AN ELEMENTARY DEDUCTION OF THE TOPOLOGICAL RADON THEOREM FROM BORSUK-ULAM

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Abstract. The Topological Radon Theorem states that, for every continuous function from the boundary of a $(d+1)$-dimensional simplex into $\mathbb{R}^n$, there exist a pair of disjoint faces in the domain whose images intersect in $\mathbb{R}^n$. The similarity between that result and the classical Borsuk-Ulam Theorem is unmistakeable, but a proof that the Topological Radon Theorem follows from Borsuk-Ulam is not immediate. In this note we provide an elementary argument verifying that implication.

1. Introduction

The classical Radon Theorem states that any collection $X = \{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_{d+2}\}$ of $d+2$ points in $\mathbb{R}^d$ can be divided into two disjoint sets whose convex hulls intersect. The proof is a straightforward application of elementary linear algebra. See, for example, [Ma, p.90]. An equivalent formulation of this theorem, with Δ^{d+1} denoting the $(d+1)$-dimensional simplex, is the following.

Theorem 1.1 (Radon's Theorem). *For every affine map $f : \mathbf{\Delta}^{d+1} \to \mathbb{R}^d$ there exist a pair of disjoint faces F_A and F_B of $\mathbf{\Delta}^{d+1}$ such that $f(F_A) \cap f(F_B) \neq \varnothing$.*

The equivalence of these two statements is easily deduced from the fact that every set $X = \{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_{d+2}\} \subseteq \mathbb{R}^d$ determines an affine map $f : \mathbf{\Delta}^{d+1} \to \mathbb{R}^d$ taking the vertices of $\mathbf{\Delta}^{d+1}$ to the elements of X. Under this map, the image of each face is the convex hull of the images of its vertices.

The 'topological version' of the above theorem relaxes the requirements on the function f.

Theorem 1.2 (The Topological Radon Theorem). *For every continuous function $f : \mathbf{\Delta}^{d+1} \to \mathbb{R}^d$ there exists a pair of disjoint faces F_A and F_B of $\mathbf{\Delta}^{d+1}$ such that $f(F_A) \cap f(F_B) \neq \varnothing$.*

Several proofs of this theorem may be found in the literature—each depending on an application of the Borsuk-Ulam Theorem. See for example [BB], [Wo] and [Ma, Ch 5]. The goal of this paper is to present a new and particularly elementary method for deducing the Topological Radon Theorem from Borsuk-Ulam.

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2. Background and notation

Recall that the Borsuk-Ulam Theorem guarantees that, for any continuous $g : S^d \to \mathbb{R}^d$, there exists $\mathbf{x} \in S^d$ such that $g(\mathbf{x}) = g(-\mathbf{x})$. Here S^d denotes the standard d-sphere $\{\mathbf{x} \in \mathbb{R}^{d+1} \mid \|\mathbf{x}\| = 1\}$. (Points $\mathbf{x}$ and $-\mathbf{x}$ from S^d are called *antipodal points.*)

Let $\mathbf{N} = (0, \cdots, 0, 1)$ and $\mathbf{S} = (0, \cdots, 0, -1)$ denote the *north* and *south poles* of S^d and view S^{d-1} as a subset of S^d — the intersection of S^d with the hyperplane $\mathbb{R}^d \times \mathbf{0}$. We may then view S^d as the union $S^d = \cup_{\mathbf{y} \in S^{d-1}} G_{\mathbf{y}}$ where $G_{\mathbf{y}}$ is the great semicircle with endpoints $\mathbf{S}$ and $\mathbf{N}$ intersecting S^{d-1} at the point $\mathbf{y}$. In other words, $G_{\mathbf{y}} = \left\{(\cos\theta \cdot \mathbf{y}, \sin\theta) \mid \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right\}$. Notice that for distinct $\mathbf{y}_1, \mathbf{y}_2 \in S^{d-1}$, $G_{\mathbf{y}_1}$ intersects $G_{\mathbf{y}_2}$ only in the poles $\{\mathbf{N}, \mathbf{S}\}$.

For convenience, we represent a point $(\cos\theta \cdot \mathbf{y}, \sin\theta)$ in *generalized polar form* by the expression $\langle \mathbf{y}, \theta \rangle$. This representation is unique provided $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. In this form antipodal points are easy to recognize—the antipode of $\langle \mathbf{y}, \theta \rangle$ is $\langle -\mathbf{y}, -\theta \rangle$.

Next we discuss simplexes, their faces, and their boundaries. Let $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_{d+1}$ be the points $(1, 0, 0, \cdots, 0)$, $(0, 1, 0, \cdots, 0), \cdots, (0, 0, 0, \cdots, 1)$ in $\mathbb{R}^{d+1}$. The d-dimensional simplex $\mathbf{\Delta}^d$ is the convex hull of $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_{d+1}\}$. Thus,

$$\mathbf{\Delta}^d = \left\{ \sum_{i=1}^{d+1} a_i \mathbf{v}_i \mid a_i \geq 0 \text{ and } \sum_{i=1}^{d+1} a_i = 1 \right\}.$$

We call $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_{d+1}$ the *vertices* of $\mathbf{\Delta}^d$. The coefficient a_i of a given point is called its i^{th} *barycentric coordinate*. The point in $\mathbf{\Delta}^d$ with barycentric coordinates uniformly equal to $\frac{1}{d+1}$ is called the *barycenter* of $\mathbf{\Delta}^d$; it will be denoted $\mathbf{b}_d$

Notice that, for any $k \leq d$, the simplex $\mathbf{\Delta}^k$ may be viewed as a subset of $\mathbf{\Delta}^d$. More generally, if $A \subseteq \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_{d+1}\}$, we call the convex hull of A, denoted F_A, a *face* of $\mathbf{\Delta}^d$. When A contains exactly $k+1$ elements, then F_A is an isometric copy of $\mathbf{\Delta}^k$. Faces F_A and F_B are disjoint if and only if $A \cap B = \varnothing$. The *boundary* of a simplex $\mathbf{\Delta}^d$, denoted $\partial \mathbf{\Delta}^d$, is the union of all proper faces of $\mathbf{\Delta}^d$.

In preparation for our theorem, we express $\partial \mathbf{\Delta}^d$ as a union of subsets, each made up of a pair of line segments. Let $\partial^+ \mathbf{\Delta}^d$ denote the union of all proper faces of $\mathbf{\Delta}^d$ except for $\mathbf{\Delta}^{d-1}$. Then each $\mathbf{p} \in \partial^+ \mathbf{\Delta}^d$ lies on a line segment $K_{\mathbf{q}}$ connecting a point $\mathbf{q} \in \partial \mathbf{\Delta}^{d-1}$ to the vertex $\mathbf{v}_{d+1}$. That segment is unique, unless $\mathbf{p} = \mathbf{v}_{d+1}$. Similarly, each $\mathbf{p} \in \mathbf{\Delta}^{d-1}$ lies on a segment $L_{\mathbf{q}}$ connecting a point $\mathbf{q} \in \partial \mathbf{\Delta}^{d-1}$ to the barycenter $\mathbf{b}_{d-1}$ of $\mathbf{\Delta}^{d-1}$. Let $M_{\mathbf{q}} = K_{\mathbf{q}} \cup L_{\mathbf{q}}$, a 'bent segment' connecting $\mathbf{b}_{d-1}$ to $\mathbf{v}_{d+1}$.

The $M_{\mathbf{q}}$'s in $\partial \mathbf{\Delta}^d$ are analogous to the great semi-circles $G_{\mathbf{y}}$ in S^{d-1} with $\{\mathbf{v}_{d+1}, \mathbf{b}_{d-1}\}$ analogous to $\{\mathbf{N}, \mathbf{S}\}$. In particular, $\partial \mathbf{\Delta}^d = \cup_{\mathbf{q} \in \partial \mathbf{\Delta}^{d-1}} M_{\mathbf{q}}$, with $M_{\mathbf{q}}$ intersecting $M_{\mathbf{q}'}$ precisely in $\{\mathbf{v}_{d+1}, \mathbf{b}_{d-1}\}$ whenever $\mathbf{q} \neq \mathbf{q}'$. See Figure 1.

3. Proofs

The Topological Radon Theorem is an easy consequence of the following:

Proposition 3.1. *For every $d \geq 0$, there exists a continuous function $\lambda_d : S^d \to \partial \mathbf{\Delta}^{d+1}$ such that, for any $\mathbf{x} \in S^d$, $\lambda_d(\mathbf{x})$ and $\lambda_d(-\mathbf{x})$ lie in disjoint faces of $\partial \mathbf{\Delta}^{d+1}$.*

Proof of Theorem 1.2 from Proposition 3.1. Given a continuous function $f : \mathbf{\Delta}^{d+1} \to \mathbb{R}^d$, consider $f \circ \lambda_d : S^d \to \mathbb{R}^d$. By the Borsuk-Ulam Theorem, these exists $\mathbf{x} \in S^d$ such

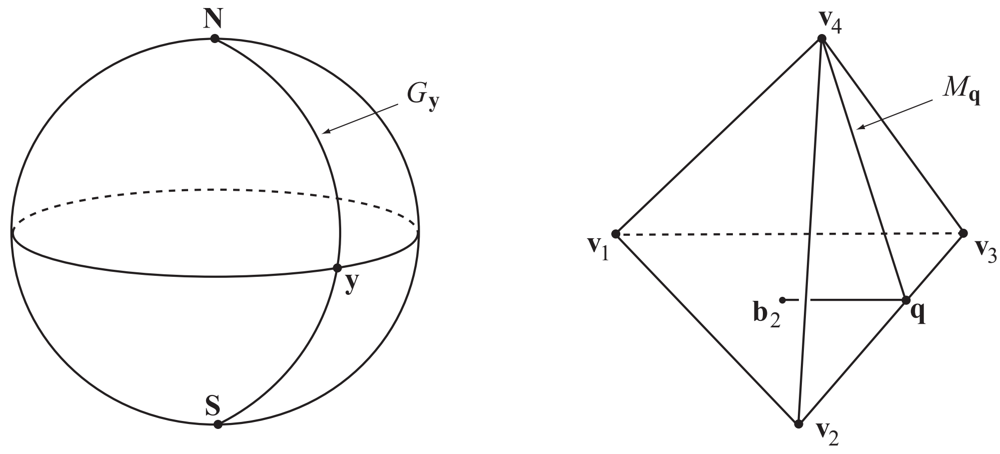

FIGURE 1. A great semicircle $G_{\mathbf{y}}$ in S^2 and a 'bent segment' $M_{\mathbf{q}}$ in $\partial\Delta^3$.

that $f \circ \lambda_d(\mathbf{x}) = f \circ \lambda_d(-\mathbf{x})$. By Proposition 3.1, there exist disjoint faces F_A and F_B of $\partial\mathbf{\Delta}^{d+1}$ containing $\lambda_d(\mathbf{x})$ and $\lambda_d(-\mathbf{x})$, respectively. Then $f(F_A) \cap f(F_B) \neq \varnothing$. $\square$

Proof of Proposition 3.1. For $d = 0$, $S^0 = \{-1, 1\}$ and $\partial\mathbf{\Delta}^1 = \{\mathbf{v}_1, \mathbf{v}_2\}$. Simply define $\lambda_0(-1) = \mathbf{v}_1$ and $\lambda_0(1) = \mathbf{v}_2$.

Proceeding inductively, assume that an acceptable $\lambda_k : S^k \to \partial\mathbf{\Delta}^{k+1}$ exists for some k. We show how to obtain $\lambda_{k+1} : S^{k+1} \to \partial\mathbf{\Delta}^{k+2}$.

For each $\mathbf{y} \in S^k$ define λ_{k+1} to take $G_{\mathbf{y}} \subseteq S^{k+1}$ onto $M_{\lambda_k(\mathbf{y})} \subseteq \partial\mathbf{\Delta}^{k+2}$ as follows:

$$\lambda_{k+1}(\langle \mathbf{y}, t\rangle) = \begin{cases} \mathbf{v}_{k+3} & \text{for } \frac{\pi}{4} \le t \le \frac{\pi}{2} \\ \left(1 - \frac{4t}{\pi}\right) \cdot \lambda_k(\mathbf{y}) + \left(\frac{4t}{\pi}\right) \cdot \mathbf{v}_{k+3} & \text{for } 0 \le t \le \frac{\pi}{4} \\ \lambda_k(\mathbf{y}) & \text{for } -\frac{\pi}{4} \le t \le 0 \\ -\left(1 + \frac{4t}{\pi}\right) \cdot \mathbf{b}_{k+1} + \left(2 + \frac{4t}{\pi}\right) \cdot \lambda_k(\mathbf{y}) & \text{for } -\frac{\pi}{2} \le t \le -\frac{\pi}{4} \end{cases}.$$

In words, λ_{k+1} maps the upper half of a great semicircle $G_{\mathbf{y}}$ onto the segment $K_{\lambda(\mathbf{y})}$ by squeezing the $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$-portion to the vertex $\mathbf{v}_{k+3}$ and stretching the $\left[0, \frac{\pi}{4}\right]$-portion over the entire segment. On the lower half of $G_{\mathbf{y}}$, λ_{k+1} maps the entire $\left[-\frac{\pi}{4}, 0\right]$-portion to the point $\lambda_k(\mathbf{y})$ and stretches the $\left[-\frac{\pi}{2}, -\frac{\pi}{4}\right]$-portion over the segment $L_{\lambda(\mathbf{y})}$. The continuity of λ_{k+1} follows easily from the continuity of λ_k combined with the obvious continuity of λ_{k+1} on each of the great semicircles $G_{\mathbf{y}}$.

Claim. *For any $\langle \mathbf{y}, t\rangle \in S^{k+1}$, $\lambda_{k+1}(\langle \mathbf{y}, t\rangle)$ and $\lambda_{k+1}(\langle -\mathbf{y}, -t\rangle)$ lie in disjoint faces of $\partial\mathbf{\Delta}^{k+2}$.*

Without loss of generality, we may assume $t \in \left[0, \frac{\pi}{2}\right]$.

Case 1. $t \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$.

Then $\lambda_{k+1}(\langle \mathbf{y}, t\rangle) = \mathbf{v}_{k+3}$ and $\lambda_{k+1}(\langle -\mathbf{y}, -t\rangle) = -\left(1 + \frac{4t}{\pi}\right) \cdot \mathbf{b}_{k+1} + \left(2 + \frac{4t}{\pi}\right) \cdot \lambda_k(\mathbf{y}) \in \mathbf{\Delta}^{k+1}$. Since $\{\mathbf{v}_{k+3}\}$ and $\mathbf{\Delta}^{k+1}$ are disjoint, the claim holds.

Case 2. $t \in \left[0, \frac{\pi}{4}\right]$.

By the inductive hypothesis, there exist disjoint faces F_A and F_B of $\partial\mathbf{\Delta}^{k+1}$ containing $\lambda_k(\mathbf{y})$ and $\lambda_k(-\mathbf{y})$, respectively. Applying the definition of λ_{k+1}, we see that $\lambda_{k+1}(\langle\mathbf{y},t\rangle) \in F_{A\cup\{\mathbf{v}_{k+3}\}}$ and $\lambda_{k+1}(\langle-\mathbf{y},-t\rangle) \in F_B \subseteq \partial\mathbf{\Delta}^{k+1}$. Since $A\cup\{\mathbf{v}_{k+3}\}$ and B are disjoint, so are the corresponding faces. □

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