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AN OPEN COLLAR THEOREM FOR 4-MANIFOLDS

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ABSTRACT. Let M^4 be an open 4-manifold with boundary. Conditions are given under which M^4 is homeomorphic to $\partial M \times [0, 1)$. Applications include a 4-dimensional weak h-cobordism theorem and a classification of weakly flat embeddings of 2-spheres in S^4. Specific examples of $(n-2)$-spheres embedded in S^n (including $n = 4$) are also discussed.

1. INTRODUCTION

This paper contains results in the area of 4-dimensional manifolds. The answers to the questions considered here have been known (or at least well understood) for several years in all other dimensions. As might be expected of results of this type, the work of Michael Freedman plays an essential role.

Theorem 3.3 may be considered the main result of the paper. It is an extension to dimension four of a result due to L. C. Siebenmann [31]. This result, "the open collar theorem," gives conditions under which an m-manifold M ($m \geq 5$) is homeomorphic to $\partial M \times [0, 1)$. The 4-dimensional version given here requires an extra hypothesis involving allowable fundamental groups; a fact that will not surprise those familiar with recent results in 4-dimensional topology. It should also be noted here that, like most recent 4-dimensional results, the conclusion is topological as opposed to PL or smooth.

In §4 we examine an embedding problem for 2-spheres in 4-dimensional space. The problem, originally motivated by a conjecture of Siebenmann's appearing in his "open collars" paper, was solved for 1-spheres in S^3 by R. J. Daverman [10] in 1973, and for $(n-2)$-spheres in S^n ($n \geq 5$) by T. B. Rushing and J. G. Hollingsworth [23] in 1976. These results give necessary and sufficient conditions for a codimension 2 sphere in S^n to have a complement the same as that of the standard $(n-2)$-sphere in S^n. An embedded $(n-2)$-sphere with this property is said to be "weakly flat." Theorem 4.3 makes use of the open collar theorem to extend this characterization to the case $n = 4$. The technique of proof used here is also valid for $n > 4$. This points out the

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surprising fact that the paper by Siebennman, which motivated much of the work on weakly flat codimension 2 spheres, virtually contains a solution when the ambient space is of dimension five or greater. This observation will be made more precise when we prove the result.

Section 5 contains some concrete examples of codimension 2 embeddings of spheres in S^n which illustrate the necessity and independence of the hypotheses in the characterization mentioned above. Besides lending credibility to our dimension four characterization, these examples seem to fill a void in the earlier work where very few examples are considered.

In §6 we mention a few applications of the characterization theorem for weakly flat 2-spheres in S^4; discuss an open question; and report on some recent results.

2. Preliminaries

Throughout this paper the symbols $\approx$, $\simeq$, and $\cong$ will denote homeomorphism, homotopy equivalence, and isomorphism, in that order. When we let a superscripted capital letter (e.g., M^n) denote a manifold, the superscript will represent the dimension of M^n. Thereafter M^n will often be referred to simply as M, with the dimension understood. The term *open manifold* will mean a noncompact manifold. For us the term manifold means "manifold possibly with boundary."

When dealing with a noncompact manifold M we will often talk about the *ends* of M. Though our definitions are standard, we repeat them for completeness.

2.1. **Definition.** Let M be a manifold and let ε be a collection of subsets of M such that

(i) each $g \in \varepsilon$ is a connected nonempty open set with compact boundary in M,
(ii) if $G, G' \in \varepsilon$ there exists $G'' \in \varepsilon$ with $G'' \subset G \cap G'$, and
(iii) $\bigcap \mathrm{cl}(G) = \varnothing$.

Now add to ε every nonempty connected open set $H \subset M$ with compact boundary with the property that H contains some element of ε. The new collection still satisfies (i)–(iii) and is called the *end determined by* ε.

A *neighborhood of* ε is any set containing an element of ε. An end ε is *collared* if it has a manifold neighborhood N such that $N \approx \partial N \times [0, 1)$.

We will often wish to consider "the fundamental group of an end ε." We say π_1 *is stable at* ε if there is a sequence $\{X_i\}$ of path connected neighborhoods of ε with $X_1 \supset X_2 \supset X_3 \supset \cdots$, $\bigcap X_i = \varnothing$, and such that the sequence

$$\pi_1(X_1, x_1) \overset{f_1}{\leftarrow} \pi_1(X_2, x_2) \overset{f_2}{\leftarrow} \pi_1(X_3, x_3) \overset{f_3}{\leftarrow} \cdots$$

induces a sequence of isomorphisms;

$$\mathrm{image}(f_1) \overset{f_1}{\leftarrow} \mathrm{image}(f_2) \overset{f_2}{\leftarrow} \mathrm{image}(f_3) \overset{f_3}{\leftarrow} \cdots .$$

To define $f_i \colon \pi_1(X_{i+1}, x_{i+1}) \to \pi_1(X_i, x_i)$, first choose a path α_i in X_i from x_i to x_{i+1}, and let g_i be the isomorphism $\pi_1(X_i, x_{i+1}) \to \pi_1(X_i, x_i)$ induced by α_i. Then f_i is the composition

$$\pi(X_{i+1}, x_{i+1}) \to \pi_1(X_i, x_{i+1}) \overset{g_i}{\to} \pi_1(X_i, x_i),$$

where the first homomorphism is induced by inclusion. If π_1 is stable at ε, define $\pi_1(\varepsilon)$ to be the inverse limit of the sequence $\{\pi_1(X_i, x_i), f_i\}$. This is isomorphic to image(f_i) for any i. It can be shown without much trouble (see [30]) that these definitions are independent of the sequence $\{X_i\}$ chosen.

In accordance with common practice, when a manifold has only one end we will refer to that end simply as "infinity" or ∞.

Another important concept, when dealing with open manifolds (or noncompact spaces in general), is that of "properness." A map $f: X \to Y$ is *proper* if $f^{-1}(K)$ is compact for any compact K. A map $h: X \to Y$ is a *proper homotopy equivalence* if not only h is proper, but all homotopies involved can be chosen to be proper. A cobordism (W, M_0, M_1) is a *proper h-cobordism* provided $M_0 \subset W$ and $M_1 \subset W$ are proper homotopy equivalences.

A key ingredient in our proof of the 4-dimensional open collar theorem will be a 5-dimensional proper s-cobordism theorem. Instead of spending a lot of time describing the rather complicated machinery needed to give a general statement of the theorem, we will state a very special case, tailored to meet our specific needs. Other versions will appear in [20]. A very nice development by Siebenmann for dimensions ≥ 6, much of which is now applicable in dimension five, can be found in [32].

The main problem unique to dimension five (as opposed to higher dimensional) cobordism theorems is the need for fundamental group restrictions. This comes about because of the difficulty in finding Whitney disks in 4-manifolds. Freedman's disk embedding lemma (see [19 or 20]), the heart of his monumental work in 4-manifold theory, addresses this issue. We will say that a group G is a *Freedman group* if the disk embedding lemma can be proved for 4-manifolds M with $\pi_1(M) \cong G$. At this time it is known that all poly-(finite or cyclic) groups are Freedman. So far there are no groups which are known not to be Freedman (see principal question in [20]).

Let W be a manifold with M_0, M_1 disjoint submanifolds (with boundary) of ∂W. Note that $Y = \text{cl}(\partial W - (M_0 \cup M_1))$ is a cobordism between ∂M_0 and ∂M_1. We call (W, M_0, M_1) a relative h-cobordism provided $M_0 \subset W$, $M_1 \subset W$, $\partial M_0 \subset Y$, and $\partial M_1 \subset Y$ are all homotopy equivalences.

2.2. **Theorem** (Special case of the 5-dimensional proper s-cobordism theorem). *Let (W^5, M_0, M_1) be a proper relative h-cobordism such that*

(1) $Y \approx \partial M_0 \times I$,
(2) $M_0 \approx \partial M_0 \times [0, 1)$, *and*
(3) $\pi_1(M_0)$ *is a Freedman group.*

Then W is homeomorphic (rel boundary) to $M_0 \times I$.

Remarks. (i) This result is also true for $\dim(W) > 5$ without any need for condition (3) and with the final homeomorphism being PL (resp. smooth) if all of the other information is PL (resp. smooth). In dimension 5, the final homeomorphism is simply topological, even if all other information is PL or smooth.

(ii) The proof of Theorem 2.2 best understood by this author involves a careful modification of Siebenmann's higher dimensional Proper s-Cobordism Theorem. This is the same strategy used by Freedman in §10 of [18] to prove the simply-connected version. In fact a good starting point for proving Theorem

2.2 is an understanding of both [32] and §10 of [18]. The new issues which must be faced are of course due to the existence of π_1. In general, we need to work harder to achieve desired π_1 conditions for neighborhoods of infinity. Then, of course, the assumption that $\pi_1(M_0)$ (and thus $\pi_1(\infty)$ and any of its subgroups) is Freedman is essential for completing the proof.

(iii) Those unfamiliar with [32] may be surprised that Whitehead groups play no role in Theorem 2.2. This occurs primarily because our hypotheses ensure the surjectivity of π_1 (end of $M_0) \to \pi_1(M_0)$, which in turn allows us to "push torsion problems off to infinity." More precisely, the niceness of π_1 at infinity guarantees the triviality of Siebenmann's τ' obstruction. See [32] for details.

3. A 4-DIMENSIONAL OPEN COLLAR THEOREM

In his paper "On detecting open collars" [31], Siebenmann proves the following theorem.

3.1. **Theorem.** *Let M be an m-manifold where $m \geq 5$. Then M is homeomorphic to $\partial M \times [0, 1)$ iff each of the following conditions holds*:

(a) $\partial M \subset M$ *is a homotopy equivalence.*
(b) π_1 *is stable at infinity with* $\pi_1(\infty) \to \pi_1(M)$ *an isomorphism.*

Remark. We will see later that condition (a) guarantees that M has precisely one end.

The same result has been proved in dimension three provided one assumes that M contains no fake 3-cells and that $\pi_1(M) \neq Z_2$. This result is a consequence of the dimension three "finding a boundary theorem" of Husch and Price [24] when ∂M is compact, and by work of E. M. Brown and T. W. Tucker [5] if ∂M is noncompact. If a fake 3-sphere N exists, we can find a counterexample to the general statement as follows; let B be a closed ball in N and $p \in N$ such that $p \notin B$. Then $N - (\text{int}(B) \cup \{p\})$ is a counterexample to the 3-dimensional open collar theorem, as stated for higher dimensions.

The need to include condition (b) in these theorems may not be obvious, but it is essential. Removing a Whitehead continuum from the interior of a closed 3-ball produces a manifold M with $\partial M \subset M$ a homotopy equivalence, but which is not homeomorphic to $\partial M \times [0, 1)$ (see [27]). A quick way to obtain similar examples in higher dimensions is to take a product of M with any closed manifold.

An easily proved and aesthetically pleasing consequence of Theorem 3.1 is the following.

3.2. **Theorem.** *An m-manifold M $(m \geq 5)$ is homeomorphic to $\partial M \times [0, 1)$ iff the two are proper homotopy equivalent.*

Besides being a nice characterization of certain manifolds, the open collar theorem has proven useful in solving a variety of problems. Siebenmann's original paper contains three sections of applications. Section 4 of this paper gives a prime example of the ways in which this theorem can be used.

We are now ready to state the theorem.

3.3. **Theorem** (4-dimensional open collar theorem). *Let M be a 4-manifold with $\pi_1(M)$ a Freedman group. Then M is homeomorphic to $\partial M \times [0, 1)$ iff* *each of the following conditions holds*:

(a) $\partial M \subset M$ *is a homotopy equivalence.*
(b) π_1 *is stable at infinity with* $\pi_1(\infty) \to \pi_1(M)$ *an isomorphism.*

Note. As is the case with Theorem 3.1, ∂M can be noncompact and its ends need not satisfy any special conditions.

Like the higher dimensional case we get the following consequence.

3.4. **Corollary.** *A* 4*-manifold* M *with* $\pi_1(M)$ *a Freedman group is homeomorphic to* $\partial M \times [0, 1)$ *iff the two are proper homotopy equivalent.*

Another easy corollary is the following.

3.5. **Corollary** (weak 4-dimensional h-cobordism theorem). *Let* (W^4, M_0, M_1) *be a* 4*-dimensional h-cobordism* (*assume proper if* W^4 *is not compact*) *with* $\pi_1(M_0)$ *Freedman. Then* $W^4 - M_1 \approx M_0 \times [0, 1)$.

Before beginning the proof of Theorem 3.3, we note that when $\dim(M) \geq 6$ and ∂M is compact (or at least has collarable ends), the open collar theorem is a consequence of Siebenmann's work with placing boundaries on open manifolds (see [30]). One of the main surprises in his open collar theorem is the dimension five case. It gives some special situations in which a boundary can be placed on an open 5-manifold. (Even with Freedman-Quinn technology this result cannot be obtained in its full generality with the usual handle theoretic techniques.) Similarly, Theorem 3.3 is unusual in that it describes situations in which a boundary can be placed on an open 4-manifold.

In dimension four, there are two roadblocks to generalizing Siebenmann's higher dimensional proof. The first is that his well-known procedure for finding "1-neighborhoods of infinity" fails in dimension four. (A neighborhood U of infinity is a 1-neighborhood provided it is a closed, connected manifold neighborhood of infinity with connected, bicollared boundary such that the natural homomorphism $\pi_1(\infty) \to \pi_1(U)$ is an isomorphism and such that $\partial U \subset U$ induces a π_1-isomorphism.) The second difficulty is encountered in an engulfing step. Since one cannot necessarily engulf 2-dimensional polyhedra in a 4-manifold (even if it was possible to arrange the desired homotopy conditions), it is impossible to employ an engulfing trick of Stallings' which is key in Siebenmann's proof.

The first problem can be solved satisfactorily by finding "almost nice" pairs of neighborhoods of infinity. Lemma 3.6 gives an indication of what this might mean.

The second problem is not so easily solved. We end up sidestepping the issue by using a rather indirect proof. First we will build a proper h-cobordism between M and $\partial M \times [0, 1)$. This requires some work. Then we apply Theorem 2.2 (where applicable) to guarantee the promised homeomorphism. Interestingly, much of our time is now spent in a 5-dimensional world, where we will engulf lots of 2-dimensional objects! See Lemmas 3.7 and 3.8 and Theorem 3.9 for this portion of the proof.

We begin by proving four crucial lemmas and a preliminary theorem. Although we are primarily interested in the 4-dimensional cases, we will prove these results in greater generality when possible.

3.5. **Lemma.** *Suppose M is a manifold and $\partial M \subset M$ is a homotopy equivalence. Then M has precisely one end.*

Proof. Choose a sequence $K_1 \subset K_2 \subset K_3 \subset \cdots$, of compact subsets of M such that $\bigcup K_i = M$. Note that M clearly cannot be compact, so no K_i can be all of M. We break the rest of the proof into two cases.

Case 1. (M is orientable.) Look at the long exact sequence

$$0 \to \underset{\substack{\| \\ Z}}{H^0(M)} \to H^0(M - K_i) \to H^1(M, M - K_i) \to \cdots.$$

Taking the direct limit over the K_i's gives

$$0 \to Z \to H^0_\infty(M) \to H^1_c(M) \to \cdots.$$

By duality [33, p. 342], $H^1_c(M) \cong H_{n-1}(M, \partial M) = 0$; therefore, $H^0_\infty(M)$ has rank 1, which implies that M has one end.

Case 2. (M is nonorientable.) Let $p: \widetilde{M} \to M$ be the orientable double cover of M, and let $\widetilde{A}$ represent $p^{-1}(A)$ for any $A \subset M$. Then $\{\widetilde{K}_i\}$ is a collection of compact sets such that $\bigcup \widetilde{K}_i = \widetilde{M}$, and $\partial \widetilde{M} \subset \widetilde{M}$ is a homotopy equivalence. By Case 1, $\widetilde{M}$ has exactly one end, so there exists an $n_0 \in Z_+$ such that for any $n \geq n_0$, $\widetilde{M} - \widetilde{K}_n$ has precisely one noncompact component. Then $M - K_n$ will have just one noncompact component; therefore, M has just one end.

3.6. **Lemma.** *Suppose M^m $(m \geq 4)$ is a manifold satisfying conditions* (a) *and* (b) *of Theorem* 3.1 *or* 3.3. *Then there exist arbitrarily small* (*open*) *neighborhoods of infinity U_1 and U_2 such that each of the following are satisfied.*

(i) $\mathrm{cl}(U_2) \subset U_1$.
(ii) $\pi_k(M, U_j) = 0$ *for* $k = 0$ *or* 1 *and* $j = 1$ *or* 2.
(iii) *The natural homorphism $\pi_2(M, U_2) \to \pi_2(M, U_1)$ is trivial.*

Proof. Since the conclusion is a trivial consequence of Theorem 3.1 for all other cases, we assume $m = 4$. Using Lemma 3.5 and hypothesis (b) of Theorem 3.3, we can choose an arbitrarily small neighborhood U_1 of ∞ satisfying condition (ii). Next consider the 5-manifold $M \times S^1$. Note that all hypotheses of Theorem 3.1 are present, implying that $M \times S^1 \approx \partial(M \times S^1) \times [0, 1)$. This makes it easy to find a neighborhood V of infinity in $M \times S^1$ (choose $V \approx \partial(M \times S^1) \times [0, 1)$) such that $\mathrm{cl}(V) \subset U_1 \times S^1$ and $\pi_2(M \times S^1, V) = 0$. Let $p: M \times S^1 \to M \times \{*\}$ denote the projection map, where $* \in S^1$. Choose U_2, a neighborhood of infinity in M, satisfying condition (ii) and sufficiently small that $U_2 \times \{*\} \subset V$. We need only verify (iii) for the pair U_1, U_2. Let $\alpha: I^2$ into M such that $\alpha(\partial I^2) \subset U_2$. Considering α for a moment to represent an element of $\pi_2(M \times S^1, V)$, we know that there is a homotopy $H_t: I^2 \to M \times S^1$ such that $H_0 = \alpha$, $H_t(\partial I^2) \subset V$ for all t, and $H_1(I^2) \subset V$. Then pH_t is a homotopy with $pH_0 = \alpha$, keeping ∂I^2 inside of $p(V) \subset U_1$, and such that $pH_1(I^2) \subset V \subset U_1$. Thus, α represents a trivial element of $\pi_2(M, U_1)$, and the lemma is proved.

We pause now to set up some notation and terminology which will be used in both the statements and the proofs of the following two lemmas. M will denote a PL manifold with the property that $\partial M \to M$ is a homotopy equivalence, and such that condition (b) of Theorem 3.1 or 3.3 holds. $\widehat{M}$ will always denote $M \times (0, 1)$ and if A is any subset of M then $\widehat{A}$ will denote $A \times (0, 1) \subset \widehat{M}$. An open collar C, on ∂M, will be called *nice* if $\mathrm{cl}(C) \approx \partial M \times [0, 1]$.

Important note. Work by Quinn [28 or 20] guarantees that a PL structure can be placed on any noncompact 4-manifold; therefore, in the following results, the PL assumption is superfluous when $n = 4$.

3.7. **Lemma.** *Let* M^m $(m \geq 4)$ *be as stated above,* C *be a nice open collar on* ∂M *in* M, U_1 *and* U_2 *be neighborhoods of infinity satisfying* (i)–(iii) *of Lemma* 3.6, *and* $P = (M - C) \times \{1/2\} \subset \widehat{M}$. *Then there is a homeomorphism* $h: \widehat{M} \to \widehat{M}$ *satisfying the following three conditions*:

(a) h *has compact support,*

(b) $h|_{\partial \widehat{M}} = \mathrm{id}$, *and*

(c) $h(\widehat{U}_1) \supset P$.

Proof. Choose another nice open collar C' on ∂M, with $C' \subset \mathrm{cl}(C') \subset C$. Let $N = (M - C') \times [1/4, 3/4]$ be a closed regular neighborhood of P with a triangulation T (see Figure 1). By Stallings engulfing [34], there is an engulfing homeomorphism $h_1: \mathrm{int}(\widehat{M}) \to \mathrm{int}(\widehat{M})$ with compact support and such that $h_1(\widehat{U}_1) \supset |T^{(2)}|$. Clearly we can extend h_1 to the identity on $\partial \widehat{M}$.

Now let T_0 be a finite subcomplex of T such that $|T_0| \supset N \cap (\widehat{M} - \widehat{U}_0)$, where U_0 is a neighborhood of infinity in M sufficiently small that $h_1|_{\widehat{U}_0} = \mathrm{id}$. Use Bing's radial engulfing [3] to obtain a homeomorphism $h_2: \widehat{M} - \mathrm{cl}(C') \to \widehat{M} - \mathrm{cl}(C')$ with compact support, such that $h_2(\widehat{C} - \mathrm{cl}(\widehat{C}'))$ contains the dual $(n-3)$-skeleton of T_0, and such that h_2 moves points a distance of less than $1/8$ in the "vertical direction." The control function called for in radial engulfing is

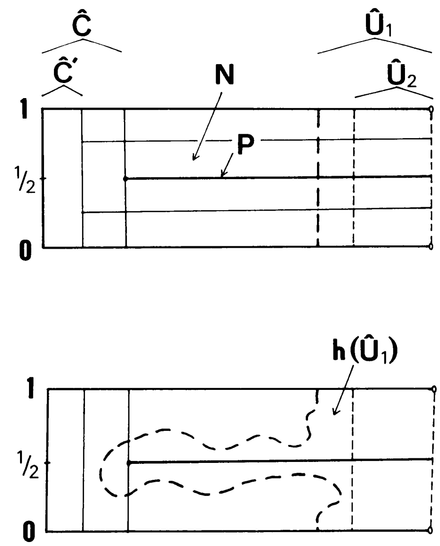

FIGURE 1. Result of applying Lemma 3.7

simply the projection $f: \widehat{M} = M \times (0, 1) \to (0, 1)$. Extend h_2 trivially, to get a homeomorphism of all of $\widehat{M}$ onto itself.

Finally, alter h_1 slightly to expand $h_1(\widehat{U})$ along the join structure of T, so that $h_1(\widehat{U}) \cup h_2(\widehat{C}) \supset N$. This implies that $h_2^{-1}h_1(\widehat{U}_1) \cup \widehat{C} \supset h_2^{-1}(N)$. Futhermore, the contols used in obtaining h_2 guarantee that $h_2^{-1}(N)$ contains P; therefore, $h_2^{-1}h_1(\widehat{U}_1)$ contains P, and letting $h = h_2^{-1}h_1$ completes the proof.

3.8. **Lemma.** *Again let M^m $(m \geq 4)$ be as stated above. Let C_1 and C_2 be nice open collars on ∂M such that $\mathrm{cl}(C_1) \subset C_2$. Let U be a connected neighborhood of infinity in M, and let*

$$V = \widehat{U} \cup [(M - \mathrm{cl}(C_1)) \times (0, 1/j)] \cup [(M - \mathrm{cl}(C_1)) \times ((j-1)/j, 1)],$$

where j is any integer larger than 2. Then there exists a homeomorphism $h: \widehat{M} \to \widehat{M}$ such that

(a) *h has compact support,*

(b) *$h|_{\partial \widehat{M}} = \mathrm{id}$, and*

(c) *$h(V) \cup \widehat{C}_2 = \widehat{M}$.*

Proof. Begin by choosing another nice open collar C_0 on ∂M, such that $C_1 \subset \mathrm{cl}(C_1) \subset C_0 \subset \mathrm{cl}(C_0) \subset C_2$. Let T be a triangulation of $\widehat{M} - \widehat{C}_0$, and note that $|T| - V$ is contained in a finite subcomplex of T (see Figure 2). Our first goal is to engulf $|T^{(2)}|$ with V.

Claim. $\pi_k(\widehat{M}, V) = 0$ for $k = 0, 1, 2$.

The $k = 0$ case is trivial. We will use the following long exact sequence for the other two cases.

$$\cdots \to \pi_2(V) \xrightarrow{\lambda_2} \pi_2(\widehat{M}) \to \pi_2(\widehat{M}, V) \to \pi_1(V) \xrightarrow{\lambda_1} \pi_1(\widehat{M}) \to \pi_1(\widehat{M}, V) \to 0.$$

It is clear that both λ_1 and λ_2 are surjective, thus $\pi_1(\widehat{M}, V)$ is trivial, telling us that we need only show λ_1 to be injective in order to get $\pi_2(\widehat{M}, V) = 0$. To do this, choose a loop β in V, based at $* \in [M - \mathrm{cl}(C_1)] \times ((j-1)/j, 1)$, which contracts in $\widehat{M}$. As in the previous lemma we use hypothesis (b) of Theorems 3.1 and 3.3 to see that $\pi_1(M, U) = 0$. This allows us to pull the portion of β lying below the $(1/j)$-level into $\widehat{U}$, while staying in V. Now push β straight upward into $[M - \mathrm{cl}(C_1)] \times ((j-1)/j, 1)$, where β must contract since $[M - \mathrm{cl}(C_1)] \times ((j-1)/j, 1)$ includes into $\widehat{M}$ as a homotopy equivalence. This finishes the claim.

We now have the necessary conditions to apply Stallings engulfing to obtain a homeomorphism with compact support, $h_1: \widehat{M} - \mathrm{cl}(\widehat{C}_1) \to \widehat{M} - \mathrm{cl}(\widehat{C}_1)$, such that $h_1(V) \supset |T^{(2)}|$. Extend h_1 via the identity to $\mathrm{cl}(\widehat{C}_1)$.

Next we go to the "other side" of $\widehat{M}$ to do some more engulfing. This time we will work in $\widehat{M} - \mathrm{cl}(\widehat{C}_0)$ and our engulfing set will be $W = \widehat{C}_2 - \mathrm{cl}(\widehat{C}_0)$. Note that the inclusion $W \subset \widehat{M} - \mathrm{cl}(\widehat{C}_0)$ is a homotopy equivalence. Let T_0 be a finite subcomplex of T such that $|T_0| \supset [[\widehat{M} - (\mathrm{cl}(\widehat{C}_0) \cup V)] \cup \mathrm{support}(h_1)]$. Let Γ denote the dual $(n-3)$-skeleton of T_0. Then $K = |\Gamma| \cap (\widehat{M} - \mathrm{cl}(\widehat{C}_0))$ is a (not necessarily compact) $(n-3)$-polyhedron in $\widehat{M} - \mathrm{cl}(\widehat{C}_0)$ such that $K - W$

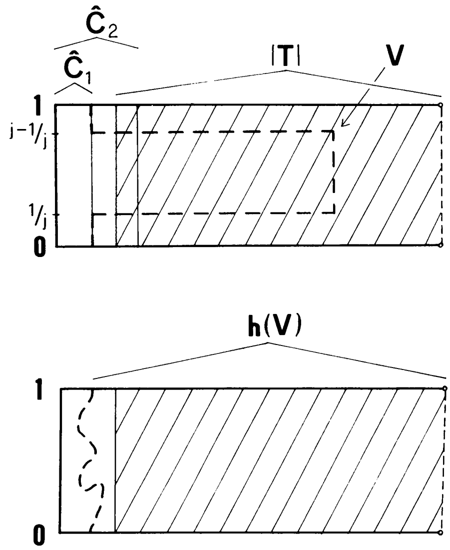

FIGURE 2. Result of applying Lemma 3.8

has compact closure in $\widehat{M} - \mathrm{cl}(\widehat{C}_0)$. Then, again by Stallings engulfing there is a homeomorphism $h_2\colon \widehat{M} - \mathrm{cl}(\widehat{C}_0) \to \widehat{M} - \mathrm{cl}(\widehat{C}_0)$, with compact support, such that $h_2(W) \supset K$. As usual, extend h_2 to the remainder of $\widehat{M}$ via the identity. Note that $h_2(\widehat{C}_2) \supset |\Gamma|$. By pushing along the join structure we can assume that $h_1(V) \cup h_2(\widehat{C}_2) = \widehat{M}$; therefore $h_2^{-1}h_1(V) \cup \widehat{C}_2 = \widehat{M}$. Letting $h = h_2^{-1}h_1$ completes the proof.

3.9. **Theorem.** *Let* M^m *be a* PL *manifold* $(m \geq 4)$ *such that*

(i) $\partial M \subset M$ *is a homotopy equivalence, and*

(ii) π_1 *is stable at infinity with* $\pi_1(\infty) \to \pi_1(M)$ *an isomorphism.*

Then there exists a proper PL *h-cobordism* (W^{m+1}, M_0, M_1) *with* $M_0 \approx \partial M \times [0, 1)$ *and* $M_1 \approx M$.

Proof. Let A be a nice open collar on ∂M, in M, and let $W = (M \times (0, 1]) \dot{\cup} (A \times \{0\})$, $M_0 = A \times \{0\}$, and $M_1 = M \times \{1\}$. Clearly we have an h-cobordism. Our job is to show that the required deformations of W onto its boundary components can be made proper. The general strategy is to choose collars on M_0 and M_1, and then to "zip" them together so that fibers near the end of M_0 are brought together with fibers near the end of M_1. The collar lines will then show the way for the desired proper deformation retractions.

The proof is broken down into five steps.

Step 1. (Setup.) By Lemma 3.6 we can write M as $\bigcup K_i$, where $U_i = M - K_i$ is a connected neighborhood of infinity for each i, and with the property that $\pi_k(M, U_i) = 0$ $(k = 0, 1)$ and the natural homomorphism $\pi_2(M, U_{i+1}) \to \pi_2(M, U_i)$ is trivial for all i. Use Urysohn's Lemma to construct a map $\mu\colon M \to (0, (1/2) - \varepsilon]$, where ε is small, so that $\mu(K_1) = \{(1/2) - \varepsilon\}$, $\mu(K_2 - \mathrm{int}(K_1)) = [1/3, (1/2) - \varepsilon]$, and $\mu(K_i - \mathrm{int}(K_{i-1})) = [1/(i+1), 1/i]$ for $i > 2$.

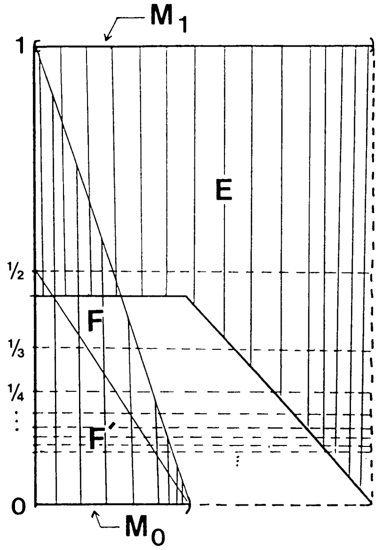

FIGURE 3. Set up for Theorem 3.9

Step 2. (Choosing special collars for M_0 and M_1.) Let E be the (closed) collar on M_1, each of whose collar lines is a set of the form $\{m\} \times [\mu(m), 1] \subset W$; where W is thought of as $M \times (0, 1] \cup A \times \{0\}$, and $m \in M_1$. To select a collar for M_0, begin by choosing a (PL) homeomorphism $\tau: M_0 \to \partial M \times [0, 1)$, and let $p: \partial M \times [0, 1) \to [0, 1)$ be the projection map. Define F to be the collar on M_0, each of whose fibers is of the form $\{m\} \times [0, 1 - p\tau(m)] \subset W$, where $m \in M_0$. Also needed will be a collar $F' \subset F$ with collar lines of the form $\{m\} \times [0, (1 - p\tau(m))/2]$ (see Figure 3).

Step 3. (An initial stretching of the collar E.) Consider a "slice" of W of the form $M \times (1/(2n+1), 1/(2n-1))$, where n is an integer > 1. Let $\widehat{M}_{2n-1}$ denote $[M \times (1/(2n+1), 1/(2n-1))] - F'$, $\widehat{C}_{2n-1}$ denote $(F - F') \cap \widehat{M}_{2n-1}$, and P_{2n-1} denote $[M \times \{1/(2n)\}] - F$. Then the triple $(\widehat{M}_{2n-1}, \widehat{C}_{2n-1}, P_{2n-1})$ is homeomorphic to a triple of the form $(\widehat{M}, \widehat{C}, P)$ as described in Lemma 3.7 (use Figure 3 as a reference). It is easy to find sets $\widehat{U}_{2n-1,1}$ and $\widehat{U}_{2n-1,2}$, contained in E, and filling the roles of $\widehat{U}_1$ and $\widehat{U}_2$ in Lemma 3.7. To do this begin with a pair U_r, U_{r+1} of the neighborhoods of infinity in M described earlier, where r is sufficiently large that $U_r \times (1/(2n+1), 1/(2n-1)) \subset E$. Let $\widehat{U}_{2n-1,1} = U_r \times (1/(2n+1), 1/(2n-1))$, and $\widehat{U}_{2n-1,2} = U_{r+1} \times (1/(2n+1), 1/(2n-1))$. Lemma 3.7 promises a homeomorphism $f_{2n-1}: \widehat{M}_{2n-1} \to \widehat{M}_{2n-1}$ such that

(a) f_{2n-1} has compact support,

(b) $f_{2n-1}|_{\partial \widehat{M}_{2n-1}} = \mathrm{id}$, and

(c) $f_{2n-1}(\widehat{U}_{2n-1,1}) \supset P_{2n-1}$.

Now define $f: W \to W$ as follows:

$$f(x) = \begin{cases} f_{2n-1}(x) & \text{if } x \in \widehat{M}_{2n-1}, \\ x & \text{elsewhere.} \end{cases}$$

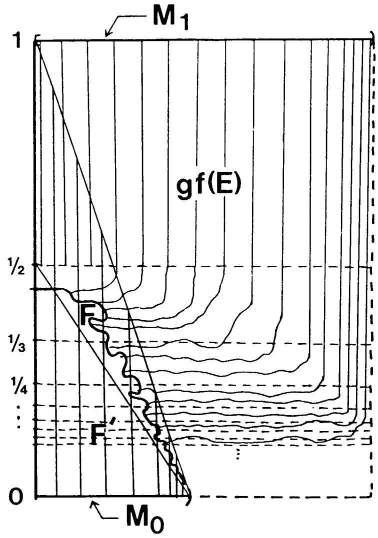

FIGURE 4. Cobordism (W, M_0, M_1) after Step 4

Condition (b) on the f_{2n-1}'s guarantees that f is continuous on F', and condition (a) gives continuity on all "horizontal seams" between the domains of the f_{2n-1}'s. Note that f satisfies the following conditions;

(i) $f|_{F' \cup M \times [1/2, 1] \cup M_0} = \mathrm{id}$,
(ii) $[f(E) \cup F] \supset M \times \{1/2n\}$ for all integers $n \geq 1$,
(iii) if $x \in M \times [1/(2n+1), 1/(2n-1)]$, then so is $f(x)$, and
(iv) for each set $M \times [1/(k+1), 1/k]$ in W, there is a neighborhood U of infinity in M such that $f|_{U \times [1/k+1, 1/k]} = \mathrm{id}$.

Step 4. (Completing the stretch of E towards F.) Here we concentrate on slices of the form $M \times (1/2n+2, 1/2n)$, where n is an integer ≥ 1. Let $\widehat{M}_{2n} = [M \times (1/2n+2, 1/2n)] - F'$ and $\widehat{C}_{2n,2} = (F - F') \cap \widehat{M}_{2n}$. We can now find an open collar $\widehat{C}_{2n,1}$ on $\partial \widehat{M}_{2n}$ contained in $\widehat{C}_{2n,2}$ and close enough to $\widehat{C}_{2n,2}$ that $\widehat{M}_{2n}$ contains a set V_{2n}, which is also contained in $f(E)$, and such that the quadruple $(\widehat{M}_{2n}, \widehat{C}_{2n,1}, \widehat{C}_{2n,2}, V_{2n})$ is homeomorphic to a quadruple $(\widehat{M}, \widehat{C}_1, \widehat{C}_2, V)$ of the form described in Lemma 3.8. Applying Lemma 3.8, for each n, gives us homeomorphisms $g_{2n}: \widehat{M}_{2n} \to \widehat{M}_{2n}$ with the properties:

(a) g_{2n} has compact support,
(b) $g_{2n}|_{\partial \widehat{M}_{2n}} = \mathrm{id}$, and
(c) $g_{2n}(V_{2n}) \cup \widehat{C}_{2n,2} = \widehat{M}_{2n}$.

We define a homeomorphism $g: W \to W$ by,

$$g(x) = \begin{cases} g_{2n}(x) & \text{if } x \in \widehat{M}_{2n}, \\ x & \text{elsewhere.} \end{cases}$$

Properties (a) and (b) ensure the continuity of g, and combine with property (c) to give the following properties for g (see Figure 4);

(i) $g|_{F'\cup M\times[1/2,1]\cup M_0} = \mathrm{id}$,
(ii) $[g(f(E))\cup F] = W$,
(iii) if $x \in M\times[1/2n+2, 1/2n]$ then so is $g(x)$, and
(iv) for each set $M\times[1/k+1, 1/k]$ in W, there is a neighborhood U of infinity in M, such that $g|_{U\times[1/k+1,1/k]} = \mathrm{id}$.

Step 5. (The deformation retractions.) Note that $gf\colon W\to W$ is a homeomorphism with the properties:

(A) $gf|_{F'\cup M\times[1/2,1]\cup M_0} = \mathrm{id}$,
(B) $[gf(E)\cup F] = W$, and
(C) if $x\in M\times[1/n+1, 1/n]$ then $gf(x)\in M\times[1/(n+3), 1/(n-3)]$ for all $n\geq 3$ (use condition (A) if $n\leq 3$),
(D) for each set $M\times[1/k+1, 1/k]$ in W, there is a neighborhood U of infinity in M such that $gf|_{U\times[1/k+1,1/k]} = \mathrm{id}$.

It is now easy to exhibit the desired proper deformation retractions. To deform W onto M_0, first push W into F along the collar lines of $gf(E)$, then use the collar lines of F to push everything down onto M_0. It is a simple matter to check that this deformation is proper. A similar pair of pushes along collar lines gives a proper deformation retraction of W onto M_1. This completes our proof.

Proof of 3.3. Theorem 3.9 combines with Theorem 2.2 to give the desired proof.

4. Characterizing weakly flat 2-spheres in S^4

An embedded k-sphere $\Sigma^k\subset S^n$ is said to be *weakly flat* provided $S^n-\Sigma^k\approx S^n-S^k$, where S^k is the standard k-sphere in S^n. There are many well-known examples of embedded spheres (even when codimension $\neq 2$) which are not weakly flat. The most famous, a codimension one example is the Alexander Horned Sphere. There are also examples of weakly flat spheres which are not flat. We will discuss some of these in §5. With this in mind, it is clear that, to classify weakly flat embeddings, some restrictions on the wildness will have to be made. At the same time, these restrictions should not be so severe that they imply local flatness (see [2, 11, 7, 4, 9, or 6] for conditions of this type).

An embedded k-sphere $\Sigma^k\subset S^n$ is said to be *globally* 1-*alg* provided each neighborhood U of Σ contains a neighborhood V of Σ such that loops which are null-homologous in $V-\Sigma$ are contractible in $U-\Sigma$. Work done in the sixties and early seventies supports this condition as the right one for studying weak flatness. The following theorem combines results of McMillan [26] in 1964 (the $k=n-1$ case), Duvall [15] in 1969 (for $2\leq k\leq n-3$), and Daverman [10] in 1973 ($k=1$).

4.1. **Theorem.** *Suppose $\Sigma^k\subset S^n$ is an embedded k-sphere with $k\neq n-2$ and $n\neq 4$. Then Σ is weakly flat iff it is globally* 1-*alg.*

Work by Hollingsworth and Rushing [23] in 1976 combined with Daverman's $k=1$ work settles the codimension 2 weak flatness problem, provided $n\neq 4$.

4.2. **Theorem.** *An $(n-2)$-sphere $\Sigma^k\subset S^n$ $(n\neq 4)$ is weakly flat iff Σ is globally* 1-*alg and $S^n-\Sigma^k$ is homotopy equivalent to S^1.*

The relationship of "weak flatness" characterizations to the open collar theorem is well documented. One application Siebenmann presents in [31] is a solution to a conjecture of Hempel and McMillan [22] confirming their belief that for $n \geq 5$ and $k \leq n-3$, k-spheres in S^n are weakly flat provided they are 1-LC at each point (a later result discovered independently by Daverman [10] and Černavskiĭ [7] made this characterization obsolete). Duvall's work improved Siebenmann's characterization by again employing the open collar theorem. The main contribution of Duvall's was in recognizing that the much weaker global 1-alg condition is sufficient for getting the right "end conditions" for $S^n - \Sigma^k$, thus allowing an application of the open collar theorem to show weak flatness. The codimension 2 work of Hollingsworth and Rushing was at least partly motivated by a conjecture of Siebenmann's in [31]. Their result confirms this conjecture, and more. Surprisingly, their proof essentially mimics that of the open collar theorem (a small modification of theirs makes their proof more elementary), showing that Siebenmann's paper virtually contained a solution to his conjecture! Our proof of Theorem 4.3 shows how this works.

As was the case with many results in geometric topology, the 4-dimensional cases of the weak flatness characterizations remained unsolved for several years. Freedman's landmark paper [18] of 1982 extended Theorem 4.1 to the case of 3-spheres in S^4. In 1987, G. Venema [35] confirmed the characterization for 1-spheres in S^4. In the remainder of this section we handle the case of 2-spheres in S^4, thus completing the program for classifying weakly flat embeddings of spheres.

4.3. **Theorem** (characterization of weakly flat 2-spheres in S^4). *A 2-sphere $\Sigma^2 \subset S^4$ is weakly flat iff it is globally 1-alg and $S^4 - \Sigma$ is homotopy equivalent to S^1.*

Proof. Let β be a PL embedded loop representing a generator of $\pi_1(S^4 - \Sigma)$. Choose a regular neighborhood N of β in $S^4 - \Sigma$, and let L denote $(S^4 - \Sigma) - \operatorname{int}(N)$. Since the complement of a standard 2-sphere in S^4 is homeomorphic to $S^1 \times \mathbf{R}^3$, and since $N \approx S^1 \times B^3$, it is clear that the theorem will be proved if we can show that $L \approx \partial L \times [0, 1)$. In view of Theorem 3.3 it will suffice to verify that

(i) $\partial L \subset L$ is a homotopy equivalence, and
(ii) π_1 is stable at the end of L with $\pi_1(\infty) \to \pi_1(L)$ an isomorphism.

Note. $\pi_1(\partial L)$ $(= \pi_1(S^1 \times S^2))$ is infinite cyclic, which is a Freedman group.

Towards verifying (i), let $\widetilde{M}$ denote the universal cover of $S^4 - \Sigma$ and for any set $A \subset S^4 - \Sigma$, let $\widetilde{A} = p^{-1}(A)$ where p is the covering projection. Since the inclusion $N \subset S^4 - \Sigma$ is a homotopy equivalence, $\widetilde{N} \subset \widetilde{M}$ must also be one. Applying Van Kampen's Theorem to the triple $\widetilde{M}$, $\widetilde{N}$, $\widetilde{L}$ tells us that $\widetilde{L}$ is simply connected, and is therefore the universal cover of L. By excision $H_k(\widetilde{L}, \partial\widetilde{L}) \cong H_k(\widetilde{M}, \widetilde{N}) = 0$ for all k. Since $\widetilde{L}$ and $\partial\widetilde{L}$ are simply connected, the Hurewicz Theorem guarantees that $\pi_k(\widetilde{L}, \partial\widetilde{L}) = 0$ for all k. Finally, this implies that $\pi_k(L, \partial L)$ is trivial for all k, so $\partial L \subset L$ is a homotopy equivalence.

To verify (ii) we use the fact that Σ is an ANR, together with the global 1-alg hypothesis, to find a sequence $\{V_i\}$ of manifold neighborhoods of Σ, missing N, with $\bigcap V_i = \Sigma$ and such that for each i; V_{i+1} deformation retracts to Σ

in V_i and null-homologous loops in $V_{i+1} - \Sigma$ contract in V_i. The sequence $\{V_i - \Sigma\}$ will be our preferred set of neighborhoods of infinity in L.

For any triple i, $i+1$, $i+2$ we have the following diagram.

$$\begin{array}{ccccccc}
H_2(V_{i+2}) & \xrightarrow{0} & H_2(V_{i+2}, V_{i+2} - \Sigma) & \xrightarrow{\partial} & H_1(V_{i+2} - \Sigma) & \rightarrow & H_1(V_{i+2}) \\
\downarrow & & \downarrow \cong & & \downarrow e_{i+1} & & \downarrow 0 \\
H_2(V_{i+1}) & \xrightarrow{0} & H_2(V_{i+1}, V_{i+1} - \Sigma) & \xrightarrow{\partial} & H_1(V_{i+1} - \Sigma) & \rightarrow & H_1(V_{i+1}) \\
\downarrow & & \downarrow \cong & & \downarrow e_i & & \downarrow 0 \\
H_2(V_i) & \xrightarrow{0} & H_2(V_i, V_i - \Sigma) & \xrightarrow{\partial} & H_1(V_i - \Sigma) & \rightarrow & H_1(V_i) \\
\downarrow & & \downarrow \cong & & \downarrow & & \downarrow \\
0 & \rightarrow & H_2(S^4, S^4 - \Sigma) & \xrightarrow{\cong} & H_1(S^4 - \Sigma) & \rightarrow & 0.
\end{array}$$

The column of isomorphisms is due to excision, the maps in the far right column are trivial by our selection of the V_i's and the triviality of the first map in each row is a result of our first two observations. This diagram tells us that L is "homologically stable at infinity," with the correct homology there. More specifically, $e_i|_{\mathrm{im}(e_{i+1})}: \mathrm{im}(e_{i+1}) \rightarrow \mathrm{im}(e_i)$ is an isomorphism, and inclusion induces an isomorphism $\mathrm{im}(e_i) \rightarrow H_1(S^4 - \Sigma)$, the latter which is isomorphic to $H_1(L)$ via inclusion.

Next we use another diagram to translate this information into π_1 information.

$$\begin{array}{ccc}
\pi_1(V_{i+1} - \Sigma) & \xrightarrow{\theta_{i+2}} & H_1(V_{i+2} - \Sigma) \\
\downarrow j_{i+1} & & \downarrow e_{i+1} \\
\pi_1(V_{i+1} - \Sigma) & \xrightarrow{\theta_{i+1}} & H_1(V_{i+1} - \Sigma) \\
\downarrow j_i & & \downarrow e_i \\
\pi_1(V_i - \Sigma) & \xrightarrow{\theta_i} & H_1(V_i - \Sigma)
\end{array}$$

The θ_i's in this diagram represent Hurewicz homomorphisms. First note that $\theta_i|_{\mathrm{im}(j_i j_{i+1})}: \mathrm{im}(j_i j_{i+1}) \rightarrow \mathrm{im}(e_i e_{i+1})$ is an isomorphism. A key point here is that nullhomologous loops in any $V_{k+1} - \Sigma$ contract in $V_k - \Sigma$. It then follows that $j_i j_{i+1}|_{\mathrm{im}(j_{i+2} j_{i+3})}: \mathrm{im}(j_{i+2} j_{i+3}) \rightarrow \mathrm{im}(j_i i_{i+1})$ is an isomorphism, so π_1 is stable at infinity and $\mathrm{im}(j_i j_{i+1}) \rightarrow \pi_1(L)$ is an isomorphism. This completes our proof.

Remarks. (1) This diagramatic proof that globally 1-alg gives the right end conditions in the complement is due to Daverman (see [12]).

(2) The above proof, and therefore the theorem, works with Σ replaced by any compactum Λ with the shape of a 2-sphere, the conclusion being that $S^4 - \Lambda \approx S^1 \times R^3$.

5. Examples

Two questions which clearly should be asked regarding the study of weak flatness are as follows:

Question 1. Do there exist embeddings $\Sigma^k \subset S^n$ which are weakly flat but not flat?

Question 2. Do there exist embeddings $\Sigma^{n-2} \subset S^n$ with $S^n - \Sigma$ homotopy equivalent to S^1 but which are not weakly flat?

Since our main concern has been the codimension 2 case, we restrict our attention to that situation for Question 1 as well as Question 2. Our goal in this section is to present examples which produce affirmative answers to both questions, for all $n \geq 3$.

Example 5.1 (A weakly flat 1-sphere $\Sigma \subset S^3$ which is not flat). In [29], Row and Walsh construct a nonshrinkable decomposition G of S^3 into points and cellular arcs such that:

(i) $J = \mathrm{cl}(\bigcup\{g \in G | g$ is not a point$\})$ is a simple closed curve which bounds a disk Q that is locally flat at each of its interior points, and

(ii) each arc contained in J is cellular.

Readers unfamiliar with the language of decomposition theory are referred to [13].

To construct the desired Σ, we begin by choosing an arc $L \subset J$ containing all but one (nontrivial) $g_0 \in G$. Let G' denote the decomposition of S^3 whose nontrivial elements are the nontrivial elements of $G - \{g_0\}$. Note that G' is still nonshrinkable. Now choose an arc A in Q connecting the end points of L and meeting J only in those two points. Let D denote the disk contained in Q which is bounded by $L \cup A$.

Claim. $\Sigma = L \cup A$ is weakly flat but not flat.

If Σ was flat then L would be also, but since G' is not shrinkable this is impossible. Seeing that Σ is weakly flat involves several observations. First, since D is locally flat away from L we can thicken D up on one side to create a 3-cell B with $D \subset \partial B$ and ∂B locally flat at all points not on L. Furthermore, each point of L is contained in a tame arc on D. (To see this requires a quick inspection of [29], where the construction is similar to that used in creating Bing's "hooked run.") This in turn implies that all points of the cellular arc L are piercing points of ∂B. We are now in position to apply a result of Garza (see [21, Theorem 2.1] and the discussion which follows) to conclude that $\partial B/L$ is a flat 2-sphere in $S^3/L \approx S^3$. This of course means that Σ/L is a flat 1-sphere in S^3/L. Since $\Sigma \subset S^3$ and $\Sigma/L \subset S^3/L$ have homeomorphic complements, then Σ is weakly flat in S^3.

Example 5.2 (A 1-sphere $\Sigma \subset S^3$ with $S^3 - \Sigma \simeq S^1$, but which is not weakly flat). This example was first cited in [14] for this purpose. We repeat it here (without proofs) for completeness and later use. Let Σ be the boundary of a Fox-Artin wild disk [17] (see Figure 5). Daverman and Rushing show that because Σ bounds a cellular disk, its complement will have infinite cyclic fundamental group. In S^3 this suffices to ensure that the complement has the homotopy type of S^1. They also show that the type of wildness present at the bad point prevents it from being weakly flat. It is an easy but interesting exercise to use the picture to find a loop which violates the global 1-alg condition.

Next we discuss a couple of methods for turning these (or other) examples into interesting examples in higher dimensions. In the following lemma, susp(A) stands for the suspension of A. Recall that the suspension of an n-sphere is an $(n+1)$-sphere.

5.3. **Lemma.** *Let $\Sigma^{n-2} \subset S^n$ be an embedded $(n-2)$-sphere such that*

$$\pi_1(S^n - \Sigma) \cong Z.$$

Then $\mathrm{susp}(\Sigma) \subset \mathrm{susp}(S^n)$ *is a globally* 1-*alg* $(n-1)$*-sphere in* S^{n+1}.

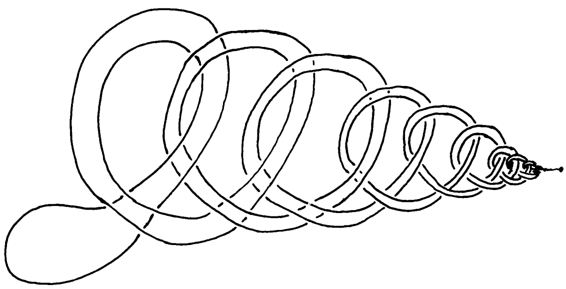

FIGURE 5. A Fox-Artin simple closed curve

Proof. Let U be a connected manifold neighborhood of Σ in S^n. The long exact sequence for the pair (S^n, U), together with excision, tells us that

$$H_1(S^n - \Sigma, U - \Sigma) = 0.$$

Since $\pi_1(S^n - \Sigma)$ is abelian, this implies that $\pi_1(S^n - \Sigma, U - \Sigma) = 0$. We will use this later. We now find an (arbitrarily small) neighborhood V of $\mathrm{susp}(\Sigma)$ in $\mathrm{susp}(S^n)$ such that nullhomologous loops in $V - \mathrm{susp}(\Sigma)$ contract in $V - \mathrm{susp}(\Sigma)$. Let p, p' denote the suspension points. Then $(\mathrm{susp}(S^n) - \{p, p'\}, \mathrm{susp}(\Sigma) - \{p, p'\}) \approx (S^n \times R^1, \Sigma \times R^1)$. This tells us that $\mathrm{susp}(\Sigma)$ has artibrarily small neighborhoods V such that

$$V - \mathrm{susp}(\Sigma) \approx ((S^n - \Sigma) \times [(-\infty, -J] \cup [J, \infty)]) \cup ((U - \Sigma) \times R^1),$$

where U is a connected manifold neighborhood of Σ in S^n. Let α be a nullhomologous loop in $V - \mathrm{susp}(\Sigma)$. Since $\pi_1(S^n - \Sigma, U - \Sigma) = 0$, we can pull the portion of α lying below the "$(-J)$-level" of $V - \mathrm{susp}(\Sigma)$ into the $U \times R^1$ portion. Now push α upward into the $(S^n - \Sigma) \times [J, \infty)$ portion. Here α must contract since $\pi_1((S^n - \Sigma) \times [J, \infty)) \cong \pi_1(S^n - \Sigma) \cong Z$ is abelian. Thus V has the desired property, and the lemma is proved.

5.4. **Lemma.** *If $\Sigma^{n-2} \subset S^n$ is an embedded $(n-2)$-sphere with $S^n - \Sigma$ homotopy equivalent to S^1, then* $\mathrm{susp}(\Sigma)$ *is weakly flat in* $\mathrm{susp}(S^n)$.

Proof. Since $\mathrm{susp}(S^n) - \mathrm{susp}(\Sigma) \approx (S^n - \Sigma) \times R^1 \simeq S^1$, the proof is immediate from Lemma 5.3 and either Theorem 4.2 or 4.3.

5.5. **Example.** (Weakly flat $(n-2)$-spheres which are not flat in S^n (for any $n \geq 3$).) Applying Lemma 5.4 to either Example 5.1 or 5.2 gives us a weakly flat 2-sphere in S^4. It is easy to see that since the original embeddings are not locally flat, their suspensions cannot be locally flat, and are therefore not flat. Continuing this process inductively gives examples with the desired properties for any $n \geq 3$.

5.6. **Example.** (A method for constructing an $(n-2)$-sphere with complement homotopy equivalent to S^1, but which is not weakly flat (for any $n \geq 3$).) We will describe a spinning construction like that used by Artin [1] to create knots in S^4 using knots in S^3. Although the technique is much more general, we will concentrate on using Example 5.2 to create a 2-sphere in S^4 with the desired properties. Inductively, one can then create an example in each dimension ≥ 3.

Let $\Sigma_{FA} \subset S^3$ denote Example 5.2 (see Figure 5). Begin by choosing a small round ball B in S^3 such that Σ_{FA} intersects B in a flat arc A which pierces ∂B in exactly two points. Next remove a point $p \in \partial B - A$ to get Σ_{FA} contained in R^3, with $\partial B - \{p\}$ a nice plane in R^3 separating Σ_{FA} into two arcs; A and $A_w = \Sigma_{FA} - \operatorname{int}(A)$. Let R^3_+ denote the closure of the component containing $\operatorname{int}(A_w)$. Note that

$$\pi_1(R^3 - A_w) \cong Z.$$

We now create R^4 by "spinning," i.e., we realize R^4 as the decomposition space $(R^3 \times S^1)/\{\{x\} \times S^1 | x \in \partial R^3_+\}$. Denote the "image" of $A_w \times S^1$ by $\operatorname{spin}(A_w)$, and note that $\operatorname{spin}(A_w) \approx S^2$. Compactifying gives us a 2-sphere in S^4. It is left to the reader to show that $S^4 - \operatorname{spin}(A_w) \simeq S^1$ and that $\operatorname{spin}(A_w)$ is not globally 1-alg in S^4.

Remark. Certainly there are many different (and more clever) ways to build examples such as these. The examples presented here were chosen mainly for their simplicity and the fact that they are very visual. They also seem to illustrate some nice uses of the different embedding conditions.

6. Final remarks

A nice set of applications of the 1-alg characterization of weakly flat codimension 2 spheres appears in [14]. With our proof of Theorem 4.3, most of their proofs can be immediately extended to the cases involving 2-spheres in S^4. One which is especially interesting, due to its analogue in knot theory, is the following.

6.1. **Theorem.** *A 2-sphere in S^4 which bounds a cellular 3-cell is weakly flat.*

We mentioned earlier that Theorem 4.3 completes the 1-alg classification scheme for weakly flat spheres. As is the case with most recent work in dimension four, the results are purely topological (as opposed to PL or smooth). In all other dimensions, the results on weak flatness (as well as most of the other results mentioned here) are also true in the PL or smooth categories. It is an open question, whether or not two weakly flat k-spheres in S^4 have PL (or smoothly) homeomorphic complements.

Finally we report that Liem and Venema [25] have used the above results to show that every globally 1-alg embedding of S^2 in S^4 is the complement of some locally flat knot. This extends the higher dimensional result of Daverman [12].

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