = \overline{\mathbb{R}^n - \mathbb{B}_i^n}$ is homeomorphic to $\mathbb{S}^{n-1} \times [i, \infty)$. If we let r be a true ray emanating from the origin and $p_i = r \cap \left(\mathbb{S}^{n-1} \times \{i\}\right)$ we get a representation of the fundamental group at infinity as

$$1 \leftarrow 1 \leftarrow 1 \leftarrow 1 \leftarrow \cdots \tag{3.2}$$

when $n \geq 3$, and when $n = 2$, we get (with a slight abuse of notation)

$$1 \leftarrow \mathbb{Z} \xleftarrow{\text{id}} \mathbb{Z} \xleftarrow{\text{id}} \mathbb{Z} \xleftarrow{\text{id}} \cdots \tag{3.3}$$

Modulo the technical issues, we have a modest application of the fundamental group at infinity—it distinguishes the plane from higher-dimensional Euclidean spaces.

Example 3.2.16 (*Fundamental group at infinity for open Newman manifolds*) Let C^n be a compact contractible n-manifold and $G = \pi_1(\partial C^n)$. By deleting ∂C^n from a collar neighborhood of ∂C^n in C^n we obtain an *open collar* neighborhood of infinity $U_0 \approx \partial C^n \times [0, \infty)$ in the open Newman manifold $\operatorname{int}C^n$. For each $i \geq 1$, let U_i be the subcollar corresponding to $\partial C^n \times [i, \infty)$ and let r to be the ray $\{p\} \times [0, \infty)$, with $p_i = p \times \{i\}$. We get a representation of the fundamental group at infinity

$$G \xleftarrow{\text{id}} G \xleftarrow{\text{id}} G \xleftarrow{\text{id}} \cdots$$

The (still-to-be-quantified) difference between this and (3.2) verifies that $\operatorname{int}C^n$ is not homeomorphic to $\mathbb{R}^n$.

Example 3.2.17 Fundamental group at infinity for Davis manifolds) To aid in obtaining a representation of the fundamental group at infinity of a Davis manifold $\mathscr{D}^n$, we use Theorem 3.2.6 to view $\mathscr{D}^n$ as the interior of $C_0 \overset{\partial}{\#} (-C_1) \overset{\partial}{\#} (C_2) \overset{\partial}{\#} (-C_3) \overset{\partial}{\#} \ldots$,

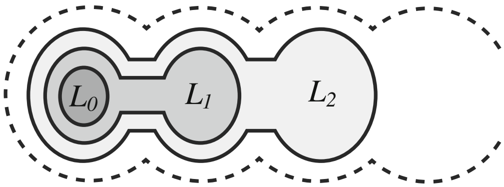


Fig. 3.4 An exhaustion of $\mathscr{D}^n$ by compact contractible manifolds

where each C_i is a copy of a fixed compact contractible n-manifold C. (Superscripts omitted to avoid excessive notation.)

Borrow the setup from Example 3.2.16 to express int$C$ as $\cup_{i=0}^{\infty} K_i$ where each $K_i \equiv \overline{\text{int}C - U_i}$ is homeomorphic to C^n. We may exhaust $\mathscr{D}^n$ by compact contractible manifolds $L_i \approx C_0 \overset{\partial}{\#} (-C_1) \overset{\partial}{\#} \cdots \overset{\partial}{\#} (\pm C_i)$ created by "tubing together" $K_i^0 \cup (-K_i^1) \cup \cdots \cup (\pm K_i^i)$, where the tubes are copies of $\mathbb{B}^{n-1} \times [-1, 1]$ and K_i^j is the copy of K_i in $\pm C_j$ See Fig. 3.4. It is easy to see that a corresponding neighborhood of infinity $V_i = \overline{\mathscr{D}^n - L_i}$ has fundamental group $G_0 * G_1 * \cdots * G_i$ where each G_i is a copy of G; moreover, the homomorphism of $G_0 * G_1 * \cdots * G_i * G_{i+1}$ to $G_0 * G_1 * \cdots * G_i$ induced by $V_{i+1} \hookrightarrow V_i$ acts as the identity on $G_0 * G_1 * \cdots * G_i$ and sends G_{i+1} to 1. With appropriate choices of base points and ray, we arrive at a representation of the fundamental group at infinity of $\mathscr{D}^n$ of the form

$$G_0 \twoheadleftarrow G_0 * G_1 \twoheadleftarrow G_0 * G_1 * G_2 \twoheadleftarrow G_0 * G_1 * G_2 * G_3 \twoheadleftarrow \cdots . \tag{3.4}$$

Example 3.2.19 (*Fundamental group at infinity for asymmetric Davis manifolds*) By proceeding as in Example 3.2.17, but not requiring $C_j \approx C_k$ for $j \neq k$, we obtain manifolds with fundamental groups at infinity represented by inverse sequences like (3.4), except that the various G_i need not be the same. By choosing different sequences of compact contractible manifolds, we can arrive at an uncountable collection of inverse sequences. Some work is still necessary in order to claim an uncountable collection of topologically distinct manifolds.

Example 3.2.20 (*Fundamental group at infinity for the Whitehead manifold*) Referring to Example 3.2.1 and Fig. 3.1, for each $i \geq 0$, let $A_i = \overline{T_i - T_{i+1}}$. Then A_i is a compact 3-manifold, with a pair of torus boundary components ∂T_i and ∂T_{i+1}. Standard techniques from 3-manifold topology allow one to show that $G = \pi_1(A_i)$ is nonabelian and that each boundary component is incompressible in A_i, i.e., $\pi_1(\partial T_i)$ and $\pi_1(\partial T_{i+1})$ inject into G. If we let A_{-1} be the solid torus $\overline{\mathbb{S}^3 - T_0}$, then

$$\mathscr{W}^3 = A_{-1} \cup A_0 \cup A_1 \cup A_2 \cup \cdots$$

where $A_i \cap A_{i+1} = T_{i+1}$ for each i. Set $U_i = A_i \cup A_{i+1} \cup A_{i+2} \cup \cdots$, for each $i \geq 0$, to obtain a nested sequence of homeomorphic neighborhoods of infinity, each having fundamental group isomorphic to an infinite free product with amalgamation

$$\pi_1(U_i) = G_i *_\Lambda G_{i+1} *_\Lambda G_{i+2} *_\Lambda \cdots$$

where $\Lambda \cong \mathbb{Z} \oplus \mathbb{Z}$. Assembling these into an inverse sequence (temporarily ignoring base ray issues) gives a representation of the fundamental group at infinity

$$G_0 *_\Lambda G_1 *_\Lambda G_2 *_\Lambda G_4 *_\Lambda \cdots \hookleftarrow G_1 *_\Lambda G_2 *_\Lambda G_3 *_\Lambda \cdots \hookleftarrow G_2 *_\Lambda G_3 *_\Lambda \cdots \hookleftarrow \cdots$$

Combinatorial group theory provides a useful observation: each bonding homomorphism is injective and none is surjective.

We will return to the calculations from this section after enough mathematical rigor has been added to make them fully applicable.

3.3 Basic Notions in the Study of Noncompact Spaces

An important short-term goal is to confront the issue of well-definedness and to clarify the role of the base ray in our above approach to the fundamental group at infinity. Until that is done, the calculations in the previous section should be viewed with some skepticism. Since we will eventually broaden our scope to spaces far more general than contractible open manifolds, we first take some time to lay out a variety general facts and definitions of use in the study of noncompact spaces.

3.3.1 Neighborhoods of Infinity and Ends of Spaces

A subset U of a space X is a *neighborhood of infinity* if $\overline{X - U}$ is compact; a subset of X is *unbounded* if its closure is noncompact. (*Note*: This differs from the metric notion of "unboundedness", which is dependent upon the metric.) We say that X *has k ends*, if k is a least upper bound on the number of unbounded components in a neighborhood of infinity. If no such k exists, we call X *infinite-ended*.

Example 3.3.1 The real line has 2 ends while, for all $n \geq 2$, $\mathbb{R}^n$ is 1-ended. A space is compact if and only if it is 0-ended. A common example of an infinite-ended space is the universal cover of $\mathbb{S}^1 \vee \mathbb{S}^1$.

Exercise 3.3.2 Show that an ANR X that admits a proper action by an infinite group G, necessarily has 1, 2, or infinitely many ends. (This is a key ingredient in an important theorem from geometric group theory. See Sect. 3.6.)

Exercise 3.3.3 Show that a contractible open n-manifold of dimension ≥ 2 is always 1-ended. *Hint*: Ordinary singular or simplicial homology will suffice.

An *exhaustion of X by compacta* is a nested sequence $K_0 \subseteq K_1 \subseteq K_2 \subseteq$ of compact subsets whose union is X; in this case the corresponding collection of neighborhoods of infinity $U_i = X - K_i$ is *cofinal*, i.e., $\cap_{i=0}^{\infty} U_i = \varnothing$.[6] A compactum K_i is *efficient* if it is connected and the corresponding U_i has only unbounded components. An exhaustion of X by efficient compacta with each $K_i \subseteq \operatorname{int} K_{i+1}$ is called an *efficient exhaustion*. The following is an elementary, but nontrivial, exercise in general topology.

Exercise 3.3.4 Show that every connected ANR X admits an efficient exhaustion by compacta. *Note*: For this exercise, one can replace the ANR hypothesis with the weaker assumption of locally compact and locally path connected.

Let $\{K_i\}_{i=0}^{\infty}$ be an efficient exhaustion of X by compacta and, for each i, let $U_i = X - K_i$. Let $\mathscr{E}nds\,(X)$ be the set of all sequences $(V_0, V_1, V_2, \ldots)$ where V_i is a component of U_i and $V_0 \supseteq V_1 \supseteq \ldots$. Give $\overline{X} = X \cup \mathscr{E}nds\,(X)$ the topology generated by the basis consisting of all open subsets of X and all sets $\overline{V}_i$ where

$$\overline{V}_i = V_i \cup \{(W_0, W_1, \ldots) \in \mathscr{E}nds\,(X) \mid W_i = V_i\}.$$

Then $\overline{X}$ is separable, compact, and metrizable; it is known as the *Freudenthal compactification* of X.

Exercise 3.3.5 Verify the assertions made in the final sentence of the above paragraph. Then show that any two efficient exhaustions of X by compacta result in compactifications that are canonically homeomorphic.

Exercise 3.3.6 Show that the cardinality of $\mathscr{E}nds\,(X)$ agrees with the "number of ends of X" defined at the beginning of this section.

A *closed [open] neighborhood of infinity* in X is one that is closed [open] as a subset of X. If X is an ANR, we often prefer neighborhoods of infinity to themselves be ANRs. This is automatic for open, but not for closed neighborhoods of infinity. Call a neighborhood of infinity *sharp* if it is closed and also an ANR. Call a space X *sharp at infinity* if it contains arbitrarily small sharp neighborhoods of infinity, i.e., if every neighborhood of infinity in X contains one that is sharp.

Example 3.3.7 Manifolds, locally finite polyhedra, and finite-dimensional locally finite CW complexes are sharp at infinity—they contain arbitrarily small closed neighborhoods of infinity that are themselves manifolds with boundary, locally finite polyhedra, and locally finite CW complexes, respectively. In a similar manner, Hilbert cube manifolds are sharp at infinity by an application of Theorem 3.13.2. The existence of non-sharp ANRs can be deduced from [11, 73].

Example 3.3.8 Every proper CAT(0) space X is sharp at infinity—but this is not entirely obvious. The most natural closed neighborhood of infinity, $N_{p,r} = X - B(p; r)$, is an ANR if and only if the metric sphere $S(p; r)$ is an ANR. Surprisingly, it is not known whether this is always the case. However, we can fatten $N_{p,r}$ to an ANR by applying Exercise 3.12.8.

[6]Sometimes closed neighborhood of infinity are preferable; then we let $U_i = \overline{X - K_i}$. In many cases the choice is just a matter of personal preference.

Problem 3.3.9 In a proper CAT(0) space X, is each $S(p; r)$ an ANR? Does there exist some $p_0 \in X$ and a sequence of arbitrarily large r_i for which each $S(p_0; r_i)$ is an ANR? Does it help to assume that X is finite-dimensional or that X is a manifold?

An especially nice variety of sharp neighborhood of infinity is available in n-manifolds and Hilbert cube manifolds. A closed neighborhood of infinity $N \subseteq M^n$ in an n-manifold with compact boundary is *clean* if it is a codimension 0 submanifold disjoint from ∂M^n and $\partial N = \mathrm{Bd}_{M^n} N$ has a product neighborhood ($\approx \partial N \times [-1, 1]$) in M^n. In a Hilbert cube manifold X, where there is no intrinsic notion of boundary (recall that $\mathcal{Q}$ itself is homogeneous!), we simply require that $\mathrm{Bd}_X N$ be a Hilbert cube manifold with a product neighborhood in X. In an n-manifold with noncompact boundary a natural, but slightly more complicated, definition is possible; but it is not needed in these notes.

3.3.2 Proper Maps and Proper Homotopy Type

A map $f : X \to Y$ is proper[7] if $f^{-1}(C)$ is compact for all compact $C \subseteq Y$.

Exercise 3.3.10 Show that a map $f : X \to Y$ between locally compact metric spaces is proper if and only if the obvious extension to their 1-point compactifications is continuous.

Maps $f_0, f_1 : X \to Y$ are *properly homotopic* is there is a proper map $H : X \times [0, 1] \to Y$, with $H_0 = f_0$ and $H_1 = f_1$. We call H a *proper homotopy* between f_0 and f_1 and write $f_0 \overset{p}{\simeq} f_1$. We say that $f : X \to Y$ is a *proper homotopy equivalence* if there exists $g : Y \to X$ such that $gf \overset{p}{\simeq} \mathrm{id}_X$ and $fg \overset{p}{\simeq} Y$. In that case we say X and Y are *proper homotopy equivalent* and write $X \overset{p}{\simeq} Y$.

Remark 3.3.11 It is immediate that homeomorphisms are both proper maps and proper homotopy equivalences, but many pairs of spaces that are homotopy equivalent in the traditional sense are not *proper* homotopy equivalent. For example, whereas all contractible open manifolds (indeed, all contractible spaces) are homotopy equivalent, they are frequently distinguished by their proper homotopy types.

It would be impossible to overstate the importance of "properness" in the study of noncompact spaces. Indeed, it is useful to think in terms of the *proper categories* where the objects are spaces (or certain subclasses of spaces) and the morphisms are proper maps or proper homotopy classes of maps. In the latter case, the isomorphisms are precisely the proper homotopy equivalences. Most of the invariants defined in these notes (such as the fundamental group at infinity) can be viewed as functors on the proper homotopy category of appropriate spaces.

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[7] Yes, this is our third distinct mathematical use of the word *proper*!.

The following offers a sampling of the usefulness of proper maps in understanding noncompact spaces.

Proposition 3.3.12 *Let $f : X \to Y$ be a proper map between ANRs. Then*

(a) f induces a canonical function $f^* : \mathcal{E}nds\,(X) \to \mathcal{E}nds\,(Y)$ that may be used to extend f to a map $\overline{f} : \overline{X} \to \overline{Y}$ between Freudenthal compactifications,
(b) if $f_0, f_1 : X \to Y$ are properly homotopic, then $f_0^* = f_1^*$, and
(c) if $f : X \to Y$ is a proper homotopy equivalence, then f^* is a bijection.

Proof Begin with efficient exhaustions $\{K_i\}$ and $\{L_i\}$ of X and Y, respectively. The following simple observations make the uniqueness and well-definedness of f^* straight-forward:

- By properness, for each i, there is a k_i such that $f\left(X - K_{k_i}\right) \subseteq Y - L_i$,
- By connectedness, a given component U_i of $X - K_{k_i}$ is sent into a unique component V_i of $Y - L_i$,
- By nestedness, each entry W_j of $(W_0, W_1, \ldots) \in \mathcal{E}nds\,(X)$ determines all entries of lower index; hence every subsequence of entries determines that element.

Exercise 3.3.13 Fill in the remaining details in the proof of Proposition 3.3.12.

The following observation is a key sources of proper maps and proper homotopy equivalences.

Proposition 3.3.14 *Let $f : X \to Y$ be a proper map between connected ANRs inducing an isomorphism on fundamental groups. Then the lift $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ to universal covers is a proper map. If $f : X \to Y$ is a proper homotopy equivalence, then $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ is a proper homotopy equivalence.*

Corollary 3.3.15 *If $f : X \to Y$ is a homotopy equivalence between compact connected ANRs, then $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ is a proper homotopy equivalence.*

We prove a simpler Lemma that leads directly to Corollary 3.3.15 and contains the ideas needed for Proposition 3.3.14. A different approach and more general results can be found in [45, Sect. 10.1].

Lemma 3.3.16 *If $k : A \to B$ is a map between compact connected ANRs inducing an isomorphism on fundamental groups, then the lift $\widetilde{k} : \widetilde{A} \to \widetilde{B}$ between universal covers is proper.*

Proof (*Lemma* 3.3.16) Let G denote $\pi_1(A) \cong \pi_1(B)$. Then G acts by covering transformations (properly, cocompactly and freely) on $\widetilde{A}$ and $\widetilde{B}$ so that $\widetilde{k}$ is G-equivariant. Let $K \subseteq \widetilde{A}$ and $L \subseteq \widetilde{B}$ be compacta such that $GK = \widetilde{A}$ and $GL = \widetilde{B}$; without loss of generality, arrange that $G \cdot \operatorname{int}(L) = \widetilde{B}$ and $\widetilde{k}(K) \subseteq L$. The assertion follows easily if $\widetilde{k}^{-1}(L)$ is compact. Suppose otherwise. Then there exists a sequence $\{g_i\}_{i=1}^{\infty}$ of distinct element of G for which $g_i K \cap \widetilde{k}^{-1}(L) \neq \varnothing$. But then each $g_i L$ intersects L, contradicting properness.

Exercise 3.3.17 Fill in the remaining details for a proof for Proposition 3.3.14.

3.3.3 Proper Rays

Henceforth, we refer to any proper map $r : [a, \infty) \to X$ as a *proper ray* in X. In particular, we do not require a proper ray to be "straight" or even an embedding. A *reparametrization* r' of a proper ray r is obtained precomposing r with a homeomorphism $h : [b, \infty) \to [a, \infty)$. Note that a reparametrization of a proper ray is proper.

Exercise 3.3.18 Show that the base ray $r : [0, \infty) \to X$ described in Sect. 3.2.2 is proper. Conversely, let $s : [0, \infty) \to X$ be a proper ray, $\{K_i\}_{i=0}^{\infty}$ an efficient exhaustion of X by compacta, and for each i, $U_i = X - K_i$. Show that, by omitting an initial segment $[0, a)$ and then reparametrizing $s|_{[a,\infty)}$, we may obtain a corresponding proper ray $r : [0, \infty) \to X$ with $r([i, i + 1]) \subseteq U_i$ for each i. In this way, any proper ray in X can be used as a base ray for a representation of the fundamental group at infinity.

Declare proper rays $r, s : [0, \infty) \to X$ to be *strongly equivalent* if they are properly homotopic, and *weakly equivalent* if there is a proper homotopy $K : \mathbb{N} \times [0, 1] \to X$ between $r|_{\mathbb{N}}$ and $s|_{\mathbb{N}}$. Equivalently, r and s are weakly equivalent if there is a proper map h of the *infinite ladder* $L_{[0,\infty)} = ([0, \infty) \times \{0, 1\}) \cup (\mathbb{N} \times [0, 1])$ into X, with $h|_{[0,\infty)\times 0} = r$ and $h|_{[0,\infty)\times 1} = s$. Properness ensures that rungs near the end of $L_{[0,\infty)}$ map toward the end of X. When the squares in the ladder can be filled in with a proper collection of 2-disks in X, a weak equivalence can be promoted to a strong equivalence.

For the set of all proper rays in X with domain $[0, \infty)$, let $\mathscr{E}(X)$ be the set of weak equivalence classes and $\mathscr{SE}(X)$ the set of strong equivalence classes. There is an obvious surjection $\Phi : \mathscr{SE}(X) \to \mathscr{E}(X)$. We say that X is *connected at infinity* if $|\mathscr{E}(X)| = 1$ and *strongly connected at infinity* if $|\mathscr{SE}(X)| = 1$.

Exercise 3.3.19 Show that, for ANRs, there is a one-to-one correspondence between $\mathscr{E}(X)$ and $\mathscr{E}nds(X)$. (Hence, proper rays provide an alternative, and more geometric, method for defining the ends of a space.)

Exercise 3.3.20 Show that, for the infinite ladder $L_{[0,\infty)}$, $\Phi : \mathscr{SE}\left(L_{[0,\infty)}\right) \to \mathscr{E}\left(L_{[0,\infty)}\right)$ is not injective. In fact $\mathscr{SE}\left(L_{[0,\infty)}\right)$ is uncountable. (This is the prototypical example where $\mathscr{SE}(X)$ differs from $\mathscr{E}(X)$.)

3.3.4 Finite Domination and Homotopy Type

In addition to properness, there are notions related to homotopies and homotopy types that are of particular importance in the study of noncompact spaces. We introduce some of those here.

A space Y has *finite homotopy type* if it is homotopy equivalent to a finite CW complex; it is *finitely dominated* if there is a finite complex K and maps $u : Y \to K$

and $d : K \to Y$ such that $d \circ u \simeq \mathrm{id}_Y$. In this case, the map d is called a *domination* and we say that K dominates Y.

Proposition 3.3.21 *Suppose Y is finitely dominated with maps $u : Y \to K$ and $d : K \to Y$ satisfying the definition. Then*

(a) $H_k(Y; \mathbb{Z})$ is finitely generated for all k,
(b) $\pi_1(Y, y_0)$ is finitely presentable, and
(c) if Y' is homotopy equivalent to Y, then Y' is finitely dominated.

Proof Since d induces surjections on all homology and homotopy groups, the finite generation of $H_k(Y; \mathbb{Z})$ and $\pi_1(Y, y_0)$ are immediate. The finite presentability of the latter requires some elementary combinatorial group theory; an argument (based on [96]) can be found in [48, Lemma 2]. The final item is left as an exercise.

Exercise 3.3.22 Show that if Y' is homotopy equivalent to Y and Y is finitely dominated, then Y' is finitely dominated.

The next proposition adds some intuitive meaning to finite domination.

Proposition 3.3.23 *An ANR Y is finitely dominated if and only if there exists a self-homotopy that "pulls Y into a compact subset", i.e., $H : Y \times [0, 1] \to Y$ such that $H_0 = id_Y$ and $\overline{H_1(Y)}$ is compact.*

Proof If $u : Y \to K$ and $d : K \to Y$ satisfy the definition of finite domination, then the homotopy between id_Y and $d \circ u$ pulls Y into $d(K)$.

For the converse, begin by assuming that Y is a locally finite polyhedron. If $H : Y \times [0, 1] \to X$ such that $H_0 = \mathrm{id}_X$ and $\overline{H_1(Y)}$ is compact, then any compact polyhedral neighborhood K of $\overline{H_1(Y)}$ dominates Y, with $u = H_1$ and d the inclusion.

For the general case, we use some Hilbert cube manifold magic. By Theorem 3.13.1, $Y \times \mathcal{Q}$ is a Hilbert cube manifold, so by Theorem 3.13.2, $Y \times \mathcal{Q} \approx P \times \mathcal{Q}$, where P is a locally finite polyhedron. The homotopy that pulls Y into a compact set can be used to pull $P \times \mathcal{Q}$ into a compact subset of the form $K \times \mathcal{Q}$, where K is a compact polyhedron. It follows easily that K dominates $P \times \mathcal{Q}$. An application of Proposition 3.3.21 completes the proof.

At this point, the natural question becomes: *Does there exist a finitely dominated space Y that does not have finite homotopy type?* A version of this question was initially posed by Milnor in 1959 and answered affirmatively by Wall.

Theorem 3.3.24 (Wall's finiteness obstruction, [96]) *For each finitely dominated space Y, there is a well-defined obstruction $\sigma(Y)$, lying in the reduced projective class group $\widetilde{K}_0(\mathbb{Z}[\pi_1(Y)])$, which vanishes if and only if Y has finite homotopy type. Moreover, all elements of $\widetilde{K}_0(\mathbb{Z}[\pi_1(Y)])$ can be realized as finiteness obstructions of a finitely dominated CW complex.*

A development of Wall's obstruction is interesting and entirely understandable, but outside the scope of these notes. The interested reader is referred to Wall's original paper or the exposition in [38]. For late use, we note that $\widetilde{K}_0$ determines a functor from $\mathcal{G}roups$ to $\mathcal{A}belian\ groups$; in particular, if $\lambda : G \to H$ is a group homomorphism, then there is an induced homomorphism $\lambda_* : \widetilde{K}_0(\mathbb{Z}[G]) \to \widetilde{K}_0(\mathbb{Z}[H])$ between the corresponding projective class groups.

Example 3.3.25 Every compact ENR A is easily seen to be finitely dominated. Indeed, if $U \subseteq \mathbb{R}^n$ is a neighborhood of A and $r : U \to A$ a retraction, let $K \subseteq U$ be a polyhedral neighborhood of A, $d : K \to A$ the restriction, and u the inclusion.

Although this is a nice example, it is made obsolete by a major result of West (see Proposition 3.12.4), showing that every compact ANR has finite homotopy type.

3.3.5 Inward Tameness

Modulo a slight change in terminology, we follow [21] by defining an ANR X to be *inward tame* if, for each neighborhood of infinity N there exists a smaller neighborhood of infinity N' so that, up to homotopy, the inclusion $N' \overset{j}{\hookrightarrow} N$ factors through a finite complex K. In other words, there exist maps $f : N' \to K$ and $g : K \to N$ such that $gf \simeq j$.

Exercise 3.3.26 Show that if $X \overset{p}{\simeq} Y$ and X is inward tame, then Y is inward tame.

For the remainder of this section, our goals are as follows:

(a) to obtain a more intrinsic and intuitive characterization of inward tameness, and

(b) to clarify the (apparent) relationship between inward tameness and finite dominations.

The following is our answer to Goal (a).

Lemma 3.3.27 *An ANR X is inward tame if and only if, for every closed neighborhood of infinity N in X, there is a homotopy $S : N \times [0,1] \to N$ with $S_0 = id_N$ and $\overline{S_1(N)}$ compact (a homotopy pulling N into a compact subset).*

Proof For the forward implication, let N' be a closed neighborhood of infinity contained in $\operatorname{int}N$ so that $N' \hookrightarrow \operatorname{int}N$ factors through a compact polyhedron K. Then there is a homotopy $H : N' \times [0,1] \to \operatorname{int}N$ with H_0 the inclusion and $\overline{H_1(N')} \subseteq g(K)$. Choose an open neighborhood U of N' with $\overline{U} \cap \operatorname{Bd}_X N = \varnothing$, then let $A = \operatorname{int}N - U$ and J be the identity homotopy on A. Since $\operatorname{int}N$ is an ANR, Borsuk's Homotopy Extension Property (see Proposition 3.12.4) allows us to extend $H \cup J$ to a homotopy $S : \operatorname{int}N \times [0,1] \to \operatorname{int}N$ with $S_0 = \mathrm{id}_{\operatorname{int}N}$. This in turn may be extended via the identity over $\operatorname{Bd}_X N$ to obtain a homotopy S that pulls N into a compact subset of itself.

We will return for the converse after addressing Goal (b).

Recall that an ANR X is sharp at infinity if it contains arbitrarily small closed ANR neighborhoods of infinity.

Lemma 3.3.28 *A space X that is sharp at infinity is inward tame if and only if each of its sharp neighborhoods of infinity is finitely dominated.*

Proof Assume X is sharp at infinity and inward tame. By Lemma 3.3.27 each closed neighborhood of infinity can be pulled into a compact subset, so by Proposition 3.3.23, those which are ANRs are finitely dominated. The converse is immediate by the definitions.

Proof (*completion of Lemma* 3.3.27) Suppose that, for each closed neighborhood of infinity N in X, there is a homotopy pulling N into a compact subset. Then the same is true for $X \times \mathcal{Q}$. But $X \times \mathcal{Q}$ is sharp since it is a Hilbert cube manifold, so by Proposition 3.3.23, each ANR neighborhood of infinity in $X \times \mathcal{Q}$ is finitely dominated. By Lemma 3.3.28 and Exercise 3.3.26, X is inward tame.

We tidy up by combining the above Lemmas into a single Proposition, and adding some mild extensions. For convenience we restrict attention to spaces that are sharp at infinity.

Proposition 3.3.29 *For a space X that is sharp at infinity, the following are equivalent.*

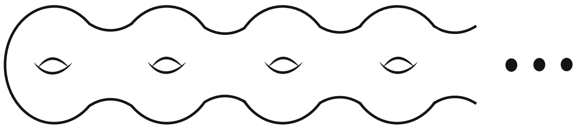
(a) *X is inward tame,*
(b) *for every closed neighborhood of infinity N, there is a homotopy $H : N \times [0, 1] \to N$ with $H_0 = id_N$ and $\overline{H_1(N)}$ compact,*
(c) *there exist arbitrarily small closed neighborhood of infinity N, for which there is a homotopy $H : N \times [0, 1] \to N$ with $H_0 = id_N$ and $\overline{H_1(N)}$ compact,*
(d) *every sharp neighborhood of infinity is finitely dominated,*
(e) *there exist arbitrarily small sharp neighborhoods of infinity that are finitely dominated.*

Proof The equivalence of (b) and (c) is by a homotopy extension argument like that found in Lemma 3.3.27. The equivalence of (d) and (e) is similar, but easier.

Remark 3.3.30 The "inward" in inward tame is motivated by conditions (2) and (3) where the homotopies are viewed as pulling the end of X inward toward the center of X. Based on the definition and conditions (4) and (5), one may also think of inward tameness as "finitely dominated at infinity". We call X absolutely inward tame if it contains arbitrarily small closed ANR neighborhoods of infinity with finite homotopy type.

Example 3.3.31 The infinite ladder $L_{[0,\infty)}$ is not inward tame, since its ANR neighborhoods of infinity have infinitely generated fundamental groups. Similarly, the infinite genus 1-ended orientable surface in Fig. 3.5 is not inward tame.

Fig. 3.5 1-ended infinite genus surface



Example 3.3.32 Although the Whitehead manifold $\mathscr{W}^3$ itself has finite homotopy type, it is not inward tame, since the neighborhoods of infinity U_i discussed in Example 3.2.20 do not have finitely generated fundamental groups (proof would require some work). The Davis manifolds, on the other hand, are absolutely inward tame. More on these observations in Sect. 3.5.3.

Exercise 3.3.33 Justify the above assertion about the Davis manifolds.

Example 3.3.34 Every proper CAT(0) space X is absolutely inward tame. For inward tameness, let $N_{p,r}$ be the complement of an open ball $B(p;r)$ and use geodesics to strong deformation retract $N_{p,r}$ onto the metric sphere $S(p;r)$. If $S(p;r)$ is an ANR, then it (and thus $N_{p,r}$) have finite homotopy type by Proposition 3.12.4. Since this is not known to be the case, more work is required. For each sharp neighborhood of infinity N (recall Example 3.3.8), choose r so that $\overline{X-N} \subseteq B(p;r)$ and let $A = \overline{N - N_{p,r}}$. Then N strong deformation retracts onto A, which is a compact ANR.

Before closing this section, we caution the reader that differing notions of "tameness" are scattered throughout the literature. Siebenmann [86] called a 1-ended open manifold *tame* if it satisfies our definition for inward tame and also has "stable" fundamental group at infinity (a concept to be discussed shortly). In [21], the definition of tame was reformulated to match our current-day definition of inward tame. Later still, [56, 82] put forth another version of "tame" in which homotopies push neighborhoods of infinity toward the end of the space—sometimes referring to that version as *forward tame* and the [21] version as *reverse tame*. In an effort to avoid confusion, this author introduced the term *inward tame,* while referring to the Quinn-Hughes-Ranicki version as *outward tame.*

Within the realm of 3-manifold topology, a *tame end* is often defined to be one for which there exists a product neighborhood of infinity $N \approx \partial N \times [0,\infty)$. Remarkably, by [94] combined with the 3-dimensional Poincaré conjecture—in the special case of 3-manifolds—this property, inward tameness, and outward tameness are all equivalent.

Despite its mildly confusing history, the concept of inward tameness (and its variants) is fundamental to the study of noncompact spaces. Throughout the reminder of these notes, its importance will become more and more clear. In Sect. 3.7.4, we will give meaning to the slogan: "an inward tame space is one that acts like a compactum at infinity".

3.4 Algebraic Invariants of the Ends of a Space: The Precise Definitions

In Sect. 3.2.2 we introduced the fundamental group at infinity rather informally. In this section we provide the details necessary to place that invariant on firm mathematical ground. In the process we begin to uncover subtleties that make this invariant even more interesting than one might initially expect.

As we progress, it will become apparent that the fundamental group at infinity (more precisely "pro-π_1") is just one of many "end invariants". By the end of the section, we will have introduced others, including pro-π_k and pro-H_k for all $k \geq 0$.

3.4.1 An Equivalence Relation on the Set of Inverse Sequences

The *inverse limit* of an inverse sequence

$$G_0 \xleftarrow{\mu_1} G_1 \xleftarrow{\mu_2} G_2 \xleftarrow{\mu_3} G_3 \xleftarrow{\mu_4} \cdots$$

of groups is defined by

$$\varprojlim \{G_i, \mu_i\} = \{(g_0, g_1, g_2, \ldots) \mid \mu_i(g_i) = g_{i-1} \text{ for all } i \geq 1\}.$$

Although useful at times, passing to an inverse limit often results in a loss of information. Instead, one usually opts to keep the entire sequence—or, more accurately, the essential elements of that sequence. To get a feeling for what is meant by "essential elements", let us look at some things that can go wrong.

In Example 3.2.13, we obtained the following representation of the fundamental group of infinity for $\mathbb{R}^3$.

$$1 \leftarrow 1 \longleftarrow 1 \longleftarrow \cdots. \tag{3.5}$$

That was done by exhausting $\mathbb{R}^3$ with a sequence $\left\{i\mathbb{B}^3\right\}$ of closed i-balls and letting $U_i = \overline{\mathbb{R}^3 - i\mathbb{B}^3} \approx \mathbb{S}^2 \times [i, \infty)$. If instead, $\mathbb{R}^3$ is exhausted with a sequence $\{T_i\}$ of solid tori where each T_j lies in T_{j+1} as shown in Fig. 3.6 and $V_i = \overline{\mathbb{R}^3 - T_i}$, the resulting representation of the fundamental group of infinity is

$$\mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{0} \cdots.$$

By choosing more complicated exhausting sequences (e.g., exhaustions by higher genus knotted handlebodies), representations with even more complicated groups can be obtained. It can also be arranged that the bonding homomorphisms are not always trivial. Yet each of these sequences purports to describe the same thing. Although it

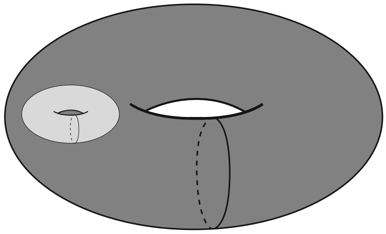


Fig. 3.6 Exhausting $\mathbb{R}^3$ with solid tori

seems clear that (3.5) is the preferred representative for the end of $\mathbb{R}^3$, in the case of an arbitrary 1-ended space, there may be no obvious "best choice". The problem is resolved by placing an equivalence relation on the set of all inverse sequences of groups. Within an equivalence class, certain representatives may be preferable to others, but each contains the essential information.

For an inverse sequence $\{G_i, \phi_i\}$, there is an obvious meaning for *subsequence*

$$G_{k_0} \xleftarrow{\phi_{k_0,k_1}} G_{k_1} \xleftarrow{\phi_{k_1,k_2}} G_{k_2} \xleftarrow{\phi_{k_2,k_3}} \cdots$$

where the bonding homomorphisms $\phi_{k_i,k_{i+1}}$ are compositions of the ϕ_i. Declare inverse sequences $\{G_i, \phi_i\}$ and $\{H_i, \psi_i\}$ to be *pro-isomorphic* if they contain subsequences that fit into a commuting "ladder diagram"

$$\begin{tikzcd}
G_{i_0} & & G_{i_1} \arrow[ll, "\lambda_{i_0,i_1}"'] \arrow[dl] & & G_{i_2} \arrow[ll, "\lambda_{i_1,i_2}"'] \arrow[dl] & & G_{i_3} \arrow[ll, "\lambda_{i_2,i_3}"'] \arrow[dl] & \cdots \\
& H_{j_0} \arrow[ul] & & H_{j_1} \arrow[ll, "\mu_{j_0,j_1}"] \arrow[ul] & & H_{j_2} \arrow[ll, "\mu_{j_1,j_2}"] \arrow[ul] & & \arrow[ll, "\mu_{j_2,j_3}"] \cdots
\end{tikzcd}$$

More broadly, define *pro-isomorphism* to be the equivalence relation on the collection of all inverse sequences of groups generated by that rule.[8]

It is immediate that an inverse sequence is pro-isomorphic to each of its subsequences; but sequences can appear very different and still be pro-isomorphic.

Exercise 3.4.2 Convince yourself that the various inverse sequences mentioned above for describing the fundamental group at infinity of $\mathbb{R}^3$ are pro-isomorphic.

Exercise 3.4.3 Show that a pair of pro-isomorphic inverse sequences of groups have isomorphic inverse limits. *Hint*: Begin by observing a canonical isomorphism between the inverse limit of a sequence and that of any of its subsequences.

The next exercise provides a counterexample to the converse of Exercise 3.4.3. It justifies our earlier assertion that passing to an inverse limit often results in loss of information.

[8]The prefix "pro" is derived from "projective". Some authors refer to inverse sequences and inverse limits as projective sequences and projective limits, respectively.

Exercise 3.4.4 Show that the inverse sequence $\mathbb{Z} \xleftarrow{\times 2} \mathbb{Z} \xleftarrow{\times 2} \mathbb{Z} \xleftarrow{\times 2} \cdots$ is not pro-isomorphic to the trivial inverse sequence $1 \leftarrow 1 \longleftarrow 1 \longleftarrow \cdots$, but both inverse limits are trivial.

Exercise 3.4.5 A more slick (if less intuitive) way to define pro-isomorphism is to declare it to be the equivalence relation generated by making sequences equivalent to their subsequences. Show that the two approaches are equivalent.

Remark 3.4.6 With a little more work, we could define *morphisms* between inverse sequences of groups and arrive at a category *pro-$\mathcal{G}$roups*, where the objects are inverse sequences of groups, in which two objects are pro-isomorphic if and only if they are isomorphic in that category.[9]

Similarly, for any category C one can build a category pro-$\mathcal{C}$ in which the objects are inverse sequences of objects and morphisms from $\mathcal{C}$ and for which the resulting relationship of pro-isomorphism is similar to the one defined above. All of this is interesting and useful, but more than we need here. For a comprehensive treatment of this topic, see [45].

3.4.2 Topological Definitions and Justification of the Pro-isomorphism Relation

A quick look at the topological setting that leads to multiple inverse sequences representing the same fundamental group at infinity provides convincing justification for the definition of pro-isomorphic.

Let $U_0 \hookleftarrow U_1 \hookleftarrow U_2 \hookleftarrow \cdots$ and $V_0 \hookleftarrow V_1 \hookleftarrow V_2 \hookleftarrow \cdots$ be two cofinal sequences of connected neighborhoods of infinity for a 1-ended space X. By going out sufficiently far in the second sequence, one arrives at a V_{k_0} contained in U_0. Similarly, going out sufficiently far in the initial sequence produces a $U_{j_1} \subseteq V_{k_0}$. (For convenience, let $j_0 = 0$.) Alternating back and forth produces a ladder diagram of inclusions

$$\begin{tikzcd}
V_{j_0} & & V_{j_1} \arrow[ll, hook'] \arrow[dl, hook'] & & V_{j_2} \arrow[ll, hook'] \arrow[dl, hook'] & & \cdots \arrow[ll, hook'] \\
& U_{k_0} \arrow[ul, hook] & & U_{k_1} \arrow[ll, hook'] \arrow[ul, hook] & & U_{k_2} \arrow[ll, hook'] \arrow[ul, hook] & \cdots
\end{tikzcd} \quad . \tag{3.6}$$

Applying the fundamental group functor to that diagram (ignoring base points for the moment) results in a diagram

[9]We are not being entirely forthright here. In the literature, *pro-Groups* usually refers to a larger category consisting of "inverse systems" of groups indexed by arbitrary partially ordered sets. We have described a subcategory, *Tow-Groups*, made up of those objects indexed by the natural numbers—also known as "towers".

$$
\begin{tikzcd}
\pi_1(V_{j_0}) & & \pi_1(V_{j_1}) \arrow[ll] \arrow[dl] & & \pi_1(V_{j_2}) \arrow[ll] \arrow[dl] & & \cdots \arrow[ll] \\
& \pi_1(U_{k_0}) \arrow[ul] & & \pi_1(U_{k_1}) \arrow[ll] \arrow[ul] & & \pi_1(U_{k_2}) \arrow[ll] \arrow[ul] & \cdots
\end{tikzcd}
\tag{3.7}
$$

showing that

$$\pi_1\,(U_0) \xleftarrow{\lambda_1} \pi_1\,(U_1) \xleftarrow{\lambda_2} \pi_1\,(U_2) \xleftarrow{\lambda_3} \cdots$$

and

$$\pi_1\,(V_0) \xleftarrow{\mu_1} \pi_1\,(V_1) \xleftarrow{\mu_2} \pi_1\,(V_2) \xleftarrow{\mu_3} \cdots$$

are pro-isomorphic.

A close look at base points and base rays is still to come, but recognizing their necessity, we make the following precise definition. For a pair (X, r) where r is a proper ray in X, let pro-$\pi_1\,(\varepsilon(X), r)$ denote the pro-isomorphism class of inverse sequences of groups which contains representatives of the form (3.1), where $\{U_i\}_{i=0}^{\infty}$ is a cofinal sequence of neighborhoods of infinity, and r has been modified (in the manner described in Exercise 3.3.18) so that $r\,([i, \infty)) \subseteq U_i$ for each $i \geq 0$. From now on, when we refer to the *fundamental group at infinity (based at r)* of a space X, we mean pro-$\pi_1\,(\varepsilon(X), r)$.

With the help of Exercise 3.4.3, we also define the *Čech fundamental group of the end of X (based at r)*, to be the inverse limit of pro-$\pi_1\,(\varepsilon(X), r)$. It is denoted by $\check{\pi}_1\,(\varepsilon(X), r)$.

Exercise 3.4.9 Fill in the details related to base points and base rays needed for the existence of diagram (3.7).

Remark 3.4.10 Now that pro-$\pi_1\,(\varepsilon(X), r)$ is well-defined and (hopefully) well-understood for 1-ended X, it is time to point out that everything done thus far works for multi-ended X. In those situations, the role of r is more pronounced. In the process of selecting base points for a sequence of neighborhoods of infinity $\{U_i\}$, r determines the component of each U_i that contributes to pro-$\pi_1\,(\varepsilon(X), r)$. So, if r and s point to different ends of X, pro-$\pi_1\,(\varepsilon(X), r)$ and pro-$\pi_1\,(\varepsilon(X), s)$ reflect information about entirely different portions of X. This observation is just the beginning; a thorough examination of the role of base rays is begun in Sect. 3.4.6.

3.4.3 Other Algebraic Invariants of the End of a Space

By now it has likely occurred to the reader that π_1 is not the only functor that can be applied to an inverse sequence of neighborhoods of infinity. For any $k \geq 1$ and proper ray r, define pro-$\pi_k\,(\varepsilon(X), r)$ in the analogous manner. By taking inverse limits we get the *Čech homotopy groups* $\check{\pi}_k\,(\varepsilon(X), r)$ of the end of X determined by r. Similarly, we may define pro-$\pi_0\,(\varepsilon(X), r)$ and $\check{\pi}_0\,(\varepsilon(X), r)$; the latter is just a set (more precisely a *pointed* set, i.e., a set with a distinguished base point), and the former an equivalence class of inverse sequences of (pointed) sets.

By applying the homology functor we obtain pro-$H_k(\varepsilon(X); R)$ and $\check{H}_k(\varepsilon(X); R)$ for each non-negative integer k and arbitrary coefficient ring R, the latter being called the *Čech homology of the end of* X. In this context, no base ray is needed!

If instead we apply the cohomology functor, a significant change occurs. The contravariant nature of H^k produces *direct sequences*

$$H^k(U_0; R) \xrightarrow{\lambda_1} H^k(U_1; R) \xrightarrow{\lambda_2} H^k(U_2; R) \xrightarrow{\lambda_3} \cdots$$

of cohomology groups. An algebraic treatment of such sequences, paralleling Sect. 3.4.1, and a standard definition of *direct limit*, allow us to define ind-$H^*(\varepsilon(X); R)$ and $\check{H}^*(\varepsilon(X); R)$.

Exercise 3.4.11 Show that for ANRs there is a one-to-one correspondence between $\mathscr{E}nds(X)$ and $\check{\pi}_0(\varepsilon(X)r)$.

3.4.4 End Invariants and the Proper Homotopy Category

In Remark 3.3.11, we commented on the importance of proper maps and proper homotopy equivalences in the study of noncompact spaces. We are now ready to back up that assertion. The following Proposition could be made even stronger with a discussion of morphisms in the category of pro-$\mathscr{G}roups$, but for our purposes, it will suffice.

Proposition 3.4.12 *Let $f : X \to Y$ be a proper homotopy equivalence and r a proper ray in X. Then*

(a) pro-$H_k(\varepsilon(X); R)$ is pro-isomorphic to pro-$H_k(\varepsilon(Y); R)$ for all k and every coefficient ring R,
(b) pro-$\pi_0(\varepsilon(X, r))$ is pro-isomorphic to pro-$\pi_0(\varepsilon(Y, f \circ r))$ as inverse sequences of pointed sets, and
(c) pro-$\pi_k(\varepsilon(X), r)$ is pro-isomorphic to pro-$\pi_k(\varepsilon(Y), f \circ r)$ for all $k \geq 1$.

Corollary 3.4.13 *A proper homotopy equivalence $f : X \to Y$ induces isomorphisms between $\check{H}_k(\varepsilon(X); R)$ and $\check{H}_k(\varepsilon(Y); R)$ for all k and every coefficient ring R. It induces a bijection between $\check{\pi}_0(\varepsilon(X), r)$ and $\check{\pi}_0(\varepsilon(Y), f \circ r)$ and isomorphisms between $\check{\pi}_k(\varepsilon(X), r)$ and $\check{\pi}_k(\varepsilon(Y), f \circ r)$ for all $k \geq 1$.*

Proof (*Sketch of the proof of Proposition* 3.4.12) Let $g : Y \to X$ be a proper inverse for f and let H and K be proper homotopies between $g \circ f$ and id_X and $f \circ g$ and id_Y, respectively. By using the properness of H and K and a back-and-forth strategy similar to the one employed in obtaining diagram (3.6), we obtain systems of neighborhoods of infinity $\{U_i\}$ in X and $\{V_i\}$ in Y that fit into a ladder diagram

$$\begin{tikzcd}
V_{j_0} & & V_{j_1} \arrow[ll, hook'] \arrow[dl] & & V_{j_2} \arrow[ll, hook'] \arrow[dl] & & \cdots \arrow[ll, hook'] \\
& U_{k_0} \arrow[ul] & & U_{k_1} \arrow[ll, hook'] \arrow[ul] & & U_{k_2} \arrow[ll, hook'] \arrow[ul] & \cdots
\end{tikzcd} . \tag{3.8}$$

Unlike the earlier case, the up and down arrows are not inclusions, but rather restrictions of f and g. Furthermore, the diagram does not commute on the nose; instead, it commutes up to homotopy. But that is enough to obtain a commuting ladder diagram of homology groups, thus verifying (a). The same is true for (b), but on the level of sets. Assertion (c) is similar, but a little additional care must be taken to account for the base rays.

3.4.5 Inverse Mapping Telescopes and a Topological Realization Theorem

It is natural to ask which inverse sequences (more precisely, pro-isomorphism classes) can occur as pro-$\pi_1(\varepsilon(X), r)$ for a space X. Here we show that, even if restricted to very nice spaces, the answer is “nearly all of them”. Later we will see that, in certain important contexts the answer becomes much different. But for now we create a simple machine for producing wide range of examples.

Let

$$(K_0, p_0) \xleftarrow{f_1} (K_1, p_1) \xleftarrow{f_2} (K_2, p_2) \xleftarrow{f_3} \cdots \tag{3.9}$$

be an inverse sequence of pointed finite CW complexes and cellular maps. For each $i \geq 1$, let M_i be a copy of the *mapping cylinder* of f_i; more specifically

$$M_i = (K_i \times [i-1, i]) \sqcup (K_{i-1} \times \{i-1\}) / \sim_i$$

where $\sim_i$ is the equivalence relation generated by the rule: $(k, i-1) \sim_i (f_i(k), i-1)$ for each $k \in K_i$. Then M_i contains a canonical copy $K_{i-1} \times \{i-1\}$ of K_{i-1} and a canonical copy $K_i \times \{i\}$ of K_i; and $M_{i-1} \cap M_i = K_{i-1} \times \{i-1\}$. The infinite union $\mathrm{Tel}\,(\{K_i, f_i\}) = \bigcup_{i=1}^{\infty} M_i$, with the obvious topology is called the *mapping telescope* of (3.9). See Fig. 3.7.

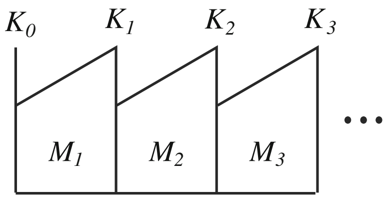


Fig. 3.7 The mapping telescope $\mathrm{Tel}\,(\{K_i, f_i\})$

For each $x \in K_i$, the (embedded) copy of the interval $\{x\} \times [i-1, i]$ in M_i is called a *mapping cylinder line*. The following observations are straightforward.

- $\operatorname{Tel}(\{K_i, f_i\})$ may be viewed as the union of infinite and dead end "telescope rays", each of which begins in $K_0 \times \{0\}$ and intersects a given M_i in a mapping cylinder line or not at all. The dead end rays and empty intersections occur only when a point $k \in K_j$ is not in the image of f_{j+1}; whereas, the infinite telescope rays are proper and in one-to-one correspondence with $\varprojlim \{K_i, f_i\}$,
- by choosing a canonical set of strong deformation retractions of the above rays to their initial points, one obtains a strong deformation retraction of $\operatorname{Tel}(\{K_i, f_i\})$ to $K_0 \times \{0\}$.
- letting $U_k = \bigcup_{i=k+1}^{\infty} M_i$ provides a cofinal sequence of neighborhoods of infinity. By a small variation on the previous observation each $K_i \times \{i\} \hookrightarrow U_i$ is a homotopy equivalence. (So $\operatorname{Tel}(\{K_i, f_i\})$ is absolutely inward tame.)
- letting r be the proper ray consisting of the cylinder lines connecting each p_i to p_{i-1}, we obtain a representation of pro-$\pi_1(\varepsilon(X), r)$ which is pro-isomorphic to the sequence

$$\pi_1(K_0, p_0) \xleftarrow{f_{1\#}} \pi_1(K_1, p_1) \xleftarrow{f_{2\#}} \pi_1(K_2, p_2) \xleftarrow{f_{3\#}} \cdots$$

- in the same manner, representations of pro-$\pi_k(\varepsilon(X), r)$ and pro-$H_k(\varepsilon(X), \mathbb{Z})$ can be obtained by applying the appropriate functor to sequence (3.9).

Proposition 3.4.16 *For every inverse sequence* $G_0 \xleftarrow{\mu_1} G_1 \xleftarrow{\mu_2} G_2 \xleftarrow{\mu_3} \cdots$ *of finitely presented groups, there exists a 1-ended, absolutely inward tame, locally finite CW complex X and a proper ray r such that pro-$\pi_1(\varepsilon(X), r)$ is represented by that sequence. If desired, X can be chosen to be contractible.*

Proof For each i, let K_i be a presentation 2-complex for G_i and let $f_i : K_i \to K_{i-1}$ be a cellular map that induces μ_i. Then let $X = \operatorname{Tel}(\{K_i, f_i\})$.

In order to make X contractible, one simply adds a trivial space $K_{-1} = \{p_{-1}\}$ to the left end of the sequence of complexes.

Example 3.4.17 An easy application of Proposition 3.4.16 produces a pro-$\pi_1(\varepsilon(X), r)$ equal to the inverse sequence $\mathbb{Z} \xleftarrow{\times 2} \mathbb{Z} \xleftarrow{\times 2} \mathbb{Z} \xleftarrow{\times 2} \cdots$ discussed in Exercise (3.4.4). For each i, let $\mathbb{S}^1_i$ be a copy of the unit circle and $f_i : \mathbb{S}^1_i \to \mathbb{S}^1_{i-1}$ the standard degree 2 map. Then $X = \operatorname{Tel}\left(\left\{\mathbb{S}^1_i, f_i\right\}\right)$ is 1-ended and has the desired fundamental group at infinity.

Proposition 3.4.18 *For every inverse sequence* $G_0 \xleftarrow{\mu_1} G_1 \xleftarrow{\mu_2} G_2 \xleftarrow{\mu_3} \cdots$ *of finitely presented groups and $n \geq 6$, there exists a 1-ended open n-manifold M^n such that pro-$\pi_1(M^n, r)$ is represented by that sequence. If a (noncompact) boundary is permitted, and $n \geq 7$, then M^n can be chosen to be contractible.*

Proof Let $X = \operatorname{Tel}(\{K_i, f_i\})$ as constructed in the previous Proposition. With some extra care, arrange for X to be a simplicial 3-complex, and choose a proper PL embedding into $\mathbb{R}^{n+1}$. Let N^{n+1} be a regular neighborhood of that embedding. It is easy to see that pro-$\pi_1(N^{n+1}, r)$ is identical to pro-π_1 $(\varepsilon(X), r)$, so if boundary is permitted, we are finished. If not, let $M^n = \partial N^{n+1}$. By general position, the base ray r may be slipped off X and then isotoped to a ray r' in M^n. Also by general position, loops and disks in N^{n+1} may be slipped off X and then pushed into M^n. In doing so, one sees that pro-π_1 $\left(M^n, r'\right)$ is pro-isomorphic to pro-π_1 $\left(N^{n+1}, r\right)$.

In the study of compact manifolds, results like Poincaré duality place significant restrictions on the topology of closed manifolds. A similar phenomenon occurs in the study of noncompact manifolds. In that setting, it is the open manifolds (and to a similar extent, manifolds with compact boundary) that are the more rigidly restricted. If an open manifold is required to satisfy additional niceness conditions, such as contractibility, finite homotopy type, or inward tameness, even more rigidity comes into play. This is at the heart of the study of noncompact manifolds, where a goal is to obtain strong conclusions about the structure of a manifold from modest hypotheses.

Exercise 3.4.19 Show that an inward tame manifold M^n with compact boundary cannot have infinitely many ends. (*Hint*: Homology with $\mathbb{Z}_2$-coefficients simplifies the algebra and eliminates issues related to orientability.) Show that this result fails if we omit the tameness hypothesis or if M^n is permitted to have noncompact boundary.

Exercise 3.4.20 Show that the inverse sequence realized in Example 3.4.17 cannot occur as pro-π_1 $(\varepsilon(M^n), r)$ for a contractible open manifold. *Hint*: A look ahead to Sect. 3.5.1 may be helpful.

The trick used in the proof of Proposition 3.4.16 for obtaining a *contractible* mapping telescope with the same end behavior as one that is homotopically nontrivial is often useful. Given an inverse sequence $\{K_i\}$ of finite CW complexes, the *augmented inverse sequence* $\{K_i, f_i\}^\bullet$ is obtained by inserting a singleton space at the beginning of $\{K_i, f_i\}$; the corresponding *contractible mapping telescope* CTel $(\{K_i, f_i\})$ is contractible, but identical to Tel $(\{K_i, f_i\})$ at infinity.

3.4.6 On the Role of the Base Ray

We now begin the detailed discussion of the role of base rays in the fundamental group at infinity—a topic more subtle and more interesting than one might expect.

As hinted earlier, small changes in base ray, such as reparametrization or deletion of an initial segment, do not alter pro-π_1 $(\varepsilon(X), r)$; this follows from a more general result to be presented shortly. On the other hand, large changes can obviously have an impact. For example, if X is multi-ended and r and s point to different ends, then pro-π_1 $(\varepsilon(X), r)$ and pro-π_1 $(\varepsilon(X), s)$ provide information about different portions of X—much as the traditional fundamental group of a non-path-connected space provides different information when the base point is moved from one component to another. When r and s point to the same end of X, it is reasonable to expect pro-$\pi_1(\varepsilon(X), r)$ and pro-$\pi_1(\varepsilon(X), s)$ to be pro-isomorphic—but this is not the case either! At the heart of the matter is the difference between the set of ends $\mathscr{E}(X)$ and the set of strong ends $\mathscr{SE}(X)$. The following requires some effort, but the proof is completely elementary.

Proposition 3.4.21 *If proper rays r and s in X are strongly equivalent, i.e., properly homotopic, then pro-$\pi_1(\varepsilon(X), r)$ and pro-$\pi_1(\varepsilon(X), s)$ are pro-isomorphic.*

Corollary 3.4.22 *If X is strongly connected at infinity, i.e., $|\mathscr{SE}(X)| = 1$, then pro-$\pi_1(\varepsilon(X))$ is a well-defined invariant of X.*

Exercise 3.4.23 Prove Proposition 3.4.21.

Remark 3.4.24 There are useful analogies between the role played by base points in the fundamental group and that played by base rays in the fundamental group at infinity:

- The fundamental group is a functor from the category of *pointed spaces*, i.e., pairs (Y, p), where $p \in Y$, to the category of groups. In a similar manner, the fundamental group at infinity is a functor from the proper category of pairs (X, r), where r is a proper ray in X, to the category pro-$\mathscr{G}$roups.
- If there is a path α in Y from p to q in Y, there is a corresponding isomorphism $\widehat{\alpha} : \pi_1(Y, p) \to \pi_1(Y, q)$. If there is a proper homotopy in X between proper rays r and s, then there is a corresponding pro-isomorphism between pro-$\pi_1(\varepsilon(X), r)$ and pro-$\pi_1(\varepsilon(X), s)$.
- Even for connected Y there may be no relationship between $\pi_1(Y, p)$ and $\pi_1(Y, q)$ when there is no path connecting p to q. Similarly, for a 1-ended space X, pro-$\pi_1(\varepsilon(X), r)$ and pro-$\pi_1(\varepsilon(X), s)$ may be very different if there is no proper homotopy from r to s.

We wish to describe a 1-ended Y with proper rays r and s for which pro-$\pi_1(\varepsilon(X), r)$ and pro-$\pi_1(\varepsilon(X), s)$ are not pro-isomorphic. We begin with an intermediate space.

Example 3.4.25 (*Another space with $\mathscr{SE}(X) \neq \mathscr{E}(X)$*) Let $X = \mathrm{CTel}\left(\left\{\mathbb{S}^1_i, f_i\right\}\right)$ where each $\mathbb{S}^1_i$ is a copy of the unit circle and $f_i : \mathbb{S}^1_i \to \mathbb{S}^1_{i-1}$ is the standard degree 2 map (see Example 3.4.17). If p_i is the canonical base point for $\mathbb{S}^1_i$ and $f_i(p_i) = p_{i-1}$ for all i, we may construct a "straight" proper ray r by concatenating the mapping cylinder lines α_i connecting p_i and p_{i-1}. Construct a second proper ray s by splicing between each α_i and α_{i+1} a loop β_i that goes once in the positive direction around $\mathbb{S}^1_i$; in other words, $s = \alpha_0 \cdot \beta_0 \cdot \alpha_1 \cdot \beta_1 \cdot \alpha_2 \cdot \cdots$. With some effort, it can be shown that r and s are not properly homotopic. That observation is also a corollary of the next example.

Example 3.4.26 For each i, let K_i be a wedge of two circles and let $g_i : K_i \to K_{i-1}$ send one of those circles onto itself by the identity and the other onto itself via the standard degree 2 map. Let $Y = \mathrm{CTel}(\{K_i, g_i\})$. This space may be viewed as the union of X from Example 3.4.25 and an infinite cylinder $\mathbb{S}^1 \times [0, \infty)$, coned off at the left end, with the union identifying the ray r with a standard ray in the product. By viewing X as a subset of Y, view r and s as proper rays in Y.

Choose neighborhoods of infinity U_i as described in Sect. 3.4.5. Each has fundamental group that is free of rank 2. If we let F_i be the free group of rank 2 with formal generators a^{2i} and b then, pro-$\pi_1(\varepsilon(Y), r)$ may be represented by

$$\langle a, b\rangle \hookleftarrow \langle a^2, b\rangle \hookleftarrow \langle a^4, b\rangle \hookleftarrow \cdots .$$

Similarly pro-$\pi_1(\varepsilon(Y), s)$ may be represented by

$$\langle a, b\rangle \xleftarrow{\lambda_1} \langle a^2, b\rangle \xleftarrow{\lambda_2} \langle a^4, b\rangle \xleftarrow{\lambda_3} \cdots .$$

where $\lambda_i\left(a^{2i}\right) = a^{2i}$ and $\lambda_i(b) = a^{2i}ba^{-2i}$. Taking inverse limits, produces $\check{\pi}_1(\varepsilon(Y), r) = \langle b\rangle \cong \mathbb{Z}$ and $\check{\pi}_1(\varepsilon(Y), s) = 1$. Hence pro-$\pi_1(\varepsilon(Y), r)$ and pro-$\pi_1(\varepsilon(Y), s)$ are not pro-isomorphic.

Exercise 3.4.27 Verify the assertions made in each of the two previous examples.

The fact that a 1-ended space can have multiple fundamental groups at infinity might lead one to doubt the value of that invariant. Over the next several sections we provide evidence to counter that impression. For example, we will investigate some properties of pro-π_1 that persist under change of base ray. Furthermore, we will see that in some of the most important situations, there is (verifiably in many cases and conjecturally in others) just one proper homotopy class of base ray—causing the ambiguity to vanish. As an example, the following important question is open.

Conjecture 3.4.28 (The Manifold Semistability Conjecture–version 1) *The universal cover of a closed aspherical manifold of dimension greater than 1 is always strongly connected at infinity.*

We stated the above problem as a conjecture because it is a special case of the following better-known conjecture. For now the reader can guess at the necessary definitions. The meaning will be fully explained in Sect. 3.6. The naming of these conjectures will be explained over the next couple of pages.

Conjecture 3.4.29 (The Semistability Conjecture-version 1) *Every finitely presented 1-ended group is strongly connected at infinity.*

3.4.7 Flavors of Inverse Sequences of Groups

When dealing with pro-isomorphism classes of inverse sequences of groups, general properties are often more significant than the sequences themselves. In this section we discuss several such properties.

Let $G_0 \xleftarrow{\mu_1} G_1 \xleftarrow{\mu_2} G_2 \xleftarrow{\mu_3} G_2 \xleftarrow{\mu_4} \cdots$ be an inverse sequence of groups. We say that $\{G_i, \mu_i\}$ is

- *pro-trivial* if it is pro-isomorphic to the trivial inverse sequence $1 \leftarrow 1 \leftarrow 1 \leftarrow 1 \leftarrow \cdots$,
- *stable* if it is pro-isomorphic to an inverse sequence $\{H_i, \lambda_i\}$ where each λ_i is an isomorphism, or equivalently, a constant inverse sequence $\{H, \mathrm{id}_H\}$,
- *semistable* (or *Mittag-Leffler,* or *pro-epimorphic*) if it is pro-isomorphic to an $\{H_i, \lambda_i\}$, where each λ_i is an epimorphism, and
- *pro-monomorphic* if it is pro-isomorphic to an $\{H_i, \lambda_i\}$, where each λ_i is a monomorphism.

The following easy exercise will help the reader develop intuition for the above definitions, and for the notion of pro-isomorphism itself.

Exercise 3.4.30 Show that an inverse sequence of non-injective epimorphisms cannot be pro-monomorphic, and that an inverse sequence of non-surjective monomorphisms cannot be semistable.

Exercise 3.4.31 Show that if $\{G_i, \mu_i\}$ is stable and thus pro-isomorphic to some $\{H, \mathrm{id}_H\}$, then H is well-defined up to isomorphism. In that case $H \cong \varprojlim \{G_i, \mu_i\}$.

A troubling aspect of the above definitions is that the concepts appear to be extrinsic, requiring a second unseen sequence, rather than being intrinsic to the given sequence. A standard result corrects that misperception.

Proposition 3.4.32 *An inverse sequence of groups $\{G_i, \lambda_i\}$ is stable if and only if it contains a subsequence for which "passing to images" results in an inverse sequence of isomorphisms, in other words: we may obtain a diagram of the following form, where all unlabeled homomorphisms are obtained by restriction or inclusion.*

$$\begin{tikzcd}
G_{i_0} & & G_{i_1} \arrow[ll, "\lambda_{i_0,i_1}"'] \arrow[dl] & & G_{i_2} \arrow[ll, "\lambda_{i_1,i_2}"'] \arrow[dl] & & G_{i_3} \arrow[ll, "\lambda_{i_2,i_3}"'] \arrow[dl] & \cdots \\
& \mathrm{Im}\left(\lambda_{i_0,i_1}\right) \arrow[ul, hook] & & \mathrm{Im}\left(\lambda_{i_1,i_2}\right) \arrow[ul, hook] \arrow[ll, "\cong"'] & & \mathrm{Im}\left(\lambda_{i_2,i_3}\right) \arrow[ul, hook] \arrow[ll, "\cong"'] & & \cdots \arrow[ll, "\cong"']
\end{tikzcd} \tag{3.10}$$

Analogous statements are true for the pro-epimorphic and pro-monomorphic sequences; in those cases we require maps in the bottom row of (3.10) to be epimorphisms, and monomorphisms, respectively.

Proof of the above is an elementary exercise, as is the following:

Proposition 3.4.34 *An inverse sequence is stable if and only if it is both pro-epimorphic and pro-monomorphic.*

Exercise 3.4.35 Prove the previous two Propositions.

3.4.8 Some Topological Interpretations of the Previous Definitions

It is common practice to characterize simply connected spaces topologically (without mentioning the word 'group'), as path-connected spaces in which every loop contracts to a point. In that spirit, we provide topological characterizations of spaces whose fundamental groups at infinity possess some of the algebraic properties discussed in the previous section.

Proposition 3.4.36 *For a 1-ended space X and a proper ray r, pro-$\pi_1(\varepsilon(X), r)$ is*

(a) pro-trivial if and only if: for any compact $C \subseteq X$, there exists a larger compact set D such that every loop in $X - D$ contracts in $X - C$,
(b) semistable if and only if: for any compact $C \subseteq X$, there exists a larger compact set D such that, for every still larger compact E, each pointed loop α in $X - D$ based on r can be homotoped into $X - E$ via a homotopy into $X - C$ that slides the base point along r, and
(c) pro-monomorphic if and only if there exists a compact $C \subseteq X$ such that, for every compact set D containing C, there exists a compact E such that every loop in $X - E$ that contracts in $X - C$ contracts in $X - D$.

Proof This is a straightforward exercise made easier by applying Proposition 3.4.32.

Note that the topological condition in part (a) of Proposition 3.4.36 makes no mention of a base ray. So (for 1-ended spaces) the property of having pro-trivial fundamental group at infinity is independent of base ray; such spaces are called *simply connected at infinity*. Similarly, the topological condition in (c) is independent of base ray; 1-ended spaces with that property are called *pro-monomorphic at infinity* (or simply *pro-monomorphic*). And despite the (unavoidable) presence of a base ray in the topological portion of (b), there does exist an elegant and useful characterization of spaces with semistable pro-π_1.

Proposition 3.4.37 *A 1-ended space X is strongly connected at infinity if and only if there exists a proper ray r for which pro-$\pi_1(\varepsilon(X), r)$ is semistable.*

Proof (*Sketch*) First we outline a proof of the reverse implication. Let r be as in the hypothesis and let s be another proper ray. By 1-endedness, there is a proper map h of the infinite ladder $L_{[0,\infty)} = ([0, \infty) \times \{0, 1\}) \cup (\mathbb{N} \times [0, 1])$ into X, with

$h|_{[0,\infty)\times 0} = r$ and $h|_{[0,\infty)\times 1} = s$. For convenience, choose an exhaustion of X by compacta $\varnothing = C_0 \subseteq C_1 \subseteq C_2 \subseteq \cdots$ with the property that the subladder $L_{[i,\infty)}$ is sent into $U_i = \overline{X - C_i}$ for each $i \geq 1$. As a simplifying hypothesis, assume that all bonding homomorphisms in the corresponding inverse sequence

$$\pi_1(X, p_0) \xleftarrow{\lambda_1} \pi_1(X - C_1, p_1) \xleftarrow{\lambda_2} \pi_1(X - C_2, p_2) \xleftarrow{\lambda_3} \cdots \tag{3.11}$$

are surjective. (For a complete proof, one should instead apply Proposition 3.4.36 inductively.)

We would like to extend h to a proper map of $[0,\infty) \times [0,1]$ into X. To that end, let $\square_i$ be the loop in X corresponding to $r_{i+1} \cup e_{i+1} \cup s_{i+1}^{-1} \cup e_i^{-1}$ in $L_{[0,\infty)}$. (Here $r_{i+1} = r|_{[i,i+1]}$ and $s_{i+1} = s|_{[i,i+1]}$; $e_j = h|_{j\times[0,1]}$, the jth "rung" of the ladder.)

If each $\square_i$ contracts in X we can use those contractions to extend h to $[0,\infty) \times [0,1]$; if each $\square_i$ contracts in $X - C_i$ the resulting extension is proper (as required). The idea of the proof is to arrange those conditions. Begin inductively with $\square_0$. If this loop does not contract in X, we make it so by rechoosing e_1 as follows: choose a loop α_1 based at p_1 so that $r_1 \cdot \alpha_1 \cdot r_1^{-1}$ is equal to $\square_0$ in $\pi_1(X, p_0)$. Replace e_1 with the rung $\hat{e}_1 = \alpha_1^{-1} \cdot e_1$. The newly modified $\square_0$ contracts in X, as desired. Now move to the correspondingly modified $\square_1$ viewed as an element of $\pi_1(X - C_1, p_1)$. If it is nontrivial, choose a loop α_2 in $X - C_2$ based at p_2 such that $\lambda_2(\alpha_2) = r_2 \cdot \alpha_2 \cdot r_2^{-1} = \square_1$. Replacing e_2 with $\hat{e}_2 = \alpha_2^{-1} \cdot e_2$ results in a further modified $\square_1$ that contracts in $X - C_1$. Continue this process inductively to obtain a proper homotopy $H : [0,\infty) \times [0,1] \to X$ between r and s.

For the reverse implication, assume that (3.11) is not semistable. One creates a proper ray s not properly homotopic to r by affixing to each vertex p_i of r a loop in β_i in $X - C_i$ that that does not lie in the image of $\pi_1(X - C_{i+1}, p_{i+1})$. More specifically

$$s = r_1 \cdot \alpha_1 \cdot r_2 \cdot \alpha_2 \cdot r_3 \cdot \alpha_3 \cdot \cdots .$$

As a result of Proposition 3.4.37, a 1-ended space X may be called *semistable at infinity* [respectively, *stable at infinity*] if pro-$\pi_1(\varepsilon(X), r)$ is semistable [respectively, stable] for some (and hence any) proper ray r. Alternatively, a 1-ended space is sometimes *defined* to be semistable at infinity (or just *semistable*) if all proper rays in X are properly homotopic. In those cases we often drop the base ray and refer to the homotopy end invariants simply as pro-$\pi_1(\varepsilon(X))$ and $\check{\pi}_1(\varepsilon(X))$.

Multi-ended spaces are sometimes called semistable if, whenever two proper rays determine the same end, they are properly homotopic; or equivalently, when $\Phi : \mathscr{SE}(X) \to \mathscr{E}(X)$ is bijective.

Remark 3.4.38 By using the sketched proof of Proposition 3.4.37 as a guide, it is not hard to see why a 1-ended space X that is not semistable will necessarily have *uncountable* $\mathscr{SE}(X)$. A method for placing $\mathscr{SE}(X)$ into an algebraic context involves the *derived limit* or '$\lim^1$ *functor*'. More generally, $\lim^1\{G_i, \mu_i\}$ is an algebraic construct that helps to recover the information lost when one passes from an inverse sequence to its inverse limit. See [45, Sect. 11.3].

3.5 Applications of End Invariants to Manifold Topology

Although a formal study of pro-homotopy and pro-homology of the ends of noncompact space is not a standard part of the education of most manifold topologists, there are numerous important results and open questions best understood in that context. In this section we discuss several of those, beginning with classical results and moving toward recent work and still-open questions.

3.5.1 Another Look at Contractible Open Manifolds

We now return to the study of contractible open manifolds begun in Sect. 3.2. We will tie up some loose ends from those earlier discussions—most of which focused on specific examples. We also present some general results whose hypotheses involve nothing more than the fundamental group at infinity.

Theorem 3.5.1 (Whitehead's Exotic Open 3-manifold) *There exists a contractible open* 3*-manifold not homeomorphic to* $\mathbb{R}^3$.

Proof We wish to nail down a proof that the Whitehead contractible 3-manifold $\mathscr{W}^3$ described in Sect. 3.2.1 is not homeomorphic to $\mathbb{R}^3$. We do that by showing $\mathscr{W}^3$ is not simply connected at infinity. Using the representation of pro-$\pi_1\left(\varepsilon(\mathscr{W}^3), r\right)$ obtained in Sect. 3.2.2 and applying the rigorous development from Sect. 3.4, we can accomplish that task with an application of Exercise 3.4.30.

Theorem 3.5.2 *The open Newman contractible n-manifolds are not homeomorphic to* $\mathbb{R}^n$*. More generally, any compact contractible n-manifold with non-simply connected boundary has interior that is not homeomorphic to* $\mathbb{R}^n$.

Proof Combine our observations from Example 3.2.16 with Exercise 3.4.31—or simply observe that the topological characterization of simply connected at infinity fails.

The next result establishes simple connectivity at infinity as the definitive property in determining whether a contractible open manifold is exotic. The initial formulation is due to Stallings [88], who proved it for PL manifolds of dimension ≥ 5; his argument is clean, elegant, and highly recommended—but outside the scope of these notes. That result was extended to all topological manifolds of dimension ≥ 5 by Luft [61]. Extending the result to dimensions 3 and 4 requires the Fields Medal winning work of Perelman and Freedman [41], respectively. The foundation for the 3-dimensional result was laid by C.H. Edwards in [33].

Theorem 3.5.3 (Stallings' Characterization of $\mathbb{R}^n$) *A contractible open n-manifold ($n \geq 3$) is homeomorphic to $\mathbb{R}^n$ if and only if it is simply connected at infinity.*

Exercise 3.5.4 Prove the following corollary to Theorem 3.5.3: If W^n is a contractible open manifold, then $W^n \times \mathbb{R} \approx \mathbb{R}^{n+1}$.

The next application of the fundamental group at infinity returns us to another prior discussion.

Theorem 3.5.5 (Davis' Exotic Universal Covering Spaces) *For $n \geq 4$, there exist closed aspherical n-manifolds whose universal covers are not homeomorphic to $\mathbb{R}^n$.*

Proof Here we provide only the punch-line to this major theorem. As noted in Sect. 3.2.1 Davis' construction produces closed aspherical n-manifolds M^n with universal covers homeomorphic to the infinite open sums described in Example 3.2.5 and Theorem 3.2.6. As observed in Example 3.2.17, pro-$\pi_1\left(\varepsilon(\widetilde{M}^n), r\right)$ may be represented by

$$G \twoheadleftarrow G * G \twoheadleftarrow G * G * G \twoheadleftarrow G * G * G * G \twoheadleftarrow \cdots, \tag{3.12}$$

a sequence that is semistable but not pro-monomorphic. An application of Exercise 3.4.30 verifies that $\widetilde{M}^n$ is not simply connected at infinity.

After Davis showed that aspherical manifolds need not be covered by $\mathbb{R}^n$, many questions remained. With the 3-dimensional version unresolved (at the time), it was asked whether the Whitehead manifold could cover a closed 3-manifold. In higher dimensions, people wondered whether a Newman contractible open manifold (or the interior of another compact contractible manifold) could cover a closed manifold. Myers [76] resolved the first question in the negative, before Wright [100] proved a remarkably general result in which the fundamental group at infinity plays the central role.

Theorem 3.5.7 (Wright's Covering Space Theorem) *Let M^n be a contractible open n-manifold with pro-monomorphic fundamental group at infinity. If M^n admits a nontrivial action by covering transformations, then $M^n \approx \mathbb{R}^n$.*

Corollary 3.5.8 *Neither the Whitehead manifold nor the interior of any compact contractible manifold with non-simply connected boundary can cover a manifold nontrivially.*

Wright's theorem refocuses attention on a question mentioned earlier.

Conjecture 3.5.9 (The Manifold Semistability Conjecture) *Must the universal cover of every closed aspherical manifold have semistable fundamental group at infinity?*

More generally we can ask:

Vague Question: *Must all universal covers of aspherical manifolds be similar to the Davis examples?*

In discussions still to come, we will make this vague question more precise. But, before moving on, we note that in 1991 Davis and Januszkiewicz [28] invented a new strategy for creating closed aspherical manifolds with exotic universal covers. Although that strategy is very different from Davis' original approach, the resulting exotic covers are remarkably similar. For example, their fundamental groups at infinity are precisely of the form (3.12).

Exercise 3.5.10 Theorem 3.5.3 suggests that the essence of a contractible open manifold is contained in its fundamental group at infinity. Show that every contractible open n-manifold W^n has the same *homology* at infinity as $\mathbb{R}^n$. In particular, show that for all $n \geq 2$, pro-H_i $(W^n; \mathbb{Z})$ is stably $\mathbb{Z}$ if $i = 0$ or $n - 1$ and pro-trivial otherwise. *Note*: This exercise may be viewed as a continuation of Exercise 3.3.3.

3.5.2 Siebenmann's Thesis

Theorem 3.5.3 may be viewed as a classification of those open manifolds that can be compactified to a closed n-ball by addition of an $(n-1)$-sphere boundary. More generally, one may look to characterize open manifolds that can be compactified to a manifold with boundary by addition of a boundary $(n-1)$-manifold. Since the boundary of a manifold P^n always has a *collar neighborhood* $N \approx \partial P^n \times [0, 1]$, an open manifold M^n allows such a compactification if and only if it contains a neighborhood of infinity homeomorphic to an *open collar* $Q^{n-1} \times [0, 1)$, for some closed $(n-1)$-manifold Q^{n-1}. We refer to open manifolds of this sort as being *collarable*.

The following shows that, to characterize collarable open manifolds, it is not enough to consider the fundamental group at infinity.

Example 3.5.11 Let M^n be the result of a countably infinite collection of copies of $\mathbb{S}^2 \times \mathbb{S}^{n-2}$ connect-summed to $\mathbb{R}^n$ along a sequence of n-balls tending to infinity (see Fig. 3.8). Provided $n \geq 4$, M^n is simply connected at infinity. Moreover, since a compact manifold with boundary has finite homotopy type, and since the addition of a manifold boundary does not affect homotopy type, this M^n admits no such compactification.

Fig. 3.8 $\mathbb{R}^n$ connect-summed with infinitely many $\mathbb{S}^2 \times \mathbb{S}^{n-2}$

For manifolds that are simply connected at infinity, the necessary additional hypothesis is as simple as one could hope for.

Theorem 3.5.12 (See [15]) *Let W^n be a 1-ended open n-manifold ($n \geq 6$) that is simply connected at infinity. Then W^n is collarable if and only if $H_*(W; \mathbb{Z})$ is finitely generated.*

For manifolds not simply connected at infinity, the situation is more complicated, but the characterization is still remarkably elegant. It is one of the best-known and most frequently applied theorems in manifold topology.

Theorem 3.5.13 (Siebenmann's Collaring Theorem) *A 1-ended n-manifold W^n ($n \geq 6$) with compact (possibly empty) boundary is collarable if and only if*

(a) *W^n is inward tame,*
(b) *pro-$\pi_1(\varepsilon(W^n))$ is stable, and*
(c) *$\sigma_\infty(W^n) \in \widetilde{K}_0\left(\mathbb{Z}\left[\check{\pi}_1(\varepsilon(X))\right]\right)$ is trivial.*

Remark 3.5.14 (1) Under the assumption of hypotheses (a) and (b), $\sigma_\infty(W^n)$ is defined to be the Wall finiteness obstruction $\sigma(N)$ of a single clean neighborhood of infinity, chosen so that its fundamental group (under inclusion) matches $\check{\pi}_1(\varepsilon(X))$. A more general definition for $\sigma_\infty(W^n)$ —one that can be used when pro-$\pi_1(\varepsilon(W^n))$ is not stable—will be introduced in Sect. 3.5.3.

(2) Together, assumptions (a) and (c) are equivalent to assuming that W^n is absolutely inward tame. That would allow for a simpler statement of the Collaring Theorem; however, the power of the given version is that it allows the finiteness obstruction to be measured on a single (appropriately chosen) neighborhood of infinity. Furthermore, in a number of important cases, $\sigma_\infty(W^n)$ is trivial for algebraic reasons. That is the case, for example, when $\check{\pi}_1(\varepsilon(X))$ is trivial, free, or free abelian, by a fundamental result of algebraic K-theory found in [5].

(3) Due to stability, no base ray needs to be mentioned in Condition (b). Use of the Čech fundamental group in Condition (c) is just a convenient way of specifying the single relevant group implied by Condition (b) (see Exercise 3.4.31).

(4) Since an inward tame manifold with compact boundary is necessarily finite-ended (see Exercise 3.4.19), the 1-ended hypothesis is easily eliminated from the above by requiring each end to satisfy (b) and (c), individually.

(5) By applying [41], Theorem 3.5.13 can be extended to dimension 5, provided $\check{\pi}_1(\varepsilon(X))$ is a "good" group, in the sense of [42]; whether the theorem holds for all 5-manifolds is an open question. Meanwhile, Kwasik and Schultz [60] have shown that Theorem 3.5.13 fails in dimension 4; partial results in that dimension can be found in [42, Sect. 11.9]. By combining the solution to the Poincaré Conjecture with work by Tucker [94], one obtains a strong 3-Dimensional Collaring Theorem—only condition (1) is necessary. For classical reasons, the same is true for $n = 2$. And for $n = 1$, there are no issues.

The proof of Theorem 3.5.13 is intricate in detail, but simple in concept. Readers unfamiliar with h-cobordisms and s-cobordisms, and their role in the topology of manifolds, should consult [83].

Proof (Siebenmann's Theorem (outline)) Since a 1-ended collarable manifold is easily seen to be absolutely inward tame with stable fundamental group at infinity, conditions (1)–(3) are necessary. To prove sufficiency, begin with a cofinal sequence $\{N_i\}_{i=0}^{\infty}$ of clean neighborhoods of infinity with $N_{i+1} \subseteq \text{int}N_i$ for all i. After some initial combinatorial group theory, a 2-dimensional disk trading argument allows us to improve the neighborhoods of infinity so that, for each i, N_i and ∂N_i have fundamental groups corresponding to the stable fundamental group $\check{\pi}_1(\varepsilon(W^n))$. More precisely, each inclusion induces isomorphisms $\pi_1(\partial N_i) \overset{\cong}{\to} \pi_1(N_i)$ and $\pi_1(N_{i+1}) \overset{\cong}{\to} \pi_1(N_i)$, with each group being isomorphic to $\check{\pi}_1(\varepsilon(W^n))$.

Under the assumption that one of these N_i has trivial finiteness obstruction, the "Sum Theorem" for the Wall obstruction (first proved in [86] for this purpose) together with the above π_1-isomorphisms, implies that all N_i have trivial finiteness obstruction. From there, a carefully crafted sequence of modifications to these neighborhoods of infinity—primarily handle manipulations—results in a further improved sequence of neighborhoods of infinity with the property that $\partial N_i \hookrightarrow N_i$ is a homotopy equivalence for each i. The resulting cobordisms $(A_i, \partial N_i, \partial N_{i+1})$, where $A_i = \overline{N_i - N_{i+1}}$ are then h-cobordisms (See Exercise 3.5.15).

A clever "infinite swindle" allows one to trivialize the Whitehead torsion of $\partial N_i \hookrightarrow A_i$ in each h-cobordism by inductively borrowing the inverse h-cobordism B_i from a collar neighborhood of ∂N_{i+1} in A_{i+1} (after which the "new" N_{i+1} is $\overline{N_{i+1} - B_i}$), until the s-cobordism theorem yields $A_i \approx \partial N_i \times [i, i+1]$, for each i. Gluing these products together completes the proof.

Exercise 3.5.15 Verify the h-cobordism assertion in the above paragraph. In particular, let N_i and N_{i+1} be clean neighborhoods of infinity with $\text{int}N_i \supseteq N_{i+1}$ satisfying the properties: *(1)* $\partial N_i \hookrightarrow N_i$ and $\partial N_{i+1} \hookrightarrow N_{i+1}$ are homotopy equivalences and *(2)* $N_{i+1} \hookrightarrow N_i$ induces a π_1-isomorphism. For $A_i = \overline{N_i - N_{i+1}}$, show that both $\partial N_i \hookrightarrow A_i$ and $\partial N_{i+1} \hookrightarrow A_i$ are homotopy equivalences.

Observe that in the absence of Condition (2), it is still possible to conclude that $(A_i, \partial N_i, \partial N_{i+1})$ is a "1-sided h-cobordism", in particular, $\partial N_i \hookrightarrow A_i$ is a homotopy equivalence.

In the spirit of the result in Exercise 3.5.4, the following may be obtained as an application of Theorem 3.5.13.

Theorem 3.5.16 ([50]) *For an open manifold M^n $(n \geq 5)$, the "stabilization" $M^n \times \mathbb{R}$ is collarable if and only if M^n has finite homotopy type.*

Proof (*Sketch*) Since a collarable manifold has finite homotopy type, and since $M^n \times \mathbb{R}$ is homotopy equivalent to M^n, it is clear that M^n must have finite homotopy in order for $M^n \times \mathbb{R}$ to be collarable. To prove sufficiency of that condition, we wish to verify that the conditions Theorem 3.5.13 are met by $M^n \times \mathbb{R}$.

Conditions (a) and (c) are relatively easy, and are left as an exercise (see below). The key step is proving stability of pro-π_1 $(\varepsilon(M^n \times \mathbb{R}), r)$. We will say just enough to convey the main idea—describing a technique that has been useful in several other contexts. Making these argument rigorous is primarily a matter of base points and base rays—a nontrivial issue, but one that we ignore for now. (See [50] for the details.)

For simplicity, assume M^n is 1-ended and $N_0 \supseteq N_1 \supseteq N_2 \supseteq \cdots$ is a cofinal sequence of clean connected neighborhoods of infinity in M^n. If $R_i = (M^n \times (-\infty, -i] \cup [i, \infty)) \cup (N_i \times \mathbb{R})$, then $\{R_i\}$ forms a cofinal sequence of clean connected neighborhoods of infinity in $M^n \times \mathbb{R}$. If $G = \pi_1(M^n)$ and $H_i = \operatorname{Im}(\pi_1(N_i) \to \pi_1(M^n))$ for each i, then $\pi_1(R_i) = G *_{H_i} G$ and pro-π_1 $(\varepsilon(M^n \times \mathbb{R}), r)$ may be represented by

$$G *_{H_0} G \twoheadleftarrow G *_{H_1} G \twoheadleftarrow G *_{H_2} G \twoheadleftarrow \cdots$$

where the bonds are induced by the identities on G factors. Notice that each H_{i+1} injects into H_i. To prove stability, it suffices to show that, eventually, H_{i+1} goes onto H_i. To that end, we argue that every loop in N_i can be homotoped into N_{i+1} by a homotopy whose tracks may go anywhere in M^n.[10] The loops of concern are those lying in $N_i - N_{i+1}$; let α be such a loop, and assume it is an embedded circle.

By the finite homotopy type of M^n (in fact, finite domination is enough), we may assume the existence of a homotopy S that pulls M^n into $M^n - N_i$. Consider the map $J = S|_{\partial N_i \times [0,1]}$. Adjust J so that it is transverse to the 1-manifold α. Then $J^{-1}(\alpha)$ is a finite collection of circles. With some extra effort we can see that at least one of those circles goes homeomorphically onto α. The strong deformation retraction of $\partial N_i \times [0, 1]$ onto $\partial N_i \times \{0\}$ composed with J pushes α into N_{i+1}.

Exercise 3.5.17 Show that for an open manifold M^n with finite homotopy type, the special neighborhoods of infinity $R_i \subseteq M^n \times \mathbb{R}$, used in the above proof, have finite homotopy type. Therefore, $M^n \times \mathbb{R}$ is absolutely inward tame.

Exercise 3.5.18 Show that if M^n (as above) is finitely dominated, but does not have finite homotopy type, then $M^n \times \mathbb{R}$ satisfies Conditions (1) and (2) of Theorem 3.5.13, but not Condition (3).

3.5.3 Generalizing Siebenmann

Siebenmann's Collaring Theorem and a "controlled" version of it found in [81] have proven remarkably useful in manifold topology; particularly in obtaining the sorts of structure and embedding theorems that symbolize the tremendous activity in high-dimensional manifold topology in the 1960 and 1970s. But the discovery of exotic universal covering spaces, along with a shift in research interests (the Borel and Novikov Conjectures in particular and geometric group theory in general) to

[10] A complete proof would do this while keeping a base point of the loop on a base ray r.

topics where an understanding of universal covers is crucial, suggests a need for results applicable to spaces with *non-stable* fundamental group at infinity. As an initial step, one may ask what can be said about open manifolds satisfying some of Siebenmann's conditions—but not π_1-stability. In Sect. 3.4.5 we described a method for constructing locally finite polyhedra satisfying Conditions (1) and (3) of Theorem 3.5.3, but having almost arbitrary pro-π_1. By the same method, we could build unusual behavior into pro-H_k. So it is a pleasant surprise that, for manifolds with compact boundary, inward tameness by itself, has significant implications.

Theorem 3.5.19 ([52, Theorem 1.2]) *If a manifold with compact (possibly empty) boundary is inward tame, then it has finitely many ends, each of which has semistable fundamental group and stable homology in all dimensions.*

Proof (*Sketch*) Finite-endedness of inward tame manifolds with compact boundary was obtained in Exercise 3.4.19. The π_1-semistability of each end is based on the transversality strategy described in Theorem 3.5.16. Stability of the homology groups is similar, but algebraic tools like duality are also needed.

Siebenmann's proof of Theorem 3.5.13 (as outlined earlier), along with the strategy used by Chapman and Siebenmann in [21] (to be discussed Sect. 3.8.2) make the following approach seem all but inevitable: Define a manifold N^n with compact boundary to be a *homotopy collar* if $\partial N^n \hookrightarrow N^n$ is a homotopy equivalence. A homotopy collar is called a *pseudo-collar* if it contains arbitrarily small homotopy collar neighborhoods of infinity. A manifold that contains a pseudo-collar neighborhood of infinity is called *pseudo-collarable*.

Clearly, every collarable manifold is pseudo-collarable, but the Davis manifolds are counterexamples to the converse (see Example 3.5.24). Before turning our attention to a pseudo-collarability characterization, modeled after Theorem 3.5.13, we spend some time getting familiar with pseudo-collars and their properties.

A cobordism $(A, \partial_- A, \partial_+ A)$ is called a *one-sided h-cobordism* if $\partial_- A \hookrightarrow A$ is a homotopy equivalence, but not necessarily so for $\partial_+ A \hookrightarrow A$. The key connection between these concepts is contained in Proposition 3.5.21. First we state a standard lemma.

Lemma 3.5.20 *Let $(A, \partial_- A, \partial_+ A)$ be a compact one-sided h-cobordism as described above. Then the inclusion $\partial_+ A \hookrightarrow A$ induces $\mathbb{Z}$-homology isomorphisms (in fact, $\mathbb{Z}[\pi_1(A)]$-homology isomorphisms) in all dimensions; in addition, $\pi_1(\partial_+ A) \to \pi_1(A)$ is surjective with perfect kernel.*

Lemma 3.5.20 is obtained from various forms of duality. For details, see [52, Theorem 2.5].

Proposition 3.5.21 (Structure of manifold pseudo-collars) *Let N^n be a pseudo-collar. Then*

(a) *N^n can be expressed as a union $A_0 \cup A_1 \cup A_2 \cup \cdots$ of one-sided h-cobordisms with $\partial_- A_0 = \partial N$ and $\partial_+ A_i = \partial_- A_{i+1} = A_i \cap A_{i+1}$ for all $i \geq 0$,*

(b) *N^n contains arbitrarily small pseudo-collar neighborhoods of infinity,*
(c) *N^n is absolutely inward tame,*
(d) *pro-H_i $(\varepsilon(N^n); \mathbb{Z})$ is stable for all i,*
(e) *pro-π_1 $(\varepsilon(N^n))$ may be represented by a sequence $G_0 \overset{\mu_1}{\twoheadleftarrow} G_1 \overset{\mu_2}{\twoheadleftarrow} G_2 \overset{\mu_3}{\twoheadleftarrow} \cdots$ of surjections, where each G_i is finitely presentable and each* $\ker(\mu_i)$ *is perfect, and*
(f) *there exists a proper map $\phi : N^n \to [0, \infty)$ with $\phi^{-1}(0) = \partial N^n$ and $\phi^{-1}(r)$ a closed $(n-1)$-manifold with the same $\mathbb{Z}$-homology as ∂N^n for all r.*

Proof Observations (a)–(c) are almost immediate, after which (d) and (e) can be obtained by straightforward applications of Lemma 3.5.20. Item (f) can be obtained by applying the (highly nontrivial) main result from [26] to each cobordism $(A_i, \partial_- A_i, \partial_+ A_i)$.

Exercise 3.5.22 Fill in the necessary details for observations (1)–(5).

Some examples are now in order.

Example 3.5.23 (*The Whitehead manifold is not pseudo-collarable*) First notice that $\mathscr{W}^3$ does contain a homotopy collar neighborhood of infinity. Let D^3 be a tame ball in $\mathscr{W}^3$ and let $N = \mathscr{W}^3 - \mathrm{int}D^3$. By excision and the Hurewicz and Whitehead theorems, N is a homotopy collar. (This argument works for all contractible open manifolds.) But since $\mathscr{W}^3$ is neither inward tame nor semistable, Proposition 3.5.21 assures that $\mathscr{W}^3$ is not pseudo-collarable.

Example 3.5.24 (*Davis manifolds are pseudo-collarable*) Non-collarable but pseudo-collarable ends are found in some of our most important examples—the Davis manifolds. It is easy to see that the neighborhood of infinity N_0 shown in Fig. 3.4 is a homotopy collar, as is N_i for each $i > 0$.

Motivated by Proposition 3.5.21 and previous definitions, call an inverse sequence of groups *perfectly semistable* if it is pro-isomorphic to an inverse sequence of finitely presentable groups for which the bonding homomorphisms are all surjective with perfect kernels. A complete characterization of pseudo-collarable n-manifolds is provided by:

Theorem 3.5.25 ([53]) *A 1-ended n-manifold W^n ($n \geq 6$) with compact (possibly empty) boundary is pseudo-collarable if and only if*

(a) *W^n is inward tame,*
(b) *pro-π_1 $(\varepsilon(W^n))$ is perfectly semistable, and*
(c) *$\sigma_\infty(W^n) \in \varprojlim \left\{ \widetilde{K}_0(\mathbb{Z}[\pi_1(N, p_i)]) \mid N \text{ a clean nbd. of infinity} \right\}$ is trivial.*

Remark 3.5.26 In (c), $\sigma_\infty(W^n)$ may be defined as $(\sigma(N_0), \sigma(N_1), \sigma(N_2), \ldots)$, the sequence of Wall finiteness obstructions of an arbitrary nested cofinal sequence of clean neighborhoods of infinity. By the functoriality of $\widetilde{K}_0$, this obstruction may be viewed as an element of the indicated inverse limit group. It is trivial if and only if each coordinate is trivial, i.e., each N_i has finite homotopy type. So just as in Theorem 3.5.13, Conditions (a) and (c) together are equivalent to W^n being absolutely inward tame.

By Theorem 3.5.19, every inward tame open manifold W^n has semistable pro-π_1 and stable pro-H_1. Together those observations guarantee a representation of pro-$\pi_1(\varepsilon(W^n))$ by an inverse sequence of surjective homomorphisms of finitely presented groups with "nearly perfect" kernels (in a way made precise in [54]). One might hope that Condition (b) of Theorem 3.5.25 is extraneous, but an example constructed in [52] dashes that hope.

Theorem 3.5.27 *In all dimensions ≥ 6 there exist absolutely inward tame open manifolds that are not pseudo-collarable.*

In light of Theorem 3.5.25, it is not surprising that Theorem 3.5.27 uses a significant dose of group theory. In fact, unravelling the group theory at infinity seems to be the key to understanding ends of inward tame manifolds. That topic is the focus of ongoing work [54]. As for our favorite open manifolds, the following is wide-open.

Question 3.5.1 Is the universal cover $\widetilde{M}^n$ of a closed aspherical n-manifold always pseudocollarable? Must it satisfy some of the hypotheses of Theorem 3.5.25? In particular, is $\widetilde{M}^n$ always inward tame? (If so, an affirmative answer to Conjecture 3.5.9 would follow from Theorem 3.5.19.)

We close this section with a reminder that the above results rely heavily on manifold-specific tools. For general locally finite complexes, Proposition 3.4.16 serves as warning. Even so, many ideas and questions discussed here have interesting analogs outside manifold topology—in the field of geometric group theory. We now take a break from manifold topology to explore that area.

3.6 End Invariants Applied to Group Theory

A standard method for applying topology to group theory is via Eilenberg–MacLane spaces. For a group G, a $K(G, 1)$ *complex* (or *Eilenberg–MacLane complex* for G or a *classifying space* for G) is an aspherical CW complex with fundamental group isomorphic to G. When the language of classifying spaces is used, a $K(G, 1)$ complex is often referred to as a BG complex and its universal cover as an EG complex. Alternatively, an EG complex is a contractible CW complex on which G acts properly and freely.

Exercise 3.6.1 Show that a CW complex X is aspherical if and only if $\widetilde{X}$ is contractible.

It is a standard fact that, for every group G: **(a)** there exists a $K(G, 1)$ complex, and **(b)** any two $K(G, 1)$ complexes are homotopy equivalent. Therefore, any homotopy invariant of a $K(G,1)$ complex is an invariant of G. In that way we define the *(co)homology of G with constant coefficients* in a ring R, denoted $H_*(G;R)$ and $H^*(G;R)$, to be $H_*(K(G,1);R)$ and $H^*(K(G,1);R)$, respectively.

At times it is useful to relax the requirement that a BG or an EG be a CW complex. For example, an aspherical manifold or a locally CAT(0) space with fundamental group G, but with no obvious cell structure might be a used as a BG. Provided the space in question is an ANR, there is no harm in allowing it, since all of the key facts from algebraic topology (for example, Exercise 3.6.1) still apply. Moreover, by Proposition 3.12.4, ANRs are homotopy equivalent to CW complexes, so, if necessary, an appropriate complex can be obtained.

3.6.1 Groups of Type F

We say that G *has type* F if $K(G,1)$ complexes have finite homotopy type or, equivalently, there exits a finite $K(G,1)$ complex or a compact ANR $K(G,1)$ space. Note that if K is a finite $K(G,1)$ complex, then $\widetilde{K}$ is locally finite and the G-action is cocompact; then we call $\widetilde{K}$ a cocompact EG complex.

Example 3.6.2 All finitely generated free and free abelian groups have type F, as do the fundamental groups of all closed surfaces, except for $\mathbb{R}P^2$. In fact, the fundamental group of every closed aspherical manifold has type F. No group that contains torsion can have type F (see [45, Proposition 7.2.12]), but every torsion-free $CAT(0)$ or δ-hyperbolic group has type F.

For groups of type F, there is an immediate connection between group theory and topology at the ends of noncompact spaces. If G is nontrivial and K_G is a finite $K(G,1)$ complex, $\widetilde{K}_G$ is contractible, locally finite, and noncompact, and by Corollary 3.3.15, all other finite $K(G,1)$ complexes (or compact ANR classifying spaces) have universal covers proper homotopy equivalent to $\widetilde{K}_G$. So the end invariants of $\widetilde{K}_G$, which are well-defined up to proper homotopy equivalence, may be attributed directly to G. For example, one may discuss: the *number of ends* of G; the *homology and cohomology at infinity* of G (denoted by pro-$H_*(\varepsilon(G);R)$, $\check{H}_*(\varepsilon(G);R)$ and $\check{H}^*(\varepsilon(G);R)$); and the homotopy behavior of the end(s) of G—properties such as simple connectedness, stability, semistability, or pro-monomorphic at infinity. In cases where $\widetilde{K}_G$ is 1-ended and semistable, pro-$\pi_*(\varepsilon(G))$ and $\check{\pi}_*(\varepsilon(G))$ are defined similarly. The need for semistability is, of course, due to base ray issues. Although $\widetilde{K}_G$ is well-defined up to proper homotopy type, there is no canonical choice base ray; in the presence of semistability that issue goes away. We will return to that topic shortly.

3.6.2 Groups of Type F_k

In fact, the existence of a finite $K(G,1)$ is excessive for defining end invariants like pro-$H_*(\varepsilon(G);R)$) and $\check{H}_*(\varepsilon(G);R)$. If G admits a $K(G,1)$ complex K with a finite k-skeleton (in which case we say G *has type* F_k), then all j-dimensional homology and homotopy end properties of the (locally finite) k-skeleton $\widetilde{K}_G^{(k)}$ of $\widetilde{K}_G$ can be directly attributed to G, provided $j<k$. The proof of invariance is rather intuitive. If L is any other $K(G,1)$ with finite k-skeleton, choose a cellular homotopy equivalence $f:K\to L$ and a homotopy inverse $g:L\to K$. These lift to homotopy equivalences $\tilde{f}:\widetilde{K}\to\widetilde{L}$ and $\tilde{g}:\widetilde{L}\to\widetilde{K}$, which cannot be expected to be proper. Nevertheless, the restrictions of $\tilde{g}\circ\tilde{f}$ and $\tilde{f}\circ\tilde{g}$ to the $(k-1)$-skeletons of $\widetilde{K}$ and $\widetilde{L}$ can be proven properly homotopic to inclusions $\widetilde{K}^{(k-1)}\hookrightarrow\widetilde{K}^{(k)}$ and $\widetilde{L}^{(k-1)}\hookrightarrow\widetilde{L}^{(k)}$. This is enough for the desired result.

As another example of the above, the number of ends, viewed as (the cardinality of) $\check{\pi}_0\left(\widetilde{K}_G^{(1)}\right)$, is a well-defined invariant of a finitely generated group, i.e., group of type F_1.

Exercise 3.6.3 Alternatively, one may define the number of ends of a finitely generated G to be the number of ends of a corresponding Cayley graph. Explain why this definition is equivalent to the above.

Remark 3.6.4 There are key connections between pro-$H_*(\varepsilon(G);R)$) and $\check{H}_*(\varepsilon(G);R)$ and the cohomology of G with RG coefficients (as presented, for example, in [16]). We have chosen not to delve into that topic in these notes. The interested reader is encouraged to read Chaps. 8 and 13 of [45].

3.6.3 Ends of Groups

In view of earlier comments, the following iconic result may be viewed as an application of $\check{\pi}_0(\varepsilon(G))$.

Theorem 3.6.5 (Freudenthal-Hopf-Stallings) *Every finitely generated group G has 0, 1, 2, or infinitely many ends. Moreover*

(a) G is 0*-ended if and only if it is finite,*
(b) G is 2*-ended if and only if it contains an infinite cyclic group of finite index, and*
(c) G is infinite-ended if and only if

- $G=A*_C B$ *(a free product with amalgamation), where C is finite and has index* ≥ 2 *in both A and B with at least one index being* ≥ 3*, or*
- $G=A*_\phi$ *(an HNN extension[11]), where* ϕ *is an isomorphism between finite subgroups of A each having index* ≥ 2.

[11] Definitions of *free product with amalgamation* and *HNN extension* can be found in [45, 85], or any text on combinatorial group theory.

Proof (*small portions*) The opening line of Theorem 3.6.5 is essentially Exercise 3.3.2; item (a) is trivial and item (b) is a challenging exercise. Item (c) is substantial [89], but pleasantly topological. Complete treatments can be found in [85] or [45].

3.6.4 The Semistability Conjectures

If G is finitely presentable, i.e., G has type F_2, and K is a corresponding presentation 2-complex (or any finite 2-complex with fundamental group G), then K may be realized as the 2-skeleton of a $K(G,1)$. That is accomplished by attaching 3-cells to K to kill $\pi_2(K)$ and proceeding inductively, attaching $(k+1)$-cells to kill the kth homotopy group, for all $k \geq 3$. It follows that pro-$H_1\left(\varepsilon(\widetilde{K}); R\right)$ and $\check{H}_1\left(\varepsilon(\widetilde{K}); R\right)$ represent the group invariants pro-$H_1(\varepsilon(G); R)$ and $\check{H}_1(\varepsilon(G); R)$, as discussed in Sect. 3.6.2. And by the same approach used there, when G (in other words $\widetilde{K}$) is 1-ended, properties such as simple connectivity at infinity, stability, semistability and pro-monomorphic at infinity can be measured in $\widetilde{K}$ and attributed directly to G. In an effort to go further with homotopy properties of the end of G, we are inexorably led back to the open problem:

Conjecture 3.6.6 (Semistability Conjecture–with explanation) *Every 1-ended finitely presented group G is semistable. In other words, the universal cover $\widetilde{K}$ of every finite complex with fundamental group G is strongly connected at infinity; equivalently, pro-$\pi_1\left(\widetilde{K}, r\right)$ is semistable for some (hence all) proper rays r.*

The fundamental nature of the Semistability Conjecture is now clear. We would like to view pro-$\pi_1\left(\varepsilon\left(\widetilde{K}\right); r\right)$ and $\check{\pi}_1\left(\varepsilon\left(\widetilde{K}\right); r\right)$ as group invariants pro-$\pi_1(\varepsilon(G))$ and $\check{\pi}_1(\varepsilon(G))$. Unfortunately, there is the potential for these to depend on base rays. A positive resolution of the Semistability Conjecture would eliminate that complication once and for all. The same applies to pro -$\pi_j\left(\varepsilon\left(\widetilde{K}\right); r\right)$and $\check{\pi}_j\left(\varepsilon\left(\widetilde{K}\right); r\right)$ when G is of type F_k and $j < k$.

The extension of Conjecture 3.6.6 to groups with arbitrarily many ends makes sense—the conjecture is that $\widetilde{K}$ is semistable (defined for multi-ended spaces near the end of Sect. 3.4.8). But this situation is simpler than one might expect: for 0-ended groups there is nothing to discuss, and 2-ended groups are known to be simply connected at each end (see Exercise 3.6.7 below); moreover, Mihalik [68] has shown that an affirmative answer for 1-ended groups would imply an affirmative answer for all infinite-ended groups.

Exercise 3.6.7 Let G be a group of type F_k. Show that every finite index subgroup H is of type F_k and the two groups share the same end invariants through dimension $k-1$. Use Theorem 3.6.5 to conclude that every 2-ended group is simply connected at each end.

Evidence for the Semistability Conjecture is provided by a wide variety of special cases; here is a sampling.

Theorem 3.6.8 *A finitely presented group satisfying any one of the following is semistable.*

(a) *G is the extension of an infinite group by an infinite group,*
(b) *G is a one-relator group,*
(c) *$G = A *_C B$ where A and B are finitely presented and semistable and C is infinite,*
(d) *$G = A*_C$ where A is finitely generated and semistable and C is infinite,*
(e) *G is δ-hyperbolic,*
(f) *G is a Coxeter group,*
(g) *G is an Artin group.*

References include: [67, 69–71, 90].

There is a variation on the Semistability Conjecture that is also open.

Conjecture 3.6.9 (H_1-semistability Conjecture) *For every 1-ended finitely presented group G, pro-H_1 $(\varepsilon(G); \mathbb{Z})$ is semistable.*

Since pro-H_1 $(\varepsilon(G); \mathbb{Z})$ can be obtained by abelianization of any representative of pro-π_1 $\left(\varepsilon(\widetilde{K}), r\right)$, for any presentation 2-complex K and base ray r, it is clear that the H_1-semistability Conjecture is weaker than the Semistability Conjecture. Moreover, the H_1-version of our favorite special case of the Semistability Conjecture—the case where G is the fundamental group of an aspherical manifold—is easily solved in the affirmative, by an application of Exercise 3.5.10. This provides a ray of hope that the Manifold Semistability Conjecture is more accessible that the general case.

Remark 3.6.10 The Semistability Conjectures presented in this section were initially formulated by Ross Geoghegan in 1979. At the time, he simply called them "questions", expecting the answers to be negative. Their long-lasting resistance to solutions, combined with an accumulation of affirmative answers to special cases, has gradually led them to become known as conjectures.

3.7 Shape Theory

Shape theory may be viewed as a method for studying bad spaces using tools created for the study of good spaces. Although more general approaches exist, we follow the classical (and the most intuitive) route by developing shape theory only for compacta. But now we are interested in arbitrary compacta—not just ANRs. A few examples to be considered are shown in Fig. 3.9.

The abrupt shift from noncompact spaces with nice local properties to compacta with bad local properties may seem odd, but there are good reasons for this temporary shift in focus. First, the tools we have already developed for analyzing the ends of manifolds and complexes are nearly identical to those used in shape theory; understanding and appreciating the basics of shape theory will now be quite easy.

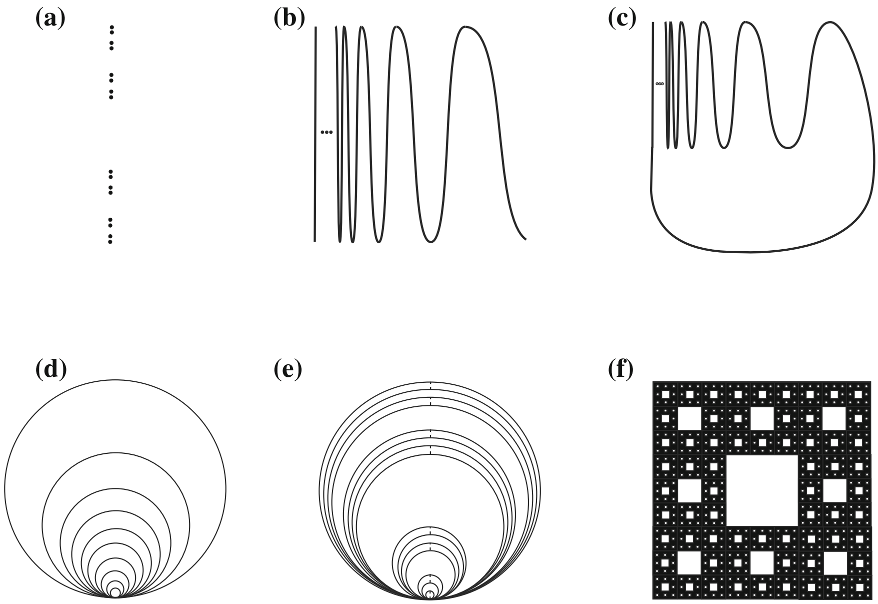


Fig. 3.9 **a** Cantor set. **b** Topologist's sine curve. **c** Warsaw circle. **d** Hawaiian earring. **e** Cantor Hawaiian earring. **f** Sierpinski carpet

More importantly, certain aspects of the study of ends are nearly impossible without shapes—if the theory did not already exist, we would be forced to invent it.

For more comprehensive treatments of shape theory, the reader can consult [13] or [31].

3.7.1 Associated Sequences, Basic Definitions, and Examples

In shape theory, the first step in studying a compactum A is to choose an *associated inverse sequence* $K_0 \xleftarrow{f_1} K_1 \xleftarrow{f_2} K_2 \xleftarrow{f_3} \cdots$ of finite polyhedra and simplicial maps. There are several ways this can be done. We describe a few of them.

Method 1: If A is finite-dimensional, choose an embedding $A \hookrightarrow \mathbb{R}^n$, and let $K_0 \supseteq K_1 \supseteq K_2 \supseteq \cdots$ be a sequence of compact polyhedral neighborhoods intersecting in A. Since it is impossible to choose triangulations under which all inclusion maps are simplicial, choose progressively finer triangulations for the K_i and let the f_i be simplicial approximations to the inclusion maps.

Method 2: Choose a sequence $\mathscr{U}_0, \mathscr{U}_1, \mathscr{U}_2, \cdots$ of finite covers of A by ε_i-balls, where $\varepsilon_i \to 0$ and each $\mathscr{U}_{i+1}$ refines $\mathscr{U}_i$. Let K_i be the nerve of $\mathscr{U}_i$ and $f_i : K_i \to K_{i-1}$ a simplicial map that takes each vertex $U \in \mathscr{U}_i$ to a vertex $V \in \mathscr{U}_{i-1}$ with $U \subseteq V$.

Method 3: If A can be expressed as the inverse limit of an inverse sequence $K_0 \xleftarrow{g_1} K_1 \xleftarrow{g_2} K_2 \xleftarrow{g_3} \cdots$ of finite polyhedra,[12] then that sequence itself may be associated to A, after each map is approximated by one that is simplicial.

Remark 3.7.1 (a) At times, it will be convenient if each K_i in an associated inverse sequence has a preferred vertex p_i with each f_{i+1} taking p_{i+1} to p_i. That can easily be arranged; we refer to the result as a *pointed inverse sequence*.

(b) Our requirement that the bonding maps in associated inverse sequences be simplicial, will soon be seen as unnecessary. But, for now, there is no harm in including that additional niceness condition.

(c) When A is infinite-dimensional, a variation on Method 1 is available. In that case, A is embedded in the Hilbert cube and a sequence $\{N_i\}$ of closed Hilbert cube manifold neighborhoods of A is chosen. By Theorem 3.13.2, each N_i has the homotopy type of a finite polyhedron K_i. From there, an associated inverse sequence for A is readily obtained.

The choice of an associated inverse sequence for a compactum A should be compared to the process of choosing a cofinal sequence of neighborhoods of infinity for a noncompact space X. In both situations, the terms in the sequences can be viewed as progressively better approximations to the object of interest, and in both situations, there is tremendous leeway in assembling those approximating sequences. In both contexts, that flexibility raises well-definedness issues. In the study of ends, we introduced an equivalence relation based on ladder diagrams to obtain the appropriate level of well-definedness. The same is true in shape theory.

Proposition 3.7.2 *For a fixed compactum A, let $\{K_i, f_i\}$ and $\{L_i, g_i\}$ be a pair of associated inverse sequences of finite polyhedra. Then there exist subsequences, simplicial maps, and a corresponding ladder diagram*

$$\begin{tikzcd}
K_{i_0} & & K_{i_1} \arrow[ll, "f_{i_0,i_1}"'] \arrow[dl] & & K_{i_2} \arrow[ll, "f_{i_1,i_2}"'] \arrow[dl] & & K_{i_3} \arrow[ll, "f_{i_2,i_3}"'] \arrow[dl] & \cdots & \\
& L_{j_0} \arrow[ul] & & L_{j_1} \arrow[ll, "g_{j_0,j_1}"'] \arrow[ul] & & L_{j_2} \arrow[ll, "g_{j_1,j_2}"'] \arrow[ul] & & & \cdots \arrow[ll, "g_{j_2,j_3}"']
\end{tikzcd}$$

in which each triangle of maps homotopy commutes. If desired, we may require that those homotopies preserve base points.

Exercise 3.7.3 Prove some or all of Proposition 3.7.2. Start by comparing any pair of sequences obtained using the same method, then note that Method 1 is a special case of Method 3.

Define a pair of inverse sequences of finite polyhedra and simplicial maps to be pro-*homotopy equivalent* if they contain subsequences that fit into a homotopy

[12] By definition, $\varprojlim \{K_i, f_i\}$ is viewed as a subspace of the infinite product space $\prod_{i=0}^{\infty} K_i$ and is topologized accordingly.

commuting ladder diagram, as described in Proposition 3.7.2. Compacta A and A' are *shape equivalent* if some (and thus every) pair of associated inverse sequences of finite polyhedra are pro-homotopy equivalent. In that case we write $\mathscr{S}h(A) = \mathscr{S}h(A')$ or sometimes $A' \in \mathscr{S}h(A)$.

Remark 3.7.4 If $\{K_i, f_i\}$ is an associated inverse sequence for a compactum A, it is not necessarily the case that $\varprojlim \{K_i, f_i\} \approx A$. But it is immediate from the definitions that the two spaces have the same shape.

Exercise 3.7.5 Show that the Topologist's Sine Curve has the shape of a point and the Warsaw Circle has the shape of a circle (see Fig. 3.9). Note that neither space is homotopy equivalent to its nicer shape version.

Exercise 3.7.6 Show that the Whitehead Continuum (see Example 3.2.1) has the shape of a point. Spaces with the shape of a point are often called *cell-like*.

Exercise 3.7.7 Show that the Sierpinski Carpet is shape equivalent to a Hawaiian Earring.

Exercise 3.7.8 Show that the Cantor Hawaiian Earring is shape equivalent to a standard Hawaiian Earring. (An observation that once prompted the reaction: "I demand a recount!")

When considering the shape of a compactum A, the space A itself becomes largely irrelevant after an associated inverse sequence has been chosen. In a sense, shape theory is just the study of pro-homotopy classes of inverse sequences of finite polyhedra. Nevertheless, there is a strong correspondence between inverse sequences of finite polyhedra and compact metric spaces themselves. If A is the inverse limit of an inverse sequence $\{K_i, f_i\}$ of finite polyhedra, then applying any of the three methods mentioned earlier to the space A yields an inverse sequence of finite polyhedra pro-homotopy equivalent to the original $\{K_i, f_i\}$. In other words, passage to an inverse limit preserves all relevant information. As we saw in Exercise 3.4.4, that is not the case with inverse sequences of groups. This phenomenon is even more striking when studying ends of spaces. If $N_0 \hookleftarrow N_1 \hookleftarrow N_2 \hookleftarrow \cdots$ is a cofinal sequence of neighborhoods of infinity of a space X, the inverse limit of that sequence is clearly the empty set. In some sense, the study of ends is a study of an imaginary "space at infinity". By using shape theory, we can sometimes make that space a reality.

Exercise 3.7.9 Prove that an inverse sequence of nonempty finite polyhedra (or more generally, an inverse sequence of nonempty compacta) is never the empty set.

Exercise 3.7.10 So far, our discussion of shape has focused on exotic compacta; but nice spaces, such as finite polyhedra, are also part of the theory. Show that finite polyhedra K and L are shape equivalent if and only if they are homotopy equivalent. *Hint*: Choosing trivial associated inverse sequences $K \xleftarrow{\text{id}} K \xleftarrow{\text{id}} \cdots$ and $L \xleftarrow{\text{id}} L \xleftarrow{\text{id}} \cdots$ makes the task easier. A more general observation of this sort will be made shortly.

3.7.2 The Algebraic Shape Invariants

In the spirit of the work already done on ends of spaces, we define a variety of algebraic invariants for compacta. Given a compactum A and any associated inverse sequence $\{K_i, f_i\}$, define pro-$H_*(A; R)$ to be the pro-isomorphism class of the inverse sequence

$$H_*(K_0; R) \xleftarrow{f_{1*}} H_*(K_1; R) \xleftarrow{f_{2*}} H_*(K_2; R) \xleftarrow{f_{3*}} \cdots$$

and $\check{H}_*(A; R)$ to be its inverse limit. By reversing arrows and taking a direct limit, we also define ind-$H^*(A; R)$ and $\check{H}^*(A; R)$. The groups $\check{H}_*(A; R)$ and $\check{H}^*(A; R)$ are know as the *Čech homology and cohomology groups* of A, respectively. If we begin with a pointed inverse sequence $\{(K_i, p_i), f_i\}$ we obtain pro-$\pi_*(A, p)$ and $\check{\pi}_*(A, p)$, where p corresponds to $(p_0, p_1, p_2, \cdots)$. Call $\check{\pi}_*(A, p)$ the *Čech homotopy groups* of A, or sometimes, the *shape groups* of A.

Čech cohomology is known to be better-behaved than Čech homology, in that there is a full-blown Čech cohomology theory satisfying the Eilenberg–Steenrod axioms. Although the Čech homology groups of A do not fit into such a nice theory, they are are still perfectly good topological invariants of A. For reasons we have seen before, pro-$H_*(A; R)$ and pro-$\pi_*(A, p)$ tend to carry more information than the corresponding inverse limits.

Exercise 3.7.11 Observe that, for the Warsaw circle W, the first Čech homology and the first Čech homotopy group are not the same as the first singular homology and traditional fundamental group of W.

Remark 3.7.12 Another way to think about the phenomena that occur in Exercise 3.7.11 is that, for an inverse sequence of spaces $K_0 \xleftarrow{g_1} K_1 \xleftarrow{g_2} K_2 \xleftarrow{g_3} \cdots$, the homology [homotopy] of the inverse limit is not necessarily the same as the inverse limit of the homologies (homotopies). It is the point of view of shape theory that the latter inverse limits often do a better job of capturing the true nature of the space.

3.7.3 Relaxing the Definitions

Now that the framework for shape theory is in place, we make a few adjustments to the definitions. These changes will not nullify anything done so far, but at times they will make the application of shape theory significantly easier.

Previously we required bonding maps in associated inverse sequences to be simplicial. That has some advantages; for example, pro-$H_*(A; R)$ and $\check{H}^*(A; R)$ can be defined using only simplicial homology. But in light of the definition of pro-homotopy equivalence, it is clear that only the homotopy classes of the bonding maps really matters. So, adjusting a naturally occurring bonding map to make it simplicial is unnecessary. From now on, we only require bonding maps to be continuous.

In a similar vein, a finite polyhedron K_i in an inverse sequence corresponding to A can easily be replaced by a finite CW complex. More generally, any compact ANR is acceptable as an entry in that inverse sequence (Proposition 3.12.4 is relevant here). Of course, once these changes are made, we must use cellular or singular (as opposed to simplicial) homology for defining the algebraic shape invariants of the previous section.

With the above relaxation of definitions in place, the following fundamental facts becomes elementary.

Proposition 3.7.13 *Let A and B be compact ANRs. Then $\mathscr{S}\mathrm{h}(A) = \mathscr{S}\mathrm{h}(B)$ if and only if $A \simeq B$.*

Proof An argument like that used in Exercise 3.7.10 can now be applied here.

Proposition 3.7.14 *If A is a compact ANR, then pro-$H_*(A; R)$ and pro-$\pi_*(A, p)$ are stable for all $*$ with $\check{H}_*(A; R)$ and $\check{\pi}_*(A, p)$ being isomorphic to the singular homology groups $H_*(A; R)$ and the traditional homotopy groups $\pi_*(A, p)$, respectively.*

Proof Choose the trivial associated inverse sequence $A \xleftarrow{\mathrm{id}} A \xleftarrow{\mathrm{id}} \cdots$.

Corollary 3.7.15 *If B is a compactum that is shape equivalent to a compact ANR A, then pro-$H_*(B; R)$ and pro-$\pi_*(B, p)$ are stable for all $*$ with $\check{H}_*(B; R) \cong H_*(B; R)$ and $\check{\pi}_*(B, p) \cong \pi_*(A, p)$. In particular, $\check{H}_*(B; R)$ is finitely generated, for all $*$ and $\check{\pi}_1(B, p)$ is finitely presentable.*

Example 3.7.16 Compacta (a), (d), (e), and (f) from Fig. 3.9 do not have the shapes of compact ANRs.

Taken together, Propositions 3.7.13 and 3.7.14 form the foundation of the true slogan: *When restricted to compact ANRs, shape theory reduces to (traditional) homotopy theory.* Making that slogan a bona fide theorem would require a development of the notion of "shape morphism" and a comparison of those morphisms to homotopy classes of maps. We have opted against providing that level of detail in these notes. We will, however, close this section with a few comments aimed at giving the reader a feel for how that can be done.

Let *pro-$\mathscr{H}$omotopy* denote the set of all pro-homotopy classes of inverse sequences of compact ANRs and continuous maps. If *$\mathscr{S}$hapes* denotes the set of all shape classes of compact metric spaces, then there is a natural bijection Θ : *pro-$\mathscr{H}$omotopy* $\to$ *$\mathscr{S}$hapes* defined by taking inverse limits; Methods 1–3 in Sect. 3.7.1 determine Θ^{-1}. With some additional work, one can define morphisms in *pro-$\mathscr{H}$omotopy* as certain equivalence classes of sequences of maps, thereby promoting *pro-$\mathscr{H}$omotopy* to a full-fledged category. From there, one can use Θ to (indirectly) define morphisms in *$\mathscr{S}$hapes*, thereby obtaining the shape category. In that case, it can be shown that each continuous function $f : A \to B$ between compacta determines a unique shape morphism (a fact that uses some ANR theory); but unfortunately, not every shape morphism from A to B can be realized by a continuous map. This is not as surprising as it first appears: as an example, the reader should attempt to construct a map from $\mathbb{S}^1$ to the Warsaw circle that deserves to be called a shape equivalence.

Remark 3.7.17 In order to present a thorough development of the *pro-Homotopy* and $\mathscr{S}$*hapes* categories, more care would be required in dealing with base points. In fact, we would end up building a pair of slightly different categories for each—one incorporating base points and the other without base points. The differences between those categories does not show up at the level of objects (for example, compacta are shape equivalent if and only if they are "pointed shape equivalent"), but the categories differ in their morphisms. In the context of these notes, we need not be concerned with that distinction.

3.7.4 The Shape of the End of an Inward Tame Space

The relationship between shape theory and the topology of the ends of noncompact spaces goes beyond a similarity between the tools used in their studies. In this section we develop a precise relationship between shapes of compacta and ends of inward tame ANRs. In so doing, the fundamental nature of inward tameness is brought into focus.

Let Y be a inward tame ANR. By repeated application of the definition of inward tameness, there exist sequences of neighborhoods of infinity $\{N_i\}_{i=0}^{\infty}$, finite complexes $\{K_i\}_{i=1}^{\infty}$, and maps $f_i : N_i \to K_i$ and $g_i : K_i \to N_{i-1}$ with $g_i f_i \simeq \text{incl}\,(N_i \hookrightarrow N_{i-1})$ for all i. By letting $h_i = f_{i-1} g_i$, these can be assembled into a homotopy commuting ladder diagram

$$\begin{tikzcd}
N_0 & & N_1 \arrow[ll, hook'] \arrow[dl, "f_1"'] & & N_2 \arrow[ll, hook'] \arrow[dl, "f_2"'] & & N_3 \cdots \arrow[ll, hook'] \arrow[dl, "f_3"'] & \\
& K_1 \arrow[ul, "g_1"] & & K_2 \arrow[ll, "h_2"'] \arrow[ul, "g_2"] & & K_3 \arrow[ll, "h_3"'] \arrow[ul, "g_3"] & & \cdots \arrow[ll, "h_4"']
\end{tikzcd}$$

The pro-homotopy equivalence class of $K_1 \xleftarrow{h_2} K_2 \xleftarrow{h_3} K_3 \xleftarrow{h_4} \cdots$ is fully determined by Y. That is easily verified by a diagram of the form (3.6), along with the transitivity of the pro-homotopy equivalence relation. Define the *shape of the end of* Y, denoted $\mathscr{S}h\,(\varepsilon\,(Y))$, to be the shape class of $\varprojlim \{K_i, h_i\}$. A compactum $A \in \mathscr{S}h\,(\varepsilon\,(Y))$ can be viewed as a physical representative of the illusive "end of Y".

The following is immediate.

Theorem 3.7.18 *Let Y be an inward tame ANR and $A \in \mathscr{S}\mathrm{h}(\varepsilon\,(Y))$. Then*

(a) pro-$H_i\,(\varepsilon(Y); R) =$ pro-$H_i\,(A; R)$ and $\check{H}_i\,(\varepsilon(Y); R) \cong \check{H}_i\,(A; R)$ for all i and any coefficient ring R, and

(b) *if Y is 1-ended and semistable then pro-π_i $(\varepsilon(Y)) =$ pro-π_i (A) and $\check{\pi}_i(\varepsilon(Y)) \cong \check{\pi}_i(A)$ for all i.*

The existence of diagrams like (3.8) shows that $\mathscr{S}h(\varepsilon(Y))$ is also an invariant of the proper homotopy class of Y. There is also a partial converse to that statement—an assertion about the proper homotopy type of Y based only on the shape of its end. Since the topology at the end of a space does not determine the global homotopy type of that space, a new definition is required.

Spaces X and Y are *homeomorphic at infinity* if there exists a homeomorphism $h : N \to M$, where $N \subseteq X$ and $M \subseteq Y$ are neighborhoods of infinity. They are *proper homotopy equivalent at infinity* if there exist pairs of neighborhoods of infinity $N' \subseteq N$ in X and $M' \subseteq M$ in Y and proper maps $f : N \to Y$ and $g : M \to X$, with $g \circ f|_{N'} \overset{p}{\simeq} \mathrm{incl}\left(N' \hookrightarrow X\right)$ and $f \circ g|_{M'} \overset{p}{\simeq} \mathrm{incl}\left(M' \hookrightarrow Y\right)$.

Theorem 3.7.19 *Let X and Y be inward tame ANRs. Then $\mathscr{S}\mathrm{h}(\varepsilon(X)) = \mathscr{S}\mathrm{h}(\varepsilon(Y))$ if and only if X and Y are proper homotopy equivalent at infinity.*

Proof The reverse implication follows from the previous paragraphs, while the forward direction is nontrivial. A proof can be obtained by combining results from [21, 32].

In certain circumstances, the "at infinity" phrase can be removed from the above. For example, we have.

Corollary 3.7.20 *Let X and Y be contractible inward tame ANRs. Then $\mathscr{S}\mathrm{h}(\varepsilon(X)) = \mathscr{S}\mathrm{h}(\varepsilon(Y))$ if and only if X and Y are proper homotopy equivalent.*

Exercise 3.7.21 Use the Homotopy Extension Property to obtain Corollary 3.7.20 from Theorem 3.7.19.

Example 3.7.22 If $K_0 \xleftarrow{f_1} K_1 \xleftarrow{f_2} K_2 \xleftarrow{f_3} \cdots$ is a sequence of finite complexes and $A = \varprojlim \{K_i, f_i\}$, it is easy to see that A represents $\mathscr{S}h(\varepsilon(\mathrm{Tel}(\{K_i, f_i\})))$. By Theorem 3.7.19, any inward tame ANR X with $\mathscr{S}h(\varepsilon(X)) = A$ is proper homotopy equivalent at infinity to $\mathrm{Tel}(\{K_i, f_i\})$. When issues of global homotopy type are resolved, even stronger conclusions are possible; for example, if X is contractible, $X \overset{p}{\simeq} \mathrm{CTel}(\{K_i, f_i\})$. In some sense, the inverse mapping telescope is an uncomplicated model for the end behavior of an inward tame ANR.

Remark 3.7.23 In this section, we have intentionally not required spaces to be 1-ended. So, for example, $\mathscr{S}h(\varepsilon(\mathbb{R}))$ is representable by a 2-point space and the shape of the end of a ternary tree is representable by a Cantor set. For more complex multi-ended X, individual components of $A \in \mathscr{S}h(\varepsilon(X))$ may have nontrivial shapes, and to each end of X there will be a component of A whose shape reflects properties of that end.

3.8 $\mathscr{Z}$-Sets and $\mathscr{Z}$-Compactifications

While reading Sect. 3.7.4, the following question may have occurred to the reader: *For inward tame X with $\mathscr{S}h(\varepsilon(X)) = A$, is there a way to glue A to the end of X to obtain a nice compactification?* As stated, that question is a bit too simple, but it provides reasonable motivation for the material in this section.

3.8.1 Definitions and Examples

A closed subset A of an ANR X is a *$\mathscr{Z}$-set* if any of the following equivalent conditions is satisfied:

- For every $\varepsilon > 0$ there is a map $f : X \to X - A$ that is ε-close to the identity.
- There exists a homotopy $H : X \times [0, 1] \to X$ such that $H_0 = \mathrm{id}_X$ and $H_t(X) \subseteq X - A$ for all $t > 0$. (We say that H *instantly homotopes X off of A*.)
- For every open set U in X, $U - A \hookrightarrow U$ is a homotopy equivalence.

The third condition explains some alternative terminology: $\mathscr{Z}$-sets are sometimes called *homotopy negligible* sets.

Example 3.8.1 The $\mathscr{Z}$-sets in a manifold M^n are precisely the closed subsets of ∂M^n. In particular, ∂M^n is a $\mathscr{Z}$-set in M^n.

Example 3.8.2 It is a standard fact that every compactum A can be embedded in the Hilbert cube $\mathscr{Q}$. It may be embedded as a $\mathscr{Z}$-set as follows: embed A in the "face" $\{1\} \times \prod_{i=2}^{\infty} [-1, 1] \subseteq \prod_{i=1}^{\infty} [-1, 1] = \mathscr{Q}$.

Example 3.8.2 is the starting point for a remarkable characterization of shape, sometimes used as an alternative definition. We will not attempt to describe a proof.

Theorem 3.8.3 (Chapman's Complement Theorem, [20]) *Let A and B be compacta embedded as $\mathscr{Z}$-sets in $\mathscr{Q}$. Then $Sh(A) = Sh(B)$ if and only if $\mathscr{Q} - A \approx \mathscr{Q} - B$.*

A *$\mathscr{Z}$-compactification* of a space Y is a compactification $\overline{Y} = Y \sqcup Z$ with the property that Z is a $\mathscr{Z}$-set in $\overline{Y}$. In this case, Z is called a *$\mathscr{Z}$-boundary* for Y. Implicit in this definition is the requirement that $\overline{Y}$ be an ANR; and since an open subset of an ANR is an ANR, Y must be an ANR to be a candidate for $\mathscr{Z}$-compactification. By a result from the ANR theory, any compactification $\overline{Y}$ of an ANR Y, for which $\overline{Y} - Y$ satisfies any of the above bullet points, is necessarily an ANR—hence, it is a $\mathscr{Z}$-compactification. The point here is that, when attempting to form a $\mathscr{Z}$-compactification, one must begin with an ANR Y. Then it is enough to find a compactification satisfying one of the above equivalent conditions.

A nice property of a $\mathscr{Z}$-compactification is that the homotopy type of a space is left unchanged by the compactification; for example, a $\mathscr{Z}$-compactification of a contractible space is contractible. The prototypical example is the compactification of $\mathbb{R}^n$ to an n-ball by addition of the $(n-1)$-sphere at infinity; the prototypical non-example is the 1-point compactification of $\mathbb{R}^n$. Finer relationships between Y, $\overline{Y}$, and Z can be understood via shape theory and the study of ends. Before moving in that direction, we add to our collection of examples.

Example 3.8.4 In manifold topology, the most fundamental $\mathcal{Z}$-compactification is the addition of a manifold boundary to an open manifold, as discussed in Sect. 3.5.2.

Not all $\mathcal{Z}$-compactifications of open manifolds are as simple as the above.

Example 3.8.5 Let C^n be a Newman contractible n-manifold embedded in $\mathbb{S}^n$ (as it is by construction). A non-standard $\mathcal{Z}$-compactification of $\text{int}\mathbb{B}^{n+1}$ can be obtained by crushing C^n to a point. In this case, the quotient $\mathbb{S}^n/C^n$ is a $\mathcal{Z}$-set in $\mathbb{B}^{n+1}/C^n$. Note that $\mathbb{S}^n/C^n$ is not a manifold!

For those who prefer lower-dimensional examples, a similar $\mathcal{Z}$-compactification of $\text{int}\mathbb{B}^4$ can be obtained by crushing out a wild arc or a Whitehead continuum in $\mathbb{S}^3$. In terms of dimension, that is as low as it gets. As a result of Corollary 3.10.8 (still to come), for $n \leq 2$, a $\mathcal{Z}$-boundary of $\mathbb{B}^{n+1}$ is necessarily homeomorphic to $\mathbb{S}^n$.

Example 3.8.6 Let ΣC^n be the suspension of a Newman compact contractible n-manifold. The suspension of ∂C^n is a $\mathcal{Z}$-set in ΣC^n, and its complement, $\text{int}C^n \times (-1, 1)$, is homeomorphic to $\mathbb{R}^{n+1}$ by Exercise 3.5.4. So this is another nonstandard $\mathcal{Z}$-compactification of $\mathbb{R}^{n+1}$.

Exercise 3.8.7 Verify the assertions made in Examples 3.8.5 and 3.8.6.

Often a manifold that cannot be compactified by addition of a manifold boundary is, nevertheless, $\mathcal{Z}$-compactifiable—a fact that is key to the usefulness of $\mathcal{Z}$-compactifications. Davis manifolds are the ideal examples.

Example 3.8.8 The 1-point compactification of the infinite boundary connected sum $C_0^n \overset{\partial}{\#} \left(-C_1^n\right) \overset{\partial}{\#} \left(C_2^n\right) \overset{\partial}{\#} \left(-C_3^n\right) \overset{\partial}{\#} \cdots$ shown at the top of Fig. 3.3 is a $\mathcal{Z}$-compactification. More significantly the point at infinity together with the original manifold boundary form a $\mathcal{Z}$-boundary for the corresponding Davis manifold $\mathcal{D}^n$. It is interesting to note that $\mathcal{D}^n$ cannot admit a $\mathcal{Z}$-compactification with $\mathcal{Z}$-boundary a manifold (or even an ANR) since pro-$\pi_1\left(\varepsilon\left(\mathcal{D}^n\right)\right)$ is not stable. This will be explained soon.

Example 3.8.9 In geometric group theory, the prototypical $\mathcal{Z}$-compactification is the addition of the visual boundary $\partial_\infty X$ to a proper CAT(0) space X. Indeed, if $\partial_\infty X$ is viewed as the set of end points of all infinite geodesic rays emanating from a fixed $p_0 \in X$, a homotopy pushing inward along those rays verifies the $\mathcal{Z}$-set property.

Example 3.8.10 In [1], an equivariant CAT(0) metric is placed on many of the original Davis manifolds. In [28] an entirely different construction produces locally CAT(0) closed aspherical manifolds, whose CAT(0) universal covers are similar to Davis' earlier examples. These objects with their $\mathcal{Z}$-compactifications and

$\mathcal{Z}$-boundaries provide interesting common ground for manifold topology and geometric group theory.

At the expense of losing the isometric group actions, [39] places CAT(-1) metrics on the Davis manifolds in such a way that the visual sphere at infinity is homogeneous and nowhere locally contractible. Their method can also be used to place CAT(-1) metrics on the asymmetric Davis manifolds from Example 3.2.9.

Example 3.8.11 If $K_0 \xleftarrow{f_1} K_1 \xleftarrow{f_2} K_2 \xleftarrow{f_3} \cdots$ is an inverse sequence of finite polyhedra (or finite CW complexes or compact ANRs), then the inverse mapping telescope $\mathrm{Tel}\left(\{K_i, f_i\}\right)$ can be $\mathcal{Z}$-compactified by adding a copy of $\varprojlim \{K_i, f_i\}$, a space that contains one point for each of the infinite telescope rays described in Sect. 3.4.5. (Note the similarity of this to Example 3.8.9.)

In Sects. 3.9 and 3.10, we will look at applications of $\mathcal{Z}$-compactification to geometric group theory and manifold topology. Before that, we address a pair of purely topological questions:

- *When is a space $\mathcal{Z}$-compactifiable?*
- *To what extent is the $\mathcal{Z}$-boundary of a given space unique?* (Examples 3.8.5 and 3.8.6 show that a space can admit nonhomeomorphic $\mathcal{Z}$-boundaries.)

3.8.2 Existence and Uniqueness for $\mathcal{Z}$-Compactifications and $\mathcal{Z}$-Boundaries

If Y admits a $\mathcal{Z}$-compactification $\overline{Y} = Y \sqcup Z$, then as noted above, Y must be an ANR; and since $Y \hookrightarrow \overline{Y}$ is a homotopy equivalence and $\overline{Y}$ is a compact ANR, Proposition 3.12.4 implies that Y has finite homotopy type. Prying a bit deeper, a homotopy $H : \overline{Y} \times [0, 1] \to \overline{Y}$ that instantly homotopes $\overline{Y}$ off Z can be "truncated" (with the help of the Homotopy Extension Property) to homotope arbitrarily small closed neighborhoods of infinity (in Y) into compact subsets. Hence, Y is necessarily inward tame.

By combining the results noted in Examples 3.7.22 and 3.8.11, every inward tame ANR is proper homotopy equivalent to one that is $\mathcal{Z}$-compactifiable. Unfortunately, $\mathcal{Z}$-compactifiability is not an invariant of proper homotopy type. The following result begins to make that clear.

Proposition 3.8.12 *Every $\mathcal{Z}$-compactifiable space that is sharp at infinity is absolutely inward tame.*

Proof If $\overline{Y} = Y \sqcup Z$ is a $\mathcal{Z}$-compactification and N is a closed ANR neighborhood of infinity in Y. Then $\overline{N} \equiv N \sqcup Z$ is a $\mathcal{Z}$-compactification of N, hence a compact ANR, and therefore homotopy equivalent to a finite complex K. Since $N \hookrightarrow \overline{N}$ is a homotopy equivalence, $N \simeq K$.

Remark 3.8.13 By employing the standard trick of considering $Y \times \mathcal{Q}$ (to ensure sharpness at infinity), Proposition 3.8.12 provides an alternative proof that a $\mathcal{Z}$-compactifiable ANR must be inward tame. This also uses the straightforward observation that, if $\overline{Y} = Y \sqcup Z$ is a $\mathcal{Z}$-compactification of Y, then $\overline{Y} \times \mathcal{Q} = (Y \times \mathcal{Q}) \sqcup (Z \times \mathcal{Q})$ is a $\mathcal{Z}$-compactification of $Y \times \mathcal{Q}$. That observation will be used numerous times, as we proceed.

Theorem 3.8.14 *Suppose Y admits a $\mathcal{Z}$-compactification $\overline{Y} = Y \sqcup Z$. Then $Z \in \mathcal{S}\mathrm{h}(\varepsilon(Y))$.*

Proof Arguing as in Remark 3.8.13, we may assume without loss of generality that Y is sharp at infinity. Choose a cofinal sequence $\{N_i\}$ of closed ANR neighborhoods of infinity in Y, and for each i let $\overline{N}_i$ be the compact ANR $N_i \sqcup Z$. The homotopy commutative diagram

$$\begin{tikzcd}
N_0 & & N_1 \arrow[ll, hook'] \arrow[dl, hook] & & N_2 \arrow[ll, hook'] \arrow[dl, hook] & & \cdots \arrow[ll, hook'] \\
& \overline{N}_0 \arrow[ul] & & \overline{N}_1 \arrow[ll, hook'] \arrow[ul] & & \overline{N}_2 \arrow[ll, hook'] \arrow[ul] & \cdots
\end{tikzcd}$$

where each up arrow is a homotopy inverse of the corresponding $N_i \hookrightarrow \overline{N}_i$, shows that the lower sequence defines $\mathcal{S}h(\varepsilon(Y))$. But, since the inverse limit of that sequence is Z (since $\cap \overline{N}_i = Z$), the sequence also defines the shape of Z.

Corollary 3.8.15 (Uniqueness of $\mathcal{Z}$-boundaries up to shape) *All $\mathcal{Z}$-boundaries of a given space Y are shape equivalent. Even more, if Y and Y' are $\mathcal{Z}$-compactifiable and proper homotopy equivalent at infinity, then each $\mathcal{Z}$-boundary of Y is shape equivalent to each $\mathcal{Z}$-boundary of Y'.*

Proof Combine the above theorem with Theorem 3.7.19.

Example 3.8.16 We can now verify the comment at the end of Example 3.8.8. For any $\mathcal{Z}$-boundary Z of a Davis manifold $\mathcal{D}^n$, pro-$\pi_1(Z)$ must match the nonstable pro-$\pi_1(\varepsilon(\mathcal{D}^n))$ established in Sect. 3.2.2. So, by Proposition 3.7.14, Z cannot be an ANR.

Next we examine the existence question for $\mathcal{Z}$-compactifications. By the above results we know that, for reasonably nice X, absolute inward tameness is necessary; moreover, prospective $\mathcal{Z}$-boundaries must come from $\mathcal{S}h(\varepsilon(X))$. It turns out that this is not enough. The outstanding result on this topic, is due to Chapman and Siebenmann [21]. It provides a complete characterization of $\mathcal{Z}$-compactifiable Hilbert cube manifolds and a model for more general characterization theorems.

Chapman and Siebenmann modeled their theorem on Siebenmann's Collaring Theorem for finite-dimensional manifolds—but there are significant differences. First, there is no requirement of a stable fundamental group at infinity; therefore, a more flexible formulation of $\sigma_\infty(X)$, like that developed in Theorem 3.5.25, is required. Second, unlike finite-dimensional manifolds, inward tame Hilbert cube manifolds can be infinite-ended. In fact, $\mathcal{Z}$-compactifiable Hilbert cube manifolds can be infinite-ended ($\mathbb{T}_3 \times \mathcal{Q}$ is a simple example); therefore, we do not want to be restricted to the 1-ended case. This generality requires an even more flexible approach to the definition of $\sigma_\infty(X)$. For the sake of simplicity, we delay that explanation until the final stage of the coming proof. We recommend that during the first reading, a tacit assumption of 1-endedness be included.

The third difference is the appearance of a new obstruction lying in the first derived limit of an inverse sequence of Whitehead groups. The topological meaning of this obstruction is explained within the sketched proof. For completeness, we include the algebraic formulation: For an inverse sequence $\{G_i, \lambda_i\}$ of abelian groups, the *derived limit*[13] is the quotient group:

$$\varprojlim{}^1 \{G_i, \lambda_i\} = \left(\prod_{i=0}^{\infty} G_i\right) / \{(g_0 - \lambda_1 g_1, g_1 - \lambda_2 g_2, g_2 - \lambda_3 g_3, \cdots) \mid g_i \in G_i\}.$$

Theorem 3.8.17 (The Chapman-Siebenmann $\mathcal{Z}$-compactification Theorem) *A Hilbert cube manifold X admits a $\mathcal{Z}$-compactification if and only if each of the following is satisfied.*

(a) X *is inward tame,*
(b) $\sigma_\infty(X) \in \varprojlim \left\{\widetilde{K}_0(\mathbb{Z}[\pi_1(N)]) \mid N \text{ a clean nbd. of infinity}\right\}$ *is trivial, and*
(c) $\tau_\infty(X) \in \varprojlim{}^1 \{Wh(\pi_1(N)) \mid N \text{ a clean nbd. of infinity}\}$ *is trivial.*

Remark 3.8.18 (a) By Theorem 3.13.1, every ANR Y becomes a Hilbert cube manifold upon crossing with $\mathcal{Q}$. So, reasoning as in Remark 3.8.13, conditions (a)–(c) are also necessary for $\mathcal{Z}$-compactifiability of an ANR (although Condition (b) and particularly (c) are best measured in $Y \times \mathcal{Q}$). For some time, it was hoped that (a)–(c) would also be sufficient for ANRs; but in [49], a 2-dimensional polyhedral counterexample was constructed.

(b) For those who prefer finite-dimensional spaces, Ferry [37] has shown that, if P is a k-dimensional locally finite polyhedron and $P \times \mathcal{Q}$ is $\mathcal{Z}$-compactifiable, then $P \times [0, 1]^{2k+5}$ is $\mathcal{Z}$-compactifiable. May [64] showed that, for the counterexample P_0 from [49], $P_0 \times [0, 1]$ is $\mathcal{Z}$-compactifiable. In still-to-be-published work, the author has shown that, for an open manifold M^n satisfying (a)–(c), $M^n \times [0, 1]$ is $\mathcal{Z}$-compactifiable.

The following are significant and still open.

Problem 3.8.19 Find conditions that must be added to those of Theorem 3.8.17 to obtain a characterization of $\mathcal{Z}$-compactifiability for ANRs.

[13] The definition of derived limit can be generalized to include nonableian groups (see [45, Sect. 11.3]), but that is not needed here.

Problem 3.8.20 Determine whether the conditions of Theorem 3.8.17 are sufficient in the case of finite-dimensional manifolds.

Before describing the proof of Theorem 3.8.17, we make some obvious adaptations of terminology from Sects. 3.5.2 and 3.5.3. A clean neighborhood of infinity N in a Hilbert cube manifold X is an *open collar* if $N \approx \mathrm{Bd}_X N \times [0, 1)$ and a *homotopy collar* if $\mathrm{Bd}_X N \hookrightarrow N$ is a homotopy equivalence. X is *collarable* if it contains an open collar neighborhood of infinity and *pseudo-collarable* if it contains arbitrarily small homotopy collar neighborhoods of infinity.

Proof (*Sketch of the proof of Theorem* 3.8.17) The necessity of Conditions (a) and (b) follows from Proposition 3.8.12; for the necessity of (c), the reader is referred to [21]. Here we will focus on the sufficiency of these conditions.

Assume that X satisfies Conditions (a)–(c). We show that X is $\mathscr{Z}$-compactifiable by showing that it is homeomorphic at infinity to $\mathrm{Tel}\left(\{K_i, f_i\}\right) \times \mathscr{Q}$, where $\{K_i, f_i\}$ is a carefully chosen inverse sequence of finite polyhedra. Since inverse mapping telescopes are $\mathscr{Z}$-compactifiable (Example 3.8.11), the result follows.

It is easiest to read the following argument under the added assumption that X is 1-ended. In the final step, we explain how that assumption can be eliminated.

Step 1 (Existence of a pseudo-collar structure) Choose a nested cofinal sequence $\left\{N_i'\right\}$ of clean neighborhoods of infinity. By Condition (a) each is finitely dominated, so we may represent $\sigma_\infty(X)$ by $(\sigma_0, \sigma_1, \sigma_2, \ldots)$, where σ_i is the Wall finiteness obstruction of N_i'. By (b) each $\sigma_i = 0$, so each N_i' has finite homotopy type. (Said differently, Conditions (a) and (b) are equivalent to absolute inward tameness.) For each i, choose a finite polyhedron K_i and an embedding $K_i \hookrightarrow N_i'$ that is a homotopy equivalences. By taking neighborhoods C_i of the K_i, we arrive at a sequence of Hilbert cube manifold pairs $\left(N_i', C_i\right)$, where each inclusion is a homotopy equivalence. By some Hilbert cube manifold magic it can be arranged that C_i is a $\mathscr{Z}$-set in N_i' and $\mathrm{Bd}N_i' \subseteq C_i$. From there one finds $N_i \subseteq N_i'$ for which $\mathrm{Bd}N_i$ is a copy of C_i and $\mathrm{Bd}N_i \hookrightarrow N_i$ a homotopy equivalence (see [21] for details). Thus $\{N_i\}$ is a pseudo-collar structure.

Step 2 (Pushing the torsion off the end of X) By letting $A_i = \overline{N_i - N_{i+1}}$ for each i, view the end of X as a countable union $A_0 \cup A_1 \cup A_2 \cup \cdots$ of compact 1-sided h-cobordisms $(A_i, \mathrm{Bd}N_i, \mathrm{Bd}N_{i+1})$ of Hilbert cube manifolds. (See Exercise 3.5.15.) By the triangulability of Hilbert cube manifolds (and pairs), each inclusion $\mathrm{Bd}N_i \hookrightarrow A_i$ has a well-defined torsion $\tau_i \in \mathrm{Wh}(\pi_1(N_i))$. Together these torsions determine a representative $(\tau_0, \tau_1, \tau_2, \ldots)$ of $\tau_\infty(X)$. (*Note*: Determining $\tau_\infty(X)$ requires that Step 1 first be accomplished; there is no $\tau_\infty(X)$ without Conditions (a) and (b) being satisfied.)

We would like to alter the choices of the N_i by using an infinite borrowing strategy like that employed in the proof of Theorem 3.5.13. In particular, we would like to borrow a compact Hilbert cube manifold h-cobordism B_0 from a collar neighborhood of $\mathrm{Bd}N_1$ in A_1 so that $\mathrm{Bd}N_0 \hookrightarrow A_0 \cup B_0$ has trivial torsion. Then, replacing N_1 with $\overline{N_1 - B_0}$, we would like to borrow B_1 from A_2 so that so that $\mathrm{Bd}N_1 \hookrightarrow A_1 \cup B_1$ has trivial torsion. Continuing inductively, we would like to arrive at an adjusted sequence

$N_0 \supseteq N_1 \supseteq N_2 \supseteq \cdots$ of neighborhoods of infinity for which each $\mathrm{Bd}N_i \hookrightarrow A_i$ has trivial torsion (where the A_i are redefined using the new N_i).

The derived limit, $\varprojlim^1$, is defined precisely to measure whether this infinite borrowing strategy can be successfully completed. In the situation of Theorem 3.5.13, where the fundamental group stayed constant from one side of each A_i to the other, there was no obstruction to the borrowing scheme. More generally, as long as the inclusion induced homomorphisms $\mathrm{Wh}(\pi_1(N_{i+1})) \to \mathrm{Wh}(\pi_1(N_i))$ are surjective for all i, the strategy works. But, in general, we must rely on a hypothesis that $\tau_\infty(X)$ is the trivial element of $\varprojlim^1\{Wh(\pi_1(N_i))\}$. (*Warning*: Even when $\pi_1(N_{i+1})$ surjects onto $\pi_1(N_i)$, $\mathrm{Wh}(\pi_1(N_{i+1})) \to \mathrm{Wh}(\pi_1(N_i))$ can fail to be surjective.)

Step 3 (Completion of the proof) Homotopy equivalences $K_i \hookrightarrow \mathrm{Bd}N_i$ and the deformation retractions of A_i onto $\mathrm{Bd}N_i$ determine maps $f_{i+1} : K_{i+1} \to K_i$ and homotopy equivalences of triples $(A_i, \mathrm{Bd}N_i, \mathrm{Bd}N_{i+1}) \simeq (\mathrm{Map}(f_{i+1}), K_i, K_{i+1})$. Using the fact that both $\mathrm{Bd}N_i \hookrightarrow A_i$ and $K_i \hookrightarrow \mathrm{Map}(f_{i+1})$ have trivial torsion (and through more Hilbert cube manifold magic), we obtain a homeomorphism of triples $(A_i, \mathrm{Bd}N_i, \mathrm{Bd}N_{i+1}) \simeq (\mathrm{Map}(f_{i+1}) \times \mathcal{Q}, K_i \times \mathcal{Q}, K_{i+1} \times \mathcal{Q})$. Piecing these together gives a homeomorphism $N_0 \approx \mathrm{Tel}(\{K_i, f_i\}) \times \mathcal{Q}$, and completes the proof.

As a mild alternative, we could have used the ingredients described above to construct a proper homotopy equivalence $h : N_0 \to \mathrm{Tel}(K_i, f_i)$ and used the triviality of the torsions to argue that h is an "infinite simple homotopy equivalence", in the sense of [87]. Then, by a variation on Theorem 3.13.7, $\widehat{h} : N_0 \to \mathrm{Tel}(K_i, f_i) \times \mathcal{Q}$ is homotopic to a homeomorphism.

Final Step. (Multi-ended spaces) When X is multi-ended (possibly infinite-ended), a neighborhood of infinity N_i used in defining σ_∞ and τ_∞ will have multiple (but finitely many) components. In that case, define $\widetilde{K}_0(\mathbb{Z}[\pi_1(N_i)])$ and $Wh(\pi_1(N_i))$ to be the direct sums $\bigoplus \widetilde{K}_0(\mathbb{Z}\left[\pi_1\left(C_j\right)\right])$ and $\bigoplus Wh(\pi_1\left(C_j\right))$, where $\left\{C_j\right\}$ is the collection of components of N_i. With these definitions, a little extra work, and the fact that reduced projective class groups and Whitehead groups of free products are the corresponding direct sums, the above steps can be carried out as in the 1-ended case.

Remark 3.8.21 If desired, one can arrange, in the final lines of Step 3, a homeomorphism $k : X \to \mathrm{Tel}^*(K_i, f_i) \times \mathcal{Q}$, defined on all of X. The space on the right is the previous mapping telescope with the addition of a single mapping cylinder $\mathrm{Map}(K_0 \xrightarrow{f_0} K_{-1})$. The finite complex K_{-1} and the map f_0 are carefully chosen so that X and $\mathrm{Tel}^*(K_i, f_i)$ are infinite simple homotopy equivalent.

Step 1 of the above proof provides a result that is interesting in its own right.

Theorem 3.8.22 *A Hilbert cube manifold is pseudo-collarable if and only if it satisfies Conditions (a) and (b) of Theorem 3.8.17 or, equivalently, if and only if it is absolutely inward tame.*

It is interesting to compare Theorem 3.8.22 to Theorems 3.5.25 and 3.5.27.

3.9 $\mathscr{Z}$-Boundaries in Geometric Group Theory

In this section we look at the role of $\mathscr{Z}$-compactifications and $\mathscr{Z}$-boundaries in geometric group theory.

3.9.1 Boundaries of δ-Hyperbolic Groups

Following Gromov [47], for a metric space (X, d) with base point p_0, define the *overlap function on* $X \times X$ by

$$(x \cdot y) = \frac{1}{2}(d(x, p_0) + d(y, p_0) - d(x, y)).$$

Call (X, d) δ-*hyperbolic* if there exists a $\delta > 0$ such that $(x \cdot y) \geq \min\{(x \cdot z), (y \cdot z)\} - \delta$, for all $x, y, z \in X$.

A sequence $\{x_i\}$ in a δ-hyperbolic space (X, d) is *convergent at infinity* if $\left(x_i, x_j\right) \to \infty$ as $i, j \to \infty$, and sequences $\{x_i\}$, and $\{y_i\}$ are declared to be equivalent if $(x_i, y_i) \to \infty$ as $i \to \infty$. The set ∂X of all equivalence classes of these sequences makes up the *Gromov boundary* of X. An easy to define topology on $X \sqcup \partial X$ results in a corresponding compactification $\widehat{X} = X \cup \partial X$. This boundary and compactification are well-defined in the following strong sense: if $f : X \to Y$ is a quasi-isometry between δ-hyperbolic spaces, then there is a unique extension $\widehat{f} : \widehat{X} \to \widehat{Y}$ that restricts to a homeomorphism between boundaries. This is of particular interest when G is a finitely generated group endowed with a corresponding word metric. It is a standard fact that, for any two such metrics, $G \xrightarrow{\text{id}} G$ is a quasi-isometry; so for a δ-hyperbolic group G, the *Gromov boundary* ∂G is well-defined.

Early in the study of δ-hyperbolic groups, it became clear that exotic topological spaces arise naturally as group boundaries. In addition to spheres of all dimensions, the collection of known boundaries includes: Cantor sets, Sierpinski carpets, Menger curves, Pontryagin surfaces, and 2-dimensional Menger spaces, to name a few. See [9, 29, 58]. So it is not surprising that shape theory has a role to play in this area. But, a priori, Gromov's compactifications and boundaries have little in common with $\mathscr{Z}$-compactifications and $\mathscr{Z}$-boundaries. After all, for a word hyperbolic group, the Gromov compactification adds boundary to a *discrete* topological space.

Exercise 3.9.1 Show that a countably infinite discrete metric space does not admit a $\mathscr{Z}$-compactification.

Nevertheless, in 1991, Bestvina and Mess introduced the use of $\mathscr{Z}$-boundaries and $\mathscr{Z}$-compactifications to the study of δ-hyperbolic groups. For a discrete metric space (X, d) and a constant ρ, the *Rips complex* $P_\rho(G)$ is the simplicial complex obtained by declaring the vertex set to be X and filling in an n-simplex for each collection $\left\{x_0, x_1, \ldots, x_n\right\}$ with $d\left(x_i, x_j\right) \leq \rho$, for all $0 \leq i, j \leq n$. Clearly, a Rips complex

$P_\rho(G)$ for a finitely generated group G admits a proper, cocompact G-action and $G \hookrightarrow P_\rho(G)$ is a quasi-isometry. So when G is δ-hyperbolic there is a canonical compactification $\overline{P_\rho(G)} = P_\rho(G) \cup \partial G$. Furthermore, it was shown by Rips that, for δ-hyperbolic G and large ρ, $P_\rho(G)$ is contractible. Using this and some finer homotopy properties of $P_\rho(G)$, Bestvina and Mess proved the following.

Theorem 3.9.2 ([8, Theorem 1.2]) *Let G be a δ-hyperbolic group and $\rho \geq 4\delta + 2$, then $\overline{P_\rho(G)} = P_\rho(G) \cup \partial G$ is a $\mathscr{Z}$-compactification.*

Implications of Theorem 3.9.2 are cleanest when $P_\rho(G)$ is a cocompact EG complex. Since contractibility and a proper cocompact action have already been established, only freeness is needed, and that is satisfied if and only if G is torsion-free.

Corollary 3.9.3 *Let G be a torsion-free δ-hyperbolic group. Then*

(a) every cocompact EG complex is inward tame and proper homotopy equivalent to $P_\rho(G)$,
(b) for every cocompact EG complex X, $\mathscr{S}\mathrm{h}(\varepsilon(X)) = \mathscr{S}\mathrm{h}(\partial G)$,
(c) pro-$H_*(\varepsilon(G); R)$, $\check{H}_*(\varepsilon(G); R)$ and $\check{H}^*(\varepsilon(G); R)$ are isomorphic to the corresponding invariants of ∂G,
(d) for 1-ended G, pro-$\pi_*(\varepsilon(G))$ and $\check{\pi}_*(\varepsilon(G))$ are well defined and isomorphic to the corresponding invariants of $\mathscr{S}\mathrm{h}(\partial G)$,
(e) $H^*(G; RG) \cong \check{H}^{*-1}(\partial G; R)$ for any coefficient ring R.

Proof (*Corollary*) The discussion in Sect. 3.6.1 explains why all cocompact EG complexes are proper homotopy equivalent. Since one such space, $P_\rho(G)$, is $\mathscr{Z}$-compactifiable and therefore inward tame, they are all inward tame. By Theorem 3.8.14, $\mathscr{S}h\big(\varepsilon\left(P_\rho(G)\right)\big) = \mathscr{S}h(\partial G)$, so Theorem 3.7.19 completes (b). Assertion (3) is a consequence of Theorem 3.7.18, while 4) is similar, except that Theorem 3.6.8 (a significant ingredient) is used to assure well-definedness. Assertion (e), a statement about group cohomology with coefficients in RG, requires some algebraic topology that is explained in [8]; it is a consequence of (c) and builds upon earlier work by Geoghegan and Mihalik [45, 46].

For the most part, Corollary 3.9.3 is all about the shape of ∂G and the relationship between a $\mathscr{Z}$-boundary and its complement. There are other applications of boundaries of δ-hyperbolic groups that use more specific properties of ∂G. Here is a small sampling:

- Bestvina [8] provides formulas relating the cohomological dimension of a torsion-free G to the topological dimension of ∂G. (Clearly, the latter is not a shape invariant.)
- The semistability of G was deduced by proving that ∂G has no cut points [90], and therefore is locally connected, by results from [8].
- By work from [18, 40, 43, 95], $\partial G \approx \mathbb{S}^1$ if and only if G is virtually the fundamental group of a closed hyperbolic surface.

- Bowditch [14] has obtained a JSJ-decomposition theorem for δ-hyperbolic groups by analyzing cut pairs in ∂G.
- See [58] for many more examples.

3.9.2 Boundaries of CAT(0) Groups

Another widely studied class of groups are the *CAT(0) groups*, i.e., groups G that act geometrically (properly and cocompactly by isometries) on a proper CAT(0) space. If X is such a CAT(0) space, the visual boundary $\partial_\infty X$ is called a *group boundary* for G. Since a given G may act geometrically on multiple proper CAT(0) spaces, it is not immediate that its boundary is topologically well-defined; and, in fact, it is not. The first example of this phenomenon was displayed by Croke and Kleiner [23]. Their work was expanded upon by Wilson [99], who showed that their group admits a continuum of topologically distinct boundaries. Mooney [74] discovered additional examples from the category of knot groups, and in [75] produced another collection of examples with boundaries of arbitrary dimension $k \geq 1$. This situation suggests that CAT(0) boundaries (being ill-defined) might not be useful—but that is not the case. One reason is the following "approximate well-definedness" result.

Theorem 3.9.4 (Uniqueness of CAT(0) boundaries up to shape) *All CAT(0) boundaries of a given CAT(0) group G are shape equivalent.*

Proof If G is torsion-free, then a geometric G-action on a proper CAT(0) space X is necessarily free, so X is an EG space. It follows that all CAT(0) spaces on which G acts geometrically are proper homotopy equivalent. So, by Corollary 3.8.15, all CAT(0) boundaries of G have the same shape.

If G has torsion there is more work to be done, but the idea is the same. In [78], Ontaneda showed that any two proper CAT(0) spaces on which G acts geometrically are proper homotopy equivalent, so again their boundaries have the same shape.

As an application of Theorem 3.9.4, Corollary 3.9.3 can be repeated for torsion-free CAT(0) groups, with two exceptions: **(a)** we must omit reference to the Rips complex since it is not known to be an EG for CAT(0) groups, and **(b)** in general, pro-$\pi_*(\varepsilon(G))$ and $\check{\pi}_*(\varepsilon(G))$ are not known to be well-defined since the following is open.

Conjecture 3.9.5 (CAT(0) Semistability Conjecture) *Every 1-ended CAT(0) group G is semistable.*

It is worth noting that ∂G, itself, provides an approach to this conjecture. By a result from shape theory [31, Theorem 7.2.3], G is semistable if and only if ∂G has the shape of a locally connected compactum. (This is true in much greater generality.)

Before moving away from CAT(0) group boundaries, we mention a few more applications.

- The Bestvina-Mess formulas, mentioned earlier, relating the cohomological dimension of a torsion-free G to the topological dimension of ∂G are also valid for CAT(0) G.
- Swenson [91] has shown that a CAT(0) group G with a cut point in ∂G has an infinite torsion subgroup.
- Ruane [84] has shown that for CAT(0) G, if ∂G is a circle, then G is virtually $\mathbb{Z} \times \mathbb{Z}$ or the fundamental group of a closed hyperbolic surface; and if ∂G is a suspended Cantor set, then G is virtually $\mathbb{F}_2 \times \mathbb{Z}$.
- Swenson and Papasoglu [80] have, in a manner similar to Bowditch's work on δ-hyperbolic groups, used cut pairs in ∂G to obtain a JSJ-decomposition result for CAT(0) groups.

3.9.3 A General Theory of Group Boundaries

Motivated by their usefulness in the study of δ-hyperbolic and CAT(0) groups, Bestvina [9] developed an axiomatic approach to group boundaries which unified the existing theories and provided a framework for defining group boundaries more generally. We begin with the original definition, then introduce some variations.

A *$\mathscr{Z}$-structure on a group G* is a topological pair $(\overline{X}, Z)$ satisfying the following four conditions:

(a) $\overline{X}$ is a compact ER,
(b) Z is a $\mathscr{Z}$-set in $\overline{X}$,
(c) $X = \overline{X} - Z$ admits a proper, free, cocompact G-action, and
(d) the G-action on X satisfies the following *nullity condition*: for every compactum $A \subseteq X$ and every open cover $\mathscr{U}$ of $\overline{X}$, all but finitely many G-translates of A are $\mathscr{U}$-small, i.e., are contained in some element of $\mathscr{U}$.

A pair $(\overline{X}, Z)$ that satisfies (a)–(c), but not necessarily (d) is called a *weak $\mathscr{Z}$-structure* on G, while a $\mathscr{Z}$-structure on G that satisfies the additional condition:

(e) the G-action on X extends to a G-action on $\overline{X}$,

is called an *$E\mathscr{Z}$-structure* (an *equivariant $\mathscr{Z}$-structure*) on G. A *weak $E\mathscr{Z}$-structure* is a weak $\mathscr{Z}$-structure that satisfies Condition (e).[14]

Under the above circumstances, Z is called a *$\mathscr{Z}$-boundary*, a *weak $\mathscr{Z}$-boundary*, an *$E\mathscr{Z}$-boundary*, or a *weak $E\mathscr{Z}$ -boundary*, as appropriate.

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[14]Bestvina informally introduced the definition of weak $\mathscr{Z}$-structure in [9], where he also commented on his decision to omit Condition (e) from the definition of $\mathscr{Z}$-structure. Farrell and Lafont introduced the term $E\mathscr{Z}$-structure in [35].

Example 3.9.6 (*A sampling of $\mathcal{Z}$-structures*)

(a) The $\mathcal{Z}$-compactification $\overline{P_\rho(G)} = P_\rho(G) \cup \partial G$ of Theorem 3.9.2 is an $E\mathcal{Z}$-structure whenever G is a torsion-free δ-hyperbolic group.
(b) If a torsion-free group G admits a geometric action on a finite-dimensional proper CAT(0) space X, then $\overline{X} = X \cup \partial_\infty X$ is an $E\mathcal{Z}$-structure for G.
(c) The Baumslag-Solitar group $BS(1,2) = \langle a, b \mid bab^{-1} = a^2 \rangle$ is put forth by Bestvina as an example that is neither δ-hyperbolic nor CAT(0), but still admits a $\mathcal{Z}$-structure. The $\mathcal{Z}$-structure described in [9] is also an $E\mathcal{Z}$-structure. The traditional cocompact EG 2-complex for $BS(1,2)$ is homeomorphic to $\mathbb{T}_3 \times \mathbb{R}$, where $\mathbb{T}_3$ is the uniformly trivalent tree. Given the Euclidean product metric, $\mathbb{T}_3 \times \mathbb{R}$ is CAT(0), so adding the visual boundary gives a weak $\mathcal{Z}$-structure, with a suspended Cantor set as boundary. (Since the action of $BS(1,2)$ on $\mathbb{T}_3 \times \mathbb{R}$ is not by isometries, one cannot conclude that $BS(1,2)$ is CAT(0)). This weak $\mathcal{Z}$-structure does not satisfy the nullity condition—instead it provides a nice illustration of the failure of that condition. Nevertheless, by using this structure as a starting point, a genuine $\mathcal{Z}$-structure (in fact more than one) can be obtained.
(d) Januszkiewicz and Świątkowski [57] have developed a theory of "systolic" spaces and groups that parallels, but is distinct from, CAT(0) spaces and groups. Among systolic groups are many that are neither δ-hyperbolic nor CAT(0). A delicate construction by Osajda and Przytycki in [79] places $E\mathcal{Z}$-structures on all torsion-free systolic groups.
(e) Dahmani [24] showed that, if a group G is hyperbolic relative to a collection of subgroups, each of which admits a $\mathcal{Z}$-structure, then G admits a $\mathcal{Z}$-structure.
(f) Tirel [92] showed that if G and H each admit $\mathcal{Z}$-structures (resp., $E\mathcal{Z}$-structures), then so do $G \times H$ and $G * H$.
(g) In [51], this author initiated a study of weak $\mathcal{Z}$-structures on groups. Examples of groups shown to admit weak $\mathcal{Z}$-structures include all type F groups that are simply connected at infinity and all groups that are extensions of a type F group by a type F group.

Exercise 3.9.7 Verify the assertion made in Item (b) of Example 3.9.6.

Exercise 3.9.8 For $G \times H$ in Item (f), give an easy proof of the existence of weak $\mathcal{Z}$-structures (resp., weak $E\mathcal{Z}$-structures). As with Item (c), the difficult part is the nullity condition.

Given the wealth of examples, it becomes natural to ask whether all reasonably nice groups admits $\mathcal{Z}$-structures. The following helps define "reasonably nice".

Proposition 3.9.9 *A group G that admits a weak $\mathcal{Z}$-structure must have type F.*

Proof If $(\overline{X}, Z)$ is a weak $\mathcal{Z}$-structure on G, then $X = \overline{X} - Z$ is an EG space and $X \to G\backslash X$ is a covering projection. Since being an ENR is a local property, $G\backslash X$ is an ENR; it is also compact and aspherical. By Proposition 3.12.4, $G\backslash X$ is homotopy equivalent to a finite complex K, which is a $K(G,1)$.

Question 3.9.1 (all are open) Does every group of type F admit a $\mathcal{Z}$-structure? an $E\mathcal{Z}$-structure? a weak $\mathcal{Z}$-structure? a weak $E\mathcal{Z}$-structure?

The first of the above questions was asked explicitly by Bestvina in [9], where he also mentions the version for weak $\mathcal{Z}$-structures. The latter of those two was also mentioned in [8], where the weak $E\mathcal{Z}$-version is explicitly asked. The $E\mathcal{Z}$-version, was suggested by Farrell and Lafont in [35].

In [9], Bestvina prefaced his posing of the $\mathcal{Z}$-structure Question with the warning: *"There seems to be no systematic method of constructing boundaries of groups in general, so the following is probably hopeless."* In the years since that question was posed, a general strategy has still not emerged. However, there have been successes (such as those noted in Example 3.9.6) when attention is focused on a specific group or class of groups. In private conversations and in presentations, Bestvina has suggested some additional groups for consideration; notable among these are $Out\left(\mathbb{F}_n\right)$ and the various Baumslag-Solitar groups $BS\left(m,n\right)$. Farrell and Lafont have specifically asked about $E\mathcal{Z}$-structures for torsion-free finite index subgroups of $SL_n\left(\mathbb{Z}\right)$. A less explicit, but highly important class of groups, are the fundamental groups of closed aspherical manifolds (or more generally, Poincaré duality groups)—the hope being that well-developed tools from manifold topology might provide an advantage.

Bestvina [9, Lemma 1.4] has shown that if G admits a $\mathcal{Z}$-structure $\left(\overline{X},Z\right)$, then every cocompact EG complex Y can be incorporated into a $\mathcal{Z}$-structure $\left(\overline{Y},Z\right)$. In particular, every cocompact EG complex satisfies the hypotheses of Theorem 3.8.17. So it seems natural to begin with:

Question 3.9.2 Must the universal cover of a finite aspherical complex be inward tame? absolutely inward tame?

Remarkably, nothing seems to be known here. An early version of the question goes back to [44], with more explicit formulations found in [37, 48].

Since, for fixed G, all cocompact EG spaces are proper homotopy equivalent, we can view *inward tameness* as a property possessed by some (possibly all) type F groups. Moreover, if G is inward tame, we can use Sect. 3.7.4 to define the *shape of the end of* G. Specifically, for X a cocompact EG, $\mathcal{S}h(\varepsilon\left(G\right))=\mathcal{S}h(\varepsilon\left(X\right))$. If $A\in\mathcal{S}h(\varepsilon\left(G\right))$, we might even view A as a "pre-$\mathcal{Z}$ -boundary" and (X,A) as a "pre-$\mathcal{Z}$-structure" for G.

As for applications of the various sorts of $\mathcal{Z}$-boundaries, we list a few.

- As noted in the previous paragraph, even pre-$\mathcal{Z}$-boundaries are well-defined up to shape. So a result like Corollary 3.9.3 can be stated here, with the same exceptions as noted above for CAT(0) groups.
- In [9], it is shown that the Bestvina-Mess formulas relating the cohomological dimension of a torsion-free G to the topological dimension of ∂G are again valid for $\mathcal{Z}$-boundaries. For this, the full strength of Bestvina's definition of $\mathcal{Z}$-structure is used.
- Carlsson and Pedersen [17] and Farrell and Lafont [35] have shown that groups admitting an $E\mathcal{Z}$-structure satisfy the Novikov Conjecture.

3.9.4 Further Generalizations

A pair of generalizations to the various $(E)\mathcal{Z}$-structure and boundary definitions can be found in the literature. See, for example, [30].

(i) Replace the requirement that $\overline{X}$ be an ER with the weaker requirement that it be an AR.

(ii) Drop the freeness requirement for the G-action on X.

Change (i) simply allows $\overline{X}$ to be infinite-dimensional; by itself that may be of little consequence. After all, X is still a cocompact EG, so there exists a finite $K(G, 1)$ complex K. If Z is finite-dimensional, Bestvina's boundary swapping trick ([9, Lemma 1.4]) produces a new $\mathcal{Z}$-structure $(\overline{Y}, Z)$ in which $\overline{Y}$ is an ER. This motivates the question:

Question 3.9.3 If $(\overline{X}, Z)$ is a $\mathcal{Z}$-structure on a group G in the sense of [9], except that $\overline{X}$ is only required to be an ANR, must Z still be finite-dimensional? (Compare to [91, Theorem 12], which shows that a CAT(0) group boundary is finite-dimensional, regardless of the CAT(0) space it bounds.)[15]

Change (ii) is more substantial; it allows for groups with torsion. $\mathcal{Z}$-structures of this sort are plentiful in the categories of δ-hyperbolic and CAT(0) groups, with Coxeter groups the prototypical examples; so this generalization is very natural. There are, however, complications. When G has torsion, the notion of a cocompact EG complex must be replaced by that of a cocompact (or G-finite) $\underline{E}G$ complex, where G may act with fixed points, subject to the requirement that stabilizers of all finite subgroups are contractible subcomplexes. This notion is fruitful and cocompact $\underline{E}G$ complexes, when they exist, are well-defined up to G-equivariant homotopy equivalence, and more importantly (from the point of view of these notes) up to proper homotopy equivalence.

In order to obtain the sorts of conclusions we are concerned with here, positive answers to the following, questions would be of interest.

Question 3.9.4 Suppose G admits a $\mathcal{Z}$-structure $(\overline{X}, Z)$, but with the G-action on X not required to be free. If $(\overline{X'}, Z')$ is another such $\mathcal{Z}$-structure, is $X \overset{p}{\simeq} X'$? More specifically, does there exist a cocompact $\underline{E}G$ complex and must X be proper homotopy equivalent to that complex?

3.10 $\mathcal{Z}$-Boundaries in Manifold Topology

In this section we look specifically at $\mathcal{Z}$-compactifications and $\mathcal{Z}$-boundaries of manifolds, with an emphasis on open manifolds and manifolds with compact boundary. In Sect. 3.9 we noted the occurrence of many exotic group boundaries: Cantor

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[15] Added in proof. An affirmative answer to this question was recently obtained by Molly Moran.

sets, suspended Cantor sets, Hawaiian earrings, Sierpinski carpets, and Pontryagin surfaces, to name a few. By contrast, we will see that a $\mathcal{Z}$-boundary of an n-manifold with compact boundary is always a homology $(n-1)$-manifold. That does not mean the $\mathcal{Z}$-boundary is always nice—recall Example 3.8.8—but it does mean that manifold topology forces some significant regularity on potential $\mathcal{Z}$-boundaries. Here we take a look at that result and some related applications. First, a quick introduction to homology manifolds.

3.10.1 Homology Manifolds

If N^n is an n-manifold with boundary, then each $x \in \mathrm{int}N^n$ has *local homology*

$$\widetilde{H}_*\left(N^n, N^n - x\right) \cong \widetilde{H}_*\left(\mathbb{R}^n, \mathbb{R}^n - \mathbf{0}\right) \cong \begin{cases} \mathbb{Z} & \text{if } * = n \\ 0 & \text{otherwise} \end{cases}$$

and each $x \in \partial N^n$ has local homology

$$\widetilde{H}_*\left(N^n, N^n - x\right) \cong \widetilde{H}_*\left(\mathbb{R}^n_+, \mathbb{R}^n_+ - \mathbf{0}\right) \equiv 0.$$

This motivates the notion of a "homology manifold".

Roughly speaking, X is a *homology n-manifold* if

$$\widetilde{H}_*\left(X, X - x\right) \cong \widetilde{H}_*\left(\mathbb{R}^n, \mathbb{R}^n - \mathbf{0}\right)$$

for all $x \in X$, and a *homology n-manifold with boundary* if

$$\widetilde{H}_*\left(X, X - x\right) \cong \widetilde{H}_*\left(\mathbb{R}^n, \mathbb{R}^n - \mathbf{0}\right) \quad \text{or} \quad \widetilde{H}_*\left(X, X - x\right) \cong \widetilde{H}_*\left(\mathbb{R}^n_+, \mathbb{R}^n_+ - \mathbf{0}\right) \equiv 0$$

for all $x \in X$. In the latter case we define

$$\partial X \equiv \left\{x \in X \mid \widetilde{H}_*\left(X, X - x\right) = 0\right\}$$

and call this set the *boundary of* X.[16]

The reason for the phrase "roughly speaking" in the above paragraph is because ordinary singular homology theory does not always detect the behavior we are looking for. This issue is analogous to what happened in shape theory; there, when singular theory told us that the homology of the Warsaw circle W was the same as that of a point, we developed Čech homology theory to better capture the circle-like nature of W. In the current setting, we again need to adjust our homology theory to match our goals. Without going into detail, we simply state that, for current purposes

[16]All homology here is with $\mathbb{Z}$-coefficients. With the same strategy and an arbitrary coefficient ring, we can also define *R-homology manifold* and *R-homology manifold with boundary.*

Borel–Moore homology, or equivalently Steenrod homology (see [10, 36], or [72]), should be used. Moreover, since Borel–Moore homology of a pair requires that A be closed in X, we interpret $\widetilde{H}_*(X, X - x)$ to mean $\varinjlim \widetilde{H}_*(X, X - U)$ where U varies over all open neighborhoods of x.

With the above adjustment in place, we are nearly ready to discuss the essentials of homology manifolds. Before doing so we note that there is an entirely analogous theory of *cohomology manifolds*, in which Alexander-Čech theory is the preferred cohomology theory. We also note that both Borel–Moore homology and Alexander-Čech cohomology theories agree with the singular theories when X is an ANR. An ANR homology manifold is often called a *generalized manifold*—a class of objects that plays an essential role in geometric topology.

Example 3.10.1 Let Σ^n be a non-simply connected n-manifold with the same $\mathbb{Z}$-homology as $\mathbb{S}^n$, e.g., the boundary of a Newman contractible $(n+1)$-manifold. Then $X = \text{cone}(\Sigma^n) = \Sigma^n \times [0,1] / \{\Sigma^n \times 1\}$ is a homology $(n+1)$-manifold with boundary, where $\partial X = \Sigma^n \times 0$. The double of X^{n+1}, the suspension of Σ^{n+1}, is a homology manifold that is homotopy equivalent to $\mathbb{S}^{n+1}$. Both of these are ANRs, hence generalized manifolds, but neither is an actual manifold.

Example 3.10.2 Let A be a compact proper subset of the interior of an n-manifold M^n and let $Y = M^n / A$ be the quotient space obtained by identifying A to a point. If A is cell-like (i.e., has trivial shape), then X is a generalized n-manifold. In many cases $Y \approx M^n$, for example, when A is a tame arc or ball. In other cases—for example, A a wild arc with non-simply connected complement or A a Newman contractible n-manifold embedded in M^n, Y is not a manifold.

Exercise 3.10.3 Verify the unproven assertions in the above two exercises.

Remark 3.10.4 The subject of *Decomposition Theory* is motivated by Example 3.10.2. There, the following question is paramount: Given a pairwise disjoint collection $\mathcal{G}$ of cell-like compacta in a manifold M^n satisfying a certain niceness condition (an *upper semicontinuous decomposition*), when is the quotient space $M/\mathcal{G}$ a manifold? Although the premise sounds simple and very specific, results from this area have had broad-ranging impacts on geometric topology, including: existence of exotic involutions on spheres, existence of exotic manifold factors (non-manifolds X for which $X \times \mathbb{R}$ is a manifold), existence of non-PL triangulations of manifolds, and a solution to the 4-dimensional Poincaré Conjecture. The Edwards-Quinn Manifold Recognition Theorem, which will be used shortly, belongs to this subject. References to the "Moore-Bing school of topology" usually indicate work by R.L. Moore, R.H. Bing, and their mathematical descendents in this area. See [25] for a comprehensive discussion of this topic.

Exercise 3.10.5 Here we describe a simple non-ANR homology manifold. Let H^n be non-simply connected n-manifold with the homology of a point and a boundary homeomorphic to $\mathbb{S}^{n-1}$, and let $\{B_i^n\}_{i=1}^{\infty}$ be a sequence of pairwise disjoint round n-balls in $\mathbb{S}^n$ converging to a point p. Create X by removing the interiors of the B_i^n

and replacing each with $H_i^n \approx H^n$. Topologize X so that each neighborhood of p in X contains all but finitely many of the H_i^n. The result is a homology manifold (some knowledge of Borel–Moore homology is needed to verify this fact).

Explain why X is not an ANR. Then show that X does not satisfy the definition of homology manifold if singular homology is used.

Exercise 3.10.6 Show that the $\mathscr{Z}$-boundary attached to the Davis manifold described in Example 3.8.8 is homeomorphic to the non-ANR homology manifold described in Exercise 3.10.5 (some attention must be paid to orientations).

For now, the reader may wish to treat the following theorem as a set of axioms; [2] shows how the classical literature can be woven together to obtain proofs.

Theorem 3.10.7 (Fundamental facts about (co)homology manifolds)

(a) *A space X is a homology n-manifold if and only if it is a cohomology n-manifold.*
(b) *The boundary of a (co)homology n-manifold is a (co)homology $(n-1)$-manifold without boundary.*
(c) *The union of two (co)homology n-manifolds with boundary along a common boundary is a (co)homology n-manifold.*
(d) *(Co)Homology manifolds are locally path connected.*

Corollary 3.10.8 *Let M^n be an open n-manifold (or even just an open generalized manifold) and $\overline{M^n} = M^n \cup Z$ be a $\mathscr{Z}$-compactification. Then*

(a) *$\overline{M^n}$ is a homology n-manifold with boundary,*
(b) *$\partial \overline{M^n} = Z$, and*
(c) *Z is a homology $(n-1)$-manifold.*

Proof (*Corollary*) For (a) and (b) we need only check that $H_*\left(\overline{M^n}, \overline{M^n} - z\right) \equiv 0$ at each $z \in Z$. Since $\overline{M^n}$ is an ANR, we are free to use singular homology in place of Borel–Moore theory. A closed subset of a $\mathscr{Z}$-set is a $\mathscr{Z}$-set, so $\{z\}$ is a $\mathscr{Z}$-set in $\overline{M^n}$, and hence, $\overline{M^n} - z \hookrightarrow \overline{M^n}$ is a homotopy equivalence. The desired result now follows from the long exact sequence for pairs.

Item (c) is now immediate from Theorem 3.10.7.

Exercise 3.10.9 Show that if M^n is a CAT(0) n-manifold, then every metric sphere in M^n is a homology $(n-1)$-manifold.

Remark 3.10.10 In addition to Theorem 3.10.7, it is possible to define orientation for homology manifolds and prove a version of Poincaré duality for the orientable ones. With those tools, one can also prove, for example, that any $\mathscr{Z}$-boundary of a contractible open n-manifold has the (Borel–Moore) homology of an $(n-1)$-sphere.

Before moving to applications, we state without proof one of the most significant results in this area. A nice exposition can be found in [25].

Theorem 3.10.11 (Edwards-Quinn Manifold Recognition Theorem) *Let X^n ($n \geq 5$) be a generalized homology n-manifold without boundary and suppose X^n contains a nonempty open set $U \approx \mathbb{R}^n$. Then X^n is an n-manifold if and only if it satisfies the disjoint disks property (DDP).*

A space X satisfies the DDP if, for any pair of maps $f, g : D^2 \to X$ and any $\varepsilon > 0$, there exist ε-approximations f' and g' of f and g, so that $f'\left(D^2\right) \cap g'\left(D^2\right) = \varnothing$.

3.10.2 Some Applications of $\mathscr{Z}$-Boundaries to Manifold Topology

Most results in this section come from [2]. Here we provide only the main ideas; for details, the reader should consult the original paper. For the sake of brevity, we focus on high-dimensional results. In many cases, low-dimensional analogs are true for different reasons.

Let $\overline{M^n} = M^n \cup Z$ be a $\mathscr{Z}$-compactification of an open n-manifold. Since $\overline{M^n}$ need not be a manifold with boundary, the following is a pleasant surprise.

Theorem 3.10.12 *Suppose $\overline{M^n} = M^n \cup Z$ and $\overline{N^n} = N^n \cup Z'$ are $\mathscr{Z}$ -compactification of open n-manifolds ($n > 4$) and $h : Z \to Z'$ is a homeomorphism. Then $P^n = \overline{M^n} \cup_h \overline{N^n}$ is a closed n-manifold.*

Proof (*Sketch*) Theorem 3.10.7 asserts that P^n is a homology n-manifold. From there one uses delicate properties of homology manifolds to prove that P^n is locally contractible at each point on the "seam", $Z = Z'$; hence, P^n is an ANR. Another delicate, but more straightforward, argument (this part using the fact that Z and Z' are $\mathscr{Z}$-sets) verifies the DDP for P^n. Open subsets of P^n homeomorphic to $\mathbb{R}^n$ are plentiful in the manifolds M^n and N^n, so Edwards-Quinn can be applied to complete the proof.

Corollary 3.10.13 *The double of $\overline{M^n}$ along Z is an n-manifold. If M^n is contractible, that double is homeomorphic to $\mathbb{S}^n$, and there is an involution of $\mathbb{S}^n$ with Z as its fixed set.*

Proof (*Sketch*) Double$(\overline{M^n}) \approx \mathbb{S}^n$ will follow from the Generalized Poincaré conjecture if we can show that it is a simply connected manifold with the homology of an n-sphere. The involution interchanges the two copies of $\overline{M^n}$.

That Double$(\overline{M^n})$ has the homology of $\mathbb{S}^n$ is a consequence of Mayer-Vietoris and Remark 3.10.10. Since $\overline{M^n}$ is simply connected, simple connectivity of Double$(\overline{M^n})$ would follow directly from van Kampen's Theorem if the intersection between the two copies was nice. Instead a controlled variation on the traditional proof of van Kampen's Theorem is employed. Use the fact that homology manifolds are locally path connected to divide an arbitrary loop into loops lying in one or the other copy of $\overline{M^n}$, where they can be contracted. Careful control is needed, and the fact that $\overline{M^n}$ is locally contractible is important.

Theorem 3.10.14 *If contractible open manifolds M^n and N^n $(n > 4)$ admit $\mathcal{Z}$-compactifications with homeomorphic $\mathcal{Z}$-boundaries, then $M^n \approx N^n$.*

Proof (*Sketch*) Let Z denote the common $\mathcal{Z}$-boundary. The argument used in Corollary 3.10.13 shows that the union of these compactifications along Z is $\mathbb{S}^n$. Let $W^{n+1} = \mathbb{B}^{n+1} - Z$ and note that $\partial W^{n+1} = M^n \sqcup N^n$, providing a noncompact cobordism $\left(W^{n+1}, M^n, N^n\right)$. The proof is completed by applying the Proper s-cobordism Theorem [87] to conclude that $W^{n+1} \approx M^n \times [0, 1]$. That requires some work. First show that $M^n \hookrightarrow W^{n+1}$ is a proper homotopy equivalence. (The fact that Z is a $\mathcal{Z}$-set in $\mathbb{B}^{n+1}$ is key.) Then, to establish that $M^n \hookrightarrow W^{n+1}$ is an infinite simple homotopy equivalence, some algebraic obstructions must be checked. Fortunately, there are "naturality results" from [21] that relate those obstructions to the $\mathcal{Z}$-compactifiability obstructions for M^n and W^{n+1} (as found in Theorem 3.8.17). In particular, since the latter vanish, so do the former.

The following can be obtained in a variety of more elementary ways; nevertheless, it provides a nice illustration of Theorem 3.10.14.

Corollary 3.10.15 *If a contractible open n-manifold M^n can be $\mathcal{Z}$-compactified by the addition of an* $(n-1)$*-sphere, then $M^n \approx \mathbb{R}^n$.*

The Borel Conjecture posits that closed aspherical manifolds with isomorphic fundamental groups are necessarily homeomorphic. Our interest in contractible open manifolds led to the following.

Conjecture 3.10.16 (Weak Borel Conjecture) *Closed aspherical manifolds with isomorphic fundamental groups have homeomorphic universal covers.*

Theorem 3.10.14 provides the means for a partial solution.

Theorem 3.10.17 *The Weak Borel Conjecture is true for those n-manifolds $(n > 4)$ whose fundamental groups admits $\mathcal{Z}$-structures.*

Proof Let P^n and Q^n be aspherical manifolds, and $\left(\overline{X}, Z\right)$ a $\mathcal{Z}$-structure on $\pi_1(P^n) \cong \pi_1(Q^n)$. By Bestvina's boundary swapping trick [9, Lemma 1.4], both $\widetilde{P}^n$ and $\widetilde{Q}^n$ can be $\mathcal{Z}$-compactified by the addition of a copy of Z. Now apply Theorem 3.10.14.

Remark 3.10.18 Aspherical manifolds to which Theorem 3.10.17 applies include those with hyperbolic and CAT(0) fundamental groups. We are not aware of applications outside of those categories.

Recently, Bartels and Lück [4] proved the full-blown Borel Conjecture for δ-hyperbolic groups and CAT(0) groups that act geometrically on finite-dimensional CAT(0) spaces. Not surprisingly, their proof is more complicated than that of Theorem 3.10.17.

3.10.3 $E\mathcal{Z}$-Structures in Manifold Topology

As discussed in Sect. 3.9, the notion of an $E\mathcal{Z}$-structure was formalized by Farrell and Lafont in [35]. Among their applications was a new proof of the Novikov Conjecture for δ-hyperbolic and CAT(0) groups. That result had been obtained earlier by Carlsson and Pedersen [17] using similar ideas. We will not attempt to discuss the Novikov Conjecture here, except to say that it is related to, but much broader (and more difficult to explain) than the Borel Conjecture.

For a person with interests in manifold topology, one of the more intriguing aspects of Farrell and Lafont's work is a technique they develop which takes an arbitrary $\mathcal{Z}$-structure $\left(\overline{X}, Z\right)$ on a group G and replaces it with one of the form $(\mathbb{B}^n, Z)$, where n is necessarily large, Z is a topological copy of the original $\mathcal{Z}$-boundary lying in $\mathbb{S}^{n-1}$, and the new EG is the n-manifold with boundary $\mathbb{B}^n - Z$. The beauty here is that, once the structure is established, all of the tools of high-dimensional manifold topology are available. In their introduction, they challenge the reader to find other applications of these manifold $\mathcal{Z}$-structures, likening them to the action of a Kleinian group on a compactified hyperbolic n-space.

3.11 Further Reading

Clearly, we have just scratched the surface on a number of topics addressed in these notes. For a broad study of geometric group theory with a point of view similar to that found in these notes, Geoghegan's book, *Topological methods in group theory* [45], is the obvious next step.

For those interested in the topology of noncompact manifolds, Siebenmann's thesis [86] is still a fascinating read. The main result from that manuscript can also be obtained from the series of papers [48, 52, 53], which have the advantage of more modern terminology and greater generality. Steve Ferry's *Notes on geometric topology* (available on his website) contain a remarkable collection of fundamental results in manifold topology. Most significantly, from our perspective, those notes do not shy away from topics involving noncompact manifolds. There one can find clear and concise discussions of the Whitehead manifold, the Wall finiteness obstruction, Stallings' characterization of euclidean space, Siebenmann's thesis, and much more.

The complementary articles [21, 87] fit neither into the category of manifold topology nor that of geometric group theory; but they contain fundamental results and ideas of use in each area. Researchers whose work involves noncompact spaces of almost any variety are certain to benefit from a familiarity with those papers. Another substantial work on the topology of noncompact spaces, with implications for both manifold topology and geometric group theory, is the book by Hughes and Ranicki, *Ends of complexes* [56].

For the geometric group theorist specifically interested in the interplay between shapes, group boundaries, $\mathcal{Z}$-sets, and $\mathcal{Z}$-compactifications, the papers by

Bestvina-Mess [8], Bestvina [9], and the follow-up by Dranishnikov give a quick entry into that subject; while Geoghegan's earlier article, *The shape of a group* [44], provides a first-hand account of the origins of many of those ideas. For general applications of $\mathcal{Z}$-compactifications to manifold topology, the reader may be interested in [2]; and for more specific applications to the Novikov Conjecture, [35] is a good starting point.

3.12 Appendix A: Basics of ANR Theory

Before beginning this appendix, we remind the reader that all spaces discussed in these notes are assumed to be separable metric spaces.

A locally compact space X is an ANR (*absolute neighborhood retract*) if it can be embedded into $\mathbb{R}^n$ or, if necessary, $\mathbb{R}^\infty$ (a countable product of real lines) as a closed set in such a way that there exists a retraction $r : U \to X$, where U is a neighborhood of X. If the entire space $\mathbb{R}^n$ or $\mathbb{R}^\infty$ retracts onto X, we call X an AR (absolute retract). If X is finite-dimensional, all mention of $\mathbb{R}^\infty$ can be omitted. A finite-dimensional ANR is often called an ENR (*Euclidean neighborhood retract*) and a finite-dimensional AR an ER.

Use of the word "absolute" in ANR (or AR) stems from the following standard fact: If one embedding of X as a closed subset of $\mathbb{R}^n$ or $\mathbb{R}^\infty$ satisfies the defining condition, then so do all such embeddings. An alternative definition for ANR (and AR) is commonly found in the literature. To help avoid confusion, we offer that approach as Exercise 3.12.6. Texts [12, 55] are devoted entirely to the theory of ANRs; readers can go to either for details.

With a little effort (Exercise 3.12.7) it can be shown that an AR is just a contractible ANR, so there is no loss of generality if focusing on ANRs.

A space Y is *locally contractible* if every neighborhood U of a point $y \in Y$ contains a neighborhood V of y that contracts within U. It is easy to show that every ANR is locally contractible. A partial converse gives a powerful characterization of finite-dimensional ANRs.

Theorem 3.12.1 *A locally compact finite-dimensional space X is an ANR if and only if it is locally contractible.*

Example 3.12.2 By Theorem 3.12.1, manifolds, finite-dimensional locally finite polyhedra and CW complexes, and finite-dimensional proper CAT(0) spaces are all ANRs.

Example 3.12.3 It is also true that Hilbert cube manifolds, infinite-dimensional locally finite polyhedra and CW complexes, and infinite-dimensional proper CAT(0) spaces are all ANRs. Proofs would require some additional effort, but we will not hesitate to make use of these facts.

Rather than listing key results individually, we provide a mix of facts about ANRs in a single Proposition. The first several are elementary, and the final item is a deep result. Each is an established part of ANR theory.

Proposition 3.12.4 (Standard facts about ANRs)

(a) *Being an ANR is a local property: every open subset of an ANR is an ANR, and if every element of X has an ANR neighborhood, then X is an ANR.*
(b) *If $X = A \cup B$, where A, B, and $A \cap B$ are compact ANRs, then X is a compact ANR.*
(c) *Every retract of an ANR is an ANR; every retract of an AR is an AR.*
(d) *(Borsuk's Homotopy Extension Property) Every $h : (Y \times \{0\}) \cup (A \times [0, 1]) \to X$, where A is a closed subset of a space Y and X is ANR, admits an extension $H : Y \times [0, 1] \to X$.*
(e) *(West, [97]) Every ANR is proper homotopy equivalent to a locally finite CW complex; every compact ANR is homotopy equivalent to a finite complex.*

Remark 3.12.5 Items (c) and (d) allow us to extend the tools of algebraic topology and homotopy theory normally reserved for CW complexes to ANRs. For example, Whitehead's Theorem, that a map between CW complexes which induces isomorphisms on all homotopy groups is a homotopy equivalence, is also true for ANRs. In a very real sense, this sort of result is the motivation behind ANR theory.

Exercise 3.12.6 A locally compact space X is an ANE (*absolute neighborhood extensor*) if, for any space Y and any map $f : A \to X$, where A is a closed subset of Y, there is an extension $F : U \to X$ where U is a neighborhood of A. If an extension to all of Y is always possible, then X is an AE (*absolute extensor*). Show that being an ANE (or AE) is equivalent to being an ANR (or AR). *Hint*: The Tietze Extension Theorem will be helpful.

Exercise 3.12.7 With the help of Exercise 3.12.6 and the Homotopy Extension Property, prove that an ANR is an AR if and only if it is contractible.

Exercise 3.12.8 A useful property of Euclidean space is that every compactum $A \subseteq \mathbb{R}^n$ has arbitrarily small compact polyhedral neighborhoods. Using the tools of Proposition 3.12.4, prove the following CAT(0) analog: every compactum A in a proper CAT(0) space X has arbitrarily small compact ANR neighborhoods. *Hint*: Cover A with compact metric balls. (For examples of ANRs that do not have this property, see [11, 73].)

3.13 Appendix B: Hilbert Cube Manifolds

This appendix is a very brief introduction to Hilbert cube manifolds. A primary goal is to persuade the uninitiated reader that there is nothing to fear. Although the main results from this area are remarkably strong (we sometimes refer to them as "Hilbert cube magic”), they are understandable and intuitive. Applying them is often quite easy.

The *Hilbert cube* is the infinite product $\mathcal{Q} = \prod_{i=1}^{\infty} [-1, 1]$ with metric $d\left((x_i), (y_i)\right) = \sum \frac{|x_i - y_i|}{2^i}$. A *Hilbert cube manifold* is a separable metric space X with the property that each $x \in X$ has a neighborhood homeomorphic to $\mathcal{Q}$. Hilbert cube manifolds are interesting in their own right, but our primary interest stems from their usefulness in working with spaces that are not necessarily infinite-dimensional—often locally finite CW complexes or more general ANRs. Two classic examples where that approach proved useful are:

- Chapman [19] used Hilbert cube manifolds to prove the topological invariance of Whitehead torsion for finite CW complexes, i.e., homeomorphic finite complexes are simple homotopy equivalent.
- West [97] used Hilbert cube manifolds to solve a problem of Borsuk, showing that every compact ANR is homotopy equivalent to a finite CW complex. (See Proposition 3.12.4.)

The ability to attack a problem about ANRs using Hilbert cube manifolds can be largely explained using the following pair of results.

Theorem 3.13.1 (Edwards, [34]) *If A is an ANR, then $A \times \mathcal{Q}$ is a Hilbert cube manifold.*

Theorem 3.13.2 (Triangulability of Hilbert Cube Manifolds, Chapman, [20]) *If X is a Hilbert cube manifold, then there is a locally finite polyhedron K such that $X \approx K \times \mathcal{Q}$.*

A typical (albeit, simplified) strategy for solving a problem involving an ANR A might look like this:

(A) Take the product of A with $\mathcal{Q}$ to get a Hilbert cube manifold $X = A \times \mathcal{Q}$.

(B) Triangulate X, obtaining a polyhedron K with $X \approx K \times \mathcal{Q}$.

(C) The polyhedral structure of K together with a variety of tools available in a Hilbert cube manifolds (see below) make solving the problem easier.

(D) Return to A by collapsing out the $\mathcal{Q}$-factor in $X = A \times \mathcal{Q}$.

In these notes, most of our appeals to Hilbert cube manifold topology are of this general sort. That is not to say the strategy always works—the main result of [49] (see Remark 3.8.18(a)) is one relevant example.

Tools available in a Hilbert cube manifold are not unlike those used in finite-dimensional manifold topology. We list a few such properties, without striving for best-possible results.

Proposition 3.13.3 (Basic properties of Hilbert cube manifolds) *Let X be a connected Hilbert cube manifold.*

(a) *(Homogeneity) For any pair $x_1, x_2 \in X$, there exists a homeomorphism $h : X \to X$ with $h(x_1) = x_2$.*
(b) *(General Position) Every map $f : P \to X$, where P is a finite polyhedron can be approximated arbitrarily closely by an embedding.*
(c) *(Regular Neighborhoods) Each compactum $C \subseteq X$ has arbitrarily small compact Hilbert cube manifold neighborhoods $N \subseteq X$. If C is a nicely embedded polyhedron, N can be chosen to strong deformation retract onto P.*

Exercise 3.13.4 As a special case, assertion (a) of Proposition 3.13.3 implies that $\mathcal{Q}$ itself is homogeneous. This remarkable fact is not hard to prove. A good start is to construct a homeomorphism $h : \mathcal{Q} \to \mathcal{Q}$ with $h(1, 1, 1, \ldots) = (0, 0, 0, \ldots)$. To begin, think of a homeomorphism $k : [-1, 1] \times [-1, 1]$ taking $(1, 1)$ to $(0, 1)$, and use it to obtain $h_1 : \mathcal{Q} \to \mathcal{Q}$ with $h_1(1, 1, 1, \ldots) = (0, 1, 1, \ldots)$. Complete this argument by constructing a sequence of similarly chosen homeomorphisms.

Example 3.13.5 Here is another special case worth noting. Let K be an arbitrary locally finite polyhedron—for example, a graph. Then $K \times \mathcal{Q}$ is homogeneous.

The material presented here is just a quick snapshot of the elegant and surprising world of Hilbert cube manifolds. A brief and readable introduction can be found in [20]. Just for fun, we close by stating two more remarkable theorems that are emblematic of the subject.

Theorem 3.13.6 (Toruńczyk [93]) *An ANR X is a Hilbert cube manifold if and only if it satisfies the General Position property (Assertion (2)) of Proposition 3.13.3.*

Theorem 3.13.7 (Chapman [20]) *A map $f : K \to L$ between locally finite polyhedra is an (infinite) simple homotopy equivalence if and only if $f \times id_{\mathcal{Q}} : K \times \mathcal{Q} \to L \times \mathcal{Q}$ is (proper) homotopic to a homeomorphism.*

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Chapter 4
A Proof of Sageev's Theorem on Hyperplanes in CAT(0) Cubical Complexes

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Abstract We prove that any hyperplane H in a CAT(0) cubical complex X has no self-intersections and separates X into two convex complementary components. These facts were originally proved by Sageev. Our argument shows that his theorem is a corollary of Gromov's link condition. We also give new arguments establishing some combinatorial properties of hyperplanes. We show that these properties are sufficient to prove that the 0-skeleton of any CAT(0) cubical complex is a discrete median algebra, a fact that was previously proved by Chepoi, Gerasimov, and Roller.

Keywords CAT(0) · Cubical complex · Hyperplanes

4.1 Introduction

Two theorems are central in the theory of CAT(0) cubical complexes. The first is Gromov's well-known link condition. A complete statement and proof appear in [1]. The second theorem was proved by Sageev in [15]. He showed that a group G semisplits over a subgroup H if and only if G acts on a CAT(0) cubical complex X and there is a hyperplane $J \subseteq X$ such that: (i) the action of G is essential relative to J, and (ii) the stabilizer of J (as a set) is H. We refer the reader to [15] for details and definitions. Sageev's result extends the Bass–Serre theory of groups acting on trees, which says that a group G splits over H if and only if G acts without inversion on a tree T, in which the stabilizer subgroup of some edge e is H. Moreover, just as Bass–Serre theory gives a construction of the tree T from the splitting of G over H, Sageev gives a construction of the CAT(0) cubical complex X from the semisplitting of G over H. Both theories are also alike in that they explicitly describe the algebraic splittings or semisplittings using their geometric hypotheses.

Both the forward and the reverse directions of Sageev's theorem have significant applications. The forward direction (from algebra to geometry) is used in

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[11, 16], among others. The proof of the reverse direction uses several properties of hyperplanes in CAT(0) cubical complexes (also established in [15]). Many of these properties are useful in their own right. For instance, Sageev showed that a hyperplane in a CAT(0) cubical complex X has no self-intersections and separates X into two convex complementary components [15]. This fact is essential in the proof that groups acting properly, isometrically, and cellularly on CAT(0) cubical complexes have the Haagerup property [12]. Sageev establishes the geometric properties of hyperplanes in CAT(0) cubical complexes using his own system of Reidemeister-style moves.

The main purpose of this note is to offer a new (and, we believe, simpler) proof of the following theorem, which we hereafter call "Sageev's Theorem" for the sake of brevity:

Theorem 4.1.1 ([15]) *A hyperplane H in a CAT(0) cubical complex X has no self-intersections and separates X into two open convex complementary components.*

Our proof avoids using Sageev's Reidemeister moves. The main tool is a block complex $\mathscr{B}(X)$, which is endowed with a natural projection $\pi_{\mathscr{B}} : \mathscr{B}(X) \to X$. We apply a criterion, due to Crisp and Wiest [5], for showing that a map between cubical complexes is an isometric embedding. The criterion is a generalized form of Gromov's link condition. We are thus able to conclude that the restriction of $\pi_{\mathscr{B}}$ to each connected component of $\mathscr{B}(X)$ is an isometric embedding. The full statement of Theorem 4.1.1 then follows from the definition of $\mathscr{B}(X)$ after a little more work.

We also give new proofs of some of Sageev's secondary results—see Sect. 4.5.2, especially Propositions 4.5.5 and 4.5.8. Sageev's original proofs used his Reidemeister moves. Our proofs use techniques from the theory of CAT(0) spaces, including (especially) projection maps onto closed convex subspaces.

The paper concludes with some applications. We sketch a proof of the theorem that every group G acting properly, isometrically, and cellularly on a CAT(0) cubical complex has the Haagerup property. (The first proof appeared in [12].) We also show that the 0-skeleton of a CAT(0) cubical complex is a discrete median algebra under the "geodesic interval" operation. Earlier proofs of the discrete median algebra property appear in [4, 7], and Martin Roller produced a proof in his unpublished Habilitation Thesis [14]. Our argument is intended to highlight the utility of the combinatorial lemmas collected in Sect. 4.5.1, and, in particular, to suggest that the latter lemmas are a sufficient basis for establishing all of the combinatorial properties of CAT(0) cubical complexes. (Indeed, "discrete median algebra" and "CAT(0) cubical complex" are equivalent ideas, by [7, 13, 14].) We refer the reader to [3] for elegant characterizations of the Haagerup property and property T in terms of median algebras.

We note one limitation of the general methods of this paper: our methods apply only to locally finite-dimensional cubical complexes satisfying Gromov's link condition. We need our complexes to be locally finite-dimensional so that their metrics will be complete (see [1], Exercise 7.62, p. 123). In fact, the CAT(0) property has been established only for locally finite-dimensional cubical complexes satisfying the link condition—see the passage after Lemma 2.7 in [8] for a useful discussion of this point. Although our argument is therefore slightly less general than the original one of Sageev, it still covers the cases that are most commonly encountered in practice.

Section 4.2 contains a description of the block complex. Section 4.3 describes the analogue of Gromov's theorem we need from [5]. Section 4.4 contains a proof of Sageev's theorem, Theorem 4.1.1. Section 4.5 collects some essential combinatorial lemmas. Finally, Sect. 4.6 contains applications of the main ideas, including proofs that every CAT(0) cubical complex is a set with walls and that the 0-skeleton of every CAT(0) cubical complex is a discrete median algebra.

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4.2 The Block Complex

Definition 4.2.1 A cubical complex X is *locally finite-dimensional* if the link of each vertex is a finite-dimensional simplicial complex.

Throughout the paper, "CAT(0) cubical complex" means locally finite-dimensional CAT(0) cubical complex.

Definition 4.2.2 Let $C \subseteq X$ be a cube of dimension at least one. A *marking* of C is an equivalence class of directed edges $e \subseteq C$. Two such directed edges e', e'' are said to be *equivalent*, i.e., to define the same marking, if there is a sequence of directed edges $e' = e_0, \ldots, e_k = e''$ such that, for $i \in \{0, \ldots, k-1\}$, e_i and e_{i+1} are opposite sides of a 2-cell $C_i \subseteq C$ and both point in the same direction. A *marked cube* is a cube (of dimension at least one) with a marking.

Example 4.2.3 Let $X = [0, 1]^3$, with the usual cubical structure. We let $C = X$. There are six markings of C. They are represented by the directed edges $[(0, 0, 0), (1, 0, 0)]$, $[(0, 0, 0), (0, 1, 0)]$, $[(0, 0, 0), (0, 0, 1)]$, and by the three corresponding edges with the opposite directions.

It is fairly clear from the example that a cube of dimension n has exactly $2n$ markings. Note that not every face of a marked cube is itself marked. In Fig. 4.1, the top and bottom faces are unmarked.

Definition 4.2.4 Let X be a CAT(0) cubical complex. We let $\mathscr{M}(X)$ denote the space of marked cubes of X, which is defined to be the disjoint union of all marked cubes of X. More formally, $\mathscr{M}(X)$ is the space of triples $(x, C, [e])$, where C is a cube in X, $[e]$ is a marking of C, and $x \in C$. For fixed C and $[e]$, the set

$$C_{[e]} = \{(x, C, [e]) \mid x \in C\}$$

is an isometric copy of C, and $\mathscr{M}(X)$ is the disjoint union of all such sets $C_{[e]}$. There is a natural map $\pi_{\mathscr{M}} : \mathscr{M}(X) \to X$, defined by sending $(x, C, [e])$ to x.

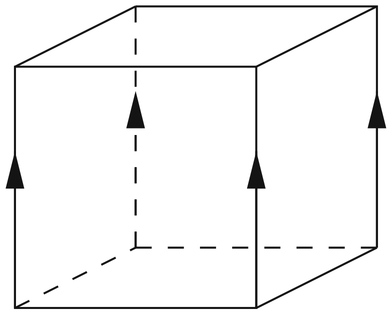

Fig. 4.1 The directed edge $[(0, 0, 0), (0, 0, 1)]$ determines the marking of the cube. The x-axis is horizontal, and the coordinate system is a right-handed one

Example 4.2.5 If $X = [0, 1]^3$, then $\mathscr{M}(X)$ is a disjoint union of 24 marked edges, 24 marked squares, and 6 marked three-dimensional cubes.

Definition 4.2.6 Let $(x_1, C_1, [e_1]), (x_2, C_2, [e_2]) \in \mathscr{M}(X)$. We write $(x_1, C_1, [e_1]) \sim (x_2, C_2, [e_2])$ if:

(a) $x_1 = x_2$, and
(b) there is a directed edge $e \in [e_1] \cap [e_2]$.

Lemma 4.2.7 *The relation $\sim$ is an equivalence relation on $\mathscr{M}(X)$.*

Proof It is already clear that $\sim$ is reflexive and symmetric.

We prove that $\sim$ is transitive. Thus, we suppose that $(x_1, C_1, [e_1]) \sim (x_2, C_2, [e_2])$ and $(x_2, C_2, [e_2]) \sim (x_3, C_3, [e_3])$. Clearly, $x_1 = x_2 = x_3$. We can express C_2 as $C_2' \times [0, 1]$, where C_2' is a cube of dimension one less than the dimension of C_2, and the second factor $[0, 1]$ is the marked one. Since $C_1 \cap C_2$ is a marked face of C_2 (because of the condition $[e_1] \cap [e_2] \neq \emptyset$), we must have $C_1 \cap C_2 = \widehat{C} \times [0, 1]$, for some non-empty face $\widehat{C} \subseteq C_2'$. Similarly, $C_2 \cap C_3 = \widetilde{C} \times [0, 1]$, for some non-empty face $\widetilde{C} \subseteq C_2'$. Now $C_1 \cap C_2 \cap C_3 \neq \emptyset$, since $x_1 \in C_1 \cap C_2 \cap C_3$. It follows that $C_1 \cap C_2 \cap C_3 = (\widehat{C} \times \widetilde{C}) \times [0, 1]$, where $\widehat{C} \times \widetilde{C}$ is a non-empty face of C_2'.

Let us suppose that the marking $[e_2]$ of C_2 is determined by the directed edge $e_2 = [(v, 0), (v, 1)]$, where v is a vertex of C_2'. It follows easily from the conditions $[e_1] \cap [e_2] \neq \emptyset$ and $[e_2] \cap [e_3] \neq \emptyset$ that the directed edge $[(v', 0), (v', 1)] \subseteq C_2$ is in $[e_1]$ (respectively, $[e_3]$) if and only if $v' \in \widehat{C}$ (respectively, $\widetilde{C}$). Thus, if v is a vertex of $\widehat{C} \cap \widetilde{C}$, then $[(v, 0), (v, 1)] \in [e_1] \cap [e_3]$. Such a vertex exists since $\widehat{C} \cap \widetilde{C} \neq \emptyset$, and this completes the proof.

Definition 4.2.8 The *block complex of* X, denoted $\mathscr{B}(X)$, is the quotient $\mathscr{M}(X)/\sim$.

Definition 4.2.9 ([5]) A map $f : X \to Y$ between cubical complexes is called *cubical* if each cube in X is mapped isometrically onto some cube in Y.

We record the following lemma, the proof of which is straightforward.

Lemma 4.2.10 *The space $\mathscr{B}(X)$ is a cubical complex with a natural cubical map $\pi_{\mathscr{B}} : \mathscr{B}(X) \to X$, defined by $\pi(x, C, [e]) = x$.* □

Example 4.2.11 We describe the cubical complex $\mathscr{B}(X)$ in a special case. Suppose that $X = \mathbb{R}^2$ with the standard cubulation. The complex $\mathscr{B}(X)$ consists of an infinite disjoint union of strips having either the form $[m, m+1] \times \mathbb{R}$ or $\mathbb{R} \times [n, n+1]$ $(m, n \in \mathbb{Z})$. The map $\pi_{\mathscr{B}} : \mathscr{B}(X) \to X$ is "inclusion". Note that there are two identical copies of each strip $[m, m+1] \times \mathbb{R}$ in $\mathscr{B}(X)$, since there are two different orientations for the edge $[m, m+1] \times \{0\}$. (There are also two copies of $\mathbb{R} \times [n, n+1]$ in $\mathscr{B}(X)$ for a similar reason.)

4.3 A Geometric Lemma

The main lemma of this section (Lemma 4.3.2) relies heavily on a theorem due to Crisp and Wiest.

Theorem 4.3.1 ([5], Theorem 1(2)) *Let X and Y be locally finite-dimensional cubical complexes and $\Phi : X \to Y$ a cubical map. Suppose that Y is locally CAT(0). The map Φ is a local isometry if and only if, for every vertex $x \in X$, the simplicial map $Lk(x, X) \to Lk(\Phi(x), Y)$ induced by Φ is injective with image a full subcomplex of $Lk(\Phi(x), Y)$.*

Proof This is exactly Theorem 1(2) from [5], except that we allow locally finite-dimensional cubical complexes, rather than only finite-dimensional ones. Since the hypotheses and conclusions are all local in nature, the proof is unchanged.

Lemma 4.3.2 *Let X and Y be locally finite-dimensional cubical complexes, let Y be CAT(0), and assume that $\Phi : X \to Y$ is a cubical map with the property that, for every vertex $x \in X$, the simplicial map $Lk(x, X) \to Lk(\Phi(x), Y)$ induced by Φ is injective with image a full subcomplex of $Lk(\Phi(x), Y)$.*

For every component $C \subseteq X$, we have:

(a) C is a CAT(0) cubical complex, and
(b) $\Phi_{|C}$ is an isometric embedding.

Proof The previous theorem shows that Φ is a local isometry. We note that both X and Y are complete metric spaces, since both are locally finite-dimensional cubical complexes (see Exercise 7.62 on p. 123 of [1]). Since Y is non-positively curved and X is locally a length space, Proposition 4.14 from p. 201 of [1] applies. It follows that X is non-positively curved, the homomorphism $\Phi_* : \pi_1(C) \to \pi_1(Y)$ is injective, and every continuous lifting $\widetilde{\Phi} : \widetilde{C} \to \widetilde{Y}$ is an isometric embedding. Since Φ_* is injective, C is simply connected, and therefore $C = \widetilde{C}$, $Y = \widetilde{Y}$, and $\widetilde{\Phi} = \Phi$. The lemma follows.

4.4 The Main Theorem

4.4.1 A Preliminary Version of Sageev's Theorem

Theorem 4.4.1 *If X is a locally finite-dimensional cubical complex, then the map $\pi_{\mathcal{B}} : \mathcal{B}(X) \to X$ embeds each connected component of $\mathcal{B}(X)$ isometrically.*

Proof By Lemma 4.3.2, we need only show that the simplicial map on links $Lk(v, \mathcal{B}(X)) \to Lk(\pi_{\mathcal{B}}(v), X)$ is injective, and that the image is a full subcomplex of $Lk(\pi_{\mathcal{B}}(v), X)$.

We choose a vertex $v \in \mathcal{B}(x)$. Such a vertex can be represented by a vertex $(x, C, [e])$ in $\mathcal{M}(X)$, where $x \in X^0$. There is a unique directed edge $e' \in [e]$ containing x. We let C' denote the (undirected) 1-cell determined by e'. It follows from the definition of $\sim$ that we can represent v by $(x, C', [e'])$.

We let $X_{C'}$ be the subcomplex of X consisting of all closed cubes C such that $C' \subseteq C$. A marked cube $C_{[e]} \subseteq \mathcal{B}(X)$ touches v if and only if $C' \subseteq C$ and $e' \in [e]$, by the definition of $\sim$. Now, for a given cube $C \subseteq X$ such that $C' \subseteq C$, there is a unique marking $[e]$ of C such that $e' \in [e]$. It follows that the closed cubes touching v in $\mathcal{B}(X)$ are in one-to-one correspondence with the closed cubes of $X_{C'}$ touching $\pi_{\mathcal{B}}(v)$. Moreover, given two marked cubes $D_{[e_1]}$ and $E_{[e_2]}$ such that $e' \in [e_1] \cap [e_2]$, the intersection $D_{[e_1]} \cap E_{[e_2]}$ is mapped isometrically to $D \cap E$ by $\pi_{\mathcal{B}}$, since $D_{[e_1]} \cap E_{[e_2]} = (D \cap E)_{[e_3]}$, where $[e_3]$ is the unique marking of $D \cap E$ determined by the property $[e_3] \subseteq [e_1] \cap [e_2]$. It follows that the union of all closed cubes in $\mathcal{B}(X)$ touching v is combinatorially identical to $X_{C'}$, and the map $\pi_{\mathcal{B}} : \mathcal{B}(X) \to X$ is locally just the inclusion $X_{C'} \to X$. Therefore, the map on links is injective.

We now consider the image in $Lk(\pi_{\mathcal{B}}(x), X)$. There is a vertex $v' \in Lk(\pi_{\mathcal{B}}(v), X)$ which is contributed by the 1-cell C'. The above description of $\pi_{\mathcal{B}}$ implies that the image of the link $Lk(v, \mathcal{B}(X))$ is the union of all simplices touching v' (i.e., the simplicial neighborhood of v'). Since $Lk(\pi_{\mathcal{B}}(v), X)$ is flag, this simplicial neighborhood is necessarily a full subcomplex.

4.4.2 Sageev's Theorem

Definition 4.4.2 Fix a component B of the block complex $\mathcal{B}(X)$. For each marked cube C of B, choose an isometric characteristic map $c : [0, 1]^n \to C$ such that the directed edge $[c(0, 0, \ldots, 0), c(0, 0, \ldots, 0, 1)]$ represents a marking of C. If $x \in C$ satisfies $x = c(t_1, \ldots, t_n)$, then the *height* of x, denoted $h(x)$, is t_n. This height function on marked cubes is easily seen to be compatible overlaps, and induces a height function $h : B \to [0, 1]$. We let $B_t = h^{-1}(t)$ for $t \in [0, 1]$.

Lemma 4.4.3 *(a) For any component B of $\mathcal{B}(X)$ and for any $t \in [0, 1]$, B_t is a closed convex subset of $\mathcal{B}(X)$. The space $\pi_{\mathcal{B}}(B_t)$ is a closed convex subset of X.*

(b) Each component B of $\mathscr{B}(X)$ factors isometrically as $B_0 \times [0, 1]$.
(c) Each B_t ($t \in [0, 1]$) is a CAT(0) cubical complex.

Proof (a) It is clear that B_t is closed.

Suppose that $x, y \in B_t$. Let $p : [0, d_B(x, y)] \to B$ be a path connecting x to y. We can factor each marked cube $C \subseteq B$ of dimension n as $C' \times [0, 1]$, where C' is a cube of dimension $n-1$ and the factor $[0, 1]$ determines the marking. There is a natural projection $\pi_t : C \to C' \times \{t\}$, and this projection doesn't increase distances. Moreover, all such projections are compatible, so in particular there is a projection $\pi_t : B \to B_t$ which fixes B_t and doesn't increase distances. It follows that $\pi_t \circ p$ is a path in B_t which is no longer than p. By the uniqueness of geodesics in CAT(0) spaces, it follows that any geodesic connecting x to y lies in B_t. Therefore, B_t is a closed convex subset of $\mathscr{B}(X)$. Since $\pi_{\mathscr{B}|B}$ is an isometric embedding, $\pi_{\mathscr{B}}(B_t)$ is a closed convex subset of X.

(b) There is a natural map $f : B \to B_0 \times [0, 1]$, where $f(x) = (\pi_0(x), h(x))$ and $\pi_0 : B \to B_0$ is the usual projection onto the closed convex subspace B_0 (see Proposition 2.4 on p. 176 of [1]).

Assume that $x, y \in B$. We need to show that

$$d_B(x, y) = \sqrt{[d_{B_0}(\pi_0(x), \pi_0(y))]^2 + |h(x) - h(y)|^2}.$$

This is clear if $\pi_0(x) = \pi_0(y)$. If $\pi_0(x) \neq \pi_0(y)$, then we consider the quadrilateral formed by the geodesic segments $[\pi_0(x), \pi_0(y)]$, $[\pi_0(x), \pi_1(x)]$, $[\pi_1(x), \pi_1(y)]$, and $[\pi_1(y), \pi_0(y)]$.

By Proposition 2.4(3) of [1], each of the four resulting Alexandrov angles measures at least $\pi/2$. It therefore follows from the Flat Quadrilateral Theorem (2.11 from p. 181 of [1]) that all of the angles in the above quadrilateral measure exactly $\pi/2$, and that the convex hull of $\pi_0(x)$, $\pi_0(y)$, $\pi_1(x)$ and $\pi_1(y)$ in B is isometric to a rectangle in Euclidean space. The desired equality now follows from the definition of the metric in Euclidean space.

(c) It is sufficient to prove this for B_0. Since $B = B_0 \times [0, 1]$ is CAT(0), it must be that each factor is CAT(0) (Exercise 1.16, p. 168 of [1]). The space B_0 is a cubical complex because the identifications in the definition of B are height-preserving.

Theorem 4.4.4 *Each hyperplane $\pi_{\mathscr{B}}(B_t)$ ($0 < t < 1$) separates X into two open convex complementary half-spaces. The hyperplane $\pi_{\mathscr{B}}(B_t)$ has no self-intersections.*

Proof We recall that $\pi_{\mathscr{B}}(B)$ is a closed convex subspace of X. We let $\pi : X \to \pi_{\mathscr{B}}(B)$ be the projection. By a slight abuse of notation, we let $h : \pi_{\mathscr{B}}(B) \to [0, 1]$ denote the height function from Definition 4.4.2.

Consider the function $h \circ \pi : X \to [0, 1]$. We claim

(a) if $[x, y]$ is any geodesic in X, then $(h \circ \pi)_{|[x,y]}$ must assume its maximum and minimum values at the endpoints, and
(b) if $h(\pi(x)) \in (0, 1)$, then $x = \pi(x)$.

We prove (2) first. Note that, if $h(\pi(x)) \in (0,1)$, then $\pi(x)$ is an interior point of $\pi_{\mathscr{B}}(B)$. This is only possible if $\pi(x) = x$.

We now prove (1). We assume the contrary. Assume that $h \circ \pi$ attains its maximum value on the geodesic $[x, y]$ at neither of the endpoints. (The case in which $h \circ \pi$ attains its minimum value at neither of the endpoints is handled in an analogous way.) We assume that $h \circ \pi$ attains its maximum value at $z \in [x, y]$, $z \notin \{x, y\}$. It follows that there is some $t \in (0, 1)$ such that

$$\max\{(h \circ \pi)(x), (h \circ \pi)(y)\} < t < (h \circ \pi)(z).$$

This implies, by the Intermediate Value Theorem, that there are points x', y' such that $(h \circ \pi)(x') = t = (h \circ \pi)(y')$, where x' lies between x and z on $[x, y]$, and y' lies between y and z. It now follows, from (2), that $x', y' \in B_t$. Since $z \in [x', y'] \subseteq B_t$, $(h \circ \pi)(z) = t$, a contradiction. This proves (1).

We now prove the first statement. Consider the sets $(h \circ \pi)^{-1}([0, t)) = B_t^-$ and $(h \circ \pi)^{-1}((t, 1]) = B_t^+$. For any $x, y \in B_t^-$, the geodesic $[x, y]$ is clearly contained in B_t^- by (1). It follows that B_t^- is convex and (therefore) connected. By similar reasoning B_t^+ is convex and connected. Both B_t^- and B_t^+ are obviously open, and they are disjoint. We note finally that $B_t^- \cup B_t^+ = X - \pi_{\mathscr{B}}(B_t)$ (since $\pi_{\mathscr{B}}(B_t) = (h \circ \pi)^{-1}(t)$, by (2)), completing the proof of the first statement.

The second statement follows from Theorem 4.4.1: the map $\pi_B : \mathscr{B}(X) \to X$ is an isometric embedding when restricted to an individual block B.

Definition 4.4.5 A *hyperplane* H in a CAT(0) cubical complex X is the image $\pi_{\mathscr{B}}(B_{1/2})$, where B is a connected component of $\mathscr{B}(X)$. We sometimes denote the complementary halfspaces H^+ and H^-.

Note 4.4.6 In what follows, we typically identify $\pi_{\mathscr{B}}(B)$ with B, and $\pi_{\mathscr{B}}(B_t)$ with B_t, for the sake of convenience in notation.

4.5 Combinatorics of Hyperplanes

Definition 4.5.1 Let X be a complete CAT(0) space. If C is a closed convex subset of X, then $\pi_{(X,C)}$ denotes the projection from X to C. If x_1, x_2, and x_3 are points in X, then $\angle^X_{x_2}(x_1, x_3)$ denotes the Alexandrov (or upper) angle in X between the geodesics $[x_2, x_1]$ and $[x_2, x_3]$. We refer the reader to [1] for the definitions, which appear on pp. 176 and 9, respectively.

4.5.1 Three Lemmas

Lemma 4.5.2 *Let H_1, H_2 be hyperplanes in X, and assume that $H_1 \cap H_2 \neq \emptyset$. The projections $\pi_{(X,H_i)} : X \to H_i$ and $\pi_{(X,H_i \cap H_j)} : X \to H_i \cap H_j$ agree on H_j, where $\{i, j\} = \{1, 2\}$.*

Proof For the sake of simplicity, we let $j = 1$ and $i = 2$. Choose a point $x \in H_1$. We consider the block B containing H_1, and the projection $\pi_{(B,B \cap H_2)} : B \to B \cap H_2$. We denote the latter projection by π. Let C be a marked cube of B containing $\pi(x)$. We note that C must be at least two-dimensional, since C meets at least two hyperplanes. We write $C = C' \times [0, 1] \times [0, 1]$, where $C' \times \{1/2\} \times [0, 1] = H_2 \cap C$ and $C' \times [0, 1] \times \{1/2\} = H_1 \cap C$.

We claim that $\pi(x) \in H_1$ (i.e., $\pi(x) \in H_1 \cap H_2$, since $\pi(x) \in H_2$ by definition). Express $\pi(x)$ as $(y, 1/2, t) \in C' \times [0, 1] \times [0, 1] = C$. Now since $x \in B_{1/2} = H_1$, we have, by the product decomposition of B (Lemma 4.4.3(2)),

$$d(x, \pi(x)) = \sqrt{D^2 + |t - 1/2|^2},$$

where D is the distance from x to $(y, 1/2, 1/2)$. Since $(y, 1/2, 1/2) \in H_2 \cap B$ and $\pi(x)$ is the point of $B \cap H_2$ closest to B, we must have $t = 1/2$. That is, $\pi(x) = (y, 1/2, 1/2)$, so $\pi(x) \in H_1$, as claimed.

Next, we claim that $\pi(x) = \pi_{(X,H_2)}(x)$. The proof of this fact uses the following characterization of the projection: if X is a complete CAT(0) space, C is a closed convex subset of X, and $x \in X - C$, then $\pi_{(X,C)}(x)$ is the unique element of C with the property that $\angle^X_{\pi_{(X,C)}(x)}(x, z) \geq \pi/2$ for all $z \in C - \pi_{(X,C)}(x)$. We choose $z \in H_2 - \{\pi(x)\}$. Since $\pi(x)$ is in the interior of B, there is some $z' \in [\pi(x), z]$, $z' \neq \pi(x)$, such that $z' \in B \cap H_2$. By the definition of $\pi(x) = \pi_{(B,B \cap H_2)}(x)$, $\angle^B_{\pi(x)}(x, z) \geq \pi/2$. Since B is a convex subset of X, $\angle^B_{\pi(x)}(x, z) = \angle^X_{\pi(x)}(x, z')$. It now follows that

$$\angle^X_{\pi(x)}(x, z) = \angle^X_{\pi(x)}(x, z') \geq \pi/2,$$

so $\pi(x) = \pi_{(X,H_2)}(x)$.

Now we argue that $\pi(x) = \pi_{(X,H_1 \cap H_2)}(x)$. If not, then there is $y \in H_1 \cap H_2$ such that $d_X(x, y) < d_X(x, \pi(x))$. This is impossible, however, since $\pi(x)$ is the closest point in H_2 to x.

Lemma 4.5.3 *Assume that H_1 and H_2 are hyperplanes, $H_1 \neq H_2$, and $H_1 \cap H_2 \neq \emptyset$. If e is a marked edge perpendicular to H_1, then $d_{H_2|_e}$ is constant.*

Proof Suppose that e is perpendicular to H_1. Let B denote the block containing the hyperplane H_1. Consider the midpoint of e; call it x. We let π denote the projection from X onto H_2. Let C be a closed marked cube of B which contains $\pi(x)$. As in the proof of Lemma 4.5.2, we write $C = C' \times [0, 1] \times [0, 1]$, where $H_1 \cap C = C' \times [0, 1] \times \{1/2\}$ and $H_2 \cap C = C' \times \{1/2\} \times [0, 1]$.

Since $\pi(x) \in H_1 \cap H_2$ by Lemma 4.5.2, one has that $[x, \pi(x)] \subseteq B_{1/2} = B_0 \times \{1/2\}$. We can express $[x, \pi(x)]$ as $[\pi_0(x), \pi_0(\pi(x))] \times \{1/2\}$, where π_0 denotes the projection from B to B_0. If y is some other point on e, then $[\pi_0(x), \pi_0(\pi(x))] \times \{h(y)\}$ is a geodesic connecting y to a point in H_2. It follows that $d_{H_2}(y) \leq d_{H_2}(x)$, for all $y \in e$.

One argues that equality always holds by the convexity of the function d_{H_2} (see Corollary 2.5 on p. 178 of [1]). Indeed, suppose that $y_1,\ y_2 \in e$, where $h(y_1) < h(x) < h(y_2)$, and $d_{H_2}(y_i) < d_{H_2}(x)$ for at least one index $i \in \{1, 2\}$. The function d_{H_2} is concave up (i.e., convex) and non-constant on the geodesic $[y_1, y_2]$, and attains a maximum value of $d_{H_2}(x)$ at the interior point x. This is a contradiction.

Lemma 4.5.4 ([6], Lemma 2.6(4)) *If H_1 and H_2 are hyperplanes, $H_1^+ \cap H_2^+$, $H_1^- \cap H_2^+$, $H_1^+ \cap H_2^-$, and $H_1^- \cap H_2^-$ are all non-empty, then $H_1 \cap H_2 \neq \emptyset$.*

Proof Assume that the four intersections in the hypothesis are all non-empty and $H_1 \cap H_2 = \emptyset$. It follows that $\{H_1^+ \cup H_2^+, H_1^- \cup H_2^-\}$ is an open cover of X. Each of the half-spaces H_1^+, H_1^-, H_2^+, and H_2^- is a convex subspace of the CAT(0) space X, and therefore contractible. Each of the four intersections in the hypothesis is contractible for the same reason.

It follows that the sets $X^+ = H_1^+ \cup H_2^+$ and $X^- = H_1^- \cup H_2^-$ are simply connected, since each is the union of two open contractible sets which intersect in an open contractible set. The intersection $X^+ \cap X^-$ is the disjoint union of two open contractible sets: $H_1^+ \cap H_2^-$ and $H_2^+ \cap H_1^-$. Let c be an arc contained in X^+, connecting $H_1^+ \cap H_2^-$ to $H_2^+ \cap H_1^-$, and meeting each in an open segment.

We apply van Kampen's theorem to the pieces $X^- \cup c$ and X^+. The first piece $X^- \cup c$ satisfies $\pi_1(X^- \cup c) \cong \mathbb{Z}$, while the second is simply connected. The intersection of these two pieces is the simply connected set $(H_1^+ \cap H_2^-) \cup (H_2^+ \cap H_1^-) \cup\ c$. It follows that $\pi_1(X^- \cup X^+) = \pi_1(X)$ is isomorphic to $\mathbb{Z}$. Since X is CAT(0), it must be contractible. This is a contradiction.

4.5.2 *Sageev's Combinatorial Results*

We cover only some basic combinatorial results in this subsection. A more advanced treatment of the combinatorial properties of hyperplanes appears in an appendix to [10].

Proposition 4.5.5 ([15]) *An edge-path p in X^1 is geodesic if and only if p crosses any given hyperplane H at most once.*

Proof We first prove the forward direction. Suppose, on the contrary, that a certain geodesic edge-path crosses some hyperplane more than once. We consider a shortest geodesic edge-path p which crosses some hyperplane multiple times. We write $p = (e_1, \ldots, e_n)$, and let $H_1, \ldots, H_n$ denote the hyperplanes crossed by the edges $e_1, \ldots, e_n$ (respectively). Since p is the shortest edge-path with the given property, we must have $H_1 = H_n$, but there are no other repetitions in the list $H_1, \ldots, H_n$ (i.e., a total of $n-1$ distinct hyperplanes are crossed by p). We let H_1^- denote the component of $X - H_1$ containing $\iota(e_1)$ and $\tau(e_n)$. Clearly the other component of $X - H_1$, denoted H_1^+, contains the edge-path $(e_2, \ldots, e_{n-1})$. We adopt the convention that $\iota(e_j) \in H_j^-$ and $\tau(e_j) \in H_j^+$, for $j \in \{2, \ldots, n-1\}$.

Consider an edge e_j, $j \in \{2, \ldots, n-1\}$. Note that $\iota(e_1) \in H_1^- \cap H_j^-$, $\iota(e_j) \in H_1^+ \cap H_j^-$, $\tau(e_j) \in H_1^+ \cap H_j^+$, and $\tau(e_n) \in H_1^- \cap H_j^+$. It follows that the hyperplanes H_1 and H_j intersect, for $j \in \{2, \ldots, n-1\}$, by Lemma 4.5.4.

We now apply Lemma 4.5.3. Since $d(\iota(e_2), H_1) = 1/2$ and d_{H_1} is constant on e_2, we must have $d_{H_1}(x) = 1/2$ for all x in e_2. We can inductively conclude that $d_{H_1}(x) = 1/2$ for all x in $(e_2, \ldots, e_{n-1})$.

It follows that the entire edge-path $p = (e_1, \ldots, e_n)$ is contained in the block B containing H_1. The edges $e_2, \ldots, e_{n-1}$ are all unmarked edges in the block $B = B_0 \times [0, 1]$. We identify $\iota(e_2)$ with a vertex $(v', 1) \in B$ and $\tau(e_{n-1})$ with a vertex $(v'', 1) \in B$. It follows that $\iota(e_1) = (v', 0)$ and $\tau(e_n) = (v'', 0)$. The edge-path $(e_2, \ldots, e_{n-1})$ connects $(v', 1)$ to $(v'', 1)$. There is a corresponding edge-path $(e_2', \ldots, e_{n-1}')$ connecting $(v', 0)$ to $(v'', 0)$. This contradicts the fact that p is geodesic.

Now suppose that p crosses any given hyperplane H at most once. It follows that the endpoints $\iota(p)$, $\tau(p)$ of p are separated by all of the hyperplanes crossed by p. If we assume that there are n such hyperplanes in all (and so p has length n), then any edge-path q from $\iota(p)$ to $\tau(p)$ must cross all n of these hyperplanes, so the length of q is at least n. It follows that p is geodesic.

Definition 4.5.6 Suppose that (e_1, e_2) is an edge-path in a CAT(0) cubical complex X such that e_1 and e_2 are perpendicular sides of a square C in X. We let e_i' denote the side of C opposite e_i, for $i = 1, 2$. The operation of replacing (e_1, e_2) by the edge-path (e_2', e_1') is called a *corner move*. Note that the edge-paths (e_1, e_2) and (e_2', e_1') have the same endpoints.

Proposition 4.5.7 *If (e_1, e_2) is an edge-path in X, e_i crosses the hyperplane H_i $(i = 1, 2)$, $H_1 \neq H_2$, and $H_1 \cap H_2 \neq \emptyset$, then the edges e_1 and e_2 are perpendicular sides of a square C in X.*

Proof Let B denote the block containing the hyperplane H_1. We write $B = B_0 \times [0, 1]$, and assume that $\iota(e_1) = (v, 0)$, for some vertex $v \in B_0$. It follows that $\tau(e_1) = (v, 1)$. Since $H_2 \cap H_1 \neq \emptyset$ and $H_1 \neq H_2$, we have that d_{H_1} is constant on e_2, by Lemma 4.5.3. In particular, $d_{H_1}(x) = 1/2$, for any x on the edge e_2, since $d(\iota(e_2), H_1) = 1/2$. It follows that e_2 has the form $[(v, 1), (v', 1)]$, where $[v, v']$ is an edge in B_0. Therefore the edge-path (e_1, e_2) forms one half of the boundary of the square $(v, v') \times [0, 1] \subseteq B$, as desired.

Proposition 4.5.8 ([15]) *If $H_1, \ldots, H_n$ satisfy $H_i \cap H_j \neq \emptyset$ for any $i, j \in \{1, \ldots, n\}$, then $H_1 \cap \cdots \cap H_n \neq \emptyset$.*

Proof The proof is by induction on n. The conclusion is obvious if $n = 2$. We suppose that $n > 2$. By induction, $H_1 \cap \cdots \cap H_{n-1} \neq \emptyset$, so we take $x \in H_1 \cap \cdots \cap H_{n-1}$. By Lemma 4.5.2, $\pi_{(X,H_n)}(x) = \pi_{(X,H_j \cap H_n)}(x)$ for $j \in \{1, \ldots, n-1\}$. It follows that $\pi_{(X,H_n)}(x) \in H_1 \cap \cdots \cap H_n$.

4.6 Applications

4.6.1 The Set-with-Walls Property

Definition 4.6.1 (*first defined in* [9]) Let S be a set. A *wall* W in S is a partition $\{W^-, W^+\}$ of S. Two points $x, y \in S$ are *separated* by the wall W if $x \in W^-$ and $y \in W^+$ (or vice versa). We say that $(S, \mathscr{W})$ is a *set with walls* if $\mathscr{W}$ is a collection of walls in S such that, for any $x, y \in S$, at most finitely many walls $W \in \mathscr{W}$ separate x from y.

If G is a group and S is a G-set, then $(S, \mathscr{W})$ is a *G-set with walls* if the natural action of G permutes the set $\mathscr{W}$.

Definition 4.6.2 If $(S, \mathscr{W})$ is a set with walls, then the *wall pseudometric* $d_{(S,\mathscr{W})}$: $S \times S \to \mathbb{R}^+$ is defined by

$$d_{(S,\mathscr{W})}(x, y) = |\{W \in \mathscr{W} \mid W \; separates \; x \; from \; y\}|.$$

If $(S, \mathscr{W})$ is a G-set with walls, then we say that G acts *properly* on $(S, \mathscr{W})$ if, for any $r > 0$ and $x \in S$, the set

$$\{g \in G \mid d_{(S,\mathscr{W})}(x, gx) < r\}$$

is finite.

Remark 4.6.3 It is straightforward to check that $d_{(S,\mathscr{W})}$ is symmetric and satisfies the triangle inequality, and that G acts isometrically on $(S, \mathscr{W})$ if the latter is a G-set with walls.

Theorem 4.6.4 *If X is a CAT(0) cubical complex, then $(X^0, \mathscr{W})$ is a set with walls, where $\mathscr{W} = \{\{H^+ \cap X^0, H^- \cap X^0\} \mid H \; is \; a \; hyperplane \; in \; X\}$. If G acts cellularly and by isometries on X, then $(X^0, \mathscr{W})$ is a G-set with walls. If G acts properly on X, then G acts properly on $(X^0, \mathscr{W})$.*

Proof (*Sketch*) The fact that $\{H^+ \cap X^0, H^- \cap X^0\}$ is a wall follows from Theorem 4.4.4; the fact that two vertices x, y are separated by at most finitely many walls $W_H = \{H^+ \cap X^0, H^- \cap X^0\} \in \mathscr{W}$ follows from the fact that a wall W_H separates x from y if and only if a geodesic edge-path from x to y crosses H. The remaining statements are similarly straightforward to check.

We note that [2] contains a proof of the converse: there is a construction of a CAT(0) cubical complex associated to any space with walls.

Definition 4.6.5 A discrete group G has the *Haagerup property* if there is a proper affine isometric action of G on a Hilbert space $\mathscr{H}$. Here "proper" means metrically proper: if $v \in \mathscr{H}$ and $r > 0$ are given, then $|\{g \in G \mid d(v, g \cdot v) < r\}| < \infty$.

Theorem 4.6.6 ([12]) *If G acts properly, cellularly, and by isometries on a CAT(0) cubical complex X, then G has the Haagerup property.*

Proof (*Sketch*) One chooses a basepoint $v \in X^0$ and orientations for all hyperplanes $H \subseteq X$. Let $\mathscr{W}^{or}$ denote the set of oriented hyperplanes. The group G acts as (infinite) signed permutation matrices on the Hilbert space $\ell^2(\mathscr{W}^{or})$. For $g \in G$, we let

$$\delta(g) = \sum \pm H,$$

where the sum is over all hyperplanes separating v from gv. Here H is counted with the plus sign if one crosses H in the direction of its given orientation when moving from v to gv, and it is counted with the minus sign otherwise.

The action $\alpha : G \times \ell^2(\mathscr{W}^{or}) \to \ell^2(\mathscr{W}^{or})$ given by $\alpha(g, v) = g \cdot v + \delta(g)$ has the desired properties.

4.6.2 The Median Algebra Property

Let $\mathscr{P}(S)$ denote the power set of S.

Definition 4.6.7 A *median algebra* is a set S together with an interval operation $[,] : S \times S \to \mathscr{P}(S)$ such that

(a) $[x, x] = \{x\}$ for $x \in S$;
(b) $[x, y] = [y, x]$ for $x, y \in S$;
(c) If $z \in [x, y]$, then $[x, z] \subseteq [x, y]$;
(d) For any $x, y, z \in S$, $[x, y] \cap [y, z] \cap [x, z]$ is a singleton set. The unique element of this singleton set, denoted $m(x, y, z)$, is called the *median* of x, y, z.

A median algebra is *discrete* if each set $[x, y]$ is finite.

Definition 4.6.8 Assume that X is a CAT(0) cubical complex. If $x, y \in X^0$, then the *geodesic interval* $[x, y]$ is the set of all vertices $z \in X^0$ that lie on some geodesic edge-path connecting x to y.

Remark 4.6.9 Note, for instance, that the geodesic interval between two integral points (a, b) and (c, d) ($a \leq c$ and $b \leq d$) in $\mathbb{R}^2$ is $\{(x, y) \mid a \leq x \leq c;\ b \leq y \leq d;\ x, y \in \mathbb{Z}\}$.

Theorem 4.6.10 *Let X be a CAT(0) cubical complex. The set of vertices X^0 is a discrete median algebra, where the interval operation $[,] : X^0 \times X^0 \to \mathscr{P}(X^0)$ is the geodesic interval.*

Proof Properties (1) and (2) are clear.

We now prove (3). Let $z \in [x, y]$. This means that there is a geodesic edge-path p connecting x to y and passing through z. We can express p as $p_1 \cup p_2$, where p_1 is a geodesic edge-path connecting x to z and p_2 is a geodesic edge-path connecting z to y. If $w \in [x, z]$, then there is a geodesic edge-path p_1' connecting x to z and passing through w. Since p_1' and p_1 have the same length, $p_1' \cup p_2$ is also a geodesic edge-path connecting x to z, and it passes through w. Therefore $w \in [x, y]$. It follows that $[x, z] \subseteq [x, y]$, proving (3).

We now prove that $[x, y]$ is always finite. If $H_1, \ldots, H_n$ are the hyperplanes separating x from y, then, by Proposition 4.5.5, an edge-path p is a geodesic edge-path connecting x to y if and only if p begins at x and crosses exactly the hyperplanes $H_1, \ldots, H_n$. However, such an edge-path is uniquely determined by the order in which the hyperplanes $H_1, \ldots, H_n$ are crossed. It follows that there are at most $n!$ geodesic edge-paths, each of which passes through only finitely many points, so $|[x, y]| < \infty$.

We now prove (4). Fix $x, y, z \in X^0$. We first show that $[x, y] \cap [y, z] \cap [x, z]$ contains at most one element. Suppose $v, w \in [x, y] \cap [y, z] \cap [x, z]$ and $v \neq w$. There is a hyperplane H separating v from w. It must be that two (or more) elements of $\{x, y, z\}$ lie in one of the complementary components of $X - H$. It follows without loss of generality (i.e., up to relabelling) that v is separated from both x and y by H. Since $v \in [x, y]$ by our assumption, there is a geodesic edge-path p from x to y passing through v. The geodesic edge-path p would necessarily cross H twice, however. This is a contradiction.

We now need to show that $[x, y] \cap [y, z] \cap [x, z]$ is non-empty. We do this by induction on $d(x, y) + d(y, z) + d(x, z)$, where d denotes the edge-path (or combinatorial) distance. The base case is trivial. For the inductive step, we need a definition. If a hyperplane H separates both x and y from z, then we say that H is an $\{x, y\}$-*hyperplane*. We can similarly define $\{x, z\}$- and $\{y, z\}$-hyperplanes. Note that any hyperplane crossed by an edge-path geodesic between any two points of $\{x, y, z\}$ must be a $\{a, b\}$-hyperplane, where $\{a, b\} \subseteq \{x, y, z\}$. If $z \in [x, y]$, $x \in [y, z]$, or $y \in [x, z]$, then the desired conclusion is clear, so we assume that none of x, y, and z is contained in the interval of the other two. We choose geodesic edge-paths p_x, p_y connecting z to x and y, respectively.

We claim that there is some $\{x, y\}$-hyperplane H that is crossed by both p_x and p_y. Indeed, p_x crosses only $\{x, y\}$- and $\{y, z\}$-hyperplanes by definition, and p_y crosses only $\{x, y\}$- and $\{x, z\}$-hyperplanes. Thus, if no $\{x, y\}$-hyperplane is crossed by both p_x and p_y, then $p_x^{-1} p_y$ crosses no hyperplane more than once, and is therefore geodesic. Since $p_x^{-1} p_y$ passes through z, we have $z \in [x, y]$, a contradiction.

Next, we claim that there are geodesic edge-paths p_x' and p_y' from z to x and y with the property that p_x' and p_y' cross all $\{x, y\}$-hyperplanes before crossing any $\{x, z\}$- or $\{y, z\}$-hyperplanes. We prove only that there is such a p_x', since the proof that there is such a p'_y is similar. To establish the existence of the desired p'_x, it is sufficient to show that, whenever p_x crosses a $\{y, z\}$-hyperplane H' before an $\{x, y\}$-hyperplane H'', $H' \cap H'' \neq \emptyset$, for then we can use corner moves to change p_x into the desired p'_x. We assume the convention that $z \in (H')^- \cap (H'')^-$. If e' is the (unique) edge of p_x crossing H', then $\tau(e') \in (H')^+ \cap (H'')^-$. If e'' is the edge of p_x crossing H'', then $\tau(e'') \in (H')^+ \cap (H'')^+$. Now note that $y \in (H')^- \cap (H'')^+$. We now have $H' \cap H'' \neq \emptyset$, by Lemma 4.5.4. This proves the claim.

We therefore have p'_x and p'_y (as above). Let H_1 be the first hyperplane crossed by p'_x. It is, of course, an $\{x, y\}$-hyperplane. We claim that we can alter p'_y to obtain a new geodesic edge-path p''_y connecting z to y, such that p''_y crosses H_1 first. (We note that p'_y must cross H_1, since H_1 separates z from y by definition.) It is enough to show that if the hyperplane $\{x, y\}$-hyperplane H_2 is crossed by p'_y before H_1, then $H_1 \cap H_2 \neq \emptyset$, for then we can alter p'_y by corner moves in order to arrive at the desired p''_y. We assume the convention that $z \in (H_1)^- \cap (H_2)^-$. If e_2 is the edge of p'_y crossing H_2, then $\tau(e_2) \in (H_1)^- \cap (H_2)^+$. If e_1 is the edge of p'_y crossing H_1, then $\tau(e_1) \in (H_1)^+ \cap (H_2)^+$. If e_x is the edge of p'_x crossing H_1 then $\tau(e_x) \in (H_1)^+ \cap (H_2)^-$. It follows from Lemma 4.5.4 that $H_1 \cap H_2 \neq \emptyset$. This proves the claim.

We've now shown that there are geodesic edge-paths p'_x, p''_y connecting z to x and y (respectively), and having the same initial edge $\hat{e}$. We assume $z = \iota(\hat{e})$. By the induction hypothesis $[\tau(\hat{e}), y] \cap [x, \tau(\hat{e})] \cap [x, y]$ is non-empty. Since

$$[\tau(\hat{e}), y] \cap [x, \tau(\hat{e})] \cap [x, y] \subseteq [z, y] \cap [x, z] \cap [x, y]$$

by (3), the induction is complete.

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Chapter 5
Simplicity of Twin Tree Lattices with Non-trivial Commutation Relations

Pierre-Emmanuel Caprace and Bertrand Rémy

Abstract We prove a simplicity criterion for certain twin tree lattices. It applies to all rank-2 Kac–Moody groups over finite fields with non-trivial commutation relations, thereby yielding examples of simple non-uniform lattices in the product of two trees.

Keywords Tree · Twinning · Moufang property · Kac-Moody group · Simplicity · Lattice

5.1 Overview

This paper deals with the construction of finitely generated (but not finitely presented) simple groups acting as non-uniform lattices on products of two twinned trees. These seem to be the first examples of *non-uniform* simple lattices in the product of two trees. They contrast with the simple groups obtained by M. Burger and Sh. Mozes [3] in a similar geometric context: the latter groups are (torsion-free) *uniform* lattices, in the product of two trees; in particular they are finitely presented. That a non-uniform lattice in a 2-dimensional CAT(0) cell complex cannot be finitely presented is a general fact recently proved by G. Gandini [7, Corollary 3.6].

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The lattices concerned by our criterion belong to the class of **twin building lattices**. By definition, a twin building lattice is a special instance of a group endowed with a **Root Group Datum** (also sometimes called **twin group datum**), i.e. a group Λ equipped with a family of subgroups $(U_\alpha)_{\alpha\in\Phi}$, called **root subgroups**, indexed by the (real) roots of some root system Φ with Weyl group W, and satisfying a few conditions called the **RGD-axioms**, see [6, 14]. Such a group Λ acts by automorphisms on a product of two buildings $X_+ \times X_-$, preserving a twinning between X_+ and X_-. The main examples arise from Kac–Moody theory, see [13, 14]. When the root groups are finite, the group Λ is finitely generated, the buildings X_+ and X_- are locally finite and the Λ-action on $X_+ \times X_-$ is properly discontinuous. In particular (modulo the finite kernel) Λ can be viewed as a discrete subgroup of the locally compact group $\mathrm{Aut}(X_+) \times \mathrm{Aut}(X_-)$. The quotient $\Lambda\backslash \mathrm{Aut}(X_+) \times \mathrm{Aut}(X_-)$ is never compact. However, if in addition the order of each root group is at least as large as the rank of the root system Φ, then Λ has finite covolume; in particular Λ is a non-uniform lattice in $\mathrm{Aut}(X_+) \times \mathrm{Aut}(X_-)$, see [6, 10]. When Λ has finite covolume in $\mathrm{Aut}(X_+) \times \mathrm{Aut}(X_-)$, it is called a **twin building lattice**.

It was proved in [6] that a twin building lattice is infinite and virtually simple provided the associated Weyl group W is irreducible and not virtually abelian. A (small) precise bound on the order of the maximal finite quotient was moreover given; in most cases the twin building lattice Λ itself happens to be simple. The condition that W is not virtually abelian was essential in loc. cit., which relied on some weak hyperbolicity property of non-affine Coxeter groups. Rank-2 root systems were thus excluded since their Weyl group is infinite dihedral, hence virtually abelian (even though many rank-2 root systems are termed *hyperbolic* within Kac–Moody theory).

The goal of this note is to provide a simplicity criterion applying to that rank-2 case. Notice that when Φ has rank 2, the twin building associated with Λ is a twin tree $T_+ \times T_-$. Moreover Λ is a lattice (then called a **twin tree lattice**) in $\mathrm{Aut}(T_+) \times \mathrm{Aut}(T_-)$ if and only if the root groups are finite; in other words, the condition on the order of the root groups ensuring that the covolume of Λ is finite is automatically satisfied in this case.

Theorem 5.1.1 *Let Λ be a group with a root group datum $(U_\alpha)_{\alpha\in\Phi}$ with finite root groups, indexed by a root system of rank 2. Suppose that Λ is center-free and generated by the root groups. Assume moreover that the following conditions hold:*

(i) There exist root groups U_ϕ, U_ψ associated with a prenilpotent pair of roots $\{\phi, \psi\}$ (possibly $\phi = \psi$) such that the commutator $[U_\phi, U_\psi]$ is non-trivial. (Equivalently the maximal horospherical subgroups of Λ are non-abelian.)

(ii) There is a constant $C > 0$ such that for any prenilpotent pair of roots whose corresponding walls are at distance $\geqslant C$, the associated root groups commute.

Then the finitely generated group Λ contains a simple subgroup Λ^0 of finite index.

We shall moreover see in Lemma 5.2.1 below that, with a little more information on the commutator $[U_\phi, U_\psi]$ in Condition (i), the maximal finite quotient Λ/Λ^0 can be shown to be abelian of very small order.

As mentioned above, the main examples of twin building lattices arise from Kac–Moody theory. Specializing Theorem 5.1.1 to that case, we obtain the following.

Theorem 5.1.2 *Let Λ be an adjoint split Kac–Moody group over the finite field $\mathscr{F}_q$ and associated with the generalized Cartan matrix $A = \begin{pmatrix} 2 & -m \\ -1 & 2 \end{pmatrix}$, with $m > 4$ coprime to q.*

Then the commutator subgroup of Λ is simple, has index $\leqslant q$ in Λ, and acts as a non-uniform lattice on the product $T_+ \times T_-$ of the twin trees associated with Λ.

The following consequence is immediate, since split Kac–Moody groups over fields of order >3 are known to be perfect.

Corollary 5.1.1 *Let Λ be an adjoint split Kac–Moody group over the finite field $\mathscr{F}_q$ and associated with the generalized Cartan matrix $A = \begin{pmatrix} 2 & -m \\ -1 & 2 \end{pmatrix}$.*

If $m > 4$ is coprime to q and $q > 3$, then Λ is simple.

Other examples of twin tree lattices satisfying the conditions from Theorem 5.1.1 can be constructed in the realm of Kac–Moody theory, as **almost split groups**. Indeed, it is possible to construct non-split Kac–Moody groups of rank 2, using Galois descent, so that some root groups are nilpotent of class 2, while all commutation relations involving distinct roots are trivial.

Here is an example among many other possibilities. Pick an integer $m \geqslant 2$ and consider the generalized Cartan matrix $A = \begin{pmatrix} 2 & -1 & -m \\ -1 & 2 & -m \\ -m & -m & 2 \end{pmatrix}$. This defines a split Kac–Moody group (over any field) whose Weyl group is the Coxeter group obtained, via Poincaré's theorem, from the tessellation of the hyperbolic plane by the (almost ideal) triangle with two vertices at infinity and one vertex of angle $\frac{\pi}{3}$. The associated twinned buildings have apartments isomorphic to the latter hyperbolic tessellation. This Weyl group is generated by the reflections in the faces of the hyperbolic triangle, and the Dynkin diagram has a (unique, involutive) symmetry exchanging the vertices corresponding to the reflections in the two edges of the hyperbolic triangle meeting at the vertex of angle $\frac{\pi}{3}$. Using [2, Theorem 1] and [9, Theorem 2], one sees that any prenilpotent pair of two roots leading to a non-trivial commutation relation between root groups is contained, up to conjugation by the Weyl group, in the standard residue of type A_2.

Suppose now that $\mathscr{G}_A(\mathscr{F}_{q^2})$ is the split Kac–Moody group of that type, defined over a finite ground field of order q^2. According to [11, Proposition 13.2.3], the non-trivial element of the Galois group of the extension $\mathscr{F}_{q^2}/\mathscr{F}_q$, composed with the symmetry of the Dynkin diagram, yields an involutory automorphism of $\mathscr{G}_A(\mathscr{F}_{q^2})$ whose centraliser, which we denote by Λ, is a quasi-split Kac–Moody group over the finite field $\mathscr{F}_q$. This quasi-split group acts on a twin tree obtained as the fixed point set of the involution acting on the twin building of the split group; the valencies are equal to $1 + q$ and $1 + q^3$, corresponding to root groups isomorphic to $(\mathscr{F}_q, +)$

and to a p-Sylow subgroup of $\mathrm{SU}_3(q)$, respectively. In particular, the root groups of order q^3 are nilpotent of step 2. Moreover for any prenilpotent pair of two distinct roots, the corresponding root groups commute to one another: this follows from the last statement in the previous paragraph. Hence Λ satisfies both conditions from Theorem 5.1.1 and is thus virtually simple. In fact, Lemma 5.2.1 below also applies to Λ, and yields the sharper conclusion that the derived group $[\Lambda, \Lambda]$ is simple and of index at most q^2 in Λ. If in addition $q > 3$, then the rank-1 subgroups of Λ are perfect. Since they generate Λ, we infer that Λ itself is perfect, hence simple.

Further examples of twin tree lattices satisfying the simplicity criterion from Theorem 5.1.1, of a more exotic nature, can be constructed as in [12]. In particular it is possible that the two conjugacy classes of root groups have coprime order.

Finitely generated Kac–Moody groups associated with the generalized Cartan matrix $\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$ or $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$, are known to be residually finite (and can in fact be identified with some S-arithmetic groups of positive characteristic). In particular it cannot be expected that the conclusions of Theorem 5.1.2 hold without any condition on the Cartan matrix A. The remaining open case is that of a matrix of the form $A_{m,n} = \begin{pmatrix} 2 & -m \\ -n & 2 \end{pmatrix}$ with $m, n > 1$. In that case Condition (ii) from Theorem 5.1.1 holds, but Condition (i) is violated. On the other hand, if the matrix $A_{m,n}$ is congruent to the matrix $A_{m',n'}$ modulo $q - 1$, then the corresponding Kac–Moody groups over $\mathscr{F}_q$ are isomorphic (see [5, Lemma 4.3]). In particular if $(m', n') = (2, 2)$ or $(m', n') = (4, 1)$, then all these Kac–Moody groups are residually finite. It follows that over $\mathscr{F}_2$, a rank-2 Kac–Moody group is either residually finite (because it is isomorphic to a Kac–Moody group of affine type), or virtually simple, by virtue of Theorem 5.1.2. The problem whether this alternative holds for rank-2 Kac–Moody groups over larger fields remains open; its resolution will require to deal with Cartan matrices of the form $A_{m,n}$ with $m, n > 1$.

5.2 Proof of the Simplicity Criterion

Virtual simplicity will be established following the Burger–Mozes strategy from [3] by combining the **Normal Subgroup Property**, abbreviated **(NSP)**, with the property of non-residual finiteness. This strategy was also implemented in [6]. The part of the work concerning (NSP) obtained in that earlier reference already included the rank-2 case, and thus applies to our current setting; its essential ingredient is the work of Bader–Shalom [1]:

Proposition 5.2.1 *Let Λ be a twin building lattice with associated root group datum $(U_\alpha)_{\alpha\in\Phi}$. Assume that Λ is generated by the root groups.*

If Φ is irreducible, then every normal subgroup of Λ is either finite central, or of finite index. In particular, if Λ is center-free (equivalently if it acts faithfully on its twin building), then Λ is just-infinite.

Proof See [6, Theorem 18].

The novelty in the present setting relies in the proof of non-residual finiteness. In the former paper [6], we exploited some hyperbolic behaviour of non-affine Coxeter groups, appropriately combined with the commutation relations of Λ. This argument cannot be applied to infinite dihedral Weyl groups. Instead, we shall use the following non-residual finiteness result for wreath products, due to Meskin [8]:

Proposition 5.2.2 *Let F, Z be two groups. Assume that Z is infinite and let Γ be the wreath product $F \wr Z \cong (\bigoplus_{i\in Z} F) \rtimes Z$.*

Then any finite index subgroup of Γ contains the subgroup $\bigoplus_{i\in Z}[F, F]$. In particular, if F is not abelian, then Γ is not residually finite.

Proof For each $i \in Z$, let F_i be a copy of F, so that $F \wr Z = (\bigoplus_{i\in Z} F_i) \rtimes Z$.

Let $\varphi\colon \Gamma \to Q$ be a homomorphism to a finite group Q. Since Z is infinite, there is some $t \in Z \setminus \{1\}$ such that $\varphi(t) = 1$. Notice that, for all $i \in Z$ and all $x \in F_i$, we have $txt^{-1} \in F_{ti} \neq F_i$, whence txt^{-1} commutes with every element of F_i. Therefore, for all $y \in F_i$, we have

$$\begin{aligned}\varphi\big([x, y]\big) &= [\varphi(x), \varphi(y)]\\ &= [\varphi(txt^{-1}), \varphi(y)]\\ &= \varphi\big([txt^{-1}, y]\big)\\ &= 1.\end{aligned}$$

This proves that $[F_i, F_i]$ is contained in $\mathrm{Ker}(\varphi)$, and so is thus $\bigoplus_{i\in Z}[F_i, F_i]$. This proves that every finite index normal subgroup Γ contains $\bigoplus_{i\in Z}[F_i, F_i]$. The desired result follows, since every finite index subgroup contains a finite index normal subgroup.

Proof (*Proof of Theorem* 5.1.1) Recall that in the case of twin trees, a pair of roots $\{\phi; \psi\}$ is prenilpotent if and only if $\phi \supseteq \psi$ or $\psi \supseteq \phi$ (where the roots ϕ and ψ are viewed as half-apartments). By (i) there exists such a pair with $[U_\phi, U_\psi] \neq \{1\}$ (possibly $\phi = \psi$). In particular the group $F = \langle U_\phi, U_\psi\rangle$ is non-abelian.

In view of (ii), the distance between the roots ϕ and ψ in the trees on which Λ acts is smaller than C. Pick an element $t \in \Lambda$ stabilising the standard twin apartment and acting on it as a translation of length $>2C$. It follows from (ii) and from the axioms of Root Group Data that the subgroup of Λ generated by F and t is isomorphic to the wreath product $F \wr \mathbf{Z}$, where the cyclic factor is generated by t.

Since F is not abelian, we deduce from Proposition 5.2.2 that Λ contains a non-residually finite subgroup, and can therefore not be residually finite. On the other hand Λ is just-infinite by Proposition 5.2.1. Therefore, we deduce from [15, Proposition 1] that the unique smallest finite index subgroup Λ^0 of Λ is a finite direct product of $m \geqslant 1$ pairwise isomorphic simple groups. It remains to show that $m = 1$. This follows from the fact that Λ acts minimally (in fact: edge-transitively) on each half of its twin tree, and so does Λ^0. But a group acting faithfully minimally on a tree cannot split non-trivially as a direct product. Hence $m = 1$ and Λ^0 is a simple subgroup of finite index in Λ.

Lemma 5.2.1 *Retain the hypotheses of Theorem 5.1.1 and assume in addition that one of the following conditions is satisfied:*

(iii-a) the commutator $[U_\phi, U_\psi]$ *contains some root group* U_γ*;*
(iii-b) we have $\pi = \psi$*, and the rank-1 group* $\langle U_\phi, U_{-\phi}\rangle$ *is either a perfect group of Lie type, or a sharply* 2*-transitive group such that the commutator subgroup* $[U_\phi, U_\phi]$ *is of even order.*

Then the maximal finite quotient Λ/Λ^0 *afforded by Theorem 5.1.1 is abelian. Moreover we have* $|\Lambda/\Lambda^0| \leqslant \max_{\alpha\in\Phi} |U_\alpha|$*, or* $|\Lambda/\Lambda^0| \leqslant (\max_{\alpha\in\Phi} |U_\alpha/[U_\alpha, U_\alpha]|)^2$ *if the second case of* (iii-b) *holds.*

Proof Retain the notation from the proof of Theorem 5.1.1. Proposition 5.2.2 ensures that every finite index normal subgroup of $F \wr \mathbf{Z}$ contains the commutator subgroup $[F, F]$. In particular, so does $N = \Lambda^0$.

Assume that (iii-a) holds, i.e. that $[U_\phi, U_\psi]$ contains some root group U_γ. Then U_γ is contained in N. Since $X_\gamma = \langle U_\gamma, U_{-\gamma}\rangle$ is a finite group acting 2-transitively on the conjugacy class of U_γ, it follows that X_γ is entirely contained in N. In particular, so is the element $r_\gamma \in X_\gamma$ acting as the reflection associated with γ on the standard twin apartment.

Let now $\alpha \in \Phi$ be any root such that $\alpha \subset \gamma$ and that the wall $\partial\alpha$ is at distance $> C/2$ away from $\partial\gamma$. Then $\alpha \subset r_\gamma(-\alpha)$, and the walls associated with the latter two roots have distance $> C$. By condition (ii), the corresponding root groups commute. Denoting by $\varphi\colon \Lambda \to \Lambda/N$ the quotient map, we deduce

$$\begin{aligned}[\varphi(U_\alpha), \varphi(U_{-\alpha})] &= [\varphi(U_\alpha), \varphi(U_{r_\gamma(-\alpha)})]\\ &= \varphi\big([U_\alpha, U_{r_\gamma(-\alpha)}]\big)\\ &= 1.\end{aligned}$$

Since $\varphi(U_\alpha)$ and $\varphi(U_{-\alpha})$ commute in the image under φ of the rank-1 group $X_\alpha = \langle U_\alpha \cup U_{-\alpha}\rangle$, and since U_α and $U_{-\alpha}$ are conjugate in X_α, we conclude that we have $\varphi(X_\alpha) = \varphi(U_\alpha) = \varphi(U_{-\alpha})$ and that the latter group identifies with an abelian quotient of U_α.

Remark finally that there are only two conjugacy classes of root groups, the union of which generates the whole group Λ. One of these conjugacy classes has trivial image under φ, since N contains the root group U_γ. The other conjugacy class contains root groups associated with roots α whose wall is far away from $\partial\gamma$. This implies that $\varphi(\Lambda) = \varphi(U_\alpha)$, which has been proved to be abelian. We are done in this case.

Assume now that condition (iii-b) holds. Again, by Proposition 5.2.2, the commutator $[U_\phi, U_\phi]$ is contained in N.

If the rank-1group $X_\phi = \langle U_\phi, U_{-\phi}\rangle$ is not a sharply 2-transitive group, then it is a perfect group of Lie type by hypothesis, and we may conclude that it is entirely contained in N. Hence the same argument as in the case (iii-a) with ϕ playing the role of γ yields the conclusion.

If the rank-1 group $X_\phi = \langle U_\phi, U_{-\phi}\rangle$ is a sharply 2-transitive group, then we have at our disposal the additional hypothesis that the commutator $[U_\phi, U_\phi]$ contains an involution. Since X_ϕ is sharply 2-transitive, all its involutions are conjugate. They must thus all be contained in N. In particular N contains some involution r_ϕ swapping U_ϕ and $U_{-\phi}$. Again, this is enough to apply the same computation as above and conclude that for each root α whose wall is far away from $\partial\phi$, the image of $\langle U_\alpha, U_{-\alpha}\rangle$ under φ is abelian and isomorphic to a quotient of U_α. We take two distinct such roots $\alpha \subset \beta$ so that there is no root γ strictly between α and β. Thus U_α and U_β commute by the axioms of Root Group Data. Moreover Λ is generated by $U_\alpha \cup U_{-\alpha} \cup U_\beta \cup U_{-\beta}$, and we have just seen that, modulo N, the root groups U_α and $U_{-\alpha}$ (resp. U_β and $U_{-\beta}$) become equal, and abelian. It follows that Λ/N is isomorphic to a quotient of the direct product $U_\alpha/[U_\alpha, U_\alpha] \times U_\beta/[U_\beta, U_\beta]$. The desired result follows.

Remark 5.2.1 Finite sharply 2-transitive groups are all known; they correspond to finite near-fields, which were classified by Zassenhaus. All of them are either Dickson near-fields, or belong to a list of seven exceptional examples. An inspection of that list shows that, in all of these seven exceptions, the root group contains a copy of $\mathrm{SL}_2(\mathscr{F}_3)$ or $\mathrm{SL}_2(\mathscr{F}_5)$ (see [4, Sect. 1.12]); in particular the commutator subgroup of a root group is always of even order in those cases. Thus condition (iii-b) from Lemma 5.2.1 only excludes certain sharply 2-transitive groups associated with Dickson near-fields.

5.3 Kac–Moody Groups of Rank 2

Let Λ be a Kac–Moody group over the finite field $\mathscr{F}_q$ of order q, associated with the generalized Cartan matrix $A_{m,n} = \begin{pmatrix} 2 & -m \\ -n & 2 \end{pmatrix}$. The Weyl group of Λ is the infinite dihedral group and Λ is a twin tree lattice; the corresponding trees are both regular of degree $q + 1$.

When $mn < 4$, the matrix A is of **finite type** and Λ is then a finite Chevalley group over $\mathscr{F}_q$. When $mn = 4$, the matrix A is of **affine type** and Λ is linear, and even S-arithmetic; in particular it is residually finite.

In order to check that the conditions from Theorem 5.1.1 are satisfied when $m > 4$ and $n = 1$, we need a sharp control on the commutation relations satisfied by the root groups. The key technical result is the following lemma, which follows from the work of Morita [9] and Billig–Pianzola [2].

Lemma 5.3.1 *Let $\Pi = \{\alpha, \beta\}$ be the standard basis of the root system Δ for Λ and set $t = r_\alpha r_\beta$. For all $i \in \mathbf{Z}$, let $\alpha_i = t^i\alpha$ and $\beta_i = t^i\beta$ and set*

$$\Phi(+\infty) = \{-\alpha_i, \beta_j \mid i, j \in \mathbf{Z}\} \quad \textit{and} \quad \Phi(-\infty) = \{\alpha_i, -\beta_j \mid i, j \in \mathbf{Z}\}.$$

Assume that $m > 4$ is coprime to q and that $n = 1$. Then for all $\phi, \psi \in \Phi(+\infty)$, either U_ϕ and U_ψ commute, or we have

$$\{\phi, \psi\} = \{-\alpha_i, -\alpha_{i+1}\} \textit{ for some } i \in \mathbf{Z} \textit{ and } [U_\phi, U_\psi] = U_{\beta_i}.$$

Similarly, for all $\phi, \psi \in \Phi(-\infty)$*, either* U_ϕ *and* U_ψ *commute, or we have*

$$\{\phi, \psi\} = \{\alpha_i, \alpha_{i+1}\} \text{ for some } i \in \mathbf{Z} \text{ and } [U_\phi, U_\psi] = U_{-\beta_i}.$$

Proof It follows from Theorem 2 in [9] and Theorem 1 in [2] that the only potentially non-trivial commutation relations between U_ϕ and U_ψ arise when $\{\phi, \psi\} = \{-\alpha_i, -\alpha_{i+1}\}$ or $\{\phi, \psi\} = \{-\alpha_i, -\alpha_{i+1}\}$. In the latter cases, the equality $[U_\phi, U_\psi] = U_{\beta_i}$ (resp. $[U_\phi, U_\psi] = U_{-\beta_i}$) holds if m is coprime to q, in view of Sect. 3.5 in [13] (while if m is not coprime to q, we have $[U_\phi, U_\psi] = 1$).

Proof (*Proof of Theorem* 5.1.2) Lemma 5.3.1 readily implies that Conditions (i) and (ii) from Theorem 5.1.1 are satisfied (we can take $C = 2$ in this case), so that Λ is virtually simple. In fact, Lemma 5.3.1 shows that some root group is equal to the commutator of a pair of prenilpotent root groups, so that condition (iii-a) from Lemma 5.2.1 is satisfied. The latter ensures that Λ^0 is the commutator subgroup of Λ, and that the quotient Λ/Λ^0 is bounded above by the maximal order of a root group. Thus the theorem holds, since all the root groups have order q in this case.

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Chapter 6
Groups with Many Finitary Cohomology Functors

Peter H. Kropholler

Abstract For a group G, we study the question of which cohomology functors commute with all small filtered colimit systems of coefficient modules. We say that the functor $H^n(G, -)$ is *finitary* when this is so and we consider the *finitary set* for G, that is the set of natural numbers for which this holds. It is shown that for the class of groups $\mathbf{LH}\mathfrak{F}$ there is a dichotomy: the finitary set of such a group is either finite or cofinite. We investigate which sets of natural numbers n can arise as finitary sets for suitably chosen G and what restrictions are imposed by the presence of certain kinds of normal or near-normal subgroups. Although the class $\mathbf{LH}\mathfrak{F}$ is large, containing soluble and linear groups, being closed under extensions, subgroups, amalgamated free products, HNN-extensions, there are known to be many not in $\mathbf{LH}\mathfrak{F}$ such as Richard Thompson's group F. Our theory does not extend beyond the class $\mathbf{LH}\mathfrak{F}$ at present and so it is an open problem whether the main conclusions of this paper hold for arbitrary groups. There is a survey of recent developments and open questions.

Keywords Cohomology of groups, Finiteness conditions, Eilenberg–Mac Lane spaces

Organizational Statement

This paper lays the foundation stones for a series of papers by the author's former student Martin Hamilton: [11–13]. As sometimes happens, this literature has not been published in the order in which it was intended to be read and for this reason I am taking the opportunity of this conference proceedings to include a survey of Hamilton's papers and a discussion of possible future directions. This survey follows and expands upon the spirit of the talk I gave at the meeting.

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6.1 Introduction

Groups of types FP, FP_∞ or FP_n have been widely explored. The properties are most often described in terms of projective resolutions. A group G has type FP_n if and only if there is a projective resolution $\cdots \to P_j \to P_{j-1} \to \cdots \to P_1 \to P_0 \twoheadrightarrow \mathbb{Z}$ of the trivial module $\mathbb{Z}$ over the integral group ring $\mathbb{Z}G$ such that P_j is finitely generated for $j \leq n$. Type FP_1 is equivalent to finite generation of the group. For finitely presented groups, type FP_n is equivalent to the existence of an Eilenberg–Mac Lane space with finite n-skeleton. These properties can also be formulated in terms of cohomology functors by using the notion of a *finitary functor*. A functor is said to be *finitary* if it preserves filtered colimits (see Sect. 6.5 of [22]; also Sect. 18 of [1]). For a group G and a natural number n we can consider whether or not the nth cohomology functor is finitary. For our purposes it is also useful to consider additive functors F between abelian categories with the property that

$$\varinjlim F(M_\lambda) = 0$$

whenever (M_λ) is a filtered colimit system satisfying $\varinjlim M_\lambda = 0$: we shall say that F is 0*-finitary* when this condition holds. Here is a classical result of Brown phrased in this language:

Theorem 6.1.1 (Corollary to Theorem 1 of [5]) *Let R be a ring and let M be an R-module. Then the following are equivalent:*

(a) M admits a resolution by finitely generated projectives.
(b) The functors $\mathrm{Ext}^n_R(M, -)$ are finitary for all n.
(c) The functors $\mathrm{Tor}^R_n(-, M)$ commute with products for all n.

In this article we are concerned with the equivalence of (a) and (b) and we shall not further investigate the connection with (c). Our interest is in the application to group rings, so our applications involve the case $R = \mathbb{Z}G$ and $M = \mathbb{Z}$. This special case of group ring and trivial module has close connections with topological applications. There are many variations on this theme. Details of Brown's contribution along with further results of Bieri and Eckmann can be found in ([4], Theorem 1.3) and ([6], VIII Theorem 4.8). The following summarizes the formulationthat generalizes Theorem 6.1.1(a) $\Longleftrightarrow$ (b) and suits our purpose of studying group cohomology.

Lemma 6.1.2 *For a group G and $n \geq 0$, the following are equivalent:*

(a) G is of type FP_n;
(b) $H^i(G, -)$ is finitary for all $i < n$;
(c) $H^i(G, -)$ is 0-finitary for all $i \leq n$;

The classical definitions and results have focussed on investigating the largest n for which the FP_n property holds. Many interesting sequences of groups have been discovered in which the nth term of the sequence is of type FP_n but not of type FP_{n+1}.

Amongst recent deep results in this genre, the work of [8] of Bux, Gramlich and Witzel stands out. The earlier work [7] of Brown already includes several interesting cases and remains a vital contribution.

Nevertheless, investigations of this kind do not touch on what seems to us to be a natural question: if a group is of type FP_n but not of type FP_{n+1} could there exist natural numbers $k > n + 1$ for which the cohomology functor $H_k(G, -)$ is finitary or 0-finitary? Obviously, if a group has finite cohomological dimension then its cohomology functors become eventually finitary in a trivial way and given the wealth of different kinds of groups of finite integral cohomological dimension it quickly becomes clear that the following definition is both natural and likely to lead to interesting investigations.

Definition 6.1.3 We write $\mathscr{F}(G)$ (resp. $\mathscr{F}_0(G)$) for the set of natural numbers $n \geq 1$ for which $H^n(G, -)$ is finitary (resp. 0-finitary).

6.2 Main Theorems

Our basic result concerns groups in the class $\mathbf{LH}\mathfrak{F}$ as described in [18, 20]. We write $\mathbb{N}^+$ for the set of natural numbers $n \geq 1$.

Theorem 6.2.1 *Let G be an $\mathbf{LH}\mathfrak{F}$-group for which $\mathscr{F}_0(G)$ is infinite. Then*

(a) $\mathscr{F}_0(G)$ is cofinite in $\mathbb{N}^+$;
(b) there is a bound on the orders of the finite subgroups of G;
(c) there is a finite dimensional model for the classifying space $\underline{E}G$ for proper group actions.

We refer the reader to [20] for a brief explanation of the classifying space $\underline{E}G$, and to Lück's survey article [23] for a comprehensive account. Our theorem shows that for any $\mathbf{LH}\mathfrak{F}$-group G the set $\mathscr{F}_0(G)$ is either finite or cofinite in the set $\mathbb{N}^+$ of positive natural numbers. It is unknown whether there exists a group G outwith the class $\mathbf{LH}\mathfrak{F}$ for which $\mathscr{F}_0(G)$ is a moiety (i.e. neither finite nor cofinite). Notice that groups of finite integral cohomological dimension all belong to $\mathbf{LH}\mathfrak{F}$ and have a cofinite invariant because almost all their cohomology functors vanish. On the other hand the theorem shows that $\mathscr{F}_0(G)$ is finite for all torsion-free $\mathbf{LH}\mathfrak{F}$ groups of infinite cohomological dimension. Both conditions (ii) and (iii) above are highly restrictive. However, the theorem does not give a characterization for cofiniteness of $\mathscr{F}_0(G)$ for groups with torsion: this turns out to be a delicate question even for abelian-by-finite groups and is studied by Hamilton in the companion article [13]. Before turning to the proof of Theorem 6.2.1 we show that $\mathscr{F}_0(G)$ can behave in any way subject to the constraints it entails.

Theorem 6.2.2 *Given any finite or cofinite subset $S \subseteq \mathbb{N}^+$ there exists a group G such that*

(a) $\mathscr{F}_0(G) = S$;
(b) G *has a finite dimensional model for the classifying space* $\underline{E}G$.

Note that all groups with finite dimensional models for $\underline{E}$ belong to $\mathbf{H}_1\mathfrak{F} \subset \mathbf{LH}\mathfrak{F}$. There is an abundance of examples of groups satisfying various homological finiteness conditions and we can select examples easily to establish Theorem 6.2.2.

Lemma 6.2.3 *(a) For each* n *there is a group* J_n *of finite integral cohomological dimension such that* $\mathscr{F}_0(J_n) = \mathbb{N}^+ \setminus \{n\}$.
(b) For each n *there is a group* H_n *with a finite dimensional classifying space for proper actions, which is of type* FP_n *and for which* $\mathscr{F}_0(H_n)$ *is finite.*

Proof For J_n we can take Bieri's example A_{n-1} of a group which is of type FP_{n-1} but not of type FP_n ([4], Proposition 2.14). This group is the kernel of the homomorphism from a direct product of n free groups to $\mathbb{Z}$ determined by sending each generator to $1 \in \mathbb{Z}$; it has cohomological dimension n and hence it has the desired properties. The example predates and is generalized by the fundamental work of Bestvina–Brady [3], and many more examples like this can be obtained using the powerful results of [3]. Bieri's example arises in one of the simplest cases, a kernel within a right-angled Artin group described by a hyper-octahedron.

For the groups H_n we may choose Houghton's examples [15] of groups which were shown to be of type FP_n but not type FP_{n+1} by Brown [7]. The group H_n is defined to be the group comprising those permutations σ of $\{0, 1, 2, \dots, n\} \times \mathbb{N}$ for which there exists $m_0, \dots, m_n \in \mathbb{N}$ (depending on σ) such that $\sigma(i, m) = (i, m + m_i)$ for all but finitely many ordered pairs (i, m). The *translation* vector $(m_0, \dots, m_n)$ is uniquely determined by σ and necessarily satisfies $m_0 + \cdots + m_n = 0$. Every vector satisfying this condition arises and so there is a group homomorphism $H_n \to \mathbb{Z}^{n+1}$ given by $\sigma \mapsto (m_0, \dots, m_n)$, whose image is free abelian of rank n. The kernel of this homomorphism consists of those permutations which fix almost all elements of $\{0, 1, 2, \dots, n\} \times \mathbb{N}$; the *finitary permutations*. Thus H_n fits into a group extension

$$T \rightarrowtail H_n \twoheadrightarrow \mathbb{Z}^n$$

where T is the group of finitary permutations. We describe an explicit construction for a finite dimensional $\underline{E}H_n$. Let $T_0 < T_1 < T_2 < \cdots < T_i < \cdots$ be a chain of finite subgroups of the locally finite group T, indexed by $i \in \mathbb{N}$ and having $\bigcup T_i = T$. Let Γ be the graph whose edge and vertex sets are the cosets of the T_i:

$$V := \bigsqcup T_i \backslash T =: E$$

and in which the terminal and initial vertices of an edge $e = T_i g$ are $\tau e = T_{i+1} g$ and $\iota e = T_i g$. Then Γ is a T-tree and its realization as a one dimensional CW-complex is a one dimensional model X for $\underline{E}T$. Now take any H_n-simplicial complex abstractly homeomorphic to $\mathbb{R}^n$ on which T acts trivially and on which the induced action of H_n/T is free. Then we can thicken the space X by replacing each vertex by a copy of T appropriately twisted by the action of H_n and replacing each higher dimensional simplex of X by the join of the trees placed at its vertices. This creates a finite dimension model for $\underline{\mathrm{E}}H_n$. This construction also shows that H_n belongs to $\mathbf{H}_1\mathfrak{F}$ and hence Theorem 6.2.1 applies. Since T is an infinite locally finite group we see that the conclusion Theorem 6.2.1(b) fails and it follows that $\mathscr{F}_0(H_n)$ is finite as required.

Lemma 6.2.4 *Suppose that G is the fundamental group of a finite graph of groups in which the edge groups are of type* FP_∞. *Then* $\mathscr{F}_0(G) = \bigcap \mathscr{F}_0(G_v)$, *the intersection of the finitary sets of vertex stabilizers* G_v *as* v *runs through a set of orbit representatives of vertices.*

Proof The Mayer–Vietoris sequence for G is a long exact sequence of the form

$$\cdots \to \prod H^{n-1}(G_e, -) \to H^n(G, -) \to \prod H^n(G_v, -) \to \prod H^n(G_e, -) \to \cdots$$

Here, e and v run through sets of orbit representatives of edges and vertices, and since G comes from a *finite* graph of groups, the product here are finite. Since the edge groups G_e are FP_∞, we find that restriction induces an isomorphism

$$\mathrm{colim}\ H^n(G, M_\lambda) \to \prod \mathrm{colim}\ H^n(G_v, M_\lambda)$$

whenever (M_λ) is a vanishing filtered colimit system of $\mathbb{Z}G$-modules. Thus if $n \notin \mathscr{F}_0(G)$ then any system (M_λ) witnessing this must also bear witness to a infinitary functor $H^n(G_v, -)$ for some v, and we see that

$$\mathscr{F}_0(G) \supseteq \bigcap \mathscr{F}_0(G_v).$$

On the other hand, if $n \notin \bigcap \mathscr{F}_0(G_v)$ then there is a v and a vanishing filtered colimit system (U_λ) of $\mathbb{Z}G_v$-modules such that

$$\mathrm{colim}\ H^n(G_v, U_\lambda) \neq 0.$$

Set $M_\lambda := U_\lambda \otimes_{\mathbb{Z}G_v} \mathbb{Z}G$. Since, *qua* $\mathbb{Z}G_v$*-module,* U_λ is a natural direct summand of M_λ we also have

$$\mathrm{colim}\ H^n(G_v, M_\lambda) \neq 0$$

and therefore from the isomorphism

$$\mathrm{colim}\ H^n(G, M_\lambda) \neq 0$$

and $n \notin \mathscr{F}_0(G)$. Thus $\mathscr{F}_0(G) \subseteq \bigcap \mathscr{F}_0(G_v)$ and the result is proved.

The simplest way to apply this is to a free product of finitely many groups. We deduce that the collection of subsets which can arise as $\mathscr{F}_0(G)$ for some G is closed under finite intersections.

Proof (Proof of Theorem 6.2.2) Suppose that S is a cofinite subset of $\mathbb{N}^+$. Then we take G to be the free product of the finitely many groups J_n, as described in Lemma 6.2.3, for which $n \notin S$. Lemmas 6.2.3 and 6.2.4 show that $\mathscr{F}_0(G) = S$.

On the other hand if S is finite then choose an $n \in \mathbb{N}^+$ greater than any element of S. Now S contained in $\mathscr{F}_0 H_n$ and $\mathscr{F}_0 H_n$ is finite. and let G be the free product of the group H_n and the finitely many groups J_m as m runs through $\mathscr{F}_0 H_n \setminus S$. Again, Lemmas 6.2.3 and 6.2.4 show that $\mathscr{F}_0(G) = S$.

That the groups constructed this way have finite dimensional models for their classifying spaces follows from the easy result below.

Lemma 6.2.5 *Let G be a finite free product $K_1 * \cdots * K_n$ where each K_i has a finite dimensional $\underline{E}K_i$. Then G also has a finite dimensional $\underline{E}G$.*

Proof Choose a G-tree T whose vertex set V is the disjoint union of the G-sets

$$K_i \backslash G := \{K_i g : \ g \in G\}$$

and so that G acts freely on the edge set E.

In order to prove Theorem 6.2.1 we shall make use of complete cohomology: we shall use Mislin's definition in terms of satellite functors. Let M be a $\mathbb{Z}G$-module. We write FM for the free module on the underlying set of non-zero elements of M. The inclusion

$$M \setminus \{0\} \to M$$

induces a natural surjection

$$FM \to M$$

whose kernel is written ΩM. Both F and Ω are functorial: for a map $\theta : M \to N$, the induced map $F\theta : FM \to FN$ carries elements $m \in M \setminus \ker\theta$ to their images $\theta m \in N$ and carries elements of $\ker\theta \setminus \{0\}$ to 0. The functor F is left adjoint to the forgetful functor from $\mathbb{Z}G$-modules to pointed sets which forgets everything save the set and zero. The advantage of working with F rather than simply using the free module on the underlying set of M is that it is 0-finitary. Our functor Ω inherits this property: it is also 0-finitary. We shall make use of these observations in proving Theorem 6.2.1. As in [25] the jth complete cohomology of G is given by the colimit:

$$\widehat{H}^j(G, M) := \varinjlim_n \ H^{j+n}(G, \Omega^n M).$$

Lemma 6.2.6 *If there is an m such that $H^j(G, F) = 0$ for all free modules F and all $j \geq m$ then the natural map*

$$H^j(G, -) \to \widehat{H}^j(G, -)$$

is an isomorphism for all $j \geq m + 1$.

Proof The connecting maps $H^{j+n}(G, \Omega^n M) \to H^{j+n+1}(G, \Omega^{n+1} M)$ in the colimit system defining complete cohomology are all isomorphisms because they fit into the cohomology exact sequence with $H^{j+n}(G, F\Omega^n M)$ and $H^{j+n+1}(G, F\Omega^n M)$ to the left and the right, and these both vanish for $j \geq m + 1$.

The next result makes use of the ring of bounded $\mathbb{Z}$-valued functions and some remarks are in order to explain why we might consider bounded functions in preference to arbitrary functions. Let G be a group. If $f : G \to \mathbb{Z}$ is a function and $g \in G$ then we may define f^g to be the function defined by $g' \mapsto f(gg')$. In this way the ring of functions becomes a (right) $\mathbb{Z}G$-module and the ring of bounded functions is a submodule. In group cohomology, the ring of *all* $\mathbb{Z}$-valued functions on G yields the *coinduced module* which is cohomologically acyclic and for this reason coinduced modules are useful in dimension-shifting arguments where their role is similar to but sometimes more transparent than that of injective modules. If the group is infinite, the coinduced module involves a subtlety: it is torsion-free as an abelian group, but *not free abelian*. The ring of bounded functions, even on an infinite set, is convenient because it is free abelian no matter what the cardinality of the set. The ring B of bounded $\mathbb{Z}$-valued functions with domain a group G yields a $\mathbb{Z}G$-module which retains at least some of the good properties of the coinduced module, in particular it contains the constant functions, while it also enjoys the useful property of having free abelian underlying additive group. The results we need are summarized as follows.

Theorem 6.2.7 *Let G be an* $\mathbf{LH}\mathfrak{F}$*-group for which the complete cohomology functors* $\widehat{H}^j(G, -)$ *are* 0*-finitary for all j. Then*

(a) The set B of bounded $\mathbb{Z}$-valued functions on G has finite projective dimension.
(b) If M is a $\mathbb{Z}G$-module whose restriction to every finite subgroup is projective then M has finite projective dimension: in fact

$$\operatorname{proj.dim} M \leq \operatorname{proj.dim} B.$$

(c) For all $n >$ proj. dim$B$, $H^n(G, -)$ vanishes on free modules.
(d) For all $n >$ proj. dim$B$, the natural map $H^n(G, -) \to \widehat{H}^n(G, -)$ is an isomorphism.
(e) $n \in \mathscr{F}_0(G)$ for all $n >$ proj. dim$B$.
(f) G has rational cohomological dimension $\leq$ proj. dim$B + 1$.
(g) There is a bound on the orders of the finite subgroups of G.
(h) There is a finite dimensional model for $\underline{E}G$.

Proof (Outline of the proof) Since $\widehat{H}^j(G, -)$ is finitary and G belongs to the class $\mathbf{LH}\mathfrak{F}$ we have the following algebraic result about the cohomology of G:

$$\widehat{H}^j(G, B) = 0 \text{ for all } j. \tag{6.1}$$

For $\mathbf{H}\mathfrak{F}$-groups of type FP_∞ this follows from ([9], Proposition 9.2) by taking the ring R to be $\mathbb{Z}G$ and taking the module M to be the trivial $\mathbb{Z}G$-module $\mathbb{Z}$. However we need to strengthen this result in two ways. Firstly we wish to replace the assumption that G is of type FP_∞ by the weaker condition that the functors $\widehat{H}^j(G, -)$ are 0-finitary for all j. This presents no difficulty because the proofs in [9] depend solely on calculations of complete cohomology rather than ordinary cohomology. The second problem is also easy to address but we need to take care. Groups of type FP_∞ are finitely generated and so $\mathbf{LH}\mathfrak{F}$-groups of type FP_∞ necessarily belong to $\mathbf{H}\mathfrak{F}$. However the weaker condition that the complete cohomology is finitary does not imply finite generation: for example, all groups of finite cohomological dimension have vanishing complete cohomology and there exists such groups of arbitrary cardinality. A priori we do not know that G belongs to $\mathbf{H}\mathfrak{F}$ and we must reprove the result that

$$\widehat{H}^*(G, B) = 0$$

from scratch. The key, which has been established [24] by Matthews, is as follows:

Lemma 6.2.8 *Let G be an group for which all the functors $\widehat{H}^j(G, -)$ are 0-finitary. Let M be a $\mathbb{Z}G$-module whose restriction to every finite subgroup of G is projective. Then*

$$\widehat{H}^j(G, M \otimes_{\mathbb{Z}H} \mathbb{Z}G) = 0$$

for all j and all $\mathbf{LH}\mathfrak{F}$-subgroups H of G.

Proof (Proof of Lemma 6.2.8) If H is an $\mathbf{H}\mathfrak{F}$-group then this can be proved by induction on the ordinal height of H in the $\mathbf{H}\mathfrak{F}$-hierarchy. The proof proceeds in exactly the same way as the proof of the Vanishing Theorem ([9], Sect. 8).

In general, suppose that H is an $\mathbf{LH}\mathfrak{F}$-group. Let (H_λ) be the family of finitely generated subgroups of H. Then we may view H as the filtered colimit $H = \varinjlim H_\lambda$.

Now suppose that G is as in the statement of Theorem 6.2.7. Lemma 6.2.8 shows that

$$\widehat{H}^0(G, B) = 0.$$

and using the ring structure on B it follows that

$$\widehat{\operatorname{Ext}}^0_{\mathbb{Z}G}(B, B) = 0.$$

This implies that B has finite projective dimension: see ([18], 4.2) for discussion and proof of the fundamental property of complete cohomology that a $\mathbb{Z}G$-module M has finite projective dimension if and only if $\widehat{\operatorname{Ext}}^0_{\mathbb{Z}G}(M, M) = 0$. Like the coinduced module, the module B contains a copy of the trivial module $\mathbb{Z}$ in the form of the constant functions. Thus Theorem 6.2.7(i) is established.

Let M be a module satisfying the hypotheses of (ii), namely that M is projective as a $\mathbb{Z}H$-module for all finite subgroups H of G. If B has projective dimension b then $\Omega^b M \otimes B$ is projective. We therefore replace M by $\Omega^b M$ and our goal is to prove that M is projective. We have reduced to the case when $M \otimes B$ is projective.

The proof that M is projective requires two steps. First we show that M is projective over $\mathbb{Z}H$ for all $\mathbf{H}\mathfrak{F}$-subgroups H of G. The argument here is essentially the same as that used to prove Theorem B of [9], using transfinite induction on the least ordinal α such that H belongs to $\mathbf{H}_\alpha\mathfrak{F}$. We consider first the case

$\alpha = 0$

This is the starting point of the inductive proof. Since $\mathbf{H}_0\mathfrak{F}$ is the class of finite groups and we are assuming that M is projective on restriction to every finite subgroup there is nothing to prove in this case. Next we consider the case

$\alpha > 0$

Let H be a subgroup of G which belongs to $\mathbf{H}_\alpha\mathfrak{F}$ and consider an action of H on a contractible finite dimensional complex X so that each isotropy subgroup belongs to $\mathbf{H}_\beta\mathfrak{F}$ with $\beta < \alpha$. Note that β may vary depending on the choice of istropy subgroup and so conceivably α is the least upper bound of the β which arise. The augmented cellular chain complex $C_* \twoheadrightarrow \mathbb{Z}$ of X is an exact sequence of finite length:

$$0 \to C_d \to C_{d-1} \to \cdots \to C_1 \to C_0 \to \mathbb{Z} \to 0$$

where d is the dimension of X. Each chain group C_i is a permutation module for H and therefore a direct sum of modules of the form $\mathbb{Z} \otimes_{\mathbb{Z}K} \mathbb{Z}H$ where K is a subgroup of H belonging to one of the classes $\mathbf{H}_\beta\mathfrak{F}$ with $\beta < \alpha$. Observe that the diagonal action of G on $M \otimes (\mathbb{Z} \otimes_{\mathbb{Z}K} \mathbb{Z}H)$ yields a module isomorphic to the induced module $M \otimes_{\mathbb{Z}K} \mathbb{Z}G$ and since M is, by the inductive hypothesis, projective over $\mathbb{Z}K$, therefore $M \otimes (\mathbb{Z} \otimes_{\mathbb{Z}K} \mathbb{Z}H)$ is projective over $\mathbb{Z}G$. thus $M \otimes C_i$ is a projective $\mathbb{Z}G$-module for each i and hence, on tensoring augmented cellular chain complex with M we obtain a projective resolution of M over $\mathbb{Z}G$:

$$0 \to M \otimes C_d \to M \otimes C_{d-1} \to \cdots \to M \otimes C_1 \to M \otimes C_0 \to M \to 0.$$

This shows that M has finite projective dimension. At this stage, the projective dimension of M appears to depend on the dimension d of the witness X. However, if we write $\overline{B}$ denote that quotient $B/\mathbb{Z}$ of B by the constant functions then we also see that $M \otimes \underbrace{\overline{B} \otimes \cdots \otimes \overline{B}}_{k}$ has a finite projective resolution for any $k \geq 0$ and always of length at most d. Note that here we use the fact that $\overline{B}$ is additively free abelian. The short exact sequence $\mathbb{Z} \rightarrowtail B \twoheadrightarrow \overline{B}$ gives rise to the short exact sequences

$$M \rightarrowtail M \otimes B \twoheadrightarrow M \otimes \overline{B}$$

$$M \otimes \overline{B} \rightarrowtail M \otimes \overline{B} \otimes B \twoheadrightarrow M \otimes \overline{B} \otimes \overline{B}$$

$$\vdots$$

$$M \otimes \underbrace{\overline{B} \otimes \cdots \otimes \overline{B}}_{k} \rightarrowtail M \otimes \underbrace{\overline{B} \otimes \cdots \otimes \overline{B}}_{k} \otimes B \twoheadrightarrow M \otimes \underbrace{\overline{B} \otimes \cdots \otimes \overline{B}}_{k+1}$$

Since M arises as a dth kernel in a projective resolution of $M \otimes \underbrace{\overline{B} \otimes \cdots \otimes \overline{B}}_{d}$ it follows that M itself is projective of $\mathbb{Z}H$. Concatenating these short exact sequences upto and including the case when $k + 1 = d$ we obtain a partial projective resolution of $M \otimes \underbrace{\overline{B} \otimes \cdots \otimes \overline{B}}_{d}$ in which M arises as a dth kernel. But since we know that $M \otimes \underbrace{\overline{B} \otimes \cdots \otimes \overline{B}}_{d}$ has projective dimension at most d, it follows that M is projective. This completes the inductive proof.

The $\mathbf{H}\mathfrak{F}$-subgroups of G account for all countable subgroups. The next step is to establish by induction on the cardinality κ that M is projective on restriction to all subgroups of G of cardinality κ. This argument can be found in the work [2] of Benson. In this way (ii) is established.

Part (iii) follows from the inequality

$$\operatorname{silp}(\mathbb{Z}G) \leq \kappa(\mathbb{Z}G)$$

as stated in Theorem C of [10]. Note that although ([10], Theorem C) is stated for $\mathbf{H}\mathfrak{F}$-groups, the given proof shows that the above inequality holds for arbitrary groups.

Lemma 6.2.6 yields (iv).

We are assuming that the complete cohomology is 0-finitary in all dimensions. Now we also know that the ordinary cohomology coincides with the complete cohomology in high dimensions. Hence (v) is established.

The trivial module $\mathbb{Q}$ is an instance of a module whose restriction to every finite subgroup has finite projective dimension, (projective dimension one in fact). Therefore the dimensional finiteness conditions imply (vi). This means in particular that

$$\widehat{H}^0(G, \mathbb{Q}) = 0.$$

Since the complete cohomology is 0-finitary we can deduce that $\widehat{H}^0(G, \mathbb{Z})$ is torsion. Being a ring with a one, it therefore has finite exponent, say m, and the argument with classical Tate cohomology used to prove ([18], Sect. 5, Proposition) shows that the orders of the finite subgroups of G must divide m and thus (vii) is established. The argument for proving (viii) can be found in [20]. Although the Theorem as stated there does not directly apply to our situation, a reading of the proof will reveal that the all the essentials to make the construction work are already contained in the conclusions (i)–(vii).

Proof (Proof of Theorem 6.2.1) We first show that the complete cohomology of G is 0-finitary in all dimensions. Recall that the jth complete cohomology of G is the colimit:

$$\widehat{H}^j(G, M) := \varinjlim_{n} H^{j+n}(G, \Omega^n M).$$

The maps $H^{j+n}(G, \Omega^n M) \to H^{j+n+1}(G, \Omega^{n+1} M)$ in this system are the connecting maps in the long exact sequence of cohomology which comes from the short exact sequence

$$\Omega^{n+1} M \rightarrowtail F\Omega^n M \twoheadrightarrow \Omega^n M.$$

Let $S = \{s \in \mathbb{N} : s + j \in \mathscr{F}_0(G)\}$. Since S is infinite, it is cofinal in $\mathbb{N}$. Hence

$$\widehat{H}^j(G, M) := \varinjlim_{s \in S} H^{j+s}(G, \Omega^s M).$$

Now, for any vanishing filtered colimit system (M_λ) of $\mathbb{Z}G$-modules we have

$$\begin{aligned}
\operatorname{colim} \widehat{H}^j(G, M_\lambda) &= \operatorname{colim} \varinjlim_{s \in S} H^{j+s}(G, \Omega^s M_\lambda) \\
&= \varinjlim_{s \in S} \operatorname{colim} H^{j+s}(G, \Omega^s M_\lambda) \\
&= \varinjlim_{s \in S} H^{j+s}(G, \operatorname{colim} \Omega^s M_\lambda) \\
&= 0.
\end{aligned}$$

Theorem 6.2.1 now follows from Theorem 6.2.7.

6.3 General Behaviour of Finitary Cohomology Functors

In this section we show how the finitary properties of one cohomology functor can influence neighbouring functors. Our arguments are based on an unpublished observation of Robert Snider. The first gives a further insight into the nature of the finite-cofinite dichotomy for the set $\mathscr{F}_0(G)$. It is a property held by many groups G that $H^n(G, -)$ vanishes on projective modules for all sufficiently large n. For example, we have the following.

Lemma 6.3.1 *If G belongs to $\mathbf{H}_1\mathfrak{F}$ and P is a projective $\mathbb{Z}G$-module then $H^n(G, P) = 0$ for all sufficiently large n.*

Proof Let X be a finite dimensional contractible G-complex to witness that G belongs to $\mathbf{H}_1\mathfrak{F}$: i.e. G acts on X with finite isotropy groups. Let d be the dimension of X. We show that $H^n(G, P) = 0$ for all $n > d$. Let

$$0 \to C_d \to C_{d-1} \to \cdots \to C_1 \to C_0 \to \mathbb{Z} \to 0$$

be the cellular chain complex of X. Then there is a first quadrant spectral sequence with $E_1^{p,q} := \operatorname{Ext}^q_{\mathbb{Z}G}(C_p, P)$ converging to $H^{p+q}(G, P)$. For each p, the chain group C_p is a direct sum $\bigoplus_\sigma \mathbb{Z} \otimes_{\mathbb{Z}G_\sigma} \mathbb{Z}G$ of induced modules where σ runs through a set of orbit representatives of p-cells in X. By using the Shapiro–Eckmann lemma we have

$$\mathrm{Ext}^q_{\mathbb{Z}G}(C_p, P) \cong \prod_\sigma H^q(G_\sigma, P).$$

Since the subgroups G_σ are all finite, it follows that $H^q(G_\sigma, -)$ vanishes on free and therefore also on projective modules for any $q > 0$. Therefore the spectral sequence collapses with $E_1^{p,q} = 0$ whenever $q > 0$. It follows that for all n, $H^n(G, P)$ is isomorphic to the nth homology of the cochain complex $\mathrm{Hom}_{\mathbb{Z}G}(C_*, P)$ and so is supported in the range 0 through to d.

When the conclusion of this lemma holds, there is a very simple proof that the finitary set is either finite or cofinite: it is a corollary of the following.

Lemma 6.3.2 *Let n be a positive integer. Suppose that G is a group such that*

(a) $H^{n-1}(G, -)$ vanishes on all projective $\mathbb{Z}G$-modules, and
(b) $H^n(G, -)$ is 0-finitary.

Then $H^{n-1}(G, -)$ is 0-finitary.

Proof Let F and Ω denote the free module and loop functors described in the proof of Theorem 6.2.1. Let (M_λ) be a vanishing filtered colimit system of $\mathbb{Z}G$-modules. From the short exact sequence

$$\Omega M_\lambda \to FM_\lambda \to M_\lambda$$

we obtain the long exact sequence

$$\cdots \to H^{n-1}(G, FM_\lambda) \to H^{n-1}(G, M_\lambda) \to H^n(G, \Omega M_\lambda) \to \cdots$$

Here the left hand group vanishes by hypothesis (i) and the right hand system vanishes on passage to colimit by hypothesis (ii). Hence

$$\mathrm{colim}\ H^{n-1}(G, M_\lambda) = 0$$

as required.

Thus, if G is a group for which the set

$$\{n : \ H^n(G, F) \text{ is non-zero for some free module } F\}$$

is bounded while the finitary set $\mathscr{F}_0(G)$ is unbounded, then the finitary set is cofinite.

We have seen that any finite or cofinite set can be realized as the 0-finitary set of some group. It is interesting to note that the existence of certain normal or near normal subgroups will impose some restrictions. The next lemma provides a way of seeing this.

Lemma 6.3.3 *Let G be a group and suppose that there is an overring $R \supset \mathbb{Z}G$ such that R is flat over $\mathbb{Z}G$ and $\mathbb{Z} \otimes_{\mathbb{Z}G} R = 0$. Let n be a positive integer. If both $H^{n-1}(G, -)$ and $H^{n+1}(G, -)$ are 0-finitary then $H^n(G, -)$ is also 0-finitary.*

Proof Let F and Ω denote the free module and loop functors described in the proof of Theorem 6.2.1. Let (M_λ) be a vanishing filtered colimit system of $\mathbb{Z}G$-modules. Then we have a short exact of vanishing filtered colimit systems:

$$\Omega M_\lambda \to FM_\lambda \to M_\lambda.$$

Applying the long exact sequence of cohomology and taking colimits we obtain the exact sequence

$$\text{colim } H^n(G, FM_\lambda) \to \text{colim } H^n(G, M_\lambda) \to \text{colim } H^{n+1}(G, \Omega M_\lambda).$$

Here we wish to prove that the central group is zero and we know that the right hand term is zero because $H^{n+1}(G, -)$ is 0-finitary. Therefore it suffices to prove that the left hand group is zero. Since the FM_λ are free we have the short exact sequence

$$FM_\lambda \to FM_\lambda \otimes_{\mathbb{Z}G} R \to (FM_\lambda \otimes_{\mathbb{Z}G} R)/FM_\lambda$$

of vanishing filtered colimit systems and hence we obtain an exact sequence

$$\text{colim } H^{n-1}(G, (FM_\lambda \otimes_{\mathbb{Z}G} R)/FM_\lambda) \to \text{colim } H^n(G, FM_\lambda) \to \text{colim } H^n(G, FM_\lambda \otimes_{\mathbb{Z}G} R).$$

We need to prove that the central group here is zero and we know that the left hand group vanishes because $H^{n-1}(G, -)$ is 0-finitary. Therefore it suffices to prove that the right hand group is zero. In fact it vanishes even before taking colimits: let F be any free $\mathbb{Z}G$-module and let $P_* \twoheadrightarrow \mathbb{Z}$ be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. Then $\text{Hom}_{\mathbb{Z}G}(P_*, F \otimes_{\mathbb{Z}G} R) \cong \text{Hom}_R(P_* \otimes_{\mathbb{Z}G} R, F \otimes_{\mathbb{Z}G} R)$ is split exact because R is flat over $\mathbb{Z}G$ and $\mathbb{Z} \otimes_{\mathbb{Z}G} R = 0$. Thus $H^*(G, F \otimes_{\mathbb{Z}G} R) = 0$.

For example, if G is a group with a non-trivial torsion-free abelian normal subgroup A then the lemma can be applied by taking R to be the localization $\mathbb{Z}G(\mathbb{Z}A \setminus \{0\})^{-1}$ and shows that for such groups there cannot be isolated members in the complement of the finitary set $\mathscr{F}_0(G)$. The condition that A is normal can be weakened and yet it can still be possible to draw similar conclusions. We conclude this paper with two further results showing how this can happen.

Two subgroups H and K of a group G are said to be commensurable if and only if $H \cap K$ has finite index in both H and K. We write $Comm_G(H)$ for the set $\{g \in G : \ H \text{ and } H^g \text{ are commensurable}\}$. This is a subgroup of G containing the normalizer of H.

Lemma 6.3.4 *Let G be a group with a subgroup H such that $Comm_G(H) = G$ and $\mathbb{Z}H$ is a prime Goldie ring. Then the set Λ of non-zero divisors in $\mathbb{Z}H$ is a right Ore* *set in $\mathbb{Z}G$. Moreover, if H is non-trivial, then the localization $R := \mathbb{Z}G\Lambda^{-1}$ satisfies the hypotheses of Lemma 6.3.3.*

Proof Before starting, recall that in a prime Goldie ring, the set of non-zero divisors is a right Ore set and the resulting Ore localization is a simple Artinian ring. We first prove that Λ is a right Ore set in $\mathbb{Z}G$. If H is normal in G then this is an easy and well known consequence of Λ being a right Ore set in $\mathbb{Z}H$. Now consider the general case. Let r be an element of $\mathbb{Z}G$ and let λ be an element of Λ. We need to find $\mu \in \Lambda$ and $s \in \mathbb{Z}G$ such that $r\mu = \lambda s$. Choose any way

$$r = g_1r_1 + \cdots + g_mr_m$$

of expressing r as a finite sum in which each r_i belongs to $\mathbb{Z}H$ and $g_i \in G$. Since all the subgroups $g_iHg_i^{-1}$ are commensurable with H we can choose a normal subgroup K of finite index in H such that

$$K \subseteq \bigcap_{i=1}^{m} g_iHg_i^{-1}.$$

The group algebra $\mathbb{Z}K$ inherits the property of being a prime Goldie ring and the set of non-zero divisors in $\mathbb{Z}K$ is

$$\Lambda_0 := \Lambda \cap \mathbb{Z}K.$$

By our initial remarks on the case of a normal subgroup, Λ_0 is a right Ore set in $\mathbb{Z}H$. Moreover, $\mathbb{Z}H\Lambda_0^{-1}$ is finitely generated over the Artinian ring $\mathbb{Z}K\Lambda_0^{-1}$. It follows *a fortiori* that $\mathbb{Z}H\Lambda_0^{-1}$ is Artinian as a ring and since every non-zero divisor in an Artinian ring is a unit, we conclude that

$$\mathbb{Z}H\Lambda_0^{-1} = \mathbb{Z}H\Lambda^{-1}.$$

Hence, there exists a t in $\mathbb{Z}H$ such that $\nu := \lambda t \in \Lambda_0$: to see this, simply choose an expression $t\nu^{-1}$ for $\lambda^{-1} \in \mathbb{Z}H\Lambda^{-1}$ in the spirit of the localization $\mathbb{Z}H\Lambda_0^{-1}$. For each i, we have $g_i^{-1}Kg_i \subseteq H$ and hence $g_i^{-1}\nu g_i \in \mathbb{Z}H$. It is straightforward to check that each $g_i\nu g_i^{-1}$ is a non-zero divisor in $\mathbb{Z}H$. Applying the Ore condition to the pair $r_i, g_i^{-1}\nu g_i$ we find $s_i \in \mathbb{Z}H$ and $\mu_i \in \Lambda$ such that

$$r_i\mu_i = g_i^{-1}\nu g_is_i.$$

It is routine that a finite list of elements in an Ore localization can be placed over a common denominator and it is therefore possible to make these choices so that the μ_i are all equal: we do this and write μ for the common element. Thus

$$r_i\mu = g_i^{-1}\nu g_is_i,$$

and

$$r\mu = \sum_i g_i r_i \mu = \sum_i g_i g_i^{-1} \nu g_i s_i = \nu \left(\sum_i g_i s_i \right) = \lambda t \left(\sum_i g_i s_i \right).$$

This establishes the Ore condition as required with $s = t\left(\sum_i g_i s_i\right)$.

Finally, assume H is non-trivial and let $\mathfrak{h}$ denote the augmentation ideal in $\mathbb{Z}H$. Then $\mathfrak{h}$ is non-zero and $\mathfrak{h}.\mathbb{Z}H\Lambda^{-1}$ is a non-zero two-sided ideal in the simple Artinian ring $\mathbb{Z}H\Lambda^{-1}$. Hence $\mathfrak{h}.\mathbb{Z}H\Lambda^{-1} = \mathbb{Z}H\Lambda^{-1}$ and $\mathbb{Z} \otimes_{\mathbb{Z}H} \mathbb{Z}H\Lambda^{-1} = 0$. It follows that $\mathbb{Z} \times_{\mathbb{Z}G} R = 0$ so R does indeed satisfy the hypotheses of Lemma 6.3.3.

The condition $Comm_G(H) = G$ has been studied by the author in cohomological contexts, see [17, 19]. The second paper [19] addresses a more general situation in which H is replaced by a set $\mathscr{S}$ of subgroups which is closed under conjugation and finite intersections: it is then shown one can define a cohomological functor $H^*(G/\mathscr{S}, -)$ on $\mathbb{Z}G$-modules and that spectral sequence arguments can be used to carry out certain calculations. Here we show, for the reader familiar with [19] how these arguments may be used to investigate when the new functors $H^*(G/\mathscr{S}, -)$ are finitary.

Lemma 6.3.5 *Let $\mathscr{S}$ and G be as above.*

(a) The functor $H^0(G/\mathscr{S},\quad)$ is 0-finitary.
(b) If G is finitely generated then the functor $H^1(G/\mathscr{S},\quad)$ is 0-finitary.
(c) More generally if n is an integer such that G has type FP_n and all members of $\mathscr{S}$ have type FP_{n-1} then the functors $H^i(G/\mathscr{S},\quad)$ are 0-finitary for all $i \le n$.

Proof We prove part (iii) by induction on n. The case $n = 0$ is easy: this is part (i) of the statement and the finitary property is inherited from ordinary cohomology. The case $n = 1$ is part (ii) of the statement and there is no need to treat this separately. Fix $n \ge 1$ and assume inductively that the result is established for numbers $< n$. In particular we may assume that $H^i(G/\mathscr{S},\quad)$ is 0-finitary when $i < n$.

Let (M_λ) be a vanishing filtered colimit system in the category Mod-$\mathbb{Z}G/\mathscr{S}$. Taking colimits of the spectral sequences of [19] we obtain the spectral sequence

$$E_2^{p,q} = \mathrm{colim}\ H^p(G/\mathscr{S}, H^q(\mathscr{S}, M_\lambda)) \implies \mathrm{colim}\ H^{p+q}(G, M_\lambda).$$

Now consider the cases when $p + q \le n$, $p \ge 0$, $q \ge 0$. When p is less than n, the inductive and originally stated finitary assumptions imply that $E_2^{p,q} = 0$ and so we have a block of zeroes on the E_2-page of the spectral sequence in the range $0 \le p \le n-1$ and $0 \le q \le n$. Therefore only the term $E_2^{n,0} = E_\infty^{n,0}$ and the cohomology colim $H^n(G, M_\lambda)$ is isomorphic to

$$\mathrm{colim}\ H^n(G/\mathscr{S}, H^0(\mathscr{S}, M_\lambda)).$$

But colim $H^n(G, M_\lambda)$ is zero by assumption and so

$$\operatorname{colim} H^n(G/\mathscr{S}, H^0(\mathscr{S}, M_\lambda)) = 0.$$

The M_λ were chosen in the subcategory so this simplifies to

$$\operatorname{colim} H^n(G/\mathscr{S}, M_\lambda) = 0.$$

This vanishing applies to any choice of system (M_λ) and thus $H^n(G/\mathscr{S},\quad)$ is 0-finitary as required.

6.4 Hamilton's Results

6.4.1 When Is Group Cohomology Finitary?

Hamilton [13] uses the results of this paper to characterize the locally (polycyclic-by-finite) groups cohomology almost everywhere finitary: these are shown to be precisely the locally (polycyclic-by-finite) groups with finite virtual cohomological dimension and in which the normalizer of every non-trivial finite subgroup is of type FP_∞. In particular this class of groups is subgroup closed. Note that the class of locally (polycyclic-by-finite) groups includes the class of abelian-by-finite groups and already, within the class of abelian-by-finite groups there are many interesting examples. The abelian group $\mathbb{Q}^+ \times C_2$ (a direct product of the additive group of rational numbers by the cyclic group of order 2 has almost all its cohomology functors infinitary. By contrast, the non-abelian extension of $\mathbb{Q}^+$ by C_2 is almost everywhere finitary even though it is infinitely generated. Hamilton finds that in general, the locally (polycyclic-by-finite) groups which have almost all cohomology functors finitary form a subgroup closed class.

In view of our Theorem 6.2.1, Hamilton naturally focusses on groups with finite virtual cohomological dimension. He shows, for example, that if G is a group with finite vcd and then G has cohomology almost everywhere finitary over the field $\mathbb{F}_p$ of p elements if and only if G has finitely many conjugacy classes of elementary abelian p-subgroups and the centralizer of each non-trivial elementary abelian p-subgroup is of type FP_∞ over $\mathbb{F}_p$.

Clearly, a natural question is whether one can generalize Hamilton's results from locally (polycyclic-by-finite) groups to other classes of soluble groups. One of the main reasons why this appears hard is that there is no clear classification of which soluble groups have type FP_∞ over a given finite field. There are satisfactory theories of soluble groups of type FP_∞ over $\mathbb{Q}$ and over $\mathbb{Z}$ but these have yet to be generalized to the case of finite fields. Hamilton's proofs make use of the deep results [14] of Henn and in particular this leads to an answer to a question raised by Leary and

Nucinkis [16], namely he shows that if G is a group of type VFP over $\mathbb{F}_p$, and P is a p-subgroup of G, then the centralizer $C_G(P)$ of P is also of type VFP over $\mathbb{F}_p$.

The most relevant questions raised by this research are as follows:

Question 6.4.1 Let G be a soluble group and let p be a prime. What are the homological and cohomological dimensions of G over $\mathbb{F}_p$? Is there a simple criterion for G to have type FP_∞ over $\mathbb{F}_p$.

One may expect soluble groups to behave similarly over $\mathbb{F}_p$ as they do over $\mathbb{Q}$ with the obvious elementary caveat that one has to take care of p-torsion. However, there is no detailed account in the literature: Bieri's notes confine analysis to the characteristic zero case subsequent authors have studied this case alone in depth.

Finally, in this paper, Hamilton proves a more general result for groups that admit a finite dimensional classifying space for proper actions. He concludes that if G is such a group and if there are just finitely many conjugacy classes of non-trivial finite subgroups for each of which the corresponding centralizers have cohomology almost everywhere finitary, then G itself has cohomology almost everywhere finitary.

Hamilton uses results [21] of Leary to show that the converse of this result fails. Leary has constructed groups of type FP_∞ which are of type VFP but which have infinitely many conjugacy classes of finite subgroups.

6.4.2 Eilenberg–Mac Lane Spaces

In a second paper [12], Hamilton studies the question of whether the property *almost everywhere finitary* impacts on the Eilenberg–Mac Lane space of a group. Hamilton's main result [12, Theorem A] includes the statement that a group G in the class $\mathbf{LH}\mathfrak{F}$ has cohomology almost everywhere finitary if and only if $G \times \mathbb{Z}$ (the direct product of G with an infinite cyclic group) admits an Eilenberg–Mac Lane space with finitely many n-cells for all sufficiently large n. There are two natural questions arising from this research.

Question 6.4.2 Does Hamilton's [12, Theorem A] hold for arbitrary groups, outwith the class $\mathbf{LH}\mathfrak{F}$?

Question 6.4.3 Can Hamilton's [12, Theorem A] be proved without the stabilization device of replacing G by $G \times \mathbb{Z}$?

It is natural so speculate that both of these questions have a positive answer but they remain open.

6.4.3 Group Actions on Spheres

In a third paper [11], Hamilton builds on [13] by showing that in a locally (polycyclic-by-finite) group with cohomology almost everywhere finitary, every finite subgroup admits a free action on some sphere. This perhaps surprising fact is proved purely algebraically by showing that the same algebraic restrictions apply to the finite subgroups in Hamilton's context as apply in the theory of group actions on spheres, namely that subgroups of order a product of two (not necessarily distinct) primes must be cyclic. So the natural questions that arise are:

Question 6.4.4 Is there a geometric explanation for the connection between Hamilton's ([11], Theorem 1.3) which explains the link with group actions on spheres? Are there similar results for a larger class of groups, for example, soluble groups of finite rank.

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