A SOLUTION TO DE GROOT'S ABSOLUTE CONE CONJECTURE

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ABSTRACT. A compactum X is an 'absolute cone' if, for each of its points x, the space X is homeomorphic to a cone with x corresponding to the cone point. In 1971, J. de Groot conjectured that each n-dimensional absolute cone is an n-cell. In this paper, we give a complete solution to that conjecture. In particular, we show that the conjecture is true for $n \leq 3$ and false for $n \geq 5$. For $n = 4$, the absolute cone conjecture is true if and only if the 3-dimensional Poincaré Conjecture is true.

1. Introduction

A compactum X is an *absolute suspension* if for any pair of points $x, y \in X$, the space X is homeomorphic to a suspension with x and y corresponding to the suspension points. Similarly, X is an *absolute cone* if, for each point $x \in X$, the space X is homeomorphic to a cone with x corresponding to the cone point.

At the 1971 Prague Symposium, J. de Groot [Gr] made the following two conjectures:

Conjecture 1. *Every n-dimensional absolute suspension is homeomorphic to the n-sphere.*

Conjecture 2. *Every n-dimensional absolute cone is homeomorphic to an n-cell.*

In 1974, Szymański [Sz] proved Conjecture 1 in the affirmative for $n = 1, 2$ or 3. Later, Mitchell [Mi1] reproved Szymański's results, and at the same time shed some light on higher dimensions, by showing that every n-dimensional absolute suspension is an ENR homology n-manifold homotopy equivalent to the n-sphere. Still, the 'absolute suspension conjecture' remains open for $n \geq 4$.

In 2005, Nadler [Na] announced a proof of Conjecture 2 in dimensions 1 and 2. In this paper we provide a complete solution to the 'absolute cone conjecture'. In particular, we verify Conjecture 2 for $n \leq 3$ and provide counterexamples for all $n \geq 5$. For $n = 4$ we show that the conjecture is equivalent to the 3-dimensional Poincaré Conjecture, that has recently been claimed by Perelman.

2. Definitions, notation, and terminology

2.1. **Cones.** For any topological space L, the *cone* on L is the quotient space

$$cone\,(L) = L \times [0, 1]\,/L \times \{0\}\,.$$

Let $q : L \times [0, 1] \to cone(L)$ be the corresponding quotient map. We refer to $q\,(L \times \{0\})$ as the *cone point* and we view L as a subspace of $cone\,(L)$ via the embedding $L \hookrightarrow L \times \{1\} \hookrightarrow cone\,(L)$. We refer to this copy of L as the *base* of the cone. For any $(z, t) \in L \times [0, 1]$, denote $q\,(z, t)$ by $t \cdot z$. Thus, $0 \cdot z$ represents the cone point and $1 \cdot z = z$ for all $z \in L$.

For each $z \in L$, the *cone line* corresponding to z is the arc

$$I_z = q\,(\{z\} \times [0, 1]) = \{t \cdot z \mid 0 \le t \le 1\}\,,$$

while the *open cone line* corresponding to z, denoted by $\overset{\circ}{I}_z$, is the set

$$\overset{\circ}{I}_z = q\,(\{z\} \times (0, 1)) = \{t \cdot z \mid 0 < t < 1\}\,.$$

For $\varepsilon \in (0, 1)$, the *subcone of radius ε* is the set

$$cone\,(L, \varepsilon) = q\,(L \times [0, \varepsilon]) = \{t \cdot z \mid z \in L \text{ and } 0 \le t \le \varepsilon\}$$

Clearly, each subcone is homeomorphic to $cone(L)$. More generally, if $\lambda : L \to (0, 1)$ is continuous, the *λ-warped subcone* is defined by

$$cone\,(L, \lambda) = \{t \cdot z \mid z \in L \text{ and } 0 \le t \le \lambda\,(z)\}\,.$$

It also is homeomorphic to $cone(L)$. In fact, the following is easy to prove.

Lemma 2.1. *Let L be a space, $\varepsilon \in (0, 1)$, and $\lambda : L \to (0, 1)$. Then there is a homeomorphism (in fact, an ambient isotopy) $f : cone(L) \to cone(L)$ fixed on $L \cup \{cone\ point\}$ such that $f\,(cone\,(L, \varepsilon)) = cone\,(L, \lambda)$.*

By applying the above lemma, or by a similar direct proof, we also have:

Lemma 2.2. *Let L be a space and suppose $t \cdot z$ and $t' \cdot z$ are points on the same open cone line of $cone\,(L)$. Then there is a homeomorphism (in fact, an ambient isotopy) $f : cone(L) \to cone(L)$ fixed on $L \cup \{cone\ point\}$ such that $f\,(t \cdot z) = t' \cdot z$.*

On occasion, we will have use for the *open cone* on L, which we view as a subspace of $cone\,(L)$. It is defined by

$$opencone\,(L) = L \times [0, 1)/L \times \{0\}\,.$$

2.2. **Suspensions and mapping cylinders.** For a topological space L, the *suspension* of L is the quotient space

$$susp\,(L) = L \times [0, 1]\,/\,\{L \times \{0\}\,, L \times \{1\}\}\,.$$

In other words, the suspension of L is obtained by separately crushing out the top and bottom levels of the product $L \times [0, 1]$. The images of these two sets under the quotient map are called the *suspension points* of $susp\,(L)$.

Given a map $f : L \to K$ between disjoint topological spaces, the *mapping cylinder* of f is the quotient space

$$Map(f) = ((L \times [0,1]) \sqcup K)/\sim$$

where $\sim$ is the equivalence relation on the disjoint union $(L \times [0,1]) \sqcup K$ induced by the rule: $(x,0) \sim f(x)$ for all $x \in L$. We view L and K as subsets of $Map(f)$ via the embeddings induced by

$$L \leftrightarrow L \times \{1\} \hookrightarrow (L \times [0,1]) \sqcup K, \text{ and}$$
$$K \hookrightarrow (L \times [0,1]) \sqcup K.$$

In addition, for each $z \in L$, the inclusion

$$\{z\} \times [0,1] \hookrightarrow (L \times [0,1]) \sqcup K$$

induces an embedding of an arc into $Map(f)$. We call the image arc a *cylinder line* and denote it by E_z.

Clearly, if the above range space K consists of a single point, then $Map(f) = cone(L)$. Similarly, $cone(L)$ can always be obtained as a quotient space of $Map(f)$ by crushing K to a point. The following lemma will be useful later. It allows us to view certain cones as mapping cylinders having one of the cone lines as the range space.

Lemma 2.3. *Let Y be a space and suppose $y \in Y$ has a k-dimensional euclidean neighborhood U in Y. Let B_0^k and B_1^k be (tame) k-cell neighborhoods of y lying in U such that $B_1^k \subseteq int(B_0^k)$. Then the pair $(cone(Y), I_y)$ is homeomorphic to $(Map(f), I_y)$ for some map $f : Y - int(B_1^k) \to I_y$. The homeomorphism may be chosen to be the identity on $(Y - int(B_1^k)) \cup I_y$.*

Proof. Choose a homeomorphism $h : B_0^k - int(B_1^k) \to S^{k-1} \times [0,1]$ taking ∂B_0^k to $S^{k-1} \times \{0\}$ and ∂B_1^k to $S^{k-1} \times \{0\}$. Then define $f : Y - int(B_1^k) \to I_y$ by

$$f(x) = \begin{cases} t \cdot y & \text{if } x \in B_0^k - int(B_1^k) \text{ and } h(x) \in S^{k-1} \times \{t\} \\ 0 \cdot y & \text{if } x \in Y - int(B_0^k) \end{cases}.$$

Since f sends all points of $Y - int(B_0^k)$ to the cone point $0 \cdot y$, we may identify the 'sub-mapping cylinder' $Map\left(f|_{Y - int(B_0^k)}\right)$ with the 'subcone' $cone(Y - int(B_1^k))$. In addition, it is easy to build a homeomorphism between the $(k+1)$-cell $Map\left(f|_{B_0^k - int(B_1^k)}\right)$ and the $(k+1)$-cell $cone(B_0^k)$ taking I_y identically onto I_y, and each cylinder line emanating from an $x \in \partial B_0^k$ identically onto the corresponding cone line. Fitting these pieces together yields the desired homeomorphism between $Map(f)$ and $cone(Y)$. See Figure 1.

$\square$

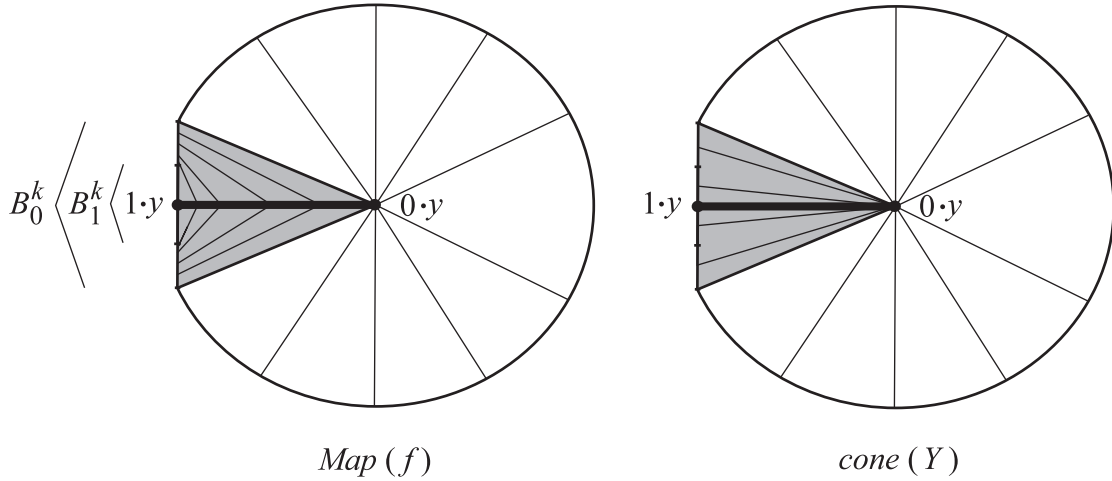$Map\,(f)$ $cone\,(Y)$

FIGURE 1.

2.3. **ENR homology manifolds.** A space is a *euclidean neighborhood retract (ENR)* if it is a retract of some open subset of euclidean space. This is equivalent to being a finite-dimensional separable metric ANR. A space that is a retract of $\mathbb{R}^n$ (for some n) is called a *euclidean retract (ER)*. This is equivalent to being a contractible ENR.

A locally compact ENR X is an *ENR homology n-manifold* if, for every $x \in X$,

$$(\dagger_n) \qquad\qquad H_*\,(X, X - x) \cong \begin{cases} \mathbb{Z} & \text{if } * = n \\ 0 & \text{otherwise} \end{cases}.$$

We call X an *ENR homology n-manifold with boundary* if, for every $x \in X$,

$$H_*\,(X, X - x) \cong \begin{cases} 0 \text{ or } \mathbb{Z} & \text{if } * = n \\ 0 & \text{otherwise} \end{cases}.$$

In this case, the *boundary of* X is the set

$$\partial X = \{\, x \in X \mid H_*\,(X, X - x) \equiv 0 \,\},$$

and the *interior of* X is the set

$$int\,(X) = X - \partial X.$$

In all of the above and throughout this paper, except where stated otherwise, homology is singular with integer coefficients.

By [Mi2], ∂X is a closed subset of X; hence, $int\,(X)$ is an ENR homology n-manifold. In addition, if Borel-Moore homology is used, ∂X satisfies the algebraic condition for being a homology $(n-1)$-manifold, i.e., ∂X satisfies $(\dagger_{n-1})$. For ENRs, Borel-Moore homology agrees with singular homology, so if ∂X is an ENR then it is an ENR homology $(n-1)$-manifold.

Remark 1. *There are interesting situations where, although X is an ENR homology manifold with boundary, ∂X is not an ENR. See, for example, [AG] or [Fi]. For the spaces of interest in this paper, existing conditions will prevent this from happening.*

3. Absolute cones

Suppose a compactum X is an absolute cone. For each $x \in X$, choose $L_x \subseteq X$ and a homeomorphism $h_x : cone\,(L_x) \to X$ which is the identity on L_x and sends the cone point to x. We will refer to L_x as the *link* of x in X. (**Note.** The choice of L_x and h_x may not be unique; however, for each x we make a choice and stick with it.) For $\varepsilon \in (0, 1)$ and $\lambda : L \to (0, 1)$ let $N\,(x, \varepsilon) = h_x\,(cone\,(L, \varepsilon))$ and $N\,(x, \lambda) = h_x\,(cone\,(L, \lambda))$. We refer to these as the ε-*cone neighborhood* and the *warped λ-cone neighborhood* of x, respectively. Clearly, each point of x has arbitrarily small ε-cone neighborhoods.

In a similar vein, for any $x \in X$ and $z \in L_x$, let $J_x\,(z)$ and $\overset{\circ}{J}_x\,(z)$ denote $h_x\,(I_z)$ and $h_x(\overset{\circ}{I}_z)$, respectively. We refer to these as *[open] cone lines of* X with respect to x.

The following proposition lists several easy properties of absolute cones.

Proposition 3.1. *Let X be a finite dimensional absolute cone, $x \in X$ and $z \in L_x$. Then*

    (1) *X is a compact ER,*
    (2) *L_x is a compact ENR,*
    (3) *$H_*\,(X, X - x) \cong \widetilde{H}_{*-1}\,(L_x)$,*
    (4) *L_z is contractible, and*
    (5) *$H_*\,(X, X - z) \equiv 0$.*

Proof. Since each point of X has arbitrarily small ε-cone neighborhoods, X is locally contractible; so by [Hu, V.7.1], X is an ENR. Since X is also contractible, it is an ER.

Since L_x is a retract of its neighborhood $X - x \approx L_x \times [0, 1)$ in X, it too is an ENR [Hu, III.7.7]. Being a closed subset of X, L_x is also compact.

To prove 3), we again use $X - x \approx L_x \times (0, 1]$. Since X is contractible, the desired isomorphisms may be obtained from the long exact sequence for the pair $(X, X - x)$.

The canonical contraction of $cone\,(L_x)$ along cone lines restricts to a contraction of $cone\,(L_x) - z$, since z lies in the base. Thus, $X - z$ is contractible. Since $X - z \approx L_z \times (0, 1]$, it follows that L_z is contractible.

Assertion 5) follows from 3) and 4). $\qquad\qquad\square$

The next proposition is a key ingredient in our understanding of absolute cones.

Proposition 3.2. *Let X be a finite dimensional absolute cone and*

$$B_X = \{\, z \in X \mid H_*\,(X, X - z) = 0 \,\}.$$

Then

    (1) *$L_x \subseteq B_X$ for all $x \in X$,*
    (2) *$X - B_X \neq \varnothing$, and*
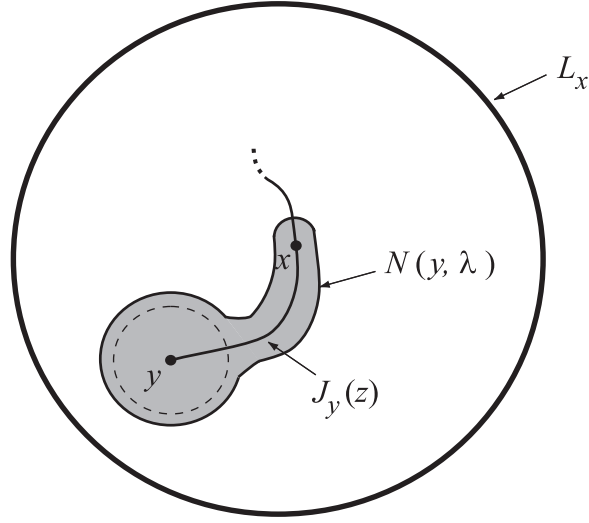   (3) *For all $x \in X - B_X$, $L_x = B_X$.*

FIGURE 2.

Proof. Assertion 1) just restates part of Proposition 3.1, while Assertion 2) is a basic fact in dimension theory. In particular, if $\dim X = n$, then there exists $x \in X$ such that $H_n(X, X - x) \neq 0$; see, for example, [Mi2, Lemma 2].

To prove 3), fix $x \in X - B_X$ and suppose $y \in X - L_x$. We must show that $y \notin B_X$, i.e., that $H_*(X, X - y)$ is non-trivial.

Choose $\varepsilon < 1$ sufficiently small that $N(y, \varepsilon) \cap L_x = \varnothing$. Choose $z \in L_y$ such that x lies on the open cone line $\overset{\circ}{J}_y(z)$. By Lemma 2.2, $H_*(X, X - x) \cong H_*(X, X - x')$ for all $x' \in \overset{\circ}{J}_y(z)$. Therefore, $\overset{\circ}{J}_y(z) \cap L_x = \varnothing$. By 'pushing out along' $\overset{\circ}{J}_y(z)$ we may expand the ε-cone neighborhood $N(y, \varepsilon)$ about y to a warped λ-cone neighborhood $N(y, \lambda)$ which contains x in its interior and is disjoint from L_x. See Figure 2.

Since the inclusions $(X, L_x) \hookrightarrow (X, X - x)$ and $(X, X - N(y, \lambda)) \hookrightarrow (X, X - y)$ are both homotopy equivalences of pairs, we have inclusion induced isomorphisms

$$H_*(X, L_x) \overset{\cong}{\longrightarrow} H_*(X, X - x), \text{ and}$$

$$H_*(X, X - N(y, \lambda)) \overset{\cong}{\longrightarrow} H_*(X, X - y)$$

The first of these can be factored via inclusions as such:

$$H_*(X, L_x) \overset{\phi}{\longrightarrow} H_*(X, X - N(y, \lambda)) \overset{\psi}{\longrightarrow} H_*(X, X - x)$$

Then ϕ is necessarily injective and ψ surjective, so $H_*(X, X - N(y, \lambda))$ is non-trivial. Thus, $H_*(X, X - y) \neq 0$. $\qquad\square$

Corollary 3.3. *For all $x \in X - B_X$, $H_*(X, X - x) \cong \widetilde{H}_{*-1}(B_X)$. This homology is finitely generated.*

Proof. Since $B_X = L_x$, the isomorphism follows from Proposition 3.1. Since $B_X = L_x$ is a compact ENR, it has the homotopy type of a finite CW complex [We]; hence, finitely generated homology. $\qquad\square$

Corollary 3.4. *For any $x, y \in X - B_X$, there exists a homeomorphism $f : X \to X$ which is the identity on B_X and sends x to y.*

Proof. Since $L_x = B_X = L_y$, we may let $f = h_y \circ h_x^{-1}$. $\qquad\square$

Theorem 3.5. *If X is an n-dimensional absolute cone, then*
   (1) *X is an ENR homology n-manifold with boundary,*
   (2) *∂X is precisely the link L_x of any point $x \in int(X)$,*
   (3) *∂X is an ENR homology $(n-1)$-manifold homotopy equivalent to S^{n-1}, and*
   (4) *for each $z \in \partial X$, L_z is a contractible ENR homology $(n-1)$-manifold with boundary.*

Proof. As above, let $B_X = \{z \in X \mid H_*(X, X - z) \equiv 0\}$. By Proposition 3.1 and Corollaries 3.3 and 3.4, $X - B_X$ is a homogeneous n-dimensional ENR with finitely generated local homology. By an application of [Bre] or [Bry], $X - B_X$ is an ENR homology n-manifold. Therefore, X is an ENR homology n-manifold with boundary, and $\partial X = B_X$.

Assertion 2) is a restatement of Proposition 3.2.

Proposition 3.1 and an application of [Mi2] (as discussed earlier) tells us that ∂X is an ENR homology $(n-1)$-manifold. Moreover, by Assertion 1) and Corollary 3.3, for any $x \in int(X)$,

$$\widetilde{H}_{k-1}(\partial X) \cong H_k(X, X - x) \cong \begin{cases} \mathbb{Z} & \text{if } k = n \\ 0 & \text{if } k \neq n \end{cases}$$

Thus, ∂X has the homology of S^{n-1}. If $n = 1, 2$ or 3 it is known that every homology $(n-1)$-manifold is an actual $(n-1)$-manifold [Wi, Ch.IX]; and in those dimensions a manifold is determined by its homology. Hence ∂X is homeomorphic to S^{n-1}. In higher dimensions we only claim a homotopy equivalence between S^{n-1} and ∂X. This may be obtained in the usual way if we can show that ∂X is simply connected. In particular, the Hurewicz Theorem would then assure us that $\pi_{n-1}(\partial X) \cong H_{n-1}(\partial X) \cong \mathbb{Z}$. A generator of $\pi_{n-1}(\partial X)$ provides a degree 1 map from S^{n-1} to ∂X. Since ∂X is an ANR—and thus has the homotopy type of a CW complex—a theorem of Whitehead shows that this map is a homotopy equivalence. We hold off proving simple connectivity of ∂X until after we verify Assertion 4).

To prove 4), note that the homeomorphism $h_z : cone(L_z) \to X$ induces a homeomorphism of $L_z \times (0, 1]$ onto $X - z$, taking $L_z \times \{1\}$ onto L_z. Since $L_z \times (0, 1]$ is an ENR homology n-manifold with boundary, L_z is an ENR homology $(n-1)$-manifold with boundary. This is an application of [Ra, Th.6]. We have already observed (Proposition 3.1) that L_z is contractible.

Lastly we complete Assertion 3) by showing that ∂X is simply connected when $n \geq 2$. Since the above mentioned homeomorphism $L_z \times (0, 1] \to X - z$ must take (homology) boundary to boundary, it follows that $\partial X - z$ has the structure of L_z

with an open collar attached to ∂L_z. Thus, $\partial X - z$ is contractible; and z has a neighborhood in ∂X homeomorphic to a cone over ∂L_z. Therefore, ∂X may be viewed as the union of open sets $\partial X - z$ and U, where $\partial X - z$ is contractible and U is homeomorphic to the open cone on ∂L_z. Since the intersection of these sets is connected, simple connectivity follows from Van Kampen's theorem. $\qquad\square$

The above proof provides some additional structure information about absolute cones which we record as:

Theorem 3.6. *If X is an n-dimensional absolute cone, then X is a contractible ENR homology n-manifold with boundary and ∂X is a locally conical ENR homology $(n-1)$-manifold; more specifically, each point of ∂X has a neighborhood in ∂X which is a cone over an ENR homology $(n-2)$-manifold with the homology of S^{n-2}.*

Proof. By the above proof, each $z \in \partial X$ has a neighborhood in ∂X homeomorphic to $cone\,(\partial L_z)$. Since this cone lies in ∂X, ∂L_z must be an ENR. Moreover, since L_z is an ENR homology $(n-1)$-manifold with boundary, then ∂L_z is an ENR homology $(n-2)$-manifold. Lastly, since ∂X has the local homology of an $(n-1)$-manifold at z, the homology type of ∂L_z must be that of an $(n-2)$-sphere. $\qquad\square$

Corollary 3.7. *If X is an n-dimensional absolute cone and $n \leq 3$, then X is an n-cell. The same is true for $n = 4$, provided the 3-dimensional Poincaré Conjecture is true.*

Proof. If $n \leq 3$, we have already observed in the proof of Theorem 3.5 that $B_X = \partial X$ is homeomorphic to S^{n-1}. Thus, $X \approx cone\,(S^{n-1}) \approx B^n$.

If $n = 4$, we have shown that ∂X is an ENR homology 3-manifold homotopy equivalent to S^3. In addition, we know that each point of ∂X has a neighborhood in ∂X homeomorphic to the cone over an ENR homology 2-manifold having the homology of S^2. As above, such ENR homology 2-manifolds are 2-spheres. Thus, ∂X is an actual 3-manifold. Assuming the 3-dimensional Poincaré Conjecture, $\partial X \approx S^3$; so X is a 4-cell. $\qquad\square$

Remark 2. *At the conclusion of the next section, we will show that if there exists a homotopy 3-sphere H^3, not homeomorphic to S^3, then $cone\,(H^3)$ is a 4-dimensional absolute cone that is not a 4-cell.*

4. Counterexamples in higher dimensions

The main goal of this section is to construct, for all $n \geq 5$, n-dimensional absolute cones which are not n-cells. In all cases, we begin with a non-simply connected k-manifold Σ^k having the same $\mathbb{Z}$-homology as S^k. Existence of such manifolds for all $k \geq 3$ is well-known. Our counterexamples are obtained by first coning over Σ^k, then suspending that cone. This section is primarily devoted to proving that the resulting spaces are absolute cones, but not cells.

For completeness, we will conclude this section by showing that—if there is a counterexample to the 3-dimensional Poincaré—then there is also a 4-dimensional absolute cone that is not a 4-cell.

We begin our construction of counterexamples in dimensions ≥ 5 with a very general lemma.

Lemma 4.1. *For any space Y, the following are homeomorphic.*

(1) $susp\,(cone\,(Y))$
(2) $cone\,(susp\,(Y))$
(3) $cone\,(cone\,(Y))$
(4) $cone\,(Y) \times [0,1]$.

Before proving this lemma, consider the map $f : cone\,(Y) \times [0,1] \to cone\,(Y) \times [0,1]$ defined by

$$f\,(t \cdot z, s) = ((st) \cdot z, s)\,.$$

This map takes $cone\,(Y) \times \{1\}$ identically onto $cone\,(Y) \times \{1\}$, and each level set $cone\,(Y) \times \{s\}$ to the subcone of radius s contained in $cone\,(Y) \times \{s\}$; finally $cone\,(Y) \times \{0\}$ is taken to the cone point of $cone\,(Y) \times \{1\}$. This map induces a level-preserving embedding of $cone\,(cone\,(Y))$ into $cone\,(Y) \times [0,1]$. The image of the embedding is a particularly nice realization of $cone\,(cone\,(Y))$ which we will denote by $\mathcal{C}^2\,(Y)$.

A similar map $g : cone\,(Y) \times [0,1] \to cone\,(Y) \times [0,1]$ can be used to induce an embedding of $susp\,(cone\,(Y))$ into $cone\,(Y) \times [0,1]$. We will denote the image of that map by $\mathcal{SC}\,(Y)$. Each of the spaces $cone\,(Y) \times [0,1]$, $\mathcal{C}^2\,(Y)$ and $\mathcal{SC}\,(Y)$ contain as a subspace $\{\text{cone point}\} \times [0,1]$, which we call the *axis* and denote by A. In addition, the points (cone point, 0), (cone point,1), and (cone point, $\frac{1}{2}$) will be denoted p_0, p_1 and p_*, respectively. See Figure 3.

Proof of Lemma 4.1. We first show that $cone\,(cone\,(Y)) \approx cone\,(Y) \times [0,1]$ by illustrating a homeomorphism from $cone\,(Y) \times [0,1]$ to $\mathcal{C}^2\,(Y)$. Each cone line I_y of $cone\,(Y)$ determines a 'square' $S_y = I_y \times [0,1]$ in $cone\,(Y) \times [0,1]$. Similarly, I_y determines a 'right triangle' T_y in $\mathcal{C}^2\,(Y)$ such that T_y and S_y have two common sides: $I_y \times \{1\}$ and A. See Figure 4. For a given y choose a homeomorphism

$$k_y : S_y \to T_y$$

which is the identity on their common sides. By using the 'same' homeomorphism for each $y \in Y$, we may combine these into a single homeomorphism.

$$k : cone\,(Y) \times [0,1] \to \mathcal{C}^2\,(Y)\,.$$

A similar strategy produces a homeomorphism of $cone\,(Y) \times [0,1]$ onto $\mathcal{SC}\,(Y)$. Thus we have $cone\,(cone\,(Y)) \approx cone\,(Y) \times [0,1] \approx susp\,(cone\,(Y))$.

Lastly, we show that $susp\,(cone\,(Y)) \approx cone\,(susp\,(Y))$ by observing that $\mathcal{SC}\,(Y)$ may be given a cone structure with base $\mathcal{S}\,(Y)$ and cone point p_*. To this end, we view $\mathcal{SC}\,(Y)$ as the union of its 'suspension triangles' intersecting in A; each triangle being the suspension of a cone line. To place a cone structure on $\mathcal{SC}\,(Y)$ we place the obvious common cone structure on each of these triangles, with p_* serving as cone point. See Figure 5.
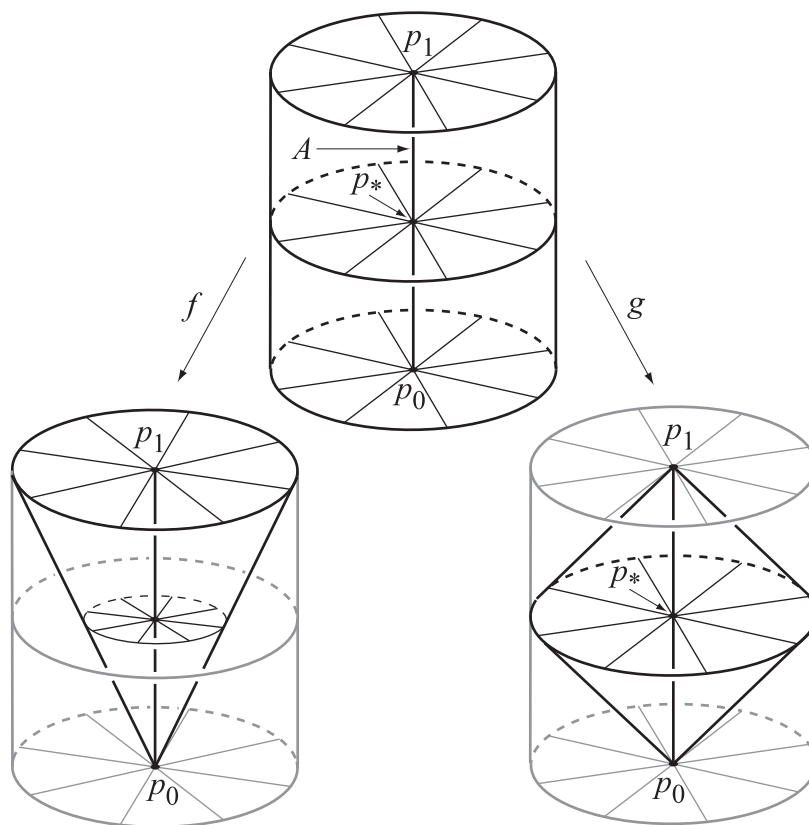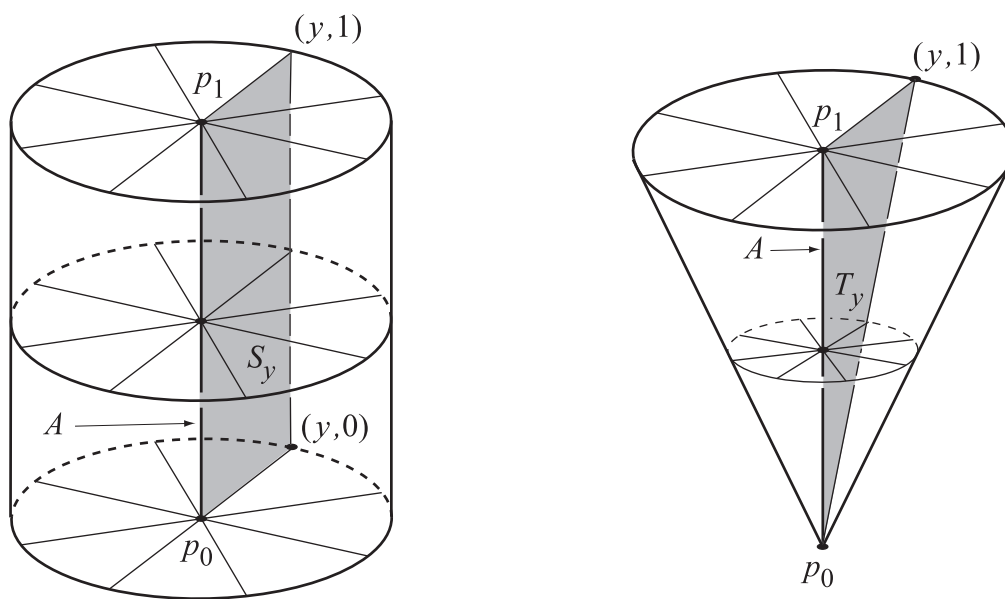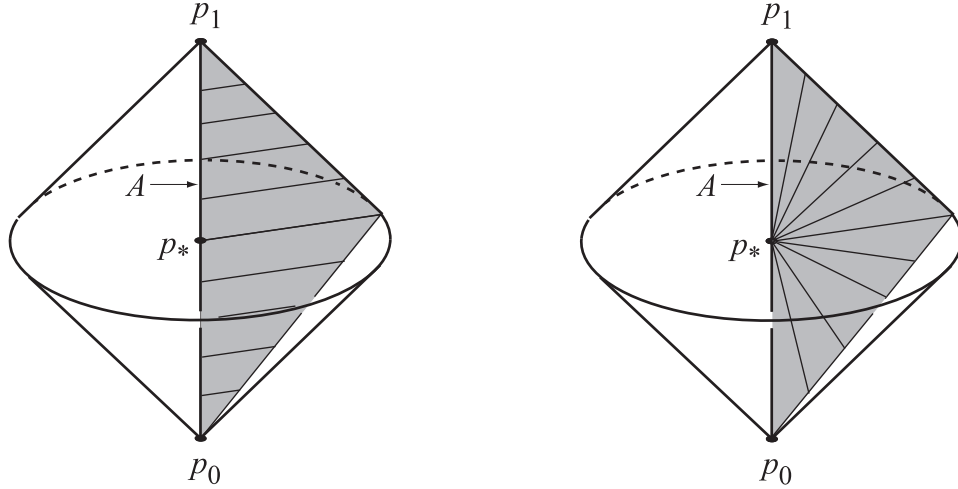
$\square$

FIGURE 3.

FIGURE 4.

FIGURE 5.

Notation 1. *For convenience, we denote the each of the homeomorphic topological spaces described by 1)-4) of Lemma 4.1 by $\Theta\left(Y\right)$.*

The following lemma will be useful later. We include it now because its proof is similar to the last part of the above argument.

Lemma 4.2. *For any map $f : Y \to K$, the pair $\left(Map\left(f\right) \times [0,1], K \times \left\{\frac{1}{2}\right\}\right)$ is homeomorphic to $\left(Map\left(F\right), K\right)$ for some map $F : \left(Y \times [0,1]\right) \cup \left(Map\left(f\right) \times \{0,1\}\right) \to K$. The homeomorphism may be chosen to take each point $\left(k, \frac{1}{2}\right) \in K \times \left\{\frac{1}{2}\right\}$ to k.*

Proof. To simplify matters, we assume that f is surjective. If it is not, the argument requires only minor adjustments.

By surjectivity, $Map\left(f\right)$ is the union of its cylinder lines $\{E_y \mid y \in Y\}$. Hence, $Map\left(f\right) \times [0,1]$ is a union of 'squares' $\{S_y \mid y \in Y\}$, where S_y denotes $E_y \times [0,1]$. Furthermore, each S_y intersects the proposed domain of our map F in three of its four boundary edges. More precisely,

$$S_y \cap \left(\left(Y \times [0,1]\right) \cup \left(Map\left(f\right) \times \{0,1\}\right)\right) = \left(\{y\} \times [0,1]\right) \cup \left(E_y \times \{0,1\}\right).$$

Due to its shape, we denote the right-hand side of the above equation by C_y. Then $\left(Y \times [0,1]\right) \cup \left(Map\left(f\right) \times \{0,1\}\right)$ is the union of the collection $\{C_y \mid y \in Y\}$ and, for all $y, y' \in Y$, $C_y \cap C_{y'} = \varnothing$ unless $f(y) = f\left(y'\right)$.

Define F by sending each point of C_y to $f\left(y\right)$. Then

$$F : \left(Y \times [0,1]\right) \cup \left(Map\left(f\right) \times \{0,1\}\right) \to K$$

is continuous, and for each $y \in Y$, the sub-mapping cylinder $Map\left(F|_{C_y}\right)$ is simply $cone(C_y)$, with cone point $f\left(y\right) \in K$. Thus, $Map\left(F\right)$ is a union of the collection $\{cone\left(C_y\right) \mid y \in Y\}$. To produce the desired homeomorphism from $Map\left(f\right) \times [0,1]$ to $Map\left(F\right)$, choose a coherent collection of homeomorphisms from the squares $\{S_y\}$

making up $Map\,(f) \times [0,1]$ to the cones $\{cone(C_y)\}$ making up $Map\,(F)$. In particular, for each y, define a homeomorphism $h_y : S_y \to cone(C_y)$ which: takes $\left(f\,(y), \frac{1}{2}\right)$ to the cone point $f\,(y)$, is the identity on C_y, and is linear on segments in between these subspaces. The union of these homeomorphisms is the desired homeomorphism from $Map\,(f) \times [0,1]$ to $Map\,(F)$. $\square$

We are now ready to prove the main theorem of this section.

Theorem 4.3. *If Σ^k is a closed k-manifold with the same $\mathbb{Z}$-homology as the k-sphere, then $\Theta\left(\Sigma^k\right)$ is a $(k+2)$-dimensional absolute cone. If Σ^k is not simply connected, then $\Theta\left(\Sigma^k\right)$ is not homeomorphic to an $(k+2)$-cell.*

Proof. We begin by observing that $\Theta\left(\Sigma^k\right)$ is ENR homology $(k+2)$-manifold with boundary. To experts on homology manifolds, this may be obvious; otherwise, proceed as follows. First note that $cone\left(\Sigma^k\right)$ is an actual $(k+1)$-manifold with boundary at all points except the cone point p. At that point

$$H_*\left(cone\left(\Sigma^k\right), cone\left(\Sigma^k\right) - p\right) \cong H_*\left(cone\left(\Sigma^k\right), \Sigma^k\right)$$
$$\cong \widetilde{H}_{*-1}\left(\Sigma^k\right)$$
$$\cong \widetilde{H}_{*-1}\left(S^k\right) \cong \begin{cases} \mathbb{Z} & \text{if } * = k+1 \\ 0 & \text{otherwise} \end{cases}$$

Thus, $cone\left(\Sigma^k\right)$ is an ENR homology $(k+1)$-manifold with boundary; moreover, $int\left(cone\left(\Sigma^k\right)\right) = opencone\left(\Sigma^k\right)$. By [Ra] or straightforward calculation, $cone\left(\Sigma^k\right) \times [0,1]$ is an ENR homology $(k+2)$-manifold with boundary, and

$$int\left(cone\left(\Sigma^k\right) \times [0,1]\right) = opencone\left(\Sigma^k\right) \times (0,1).$$

CLAIM 1. $\partial\left(\Theta\left(\Sigma^k\right)\right)$ *is not a $(k+1)$-manifold. Therefore $\Theta\left(\Sigma^k\right)$ is not an $(n+2)$-cell.*

Under the realization of $\Theta\left(\Sigma^k\right)$ as $susp(cone\left(\Sigma^k\right))$, the boundary is $susp\left(\Sigma^k\right)$. If $susp\left(\Sigma^k\right)$ were an actual $(k+1)$-manifold, then removing a finite collection of points would not change its fundamental group. But $susp\left(\Sigma^k\right)$ is simply connected, and $susp\left(\Sigma^k\right) - \{p_0, p_1\} \approx \Sigma^k \times (0,1)$ is not simply connected. The claim follows.

We now work toward showing that $\Theta\left(\Sigma^k\right)$ is an absolute cone. For each $x \in \Theta\left(\Sigma^k\right)$, we must identify a subspace L_x of $\Theta\left(\Sigma^k\right)$ and a homeomorphism of $cone\,(L_x)$ onto $\Theta\left(\Sigma^k\right)$ taking the cone point to x. The proof splits into the following three cases:

- $x \in int\left(\Theta\left(\Sigma^k\right)\right)$,
- $x = p_0$ or p_1,
- $x \in \partial\left(\Theta\left(\Sigma^k\right)\right) - \{p_0, p_1\}$

Case 1. $x \in int\left(\Theta\left(\Sigma^k\right)\right)$.

In this case we know from Theorem 3.5 that, if L_x exists, it must equal $\partial\left(\Theta\left(\Sigma^k\right)\right)$. By applying Lemma 4.1 to realize $\Theta\left(\Sigma^k\right)$ as $cone\left(susp\left(\Sigma^k\right)\right)$, we see that $\Theta\left(\Sigma^k\right)$ is indeed homeomorphic to the cone over $\partial\left(\Theta\left(\Sigma^k\right)\right)$. Let $p_* \in int\left(\Theta\left(\Sigma^k\right)\right)$ be the corresponding cone point and $h_{p_*} : cone(\partial\left(\Theta\left(\Sigma^k\right)\right)) \to \Theta\left(\Sigma^k\right)$ the homeomorphism. We must show that all other elements of $int\left(\Theta\left(\Sigma^k\right)\right)$ can be viewed similarly. The following is the essential ingredient.

CLAIM 2. $int\left(\Theta\left(\Sigma^k\right)\right)$ is an actual $(k+2)$-manifold. In fact, $int\left(\Theta\left(\Sigma^k\right)\right) \approx \mathbb{R}^{k+2}$.

As noted earlier, we may view $int\left(\Theta\left(\Sigma^k\right)\right)$ as $opencone\left(\Sigma^k\right) \times (0,1)$. That this space is a $(k+2)$-manifold is a direct application of work by J.W. Cannon and R.D. Edwards on the 'double suspension problem' [Ca], [Ed]. A nice exposition of the relevant result may be found in [Da, Cor. 24.3D]. Once we know that $int\left(\Theta\left(\Sigma^k\right)\right)$ is a manifold, an application of [St] gives us the homeomorphism to $\mathbb{R}^{k+2}$.

Now let x be an arbitrary element of $int\left(\Theta\left(\Sigma^k\right)\right)$. By a standard homogeneity argument for manifolds, there is a homeomorphism $u : int\left(\Theta\left(\Sigma^k\right)\right) \to int\left(\Theta\left(\Sigma^k\right)\right)$ taking p_* to x. Moreover, we may choose u to be the identity outside some compact neighborhood of $\{p_*, x\}$. This allows us to extend u to a homeomorphism $\overline{u}$ of $\Theta\left(\Sigma^k\right)$ to itself. The homeomorphism $\overline{u} \circ h_{p_*} : cone(\partial\left(\Theta\left(\Sigma^k\right)\right)) \to \Theta\left(\Sigma^k\right)$ now realizes x as the cone point.

Case 2. $x = p_0$ or p_1.

Use Lemma 4.1 to view $\Theta\left(\Sigma^k\right)$ as $cone\left(cone\left(\Sigma^k\right)\right)$ (or more precisely $\mathcal{C}^2\left(\Sigma^k\right)$). Then p_0 corresponds to the cone point; and the base, $cone\left(\Sigma^k\right)$, serves as L_{p_0}. Furthermore, the realizations of $\Theta\left(\Sigma^k\right)$ provided by 1) or 3) of Lemma 4.1 reveal an involution of $\Theta\left(\Sigma^k\right)$ interchanging p_0 and p_1. Thus, p_1 also may be viewed as a cone point.

Case 3. $x \in \partial\left(\Theta\left(\Sigma^k\right)\right) - \{p_0, p_1\}$.

By realization 1) of Lemma 4.1, $\Theta\left(\Sigma^k\right) - \{p_1, p_0\} \approx cone\left(\Sigma^k\right) \times (0,1)$, which by another application of Cannon-Edwards, is a $(k+2)$-manifold with boundary. That boundary corresponds to $\Sigma^k \times (0,1)$. By another standard homogeneity argument for manifolds, any two points of $\Sigma^k \times (0,1)$ can be interchanged by a homeomorphism of $cone\left(\Sigma^k\right) \times (0,1)$. This homeomorphism can be arranged to be the identity off a compact set; and, thus, extends to a self-homeomorphism of $susp\left(cone\left(\Sigma^k\right)\right)$. This means that it suffices to find a single point $x_0 \in \partial\left(\Theta\left(\Sigma^k\right)\right) - \{p_1, p_0\}$ at which $\Theta\left(\Sigma^k\right)$ is conical.

To this end, we return to the realization of $\Theta\left(\Sigma^k\right)$ as $cone\left(\Sigma^k\right) \times [0,1]$ and choose $x_0 \in \Sigma^k \times \{\frac{1}{2}\}$. Choose nice k-cell neighborhoods $B_1^k \subseteq B_0^k$ of x_0 in $\Sigma^k \times \{\frac{1}{2}\}$ and, by a mild abuse of notation, let I_{x_0} denote the corresponding cone line as a subset of $cone(\Sigma^k) \times \{\frac{1}{2}\}$. By a combination of Lemmas 2.3 and 4.2,

$$\left(cone\left(\Sigma^k\right) \times [0,1], I_{x_0}\right) \approx \left(Map\left(F\right), I_{x_0}\right)$$

for some map

$$F : ((\Sigma^k - int(B_1^k)) \times [0, 1]) \cup (cone\,(\Sigma^k) \times \{0, 1\}) \to I_{x_0}.$$

Crushing out the range end of the mapping cylinder yields the cone over its domain. Hence,

(‡) $\,cone\,(\Sigma^k) \times [0, 1]\,/I_x \approx cone\,(((\Sigma^k - int(B^k)) \times [0, 1]) \cup (cone\,(\Sigma^k) \times \{0, 1\}))\,.$

Note that, as a subspace of the manifold with boundary $cone\,(\Sigma^k) \times (0, 1)$, I_{x_0} is a tame arc intersecting the boundary in a single end point. (Local tameness for this arc is obvious at all except the interior end point of I_x. This can be seen from the nice tubular neighborhood provided by the obvious structure of $cone\,(\Sigma^k) \times [0, 1]$. But, since our manifold has dimension ≥ 4, I_{x_0} cannot have a wild set consisting of a single point (see [CE] or [Ru, Th.3.2.1]). Thus I_{x_0} is tame in $cone\,(\Sigma^k) \times (0, 1)$.)

The tameness of I_{x_0} in $cone\,(\Sigma^k) \times [0, 1]$ ensures that

$$cone\,(\Sigma^k) \times [0, 1]\,/I_{x_0} \approx cone\,(\Sigma^k) \times [0, 1]\,.$$

This homeomorphism can to chosen to take the equivalence class of I_{x_0} to x_0. Combining this homeomorphism with (‡) induces the necessary cone structure on $cone\,(\Sigma^k) \times [0, 1]$ with x_0 corresponding to the cone point and

$$L_{x_0} = cone\,(((\Sigma^k - int(B^k) \times [0, 1]) \cup (cone\,(\Sigma^k) \times \{0, 1\}))\,.$$

Note. By the equivalence of tame $(k + 1)$-cells in a $(k + 1)$-manifold, L_{x_0} may be more easily visualized as the complement of any tame open $(k + 1)$-cell neighborhood of x_0 in the manifold portion of $\partial\,(\Theta\,(\Sigma^k))$.   □

Lastly, we show that the 4-dimensional version of the absolute cone conjecture is equivalent to the 3-dimensional Poincaré Conjecture.

Theorem 4.4. *Every* 4-*dimensional absolute cone is the cone over a homotopy* 3-*sphere; moreover, if* H^3 *is a* 3-*manifold homotopy equivalent to* S^3, *then* $cone\,(H^3)$ *is an absolute cone. If* H^3 *is not homeomorphic to* S^3, *then* $cone\,(H^3)$ *is not a* 4-*cell.*

Proof. We have already shown that, if X is a 4-dimensional absolute cone, then ∂X is a 3-manifold homotopy equivalent to S^3; thus X is homeomorphic to the cone over a homotopy 3-sphere. The last statement of the theorem is obvious. Therefore, it remains only to show that $cone\,(H^3)$ is, indeed, an absolute cone. Our proof utilizes Freedman's breakthrough work on 4-dimensional manifolds. Aside from that application, the proof is just a simpler version of work we have already done.

CLAIM. $cone\,(H^3)$ *is an absolute cone.*
By [Fr, Cor.1.3], $H^3 \times \mathbb{R} \cong \mathbb{S}^3 \times \mathbb{R}$. It follows that $cone\,(H^3)$ is a 4-manifold with boundary—that boundary being the base H^3. By a homogeneity argument similar to one used earlier, any point of the interior, $opencone\,(H^3)$, may be realized as a cone point with H^3 as its link.
If $x_0 \in H^3$, use Lemma 2.3 to realize $cone\,(H^3)$ as a mapping cylinder with I_{x_0} corresponding to the 'range end'. As before, this arc is tame in the manifold $cone\,(H^3)$.

Therefore, $cone\left(H^3\right)/I_{x_0} \approx cone\left(H^3\right)$; moreover, $cone\left(H^3\right)/I_{x_0}$ is also homeomorphic to a cone with the equivalence class of I_{x_0} as its cone point (and the complement of an open 3-cell as its base). These homeomorphisms yield a cone structure on $cone\left(H^3\right)$ with x_0 as the cone point. $\square$

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