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A non-$\mathcal{Z}$-compactifiable polyhedron whose product with the Hilbert cube is $\mathcal{Z}$-compactifiable

by

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Abstract. We construct a locally compact 2-dimensional polyhedron X which does not admit a $\mathcal{Z}$-compactification, but which becomes $\mathcal{Z}$-compactifiable upon crossing with the Hilbert cube. This answers a long-standing question posed by Chapman and Siebenmann in 1976 and repeated in the 1976, 1979 and 1990 versions of *Open Problems in Infinite-Dimensional Topology*. Our solution corrects an error in the 1990 problem list.

1. Introduction. In 1976, T. A. Chapman and L. S. Siebenmann proved a Hilbert cube manifolds version of Siebenmann's famous thesis [Si], in which necessary and sufficient conditions were given for an open n-manifold ($n > 5$) to be compactifiable by addition of a boundary $(n-1)$-manifold. Since Hilbert cube manifolds do not have intrinsically defined boundaries (the Hilbert cube, $\mathcal{Q} = [-1,1]^{\infty}$, is itself homogeneous!), Chapman and Siebenmann's first task was to determine an appropriate notion for the "boundary" of a Hilbert cube manifold. The concept they arrived at involved $\mathcal{Z}$-sets. A closed subset A of a compact ANR Y is a *$\mathcal{Z}$-set* if either of the following equivalent conditions is satisfied:

- There is a homotopy $H : Y \times I \to Y$ with $H_0 = \mathrm{id}_Y$ and $H_t(Y) \cap A = \emptyset$ for all $t > 0$.
- For every open set U of Y, $U \setminus A \hookrightarrow U$ is a homotopy equivalence.

If X is a non-compact ANR, a *$\mathcal{Z}$-compactification* of X is a compact ANR $\widehat{X}$ containing X as an open subset and having the property that $\widehat{X} \setminus X$ is a $\mathcal{Z}$-set in $\widehat{X}$.

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Remark 1. Actually, one need not assume that $\widehat{X}$ is an ANR in the above definition. As noted in Appendix 3 of [CS], a compactification $\widehat{X}$ of an ANR X such that $\widehat{X}\backslash X$ satisfies the $\mathcal{Z}$-set criteria is necessarily an ANR.

Evidence that $\mathcal{Z}$-sets and $\mathcal{Z}$-compactifications are the correct analogs of n-manifold boundaries and boundary compactifications is plentiful—the boundary of a compact n-manifold is a $\mathcal{Z}$-set in that manifold, a $\mathcal{Z}$-compactification of a Hilbert cube manifold always yields a Hilbert cube manifold, and a $\mathcal{Z}$-set in a Hilbert cube manifold is always contained in a collared subset. But the ultimate proof that $\mathcal{Z}$-sets are the correct "boundaries" for Hilbert cube manifolds lies in the beautiful characterization provided by Chapman and Siebenmann.

Theorem 1.1 (see [CS, Ths. 3 and 4]). *A Hilbert cube manifold X admits a $\mathcal{Z}$-compactification iff each of the following is satisfied*:

(a) *X is inward tame at infinity.*
(b) $\sigma_\infty(X) \in \varprojlim\{\widetilde{K}_0(\pi_1(X \setminus A)) \mid A \subset X$ *compact*$\}$ *is zero.*
(c) $\tau_\infty(X) \in \varprojlim^1\{\mathrm{Wh}(\pi_1(X \setminus A)) \mid A \subset X$ *compact*$\}$ *is zero.*

We will provide definitions of the above terminology and notation in the following section. For now, we note that conditions (a) and (b) are direct analogs of conditions found in Siebenmann's thesis, and condition (c) was automatically satisfied there due to an assumption (not needed for Theorem 1.1) that the fundamental group be "stable at infinity".

At the end of [CS], the authors point out that the notion of $\mathcal{Z}$-compactification can be applied to any locally compact ANR. A theorem of R. D. Edwards [Ed] guarantees that the product of any locally compact ANR with the Hilbert cube is a Hilbert cube manifold. Hence it is observed that, in order for a locally compact ANR X to be $\mathcal{Z}$-compactifiable, it is necessary that $X \times \mathcal{Q}$ be $\mathcal{Z}$-compactifiable. Chapman and Siebenmann conclude their paper with the question: Is it sufficient? Their question was repeated and amplified in *Open Problems in Infinite-Dimensional Topology*, an appendix to Chapman's book on Hilbert cube manifolds, where it appears in two equivalent versions:

Question (see [Ch, p. 123]). *In order for a locally compact ANR X to be $\mathcal{Z}$-compactifiable, it is necessary that conditions* (a)–(c) *of Theorem* 1.1 *be satisfied. Is this sufficient?*

Equivalent Question. *If $X \times \mathcal{Q}$ is $\mathcal{Z}$-compactifiable, is X itself $\mathcal{Z}$-compactifiable?*

These questions reappeared as "Problem QM 8" in the 1979 and 1990 revisions of *Open Problems in Infinite-Dimensional Topology* (see [Ge] and [We]). Included in the latter problem list is an apparently incorrect proof that the answer to these questions is affirmative. We will show that the answer is, in fact, negative by proving the following:

THEOREM 1.2 (Main Theorem). *There exists a locally compact* 2*-dimensional polyhedron X with the property that $X \times \mathcal{Q}$ is $\mathcal{Z}$-compactifiable but X is not.*

Although the proof is complicated, the space X is quite simple. It is just the "infinite mapping telescope" of a direct sequence $S^1 \xrightarrow{\theta} S^1 \xrightarrow{\theta} S^1 \xrightarrow{\theta} \dots$ where θ is a degree 1 map which wraps the circle around itself twice counterclockwise, then once back in the clockwise direction. By contrast, the proof of Theorem 1.1 relies heavily on the fact that an infinite mapping telescope of an *inverse* sequence of finite polyhedra is always $\mathcal{Z}$-compactifiable. Hence, the "double infinite mapping telescope" Y of the system $\dots \xrightarrow{\theta} S^1 \xrightarrow{\theta} S^1 \xrightarrow{\theta} S^1 \xrightarrow{\theta} \dots$ (a space which occurs naturally as an infinite cyclic cover of a finite aspherical 2-complex) has two ends—one that admits a $\mathcal{Z}$-compactification and one that does not.

In concluding this section, we note that interest in $\mathcal{Z}$-compactifications of ANRs (and especially finite-dimensional polyhedra) has experienced a recent rebirth. See for example [BM], [Be], [FW], [CP] and [AG]. In light of our Main Theorem and these current trends, a closely related open question (also found in *Open Problems in Infinite Dimensional Topology*) seems particularly relevant:

QUESTION (CMP 1 in [Ch] and QM 7 in [Ge] and [We]). *When is a locally compact polyhedron $\mathcal{Z}$-compactifiable? In other words, is there a condition that can be added to conditions* (a)–(c) *which implies $\mathcal{Z}$-compactifiability for polyhedra?*

2. Some definitions and terminology. Let $\mathcal{Q}$ denote the *Hilbert cube*, $[-1,1]^\infty$. A *Hilbert cube manifold* is a separable metric space with the property that each of its points has a closed neighborhood homeomorphic to the Hilbert cube.

A locally compact separable metric space X is an *absolute neighborhood retract*, or *ANR*, if it may be embedded as a closed subset of $\mathbb{R}^\infty$ so that there exists a retraction $r : U \to X$, where U is a neighborhood of X in $\mathbb{R}^\infty$. If a retraction $r : \mathbb{R}^\infty \to X$ exists, then X is an *absolute retract*, or *AR*. It is well known [Mi, Th. 5.2.15] that an ANR is an AR if and only if it is contractible. When X is finite-dimensional, we may replace $\mathbb{R}^\infty$ in the above definitions with a finite-dimensional Euclidean space $\mathbb{R}^n$ (for n sufficiently large). An important characterization [Mi, Th. 5.5.7] states that a finite-dimensional locally compact separable metric space is an ANR if and only if it is locally contractible.

A non-compact ANR X is *inward tame at infinity* if, for any compact set $A \subset X$, there exists a homotopy $H : (X \setminus A) \times I \to X \setminus A$ so that $H_0 = \mathrm{id}$ and $\mathrm{cl}_X(H_1(X \setminus A))$ is compact. Equivalently, one may require that X contain arbitrarily large compact subsets A such that $X \setminus A$ is finitely dominated. If this condition holds, one may define an algebraic invariant $\sigma_\infty(X) \in \varprojlim\{\widetilde{K}_0(\pi_1(X \setminus A)) \mid A \subset X \text{ compact}\}$. Here $\widetilde{K}_0$ is the projective class group functor and all bonding maps are induced by inclusion. The individual "coordinates" of $\sigma_\infty(X)$ are the Wall finiteness obstructions for the $(X \setminus A)$'s (see [Wa1] and [Wa2]). Then $\sigma_\infty(X)$ vanishes iff X contains arbitrarily small neighborhoods of infinity having finite homotopy type. (A subset of X is a *neighborhood of infinity* if the closure of its complement is compact.) When $\sigma_\infty(X)$ vanishes, we may define the second algebraic invariant $\tau_\infty(X) \in \varprojlim^1\{\mathrm{Wh}(\pi_1(X \setminus A)) \mid A \subset X \text{ compact}\}$. Here $\varprojlim^1$ denotes the *first derived limit* (see [CS]) and Wh is the Whitehead group functor (see [Co]). Again, bonding maps are induced by inclusion.

Remark 2. In defining $\sigma_\infty(X)$ and $\tau_\infty(X)$ for an ANR, it is convenient to work with the Hilbert cube manifold, $X \times \mathcal{Q}$. Since Hilbert cube manifolds can be triangulated [Ch, Th. 37.2], this allows one to work in the category of CW-complexes where the Wall finiteness obstruction and Whitehead torsion are normally defined. For details and more definitions, see [CS].

Throughout this paper, $\approx$ will indicate a homeomorphism and $\simeq$ will indicate homotopic maps or homotopy equivalent spaces. A submanifold M of a finite-dimensional manifold N is *properly embedded* if $M \cap \partial N = \partial M$. In particular, an arc in N is properly embedded when it meets ∂M precisely in its endpoints.

3. The basic building block. In this section we describe, and then prove some simple properties of, a polyhedron K which is the basic building block for the example promised in the Main Theorem.

Let $\theta : S^1 \to S^1$ be a degree 1 map which wraps the unit circle twice around itself in the counterclockwise direction, then once back in the clockwise direction. Let K be a polyhedron with subpolyhedra L and L' so that K is homeomorphic to the mapping cylinder $\mathrm{Map}(\theta)$ with L corresponding to the domain end and L' corresponding to the range end. In particular, we may realize K as the quotient space of the rectangle $[0,3] \times [0,1]$ generated by the equivalences: $(0,y) \sim (3,y)$ for all $y \in [0,1]$, and $(x,0) \sim (1+x,0) \sim (3-x,0)$ for each $x \in [0,1]$. Let $q' : [0,3] \times [0,1] \to K$ be the corresponding quotient map. See Figure 1, in which the arrows indicate the identifications to be made.

Let Σ^1 be the "triangle" obtained as the quotient space $[0,3]/\{0 \sim 3\}$. Then we may factor the quotient map q' as $q \circ \pi$ where $\pi : [0,3] \times [0,1] \to$

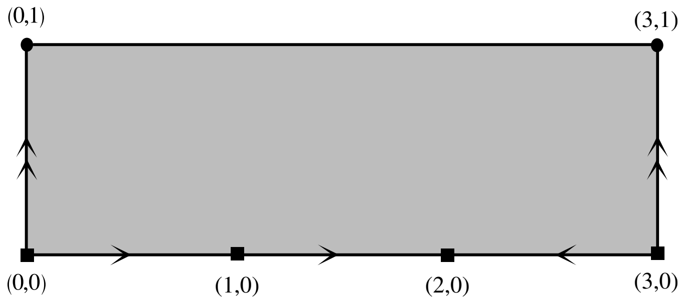

Fig. 1. A gluing diagram for K

$\Sigma^1 \times [0,1]$ identifies $(0,y)$ with $(3,y)$ for all $y \in [0,1]$ and $q : \Sigma^1 \times [0,1] \to K$ is the quotient map induced by the diagram

$$\begin{tikzcd} {[0,3] \times [0,1]} \arrow[r, "\pi"] \arrow[dr, "q'"'] & \Sigma^1 \times [0,1] \arrow[d, "q"] \\ & K \end{tikzcd}$$

Let $I_1 = \pi([0,1] \times \{0\})$, $I_2 = \pi([1,2] \times \{0\})$ and $I_3 = \pi([2,3] \times \{0\})$ denote the "sides" of $\Sigma^1 \times \{0\}$ with I_1 and I_2 oriented naturally (in the same direction as their preimages) and I_3 in the opposite direction. Then q identifies I_1, I_2 and I_3 according to these orientations. Let $e^1 = I_3 \cap I_1$, $e^2 = I_1 \cap I_2$ and $e^3 = I_2 \cap I_3$ be the "vertices" of $\Sigma^1 \times \{0\}$. Note that for each $i = 1, 2, 3$, $q|_{I_i}$ takes I_i onto L' by identifying the endpoints of I_i. The common image under q of e^1, e^2 and e^3 has some important geometric properties in K. We refer to this point as the *eye* of K and denote it by $\overline{e}$. See Figure 2.

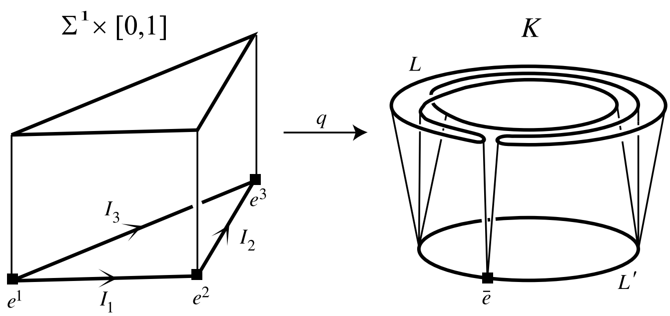

Fig. 2

For each point $\overline{a} \in L' \setminus \{\overline{e}\}$, $q^{-1}(\overline{a})$ contains three points—one in the interior of each of the I_i's. By a *triple of equivalent points* we mean a set

$\{a^1, a^2, a^3\}$ with $a^1 \in \text{int}(I_1)$, $a^2 \in \text{int}(I_2)$ and $a^3 \in \text{int}(I_3)$ such that $q(a^1) = q(a^2) = q(a^3)$. (For later convenience, we treat $\{e^1, e^2, e^3\}$ as a separate special case, and do not refer to it as a triple of equivalent points.)

Notice that the "top end" of the mapping cylinder K (the circle L) is the homeomorphic image under q of $\Sigma^1 \times \{1\}$. The images under q of the segments $\{x\} \times [0,1]$ may be viewed as the mapping cylinder lines of K.

For later use we place a partial ordering "$\prec$" on the set $\bigcup_{i=1}^3 \text{int}(I_i)$. Under $\prec$ each interval $\text{int}(I_i)$ is ordered according to its orientation, hence, I_1 and I_2 inherit the natural ordering induced by the map π and I_3 receives the reverse of the ordering suggested by π. Points which lie in different intervals are not comparable under $\prec$. Notice that if $\{a^1, a^2, a^3\}$ and $\{b^1, b^2, b^3\}$ are triples of equivalent points, then $a^i \prec b^i$ for some i if and only if $a^i \prec b^i$ for each $i = 1, 2, 3$.

LEMMA 3.1. *Let K, L, L' and $q' : \Sigma^1 \times [0,1] \to K$ be as described above. Then each of the following maps is a homotopy equivalence*:

(1) $L' \hookrightarrow K$,
(2) $L \hookrightarrow K$, *and*
(3) $q : \Sigma^1 \times [0,1] \to K$.

Proof. Since $K \approx \text{Map}(\theta)$ with L' as the range end of the mapping cylinder, one can strong deformation retract K onto L' sliding along the mapping cylinder lines. Hence, $L' \hookrightarrow K$ is a homotopy equivalence.

Let $r : K \to L'$ be the end result of the strong deformation retraction described above. Then r is a homotopy equivalence, and since θ is a homotopy equivalence and the following diagram commutes, then $L \hookrightarrow K$ is also a homotopy equivalence.

$$\begin{tikzcd} L \arrow[r, hook] \arrow[dr, "\theta"'] & K \arrow[d, "r"] \\ & L' \end{tikzcd}$$

Since the other three maps in the following diagram are homotopy equivalences, so is q.

$$\begin{tikzcd} \Sigma^1 \times \{1\} \arrow[r, "q", "\approx"'] \arrow[d, "\cap"'] & L \arrow[d, "\cap"'] \\ \Sigma^1 \times [0,1] \arrow[r, "q"] & K \end{tikzcd} \quad \blacksquare$$

The next result shows some symmetry in the map $q : \Sigma^1 \times [0,1] \to K$, which will be used to simplify several later arguments.

LEMMA 3.2 (Symmetry Property). *There is an involution $\iota : \Sigma^1 \times [0,1] \to \Sigma^1 \times [0,1]$ for which $\iota(e^1) = e^3$, $\iota(e^2) = e^2$, $\iota(I_1) = I_2$, $\iota(I_3) = I_3$, and which induces an involution $\bar{\iota} : K \to K$ so that the following diagram commutes*:

$$\begin{tikzcd} \Sigma^1 \times [0,1] \arrow[r, "\iota"] \arrow[d, "q"'] & \Sigma^1 \times [0,1] \arrow[d, "q"'] \\ K \arrow[r, "\bar{\iota}"] & K \end{tikzcd}$$

Proof. For each $y \in [0,1]$, view $\Sigma^1 \times \{y\}$ as an equilateral triangle with vertices $\pi(0,y)$, $\pi(1,y)$ and $\pi(2,y)$ and let ℓ_y be the line which contains the vertex $\pi(1,y)$ and bisects the opposite side. The involution ι acts by reflecting each triangle $\Sigma^1 \times \{y\}$ about the line ℓ_y. ■

Before proving the final lemma of this section, we describe another useful picture—a "parallelogram diagram" of the space K. A parallelogram diagram for K is obtained by cutting $\Sigma^1 \times [0,1]$ open along any properly embedded arc β with one endpoint in each boundary component. Let P_β be the resulting "parallelogram" and $q_\beta : P_\beta \to K$ the obvious quotient map. Normally β will not intersect $\Sigma^1 \times \{0\}$ at e^1, e^2 or e^3, so one of the intervals I_i ($i = 1, 2$ or 3) will be separated by β. In this case we let I_i' denote the initial portion of I_i and I_i'' denote the final portion. Here "initial" and "final" are determined by the direction of the arrow. See Figure 3.

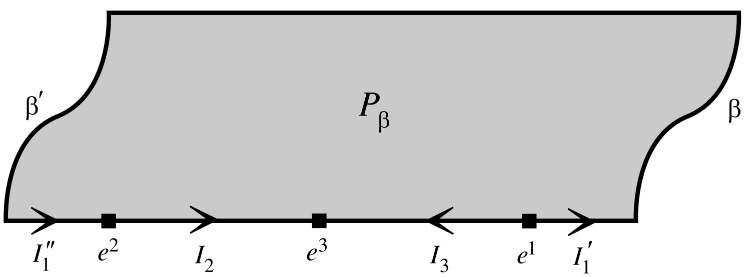

Fig. 3. A parallelogram diagram

LEMMA 3.3. *Let $\{a^1, a^2, a^3\}$ be a triple of equivalent points from $\Sigma^1 \times \{0\}$ and let α be a properly embedded arc in $\Sigma^1 \times [0,1]$ from a^i to a^j with $i \neq j$. Then the loop $\tau = q(\alpha)$ does not contract in $K \setminus \{\bar{e}\}$.*

Proof. Recall that, by definition, α is disjoint from $q^{-1}(\bar{e}) = \{e^1, e^2, e^3\}$.

Since α cannot separate $\Sigma^1 \times \{0\}$ from $\Sigma^1 \times \{1\}$, we may choose a properly embedded arc $\beta \subset \Sigma^1 \times [0,1]$ which is disjoint from $\alpha \cup \{e^1, e^2, e^3\}$, and which has one endpoint in each boundary component of $\Sigma^1 \times [0,1]$.

Case 1: *β meets $\Sigma^1 \times \{0\}$ at $b^1 \in I_1$.* Cut $\Sigma^1 \times [0,1]$ open along β to obtain a parallelogram diagram $q_\beta : P_\beta \to K$, as described above. Let $b^1, b^{1'}, \beta^1$ and $\beta^{1'}$ denote the two copies of b^1 and β in ∂P_β. Let β^2 and β^3 be arcs in P_β which are "parallel copies of β^1" beginning at b^2 and b^3, respectively, let γ denote the top edge of P_β, and let $B = (\bigcup_{i=1}^3 \beta^i) \cup \beta^{1'} \cup \gamma$. It is easy to construct a strong deformation retraction H_t of $P_\beta \setminus \{e^1, e^2, e^3\}$ onto B in such a way that $q_\beta \circ H_t$ is constant on $(q_\beta)^{-1}(\{\overline{y}\})$ for all $\overline{y} \in K$ and $t \in [0,1]$. See Figure 4(a). Thus we get an induced deformation retraction $\widetilde{H} : (K \setminus \{\overline{e}\}) \times [0,1] \to K$ which deformation retracts $K \setminus \{\overline{e}\}$ onto the graph $A = q_\beta(B)$ shown in Figure 4(b).

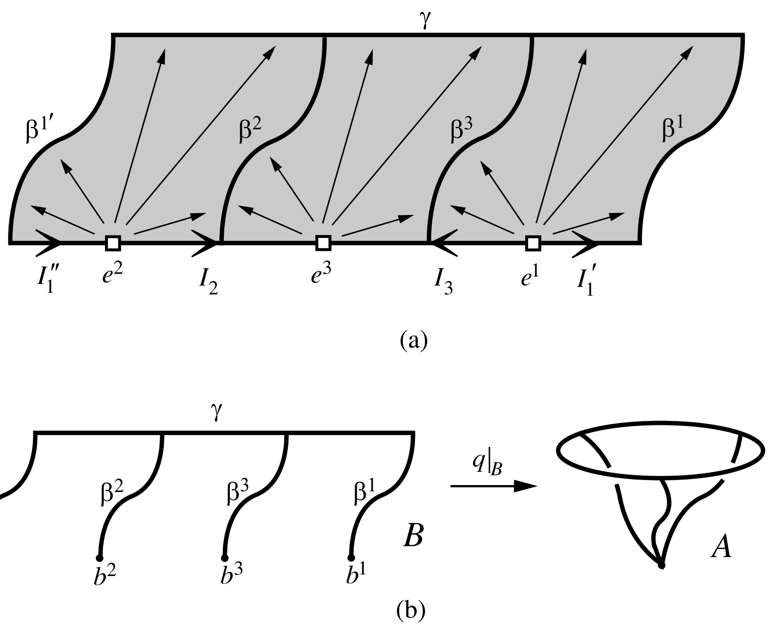

Fig. 4. A deformation retraction of $K \setminus \{\bar{e}\}$

Now, since α must pass over one or more of the points e^1, e^2, e^3 in P_β, the homotopy H pushes α to a path $\alpha' \subset B$ which (up to homotopy) rises up through a β^i (or $\beta^{1'}$), then travels in γ, before coming back down a different β^j (or $\beta^{1'}$). It is easy to see that $\tau' = q_\beta(\alpha')$ does not contract in A, hence, it does not contract in $K \setminus \{\overline{e}\}$. Moreover, $\tau' \simeq \tau$ in $K \setminus \{\overline{e}\}$ (via $\widetilde{H}$), so τ does not contract in $K \setminus \{\overline{e}\}$.

Remaining cases. If β intersects $\Sigma^1 \times [0,1]$ in I_2, we can use the Symmetry Property to revert to the above case. If β intersects $\Sigma^1 \times [0,1]$ in I_3, the picture is slightly different, but the proof is essentially the same. ■

4. Construction of X. Let $\{(K_i, L_i, L'_i)\}_{i=1}^{\infty}$ be a collection of pairwise disjoint copies of the triple (K, L, L'). For each i, let $h_i : L'_i \to L_{i+1}$ be a piecewise linear homeomorphism. Define $X = (\bigcup_{i=1}^{\infty} K_i)/\{x \sim h_i(x)$ for each $x \in L'_i,\ i = 1, 2, \ldots\}$. In other words, X is obtained from a countable collection of copies of K by gluing the range end of each mapping cylinder K_i to the domain end of K_{i+1}. Our ultimate goal in this paper is to show that X has the properties promised in the Main Theorem.

Since each includes into X as an embedding, we view the K_i's and L_i's as subsets (in fact subpolyhedra) of X. Since each L'_i has been identified with L_{i+1}, we no longer need the L'_i labels, and we refer to the triple (K_i, L_i, L'_i) as (K_i, L_i, L_{i+1}). For each $i \in \mathbb{Z}$, we let $\overline{e}_i$ denote the eye of K_i (hence, $\overline{e}_i \in L_{i+1}$) and $q_i : \varSigma^1 \times [0,1] \to K_i$ be the usual quotient map. For each $1 \le p \le q$, let $X_{p,q} = \bigcup_{i=p}^{q} K_i$ and let $X_{p,\infty} = \bigcup_{i=p}^{\infty} K_i$. Notice that for each p, $X_{p,p} = K_p$ and that $X_{1,\infty} = X$. The following is obvious.

Lemma 4.1. *Let $1 \le p \le q$ be integers. Then*

(1) *$X_{p,q}$ is homeomorphic to $X_{1,q-p+1}$, and*
(2) *$X_{p,\infty}$ is homeomorphic to X.*

The next lemma compiles a few (of many) easy homotopy properties of X and its subspaces. The primary ingredient in the proofs (which are left to the reader) is Lemma 3.1.

Lemma 4.2. *Let $1 \le p \le q$ be integers and X, K_i, L_i, $X_{p,q}$ and $X_{p,\infty}$ be as defined above. Then each of these spaces has the homotopy type of a circle, and the following inclusions are all homotopy equivalences:*

(1) $L_p \hookrightarrow X_{p,q}$,
(2) $L_{q+1} \hookrightarrow X_{p,q}$,
(3) $L_p \hookrightarrow X_{p,\infty}$,
(4) $X_{p,\infty} \hookrightarrow X$,
(5) $X_{p,q} \hookrightarrow X$.

Remark 3. Since the spaces involved are all ANRs (in fact, polyhedra), each homotopy equivalence in Lemma 3.1 ((1) and (2)) and Lemma 4.2 (parts (1)–(5)) implies the existence of a strong deformation retraction of the larger space onto the smaller.

The last two results of this section expand upon Lemma 3.3.

Lemma 4.3. *Let $p \le q$ be integers, let $E_{p,q} = \{\overline{e}_i \mid p \le i \le q\}$, and let $E' \subset E_{p,q}$. Then the inclusion induced map $\pi_1(X_{p,q} \setminus E') \to \pi_1(X \setminus E')$ is injective. Hence, if $r \le p$ and $s \ge q$, then $\pi_1(X_{p,q} \setminus E') \to \pi_1(X_{r,s} \setminus E')$ is also injective.*

Proof. First note that $X = X_{1,p-1} \cup_{L_p} X_{p,\infty}$, and by Lemma 4.2 and Remark 3, there is a strong deformation retraction of $X_{1,p-1}$ onto L_p. This may be extended via the identity to a strong deformation retraction of $X \setminus E'$ onto $X_{p,\infty} \setminus E'$. The remainder of the proof breaks into two cases.

Case 1: $\overline{e}_q \notin E'$. By Lemma 4.2 and Remark 3, there is a strong deformation retraction of $X_{q+1,\infty}$ onto L_{q+1}. Since $X_{p,\infty} = X_{p,q} \cup X_{q+1,\infty}$ and $E' \cap X_{q+1,\infty} = \emptyset$, we may combine this deformation retraction with the one mentioned above to produce a strong deformation retraction of $X \setminus E'$ onto $X_{p,q} \setminus E'$. Hence, the inclusion induced map on fundamental groups is an isomorphism.

Case 2: $\overline{e}_q \in E'$. Consider the inclusion induced maps

$$\pi_1(X_{p,q} \setminus E') \to \pi_1(X_{p,\infty} \setminus E') \to \pi_1(X \setminus E').$$

By our initial observation, the second of these maps is an isomorphism. To understand the first map, view $X_{p,\infty} \setminus E'$ as $(X_{p,q} \setminus E') \cup (X_{q+1,\infty} \setminus E')$ where the latter two spaces intersect in the contractible space $L_{q+1} \setminus \{\overline{e}_q\}$. An application of van Kampen's theorem (after noting that $L_{q+1} \setminus \{\overline{e}_q\}$ has "nice" neighborhoods in both $X_{p,q} \setminus E'$ and $X_{q+1,\infty} \setminus E'$) then shows that the first map is injective, thus completing the proof. ■

Corollary 4.4. *Let $\{a^1, a^2, a^3\}$ be a triple of equivalent points from $\Sigma^1 \times \{0\}$ and let α be a properly embedded arc in $\Sigma^1 \times [0,1]$ from a^j to a^k with $j \neq k$. Then the loop $\tau = q_i(\alpha)$ does not contract in $X \setminus \{\overline{e}_i\}$. Hence, if $p \leq i \leq q$, then τ does not contract in $X_{p,q} \setminus \{\overline{e}_i\}$.*

Proof. Combine Lemmas 3.3 and 4.3. ■

5. $\mathcal{Z}$-Compactifiability of $X \times \mathcal{Q}$. In this section we do the easy part of the Main Theorem. In particular, we offer two short proofs of the following:

Theorem 5.1. *$X \times \mathcal{Q}$ is $\mathcal{Z}$-compactifiable.*

First proof. In this (more algebraic) proof, we simply verify that X satisfies (a)–(c) of Theorem 1.1.

Notice that $\{X_{p,\infty}\}_{p=1}^{\infty}$ is a decreasing sequence containing arbitrarily small neighborhoods of infinity in X. Then condition (a) follows easily from Lemma 4.2 (conclusion (3)) and Remark 3. Since each $X_{p,\infty}$ has the homotopy type of a circle, conditions (b) and (c) follow from the well known calculations: $\widetilde{K}_0(\mathbb{Z}) = 0$ and $\mathrm{Wh}(\mathbb{Z}) = 0$ (see [Wa1, p. 67] and [Co, Th. 11.2]). ■

Second proof. We now offer a proof which is more direct and more geometric.

Recall that $K \approx \mathrm{Map}(\theta)$, where $\theta : S^1 \to S^1$ is homotopic to id_{S^1}. A well known theorem from the study of simple homotopy types (see [Co, Th. 5.5]) implies that $\mathrm{Map}(\theta)$ is equivalent via a finite sequence of elementary cellular expansions and collapses (rel ends) to $\mathrm{Map}(\mathrm{id}_{S^1}) \approx S^1 \times [0,1]$.

Applying the above observation to each $K_i \subset X$ shows that X is (infinite) simple homotopy equivalent to $S^1 \times [0, \infty)$. Hence, by [Ch, Th. 39.1], $X \times \mathcal{Q} \approx (S^1 \times [0, \infty)) \times \mathcal{Q}$. Since the latter space is clearly $\mathcal{Z}$-compactifiable, so is the former. ∎

6. Embeddings of K in $S^1 \times B^n$. By choosing a sufficiently nice triangulation of $\Sigma^1 \times [0, 1]$ we may induce a triangulation of K so that the quotient map $q : \Sigma^1 \times [0, 1] \to K$ is simplicial and L, L' and $\{\overline{e}\}$ correspond to subcomplexes. Moreover, every subdivision of this triangulation of K induces a natural triangulation of $\Sigma^1 \times [0, 1]$ so that q is still simplicial and non-degenerate, i.e., i-simplices are taken onto i-simplices. Throughout the remainder of this paper we assume that all triangulations of K and $\Sigma^1 \times [0, 1]$ are of this sort.

In this section we will study maps of K into $S^1 \times B^n$. For convenience we realize $S^1 \times B^n$ as a specific polyhedral subset of $\mathbb{R}^{n+1}$. Let

$$S^1 \times B^1 = ([-3, 3] \times [-3, 3]) \setminus ((-1, 1) \times (-1, 1)) \subset \mathbb{R}^2,$$

and for $n \geq 2$, let

$$S^1 \times B^n = S^1 \times B^1 \times [-1, 1]^{n-1} \subset \mathbb{R}^{n+1}.$$

For a given n, let $B^-, B^+ \subset S^1 \times B^n$ denote the meridional n-disks, $[-3, -1] \times \{0\} \times [-1, 1]^{n-1}$ and $[1, 3] \times \{0\} \times [-1, 1]^{n-1}$, respectively. See Figure 5.

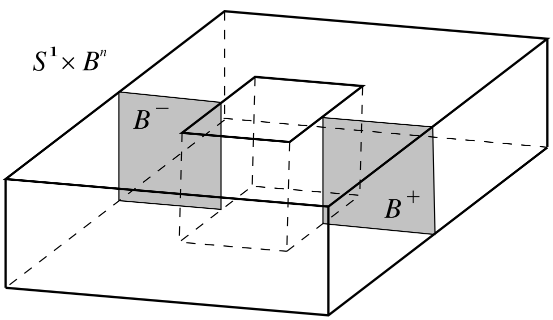

Fig. 5

Give $\mathbb{R}^{n+1}$ the usual Euclidean metric and let

$$\mathbb{H}^+ = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1 \geq 0\}.$$

Then $B^+ \subset \mathbb{H}^+$, $B^- \subset \mathbb{R}^{n+1} \setminus \mathbb{H}^+$, $\operatorname{dist}(B^-, \mathbb{H}^+) = 1$ and $\operatorname{dist}(B^+, \mathbb{R}^{n+1} \setminus \mathbb{H}^+) = 1$.

Let $n \geq 4$, and $\phi : K \to \operatorname{int}(S^1 \times B^n)$ be a piecewise linear (p.l.) embedding which is a homotopy equivalence. Since $S^1 \times B^n \subset \mathbb{R}^{n+1}$, “p.l.”

simply means that ϕ is linear on simplices of some triangulation $\mathbf{T}$ of K. By adjusting the embedding an arbitrarily small amount, we may assume that $\phi(\mathbf{T}^{(0)})\cap(B^-\cup B^+)=\emptyset$. Then for each 1-simplex $\sigma^1\in\mathbf{T}$, $\sigma^1\cap\phi^{-1}(B^-\cup B^+)$ is either empty or a single point in $\mathrm{int}(\sigma^1)$, and for each 2-simplex $\sigma^2\in\mathbf{T}$, $\sigma^2\cap\phi^{-1}(B^-\cup B^+)$ will be empty or a straight segment properly embedded in σ^2 with endpoints in the interiors of 1-dimensional faces. In this situation we say the embedding is *transverse* to $B^-\cup B^+$. Clearly, $\phi^{-1}(B^-)$ and $\phi^{-1}(B^+)$ are disjoint 1-dimensional polyhedra in K and neither intersects $\{\overline{e}\}$. If $\mathbf{S}$ is the corresponding triangulation of $\Sigma^1\times[0,1]$, then $q\phi:\Sigma^1\times[0,1]\to S^1\times B^n$ will satisfy the same transversality properties on simplices. Moreover, since $\Sigma^1\times[0,1]$ is a manifold with boundary, $(q\phi)^{-1}(B^-)$ and $(q\phi)^{-1}(B^+)$ will be disjoint properly embedded compact 1-manifolds, i.e., disjoint collections of properly embedded arcs and circles in $\Sigma^1\times[0,1]$. Furthermore, neither set intersects $\{e^1,e^2,e^3\}$. An understanding of the arcs that arise in this manner will be crucial to our proof of the Main Theorem. We spend the remainder of this section studying these arcs.

Let α be a properly embedded arc in $\Sigma^1\times[0,1]$ with endpoints a and b not intersecting the set $\{e^1,e^2,e^3\}$. We say that α is of *Type I* if one endpoint lies in each component of $\Sigma^1\times\{0,1\}$. If both endpoints lie in $\Sigma^1\times\{1\}$, then α is of *Type II*. If both a and b lie in $\Sigma^1\times\{0\}$ and $q(a)=q(b)$, then α is of *Type III*. If both a and b lie in $\Sigma^1\times\{0\}$ and $q(a)\neq q(b)$, then α is of *Type IV.*

Notice that each arc α of Type II, III or IV "cuts off a disk" $D_\alpha\subset\Sigma^1\times[0,1]$. More precisely, there is a unique arc $J\subset\Sigma^1\times\{0,1\}$ and a unique disk $D_\alpha\subset\Sigma^1\times[0,1]$ such that $\partial D_\alpha=\alpha\cup J$. We say that α is *free* if $q|_{D_\alpha}$ is injective, i.e., if no points of D_α get identified under q.

Type IV arcs play a particularly important role in this paper. We subdivide this class of arcs as follows: α is of *Type IV.0* if $D_\alpha\cap\{e^1,e^2,e^3\}=\emptyset$, α is of *Type IV.1* if $D_\alpha\cap\{e^1,e^2,e^3\}$ contains a single point, α is of *Type IV.2* if $D_\alpha\cap\{e^1,e^2,e^3\}$ contains two points, and α is of *Type IV.3* if $D_\alpha\supset\{e^1,e^2,e^3\}$. In Figure 6 a parallelogram diagram is used to illustrate a variety of arcs.

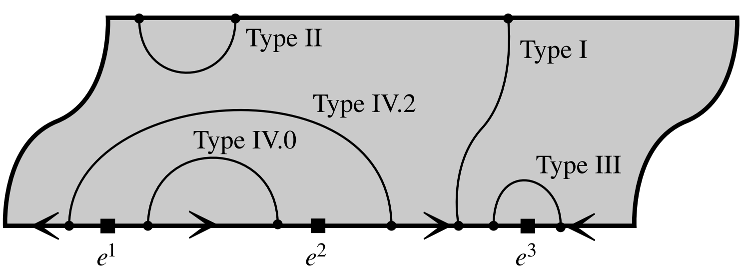

Fig. 6. Arcs in $\Sigma^1\times[0,1]$

The following easy lemma utilizes the partial ordering $\prec$ defined in Section 3. We leave the proof for the reader to check.

LEMMA 6.1. *All Type II and Type IV.0 arcs are free. No Type III, IV.2 or IV.3 arcs are free. A Type IV.1 arc α is free if and only if $D_a \cap \{e^1, e^2, e^3\} = \{e^2\}$ and the endpoints $a_1 \in I_1$ and $b_2 \in I_2$ of α determine the inequality $b_2 \prec a_2$.* ∎

In each of the remaining lemmas in this section we begin with a p.l. embedding $\phi : K \to \mathrm{int}(S^1 \times B^n)$ which is a homotopy equivalence and is transverse to $B^- \cup B^+$. Throughout the section we let $A^- = \phi^{-1}(B^-)$, $A^+ = \phi^{-1}(B^+)$, $C^- = (\phi q)^{-1}(B^-)$ and $C^+ = (\phi q)^{-1}(B^+)$.

LEMMA 6.2. *In the above setup*:

(1) *If $\{a^1, a^2, a^3\}$ is a triple of equivalent points in $\Sigma^1 \times \{0\}$ and one of these points lies in C^- [resp. C^+], then so do the other two.*

(2) *Each loop in A^- or A^+ contracts in K.*

(3) *C^- and C^+ each contain an arc of Type I.*

Proof. The first assertion is obvious. For the second, let τ be a loop in A^- or A^+. Then $\phi(\tau)$ lies in $B^- \cup B^+$ and is therefore null homotopic in $S^1 \times B^n$. Since ϕ is a homotopy equivalence, τ is null homotopic in K.

For the third assertion, assume that C^- contains no Type I arc. Then we may find a loop ω in $\Sigma^1 \times [0,1]$ disjoint from C^- so that $\omega \hookrightarrow \Sigma^1 \times [0,1]$ is a homotopy equivalence. But then $\phi q(\omega) \subset (S^1 \times B^n) \setminus B^-$, so $\phi q(\omega)$ is null homotopic in $S^1 \times B^n$. Since ϕq is a homotopy equivalence (see Lemma 3.1), this is impossible. The same argument guarantees that C^+ also contains a Type I arc. ∎

The next lemmas show that when certain (geometrically motivated) hypotheses are in place, the configuration of arcs found in C^+ (or C^-) is significantly limited. To make the proofs more readable, we use the following notational conventions. Points of $C^- \cap (\Sigma^1 \times \{0\})$ and $C^+ \cap (\Sigma^1 \times \{0\})$ will be labelled as a^i, b^i, c^i or d^i with $i = 1, 2$ or 3 denoting the interval I_i in which the point lies. In this case a symbol α^i, β^i, γ^i or δ^i will denote the corresponding arc of C^- and C^+. Of course, the label for an arc is not unique—an arc with endpoints a^1 and c^3 could be labelled as α^1 or γ^3. An arc of C^- or C^+ with undetermined endpoints will be given a neutral label such as λ or μ.

LEMMA 6.3. *Given the original setup, assume that C^+ [resp. C^-] contains no arcs of Type IV.0. Then for every component arc λ of C^+ [resp. C^-] which is of Type IV.1, $D_\lambda \cap \{e^1, e^2, e^3\} = \{e^2\}$.*

Proof. Suppose that $\lambda \subset C^+$ is of Type IV.1 and that $D_\lambda \cap \{e^1, e^2, e^3\} \neq \{e^2\}$.

Case 1: $D_\lambda \cap \{e^1, e^2, e^3\} = \{e^1\}$. Then λ has endpoints $a^3 \in I_3$ and $b^1 \in I_1$. Since a^3 and b^1 are not in the same equivalence class, we have either $a^1 \prec b^1$ or $b^1 \prec a^1$.

If $a^1 \prec b^1$ then $a^1 \in D_\lambda$, and since C^+ contains no Type IV.0 arcs, a^1 lies on a Type IV.1 arc α^1 which begins at a^1, ends at $c^3 \in I_3$, and cuts off a smaller disk $D_{\alpha^1} \subset D_\lambda$. Since $c^3 \prec a^3$ it follows that $c^1 \prec a^1$, and we may apply the same argument to find yet another Type IV arc inside of D_{α^1}. This process never ends, and since C^+ contains finitely many components we have a contradiction. See Figure 7.

If $b^1 \prec a^1$ then $b^3 \prec a^3$, so $b^3 \in D_\lambda$, and we may proceed as above.

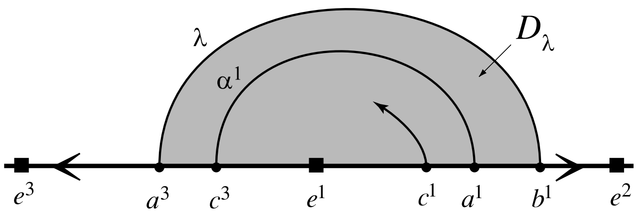

Fig. 7

Case 2: $D_\lambda \cap \{e^1, e^2, e^3\} = \{e^3\}$. By symmetry (Lemma 3.2), the above argument also rules out this case. ■

Remark 4. A precise way in which symmetry may be used to handle Case 2 is the following. Replace ϕ by the p.l. embedding $\phi\bar{\iota} : K \to \Sigma^1 \times [0,1]$. Then, by Lemma 3.2, $((\phi\bar{\iota})q)^{-1} = (\phi(q\iota))^{-1} = \iota(\phi q)^{-1}$. Since ι transposes e^1 and e^3 while leaving e^2 fixed, the original Case 2 is transformed into Case 1.

Lemma 6.4. *Given the original setup, assume that C^+ [resp. C^-] contains no arcs of Type III or Type IV.0. Then all Type IV arcs in C^+ [resp. C^-] are of Type IV.1 and free.*

Proof. We break the proof into three claims.

Claim 1. *Each Type IV.1 arc $\lambda \subset C^+$ is free.*

By Lemma 6.3, $D_\lambda \cap \{e^1, e^2, e^3\} = \{e^2\}$, thus, λ has endpoints $a^1 \in I_1$ and $b^2 \in I_2$. By Lemma 6.1, we need only show that $b^2 \prec a^2$. Suppose to the contrary that $a^2 \prec b^2$. Then $a^1 \prec b^1$ and, since C^+ contains no arcs of Type III or IV.0, b^1 is the endpoint of an arc β^1 whose other endpoint lies in I_2. This other endpoint c^2 of β^1 cannot equal a^2, otherwise we would have a loop $q(\lambda \cup \beta^1) \subset A^+$ which is easily seen not to contract in K, thus violating Lemma 6.2. Therefore, $a^2 \prec c^2 \prec b^2$ or $c^2 \prec a^2$.

Assume that $a^2 \prec c^2 \prec b^2$ (the situation shown in Figure 8). Then there exists another Type IV.1 arc γ^1 beginning at c^1 (where $a^1 \prec c^1 \prec b^1$) and ending at d^2 with $c^2 \prec d^2 \prec b^2$. This gives rise to yet another Type IV.1 arc δ^1 beginning at d^1 and ending between c^2 and d^2. Clearly, an infinite pattern has been established, and since C^+ contains only finitely many arcs we have a contradiction.

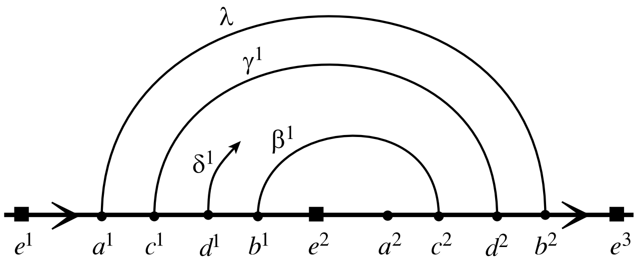

Fig. 8

The remaining possibility is that $c^2 \prec a^2$. But then a^2 is the endpoint of a Type IV.1 arc whose other endpoint d^1 satisfies $a^1 \prec d^1 \prec b^1$. We are now back to the (impossible) situation of the previous case, thus completing the proof of Claim 1.

Claim 2. *C^+ contains no arc of Type IV.2.*

Suppose that μ is a Type IV.2 arc in C^+. Assume further that μ has been chosen to be innermost with this property, i.e., D_μ contains no other Type IV.2 arcs from C^+.

Case (i): $D_\mu \cap \{e^1, e^2, e^3\} = \{e^1, e^2\}$. Then μ has endpoints $a^3 \in I_3$ and $b^2 \in I_2$, and D_μ contains I_1.

If $b^3 \prec a^3$, then $b^3 \in D_\mu$ and we may obtain a contradiction. Indeed, b^3 must be the endpoint of a Type IV arc lying in D_μ. If the other endpoint lies in I_1, we contradict Lemma 6.3, and if it lies in I_2, we contradict the choice of μ as "innermost". See Figure 9(a).

Now suppose that $a^3 \prec b^3$. Then $a^2 \prec b^2$, so a^2 and a^1 both lie in D_μ. Let α^1 be the arc of C^+ containing a^1. By hypotheses and Lemma 6.3, the other endpoint c^2 of α^1 must lie in I_2. By Claim 1, $c^2 \prec a^2$. But now a^2 also lies on a Type IV arc α^2 of C^+. If α^2 ends in I_3 we contradict the choice of μ as innermost, otherwise α^2 ends at $d^1 \in I_1$ with $d^1 \prec a^1$ and this contradicts Claim 1. See Figure 9(b). Therefore, we have ruled out Case (i).

Case (ii): $D_\mu \cap \{e^1, e^2, e^3\} = \{e^1, e^3\}$. Then μ has endpoints $a^2 \in I_2$ and $b^1 \in I_1$, and D_μ contains I_3. Therefore a^3 lies in D_μ and is the endpoint of an arc $\alpha^3 \subset C^+$. By hypothesis α^3 cannot be of Type III or Type IV.0 so, since it lies in D_μ, it must be of Type IV.1 (see Figure 9(c)). But this violates Lemma 6.3 since $e^2 \notin D_\mu$, thus ruling out Case (ii).

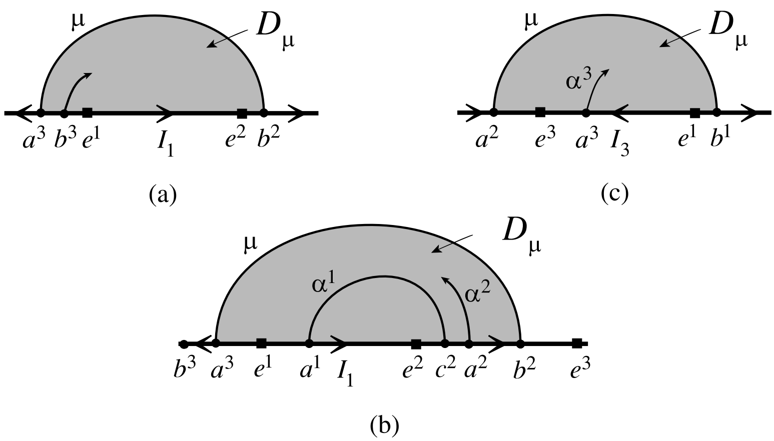

Fig. 9

CASE (iii): $D_\mu \cap \{e^1, e^2, e^3\} = \{e^2, e^3\}$. By symmetry (Lemma 3.2), this is equivalent to Case (i) and therefore impossible.

CLAIM 3. *C^+ contains no arc of Type IV.3.*

Suppose to the contrary that C^+ contains a Type IV.3 arc μ, and assume that μ has been chosen to be innermost with this property. One of the intervals I_1, I_2 or I_3 will contain both endpoints of μ.

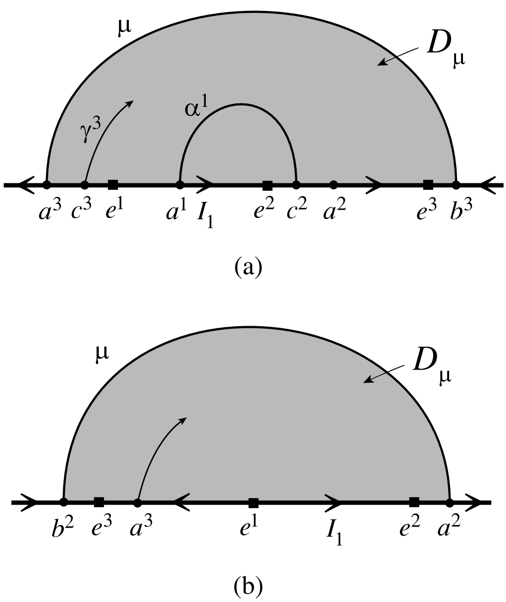

Fig. 10

CASE (i): *The endpoints of μ lie in I_3.* Let $a^3 \prec b^3$ be the endpoints of μ. Then, by Claims 1 and 2, a^1 is an endpoint of a free Type IV.1 arc $\alpha^1 \subset D_\mu$. Hence the other endpoint c^2 of α^1 must lie in I_2 with $c^2 \prec a^2$. Then $c^3 \prec a^3$, so c^3 lies on a Type IV arc $\gamma^3 \subset D_\mu$ (see Figure 10(a)). By hypothesis, Lemma 6.3, and Claim 2, γ^3 cannot be of Type IV.0, IV.1 or IV.2. Furthermore, since μ is innermost, γ^3 cannot be of Type IV.3, thus proving the impossibility of this case.

Case (ii): *The endpoints of μ lie in I_2.* Let $a^2 \prec b^2$ denote the endpoints of μ. Then $a^3 \in D_\mu$, however, our hypotheses along with Lemma 6.3 and Claim 2 rule out all components of C^+ which could possibly contain a^3 (see Figure 10(b)), rendering this case impossible.

Case (iii): *The endpoints of μ lie in I_1.* By symmetry, this case is ruled out by Case (ii), completing the proof of Claim 3 and hence of the lemma. ∎

Lemma 6.5. *Given the original setup, assume that*

(1) *C^+ contains no Type III arcs, and*
(2) *C^+ contains no arcs of Type IV.0.*

Then, for every triple $\{a^1, a^2, a^3\}$ of equivalent points lying in C^+, a^3 and at least one of a^1 or a^2 are endpoints of Type I arcs in C^+. An equivalent result may be obtained by replacing each "+" with a "−".

Proof. By our hypotheses and Lemma 6.4, each point of $\{a^1, a^2, a^3\}$ is the endpoint of an arc of Type I or a free arc of Type IV.1. Since each free arc of Type IV.1 has endpoints in I_1 and I_2 (see Lemma 6), a^3 must lie on a Type I arc, α^3.

It remains to show that a^1 and a^2 cannot both lie on a free arc of Type IV.1. Suppose a^1 is the endpoint of a free Type IV.1 arc α^1. Then the other endpoint b^2 of α^1 lies in I_2 and $b^2 \prec a^2$. Cut $\Sigma^1 \times [0,1]$ open along

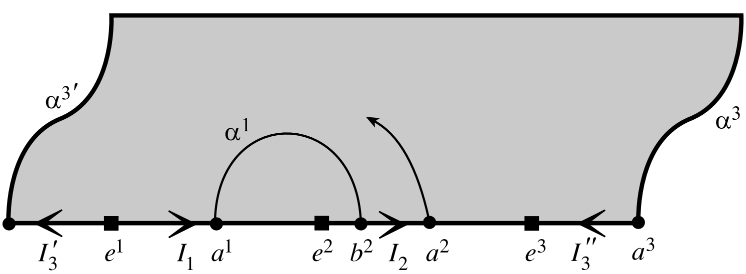

Fig. 11

α^3 to obtain Figure 11. It is now impossible for a^2 to lie on a free arc of Type IV.1 since the only points in I_1 which are accessible from a^2 are less than a^1. ∎

Lemma 6.6. *Given the original setup, assume that*

(1) C^+ *contains no Type III arcs*,

(2) *neither* C^- *nor* C^+ *contains an arc of Type IV.0*, *and*

(3) *if* C^- *contains a free Type IV.1 arc* λ *then* D_λ *contains an arc component of* C^+.

Then for every triple $\{a^1, a^2, a^3\}$ *of equivalent points lying in* C^-, *at least one element of* $\{a^1, a^2, a^3\}$ *is the endpoint of a Type I arc in* C^-.

An equivalent result may be obtained by reversing the roles of "+" *and* "−".

Proof. By Lemma 6.2, C^+ contains a Type I arc, so there is a triple of equivalent points $\{c^1, c^2, c^3\} \subset C^+$. By Lemma 6.5 and symmetry, we may assume without loss of generality that c^2 and c^3 are endpoints of Type I arcs $\gamma^2, \gamma^3 \subset C^+$.

Case 1: c^1 *is also the endpoint of a Type I arc* γ^1. Then γ^1, γ^2 and γ^3 cut $\Sigma^1 \times [0,1]$ into three "chambers" as shown in Figure 12(a).

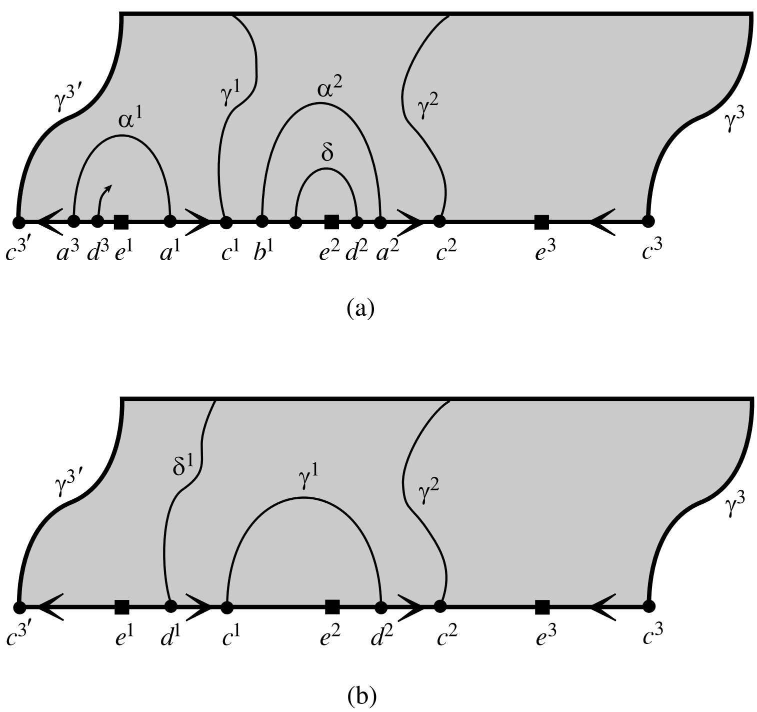

Fig. 12

Subcase (1a): $a^1 \prec c^1$. Then a^1 and a^3 each lie in the left-hand chamber of our parallelogram diagram $q'' : P_{\gamma^3} \to K$. By Lemma 6.3, this chamber contains no Type IV arcs. Therefore, a^1 and a^3 are each endpoints of

Type I arcs (and thus, our conclusion is satisfied) or else a^1 and a^3 are the endpoints of a Type III arc α^1. In this latter case, consider a^2 which lies in the middle chamber. If a^2 lies on a Type I arc, our desired conclusion is satisfied. Otherwise, a^2 is the endpoint of a Type IV.1 arc α^2 with second endpoint $b^1 \in I_1$ and $c^1 \prec b^1$. Then $a^2 \prec c^2 \prec b^2$, so α^2 is free. Furthermore, if D_{α^2} contains an arc component $\delta \subset C^+$, then δ is of Type IV.1, and therefore has an endpoint $d^2 \in I_2$ with $d^2 \prec a^2$. But then $d^3 \in C^+$ lies in D_{α^1}, and this is impossible by hypotheses (1) and (2) and Lemma 6.3. Therefore δ cannot exist. But this violates hypothesis (3), completing the proof of Subcase (1a). This scenario is pictured in Figure 12(a).

Subcase (1b): $c^1 \prec a^1$. Then a^2 and a^3 both lie in the right-hand chamber of our parallelogram and a^1 lies in the middle chamber. The argument just used in Subcase (1a) may now be used.

Case 2: *c^1 is not the endpoint of a Type I arc.* Then by Lemma 6.4, c^1 is the endpoint of a free Type VI.1 arc $\gamma^1 \subset C^+$. Let $d^2 \in I_2$ be the other endpoint of this arc. By Lemma 6.5, $d^1 \in I_1$ is the endpoint of a Type I arc $\delta^1 \subset C^+$. We now break the remainder of the proof into three subcases, each beginning with the parallelogram diagram in Figure 12(b).

Subcase (2a): $a^1 \prec d^1$. Then both a^1 and a^3 lie to the left of δ^1 in Figure 12(b), and a^2 lies beneath γ^1. We may now apply the same argument used in Subcase (1a).

Subcase (2b): $d^1 \prec a^1 \prec c^1$. Then a^3 lies to the left of δ^1 in Figure 12(b), and a^1 and a^2 both lie to the right of δ^1. By Lemma 6.3, a^3 must lie on a Type I arc.

Subcase (2c): $c^1 \prec a^1$. Then a^2 and a^3 each lie to the right of γ^2 in our diagram and the situation is the same as in Subcase (2a). ■

7. Embeddings of $X_{p,q}$ in $S^1 \times B^n$. In this section we expand our focus to study embeddings of $X_{p,q} = \bigcup_{i=p}^q K_i$ into $S^1 \times B^n$. Since $X_{p,q} \approx X_{1,q-p+1}$, we simplify notation by working with the spaces $X_{1,j} = \bigcup_{i=1}^j K_i$. Recall that for each $i = 1, \ldots, j$, $\overline{e}_i$ denotes the eye of K_i and $q_i : \Sigma^1 \times [0,1] \to K_i$ is the usual quotient map. We may choose triangulations of $X_{1,j}$ so that each K_i, L_i and $\{\overline{e}_i\}$ is a subcomplex, and so that each $q_i : \Sigma^1 \times [0,1] \to K_i$ is a non-degenerate simplicial map. (We may have a different triangulation of $\Sigma^1 \times [0,1]$ for each i.) By Lemma 4.2, we know that $X_{1,j}$ is homotopy equivalent to a circle. For $n \geq 4$, each homotopy equivalence $\phi : X_{1,j} \to \operatorname{int}(S^1 \times B^n)$ may be adjusted to a p.l. embedding which is transverse to the disks B^- and B^+ in the sense described in Section 6. Then we let $A^-, A^+ \subset X_{1,j}$ denote the disjoint 1-dimensional polyhedra $\phi^{-1}(B^-)$ and $\phi^{-1}(B^+)$, and for each i, $A_i^- = A^- \cap K_i$ and $A_i^+ = A^+ \cap K_i$. The latter will be disjoint

1-dimensional polyhedra in K_i, neither containing $\overline{e}_i$. Also (as discussed in Section 6), each map $\phi q_i : \Sigma^1 \times [0,1] \to S^1 \times B^n$ is transverse to $B^- \cup B^+$, hence, for a given i, the sets $C_i^- = (\phi q_i)^{-1}(B^-)$ and $C_i^+ = (\phi q_i)^{-1}(B^+)$ will be disjoint properly embedded compact 1-manifolds in $\Sigma^1 \times [0,1]$. All of the above will be the starting point for the results in this section.

In addition to the standard setup just described, the results in this section each involve a partition $\{E^+, E^-\}$ of the set $E = \{\overline{e}_1, \ldots, \overline{e}_j\} \subset X_{1,j}$. Although the source of these partitions is not important yet, the reader may be interested in knowing how they will arise. For a given embedding $\phi : X_{1,j} \to \operatorname{int}(S^1 \times B^n)$, E^+ will be the collection of eyes of $X_{1,j}$ which are sent into the "right half" of $S^1 \times B^n$ and E^- will be those sent into the "left half". More precisely, $E^+ = \{\overline{e}_i \in X_{1,j} \mid \phi(\overline{e}_i) \in \mathbb{H}^+\}$ and $E^- = E \setminus E^+$. Recall Figure 5.

The first two lemmas in this section utilize geometric moves to simplify a given embedding $\phi : X_{1,j} \to \operatorname{int}(S^1 \times B^n)$ while preserving key properties of that embedding.

LEMMA 7.1. *Given the standard setup described above, suppose that $\{E^-, E^+\}$ is a partition of $\{\overline{e}_1, \ldots, \overline{e}_j\}$ with the property that loops in A^- contract in $X_{1,j} \setminus E^+$ and loops in A^+ contract in $X_{1,j} \setminus E^-$. Suppose also that*

(†) *C_1^- or C_1^+ contains an arc of Type IV.0.*

Then we may choose a new p.l. embedding $\widehat{\phi} : X_{1,j} \to \operatorname{int}(S^1 \times B^n)$ which is a homotopy equivalence, transverse to $B^- \cup B^+$, and which satisfies the following conditions:

(1) *loops in $\widehat{A}^- = (\widehat{\phi})^{-1}(B^-)$ contract in $X_{1,j} \setminus E^+$,*
(2) *loops in $\widehat{A}^+ = (\widehat{\phi})^{-1}(B^+)$ contract in $X_{1,j} \setminus E^-$,*
(3) *$\widehat{\phi}|_{L_1} = \phi|_{L_1}$, and*
(4) *$|\widehat{\phi}(L_2) \cap (B^- \cup B^+)| < |\phi(L_2) \cap (B^- \cup B^+)|$.*

Proof. We begin by assuming that $j \geq 2$. A proof of the $j = 1$ case will be contained in the $j \geq 2$ argument.

STEP 1: *Eliminating circle components of $C_1^- \cup C_1^+$.* In this step we use standard cut and paste techniques to alter ϕ so that neither C_1^- nor C_1^+ contains circle components. This will make the rest of the proof less complicated.

Let τ be a circle component of C_1^-. Then τ bounds a disk $D_\tau \subset \Sigma^1 \times (0,1)$, and $\overline{\tau} = q_1(\tau)$ is a circle component of A^- bounding the disk $\overline{D}_\tau = q_1(D_\tau) \subset K_1 - \{\overline{e}_1\}$. Let $\mathcal{N}(\overline{D}_\tau)$ be a small 2-disk neighborhood of $\overline{D}_\tau$ in K_1. Then $\phi(\partial\mathcal{N}(\overline{D}_\tau))$ is a circle lying to one side of B^- in $S^1 \times B^n$. Let D^* be a disk near to (but disjoint from) B^- with $\partial D^* = \phi(\partial\mathcal{N}(\overline{D}_\tau))$, and by general position arrange that $D^* \cap \phi(X_{1,j}) = \partial D^*$. Redefine ϕ to send $\mathcal{N}(\overline{D}_\tau)$ onto D^*. This eliminates τ from C_1^-, along with any other components of C_1^- lying in D_τ. Clearly, our new map still satisfies all of the original hypotheses. Repeat this process until all circle components of C_1^- are removed, then perform the same procedure near B^+ to eliminate all circle components from C_1^+.

STEP 2: *Main setup.* By Step 1, we may assume that neither C_1^- nor C_1^+ contains circle components. Choose a Type IV.0 arc λ which is innermost in the set $C_1^- \cup C_1^+$, i.e., choose λ so that D_λ contains no other components of $C_1^- \cup C_1^+$. Without loss of generality, assume that $\lambda \subset C_1^+$. Now $\partial D_\lambda = \lambda \cup J$ where $J \subset \Sigma^1 \times \{0\}$ is an arc contained in the interior of one of the intervals I_1, I_2, I_3, and $q_1^{-1}q_1(J) = J \cup J' \cup J''$ where J' and J'' are arcs—one in each of the remaining I_i's. A fourth "copy" of the arc J lies in the preimage of the map $q_2 : \Sigma^1 \times [0,1] \to K_2$. In particular, let $J''' = q_2^{-1}h_1(q_1(J))$, where h_1 is the homeomorphism used to glue K_1 to K_2.

Let (N, N_0) be a relative regular neighborhood of the pair $(\phi q_1(D_\lambda), \phi q_1(D_\lambda))$ in $(S^1 \times B^n, \phi(X_{1,j}))$. Then N is an $(n+1)$-ball and $N_0 = N \cap \phi(X_{1,j})$ may be realized as $\phi q_1(D \cup R' \cup R'') \cup \phi q_2(R''')$, where D, R' and R'' are regular neighborhoods of D_λ, J' and J'' in $\Sigma^1 \times [0,1]$, and R''' is a regular neighborhood of J''' in a separate copy of $\Sigma^1 \times [0,1]$. See Figure 13 where

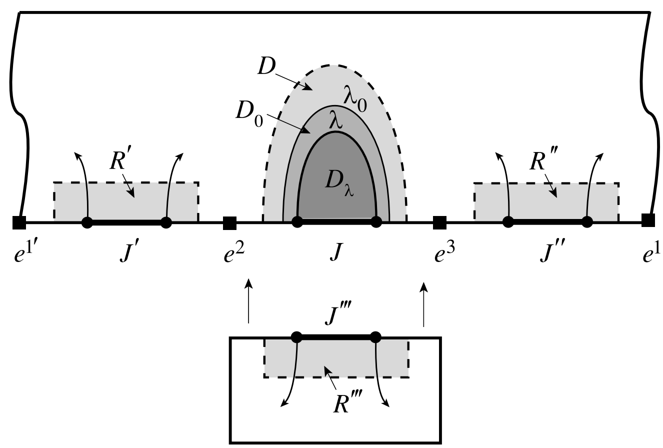

Fig. 13

we have cut each $\Sigma^1 \times [0,1]$ open to produce parallelogram diagrams. In this figure, we only show relevant parts of C^- and C^+ and we have arbitrarily placed $J \subset I_2$, $J' \subset I_1$ and $J'' \subset I_3$. In addition, we have reduced the size of the second copy of $\Sigma^1 \times [0,1]$ (the domain of q_2) to more accurately reflect the geometry after identifications.

For later use let $\overline{N}_0$ denote $\phi^{-1}(N_0)$. Thus $\overline{N}_0$ is a regular neighborhood of $q_1(D_\lambda)$ in $X_{1,j}$.

Step 3: *Construction of* $\widehat{\phi}$. To obtain the desired embedding $\widehat{\phi}$, we perform an isotopy of $S^1 \times B^n$, fixed outside of N, which rearranges N_0 to our specifications. Let λ_0 be an arc in D which is disjoint from λ and cuts off a disk $D_0 \subset D$ which contains D_λ. Let $J_0 = D_0 \cap (\Sigma^1 \times \{0\})$. Let $\Gamma : N \times [0,1] \to N$ be an isotopy, fixed on ∂N, which slides the arc $\phi q_1(J_0)$ (rel endpoints) through the disk $\phi q_1(D_0)$ and onto the arc $\phi q_1(\lambda_0)$. This isotopy squeezes $\phi q_1(D_0)$ into the disk $\phi q_1(D \setminus \mathrm{int}(D_0))$ and stretches the ϕq_1-images of "rectangles" R' and R'' and the ϕq_2-image of R''' along disks parallel to $\phi q_1(D_0)$ so that their common edge is moved "upwards" to the arc λ_0. Extend Γ via the identity outside of N, and define $\widehat{\phi} = \Gamma_1 \circ \phi$.

Step 4: *Verification of conditions* (1)–(4). The effect of the above move on the sets A^- and A^+ (now $\widehat{A}^-$ and $\widehat{A}^+$) is most easily seen back in $\widehat{C}_1^-$, $\widehat{C}_1^+$, $\widehat{C}_2^-$ and $\widehat{C}_2^+$. Roughly speaking, $\widehat{C}_2^+$ is obtained from C_2^+ by pushing λ from C_1^+ down into C_2^+. Thus the components of C_2^+ which had intersected $\Sigma^1 \times \{1\}$ at the endpoints of J''' are now joined together by an arc λ''' before they reach $\Sigma^1 \times \{1\}$. (λ''' corresponds to λ in that its image under $\widehat{\phi} q_2$ is nearly the same as the image of λ under ϕq_1.) The effect on C_1^+ is that λ has disappeared and the arc components of C_1^+ which had intersected $\Sigma^1 \times \{0\}$ at the endpoints of J' and J'', respectively, are now joined together by arcs λ' and λ'' before reaching $\Sigma^1 \times \{0\}$. Lastly, $\widehat{C}_2^-$ and $\widehat{C}_1^-$ are left unchanged. See Figure 14.

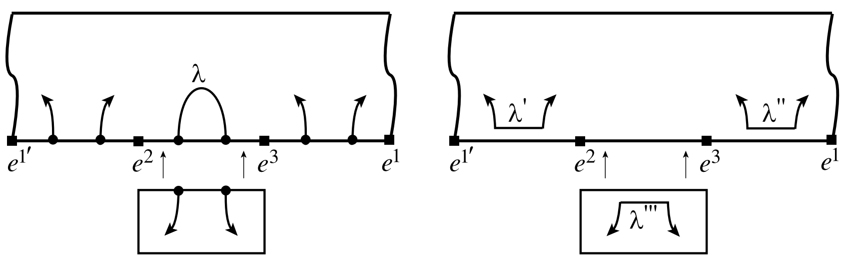

Fig. 14

By construction $\widehat{\phi}|_{L_1} = \phi|_{L_1}$, and by the above analysis, we see that $|\widehat{\phi}(L_2) \cap (B^- \cup B^+)| = |\phi(L_2) \cap (B^- \cup B^+)| - 2$, hence conditions (3) and (4) are satisfied. To check condition (1), notice that $\widehat{A}^- = A^-$.

To verify condition (2), let τ be any embedded loop in $\widehat{A}^+$. (Since $\widehat{A}^+$ is a graph, we may restrict our attention to embedded loops.) We must show that τ contracts in $X_{1,j} \setminus E^-$. Since $\widehat{\phi}|_{X_{1,j} \setminus \mathrm{int}(\overline{N}_0)} = \phi|_{X_{1,j} \setminus \mathrm{int}(\overline{N}_0)}$, we have $\tau \setminus \mathrm{int}(\overline{N}_0) \subset A^+$. By construction, $\tau \cap \overline{N}_0 \subset q_1(\lambda' \cup \lambda'' \cup \lambda''')$ and, since $\overline{N}_0$ is contractible, each arc $q_1(\lambda')$, $q_1(\lambda'')$ and $q_1(\lambda''')$ may be homotoped (rel endpoints) within $\overline{N}_0$ into $A^+ \cap \overline{N}_0$. See Figure 15. Since $\overline{e}_1 \notin \overline{N}_0$, this

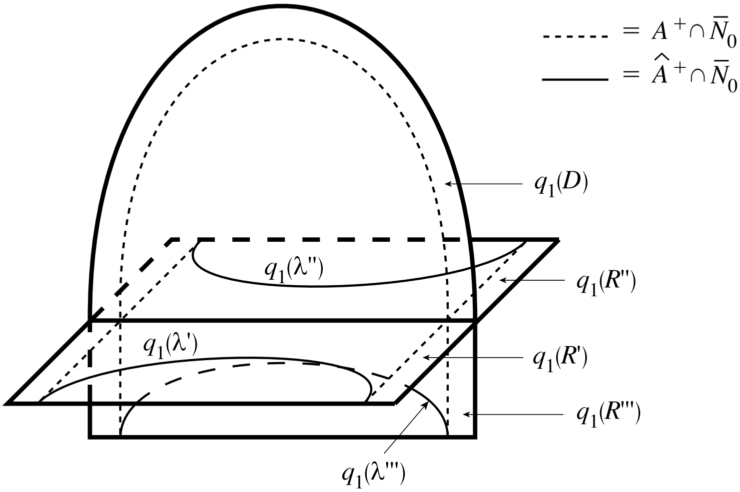

Fig. 15

shows that τ may be homotoped within $X_{1,j} \setminus E^-$ to a loop $\tau' \subset A^+$. By hypothesis τ' contracts in $X_{1,j} \setminus E^-$, hence, so does τ.

Lastly, note that for the $j = 1$ case, the same strategy works. The proof only becomes easier since $\widehat{A}_1^- = \widehat{A}^-$ and $\widehat{A}_1^+ = \widehat{A}^+$. ■

Remark 5. The map $\widehat{\phi}$ could also have been defined (without reference to an isotopy) by cutting and pasting 2-dimensional disks inside of N. Back in the domain spaces, the disk D_0 is cut from the "first copy" of $\Sigma^1 \times [0,1]$ along λ_0 and sewn to the "second copy" of $\Sigma^1 \times [0,1]$. Copies D_0' and D_0'' of D_0 are sewn to the top copy of $\Sigma^1 \times [0,1]$ along slightly lengthened copies of J' and J''. Then $\widehat{\phi}$ is defined to send these "new disks" D_0' and D_0'' to parallel copies of $\phi q_1(D_0)$ which are disjoint except that the edges λ_0' and λ_0'' are each sent to the arc $\phi q_1'(\lambda_0)$ where the new "seam" in $\phi(K_1)$ is located. This strategy, along with the effects on C_1^+ and C_2^+, is described by Figure 16.

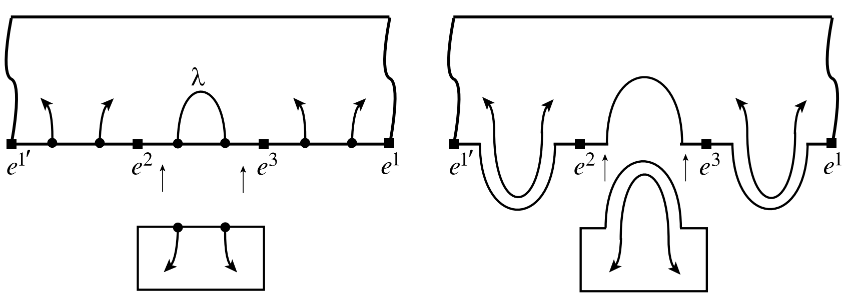

Fig. 16

The next lemma, while similar to its predecessor, has some significant differences. It involves a geometric move which takes place in a neighborhood of the eye of $K_1 \subset X_{1,j}$. For this reason, the main hypothesis (‡) is not symmetric with respect to "+" and "−" and is thus more delicate than (†).

LEMMA 7.2. *Given the standard setup, suppose that* $\{E^-, E^+\}$ *is a partition of* $\{\bar{e}_1, \ldots, \bar{e}_j\}$ *and that loops in* A^- *contract in* $X_{1,j} \setminus E^+$ *and loops in* A^+ *contract in* $X_{1,j} \setminus E^-$. *Suppose also that*

(‡) $\bar{e}_1 \in E^-$ *and* C_1^- *contains a free Type IV.1 arc* λ *with the property that* D_λ *contains no arc component of* C^+.

Then we may choose a new p.l. embedding $\widehat{\phi} : X_{1,j} \to \operatorname{int}(S^1 \times B^n)$ *which is a homotopy equivalence, transverse to* $B^- \cup B^+$, *and which satisfies the following conditions*:

(1) *loops in* $\widehat{A}^- = (\widehat{\phi})^{-1}(B^-)$ *contract in* $X_{1,j} \setminus E^+$,
(2) *loops in* $\widehat{A}^+ = (\widehat{\phi})^{-1}(B^+)$ *contract in* $X_{1,j} \setminus E^-$,
(3) $\widehat{\phi}|_{L_1} = \phi|_{L_1}$, *and*
(4) $|\widehat{\phi}(L_2) \cap (B^- \cup B^+)| < |\phi(L_2) \cap (B^- \cup B^+)|$.

The same result is true if the roles of "−" and "+" are reversed.

Proof. Begin by assuming that $j \geq 2$. As in the previous lemma, a proof of the $j = 1$ case is implicit in our argument.

STEP 1: *Eliminating circle and Type IV.0 arc components from* $C_1^- \cup C_1^+$. Circle components may be removed from $C_1^- \cup C_1^+$ by using the procedure described in Step 1 of Lemma 7.1. If $C_1^- \cup C_1^+$ contains a Type IV.0 arc component we may apply Lemma 7.1 directly to produce the desired map $\widehat{\phi}$.

STEP 2: *Main setup.* By Step 1, we may assume that neither C_1^- nor C_1^+ contains components which are circles or arcs of Type IV.0.

Let λ be the Type IV.1 arc promised by (‡). Notice that any Type IV.1 arc contained in D_λ also satisfies (‡), hence, we may assume that λ is innermost with that property, and therefore, $D_\lambda \cap (C_1^- \cup C_1^+) = \lambda$. Now $\partial D_\lambda = \lambda \cup J$, where $J \subset \Sigma^1 \times \{0\}$ is an arc containing e^2 and having endpoints $a^1 \in I_1$ and $b^2 \in I_2$ in common with λ and such that $b^2 < a^2$ (see Lemma 6). Then $q_1^{-1} q_1(J) = J \cup J' \cup J''$ where $J' \subset \Sigma^1 \times \{0\}$ contains e^1 and has endpoints b^3 and b^1, while $J'' \subset \Sigma^1 \times \{0\}$ contains e^3 and has endpoints a^2 and a^3. Another relevant arc $J''' = q_2^{-1} h_1(q_1(J))$ lies in the preimage of the map $q_2 : \Sigma^1 \times [0,1] \to K_2$.

Since $q_1|_{D_\lambda}$ is injective, $\phi q_1(D_\lambda)$ is a disk in $\operatorname{int}(S^1 \times B^n)$. Let (N, N_0) be a relative regular neighborhood of the pair $(\phi q_1(D_\lambda), \phi q_1(D_\lambda))$ in $(S^1 \times B^n, \phi(X_{1,j}))$. Then N is an $(n+1)$-ball and $N_0 = N \cap \phi(X_{1,j})$ may be realized as $\phi q_1(D \cup R' \cup R'') \cup \phi q_2(R''')$, where D, R' and R'' are regular neighborhoods of D_λ, J' and J'' in are $\Sigma^1 \times [0,1]$, and R''' is a regular neighborhood of J''' in a separate copy of $\Sigma^1 \times [0,1]$. See Figure 17. For later use let $\overline{N}_0 \subset X_{1,j}$ denote $\phi^{-1}(N_0)$.

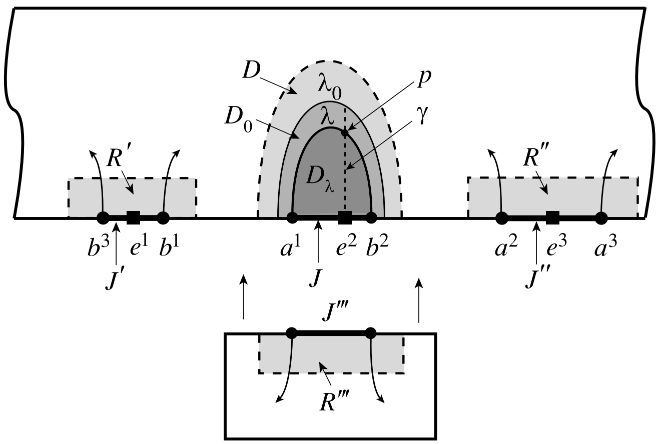

Fig. 17

Step 3: *Construction of* $\widehat{\phi}$. To obtain the desired embedding $\widehat{\phi}$, we perform an isotopy of $S^1 \times B^n$, fixed outside of N, which rearranges N_0 to our specifications. Let λ_0 be an arc in D which is disjoint from λ and cuts off a disk $D_0 \subset D$ which contains D_λ. Let $J_0 = D_0 \cap (\Sigma^1 \times \{0\})$. Let $\Gamma : N \times [0,1] \to N$ be an isotopy, fixed on ∂N, which slides the arc $\phi q_1(J_0)$ (rel endpoints) through the disk $\phi q_1(D_0)$ and onto the arc $\phi q_1(\lambda_0)$. This isotopy squeezes $\phi q_1(D_0)$ into the disk $\phi q_1(D \setminus \mathrm{int}(D_0))$ and drags the ϕq_1-images of R' and R'' and the ϕq_2-image of R''' through portions of $\phi q_1(D_0)$ so that, at the end, the arcs along which these pieces meet (which include $\overline{e}_1$) all lie on λ_0. Extend Γ via the identity outside of N, and define $\widehat{\phi} = \Gamma_1 \circ \phi$. For later use let γ denote the preimage in D_0 of the track of $\phi q_1(\overline{e}_1)$ under Γ.

Step 4: *Verification of conditions* (1)–(4). Condition (3) is clearly satisfied, and since $\widehat{A}^+$ is unchanged, condition (2) follows as well. To verify the other conditions, we must examine $\widehat{A}^-$. Begin by considering the sets $\widehat{C}_1^-$ and $\widehat{C}_2^-$. The most obvious effect of Γ is that λ has been pushed down into $\widehat{C}_2^-$. More precisely, λ has disappeared from $\widehat{C}_1^-$ and the arcs of C_2^- which previously contained the endpoints of J''' are now joined in $\widehat{C}_2^-$ by an arc λ''' before they reach $\Sigma^1 \times \{1\}$. (λ''' corresponds to λ in that its image under $\widehat{\phi} q_2$ is nearly the same as the image of λ under ϕq_1.)

To describe the remaining changes to C_1^-, recall that $\gamma \subset D_0$ is the preimage of the track of $\overline{e}_1$ under Γ. For convenience, we can arrange that γ intersect λ at a single point p, thus separating λ into subarcs λ_a and λ_b, where λ_a has endpoints a^1 and p while λ_b has endpoints b^2 and p. The arcs of C_1^- which previously contained the endpoints of J' are now joined in $\widehat{C}_1^-$ by an arc μ_b whose image in $S^1 \times B^n$ lies close to the image of the path $\lambda_b * \lambda_b^{-1}$. A similar change near J'', but with μ_a having image close to the image of $\lambda_a * \lambda_a^{-1}$, completes the transition of C_1^- into $\widehat{C}_1^-$. See Figure 18.

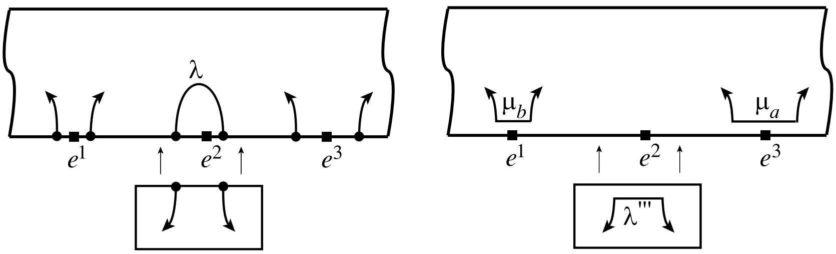

Fig. 18

NOTE. Although this figure is similar to Figure 14, the way the transition occurs in $S^1 \times B^n$ is quite different. Much of this can be seen by comparing Figures 15 and 19.

It is now clear that $|\widehat{\phi}(L_2) \cap (B^- \cup B^+)| = |\phi(L_2) \cap (B^- \cup B^+)| - 2$, hence condition (4) is satisfied, and we need only verify condition (1).

Let τ be any embedded loop in $\widehat{A}^-$. (Since $\widehat{A}^-$ is a graph, we may restrict our attention to embedded loops.) We must show that τ contracts in $X_{1,j} \setminus E^+$. Since $\widehat{\phi}|_{X_{1,j} \setminus \mathrm{int}(\overline{N}_0)} = \phi|_{X_{1,j} \setminus \mathrm{int}(\overline{N}_0)}$, we have $\tau \setminus \mathrm{int}(\overline{N}_0) \subset A^-$. By construction, $\tau \cap \overline{N}_0 \subset q_1(\mu_a \cup \mu_b \cup \lambda''')$ and, since $\overline{N}_0$ is contractible, each arc $q_1(\mu_a)$, $q_1(\mu_b)$ and $q_1(\lambda''')$ may be homotoped (rel endpoints) within $\overline{N}_0$ into $A^- \cap \overline{N}_0$. See Figure 19. These homotopies may pass through $\bar{e}_1$,

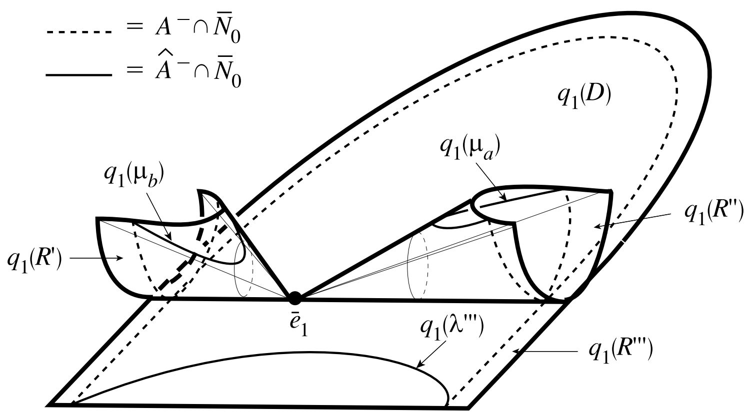

Fig. 19

but since $\overline{e}_1 \in E^-$, it follows that $\overline{N}_0 \cap E^+ = \emptyset$. Thus τ may be homotoped within $X_{1,j} \setminus E^+$ to a loop $\tau' \subset A^-$. By hypothesis τ' contracts in $X_{1,j} \setminus E^+$, hence, so does τ.

For the $j = 1$ case we use the same strategy with the proof becoming somewhat simpler. ∎

Remark 6. As with Lemma 7.1, the map $\widehat{\phi}$ just constructed could also be defined by cutting and pasting 2-dimensional disks inside of N. Back in the domain spaces, the disk D_0 is cut from the "first copy" of $\Sigma^1 \times [0,1]$ along λ_0 and sewn to the "second copy" of $\Sigma^1 \times [0,1]$. Disks D_0' and D_0'' are sewn to the top copy of $\Sigma^1 \times [0,1]$ along slightly lengthened copies of J' and J''. Then q_1 "folds" each of D_0' and D_0'' along center-lines and identifies points along the outer boundary arcs. The map $\widehat{\phi}$ now takes D_0' (after fold and identification) into $S^1 \times B^n$ parallel to the half of $\phi q_1(D_0)$ which contains the image of λ_b. In doing so, the "crease" is sent near the image of γ and the arc of identification points is sent to the ϕq_1-image of λ_b. Similarly, $\widehat{\phi}$ takes the folded and identified "disk", D_0'', near to the half of $\phi q_1(D_0)$ containing the image of λ_a, with the crease being sent near the image of γ and the arc of identification points being sent to the ϕq_1-image of λ_a. By general position, we may arrange that $\widehat{\phi}$ be an embedding. This strategy is described in Figure 20.

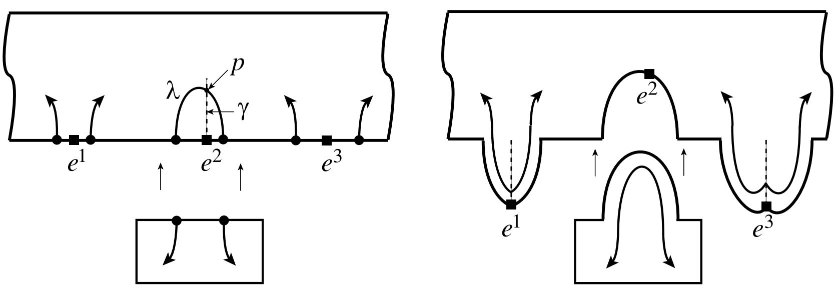

Fig. 20

The next result is the culmination of our efforts in the last two sections. It will be a key ingredient in our proof of the non-$\mathcal{Z}$-compactifiability of X.

Proposition 7.3. *Given the standard setup, suppose that* $\{E^-, E^+\}$ *is a partition of* $E = \{\overline{e}_1, \ldots, \overline{e}_j\}$ *with the property that loops in* A^- *contract in* $X_{1,j} \setminus E^+$ *and loops in* A^+ *contract in* $X_{1,j} \setminus E^-$. *Then* $|\phi(L_1) \cap B^-| \geq 2^{|E^+|}$ *and* $|\phi(L_1) \cap B^+| \geq 2^{|E^-|}$.

Proof. The proof will be by induction on j. For each j we will focus our attention on $\phi|_{K_1} : K_1 \to S^1 \times B^n$. Since ϕ and $K_1 \hookrightarrow X_{1,j}$ are homotopy equivalences (Lemma 4.2(5)), we see that $\phi|_{K_1}$ is also a homotopy equivalence.

Case 1: $j = 1$. Without loss of generality, we may assume that $E^- = \{\overline{e}_1\}$ and $E^+ = \emptyset$.

Since $\phi(L_1)$ is a non-trivial loop in $S^1 \times B^n$ it must intersect B^-, hence $|\phi(L_1) \cap B^-| \geq 1 = 2^{|E^+|}$. To see that $|\phi(L_1) \cap B^+| \geq 2$, we assume that a counterexample exists. Suppose further that this counterexample has been chosen so that $|\phi(L_2) \cap (B^- \cup B^+)|$ is minimal. Then by Lemma 7.1, C_1^+ contains no Type IV.0 arcs and, since loops in A^+ contract in $K_1 \setminus \{\overline{e}_1\}$, Lemma 3.3 implies that C_1^+ contains no Type III arcs. Hence, by an application of Lemma 6.5, C_1^+ contains at least two Type I arcs, contradicting our assumption that $|\phi(L_1) \cap B^+| < 2$.

Case 2: $j \geq 2$. Assume that our claim is true for integers less than j, but that a counterexample $\phi : X_{1,j} \to S^1 \times B^n$ exists and that, of all counterexamples, we have chosen one for which $|\phi(L_2) \cap (B^- \cup B^+)|$ is minimal. Then, by Lemma 7.1, neither C_1^- nor C_1^+ contains a Type IV.0 arc. We break the remainder of the proof into two subcases.

Subcase (2a): $\overline{e}_1 \in E^-$. By utilizing Lemma 4.3, we may apply the inductive hypothesis to $\phi|_{X_{2,j}} : X_{2,j} \to S^1 \times B^n$ to conclude that $|\phi(L_2) \cap B^-| \geq 2^{|E^+|}$ and $|\phi(L_2) \cap B^+| \geq 2^{|E^- \setminus \{\overline{e}_1\}|} = 2^{(|E^-|-1)}$. Since loops in A^+ contract in $X_{1,j} \setminus E^-$, by Corollary 4.4, C_1^+ contains no Type III arcs.

Now, for every point of $\phi(L_2) \cap B^+$ there is a triple $\{a^1, a^2, a^3\}$ of equivalent points lying in C_1^+ and, by Lemma 6.5, at least two of these points lie on Type I arcs in C_1^+. Thus,

$$|\phi(L_1) \cap B^+| \geq 2 \cdot |\phi(L_2) \cap B^+| \geq 2 \cdot 2^{|E^-|-1} = 2^{|E^-|}.$$

Next we focus on C_1^-. Since $\overline{e}_1 \in E^-$, we may not rule out the existence of Type III arcs in C_1^-; however, by Lemma 7.2 and minimality, C_1^- contains no free Type IV.1 arc λ with the property that D_λ contains no arc component of C^+. Now, for every point of $\phi(L_2) \cap B^-$ there is a triple $\{a^1, a^2, a^3\}$ of equivalent points lying in C_1^- and, by Lemma 6.6, at least one of these points lies on a Type I arc of C_1^-. Thus,

$$|\phi(L_1) \cap B^-| \geq |\phi(L_2) \cap B^-| = 2^{|E^+|}.$$

Hence ϕ was not a counterexample to our claim, so we are finished.

Subcase (2b): $\overline{e}_1 \in E^+$. The proof is the same, except that the roles of "$-$" and "$+$" are reversed. ■

8. The non-$\mathcal{Z}$-compactifiability of X. We are now ready to prove the following:

THEOREM 8.1 (Non-$\mathcal{Z}$-compactifiability of X). *The space X is not $\mathcal{Z}$-compactifiable.*

Our proof will be by contradiction. Toward that end, suppose there exists a $\mathcal{Z}$-compactification $\widehat{X} = X \cup Z$. For each p, let $\widehat{X}_{p,\infty} = X_{p,\infty} \cup Z$. Then $\widehat{X}_{p,\infty}$ is a neighborhood of Z in $\widehat{X}$; moreover, for each neighborhood U of Z in $\widehat{X}$, there exists p such that $\widehat{X}_{p,\infty} \subset U$. Easy applications of the definition of $\mathcal{Z}$-set and Lemma 4.2 imply the following:

• $X \hookrightarrow \widehat{X}$ is a homotopy equivalence (hence $\widehat{X} \simeq S^1$),

• for every $p, q \in \mathbb{Z}$ $(1 \leq p \leq q)$, the inclusions $X_{p,q} \hookrightarrow \widehat{X}$ and $\widehat{X}_{p,\infty} \hookrightarrow \widehat{X}$ are homotopy equivalences,

• for each p there is a strong deformation retraction $r_p : \widehat{X} \to \widehat{X}_{p,\infty}$.

Before beginning the proof, we introduce some more notation and then review an important fact from dimension theory.

Let $\Omega_{n,k}$ denote the cubification of $\mathbb{R}^n$ whose n-cells are of the form $\prod_{i=1}^{n}[p_i/2^k, (p_i+1)/2^k]$, where $p_1, \ldots, p_n \in \mathbb{Z}$. Note that for each r, the r-skeleton, $\Omega_{n,k}^{(r)}$, of $\Omega_{n,k}$ consists of all points $(x_1, \ldots, x_n) \in \mathbb{R}^n$ for which at least $n-r$ of the x_i's are of the form $s_i/2^k$ with $s_i \in \mathbb{Z}$. Note also that each coordinate hyperplane $\mathbb{R}^m \subset \mathbb{R}^n$ corresponds to a subcomplex of $\Omega_{n,k}$ with cubification $\Omega_{m,k}$.

For all $k \geq 0$, $S^1 \times B^n$, B^- and B^+ (as realized in Section 6) correspond to subcomplexes of $\Omega_{n+1,k}$.

Recall from dimension theory the *universal* 1*-dimensional Nöbeling space*, $\mathcal{N}_1^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid$ at most one of the x_i's is rational$\}$. This space has the important property that for any compact, 1-dimensional, separable metric space Z, the space $\mathcal{E}(Z, \mathcal{N}_1^3)$ of embeddings of X into $\mathcal{N}_1^3$ is dense in the space $\mathcal{C}(Z, \mathbb{R}^3)$ of continuous maps of Z into $\mathbb{R}^3$ (see [En, Section 1.11]).

Proof of Theorem 8.1. Suppose that X admits a $\mathcal{Z}$-compactification $\widehat{X} = X \cup Z$. We break the proof into two steps.

STEP 1: *We construct an embedding $\phi : \widehat{X} \to \operatorname{int}(S^1 \times B^4)$ with the following properties*:

(1) *ϕ is a homotopy equivalence*,

(2) *$\phi|_X$ is a p.l. embedding transverse to B^- and B^+, and*

(3) *$\phi(Z) \cap \Omega_{5,k}^{(3)} = \emptyset$ for all k.*

To get started, we will need the following:

CLAIM (a). $\dim(Z) = 1$.

By the definition of a $\mathcal{Z}$-set, for every $\varepsilon > 0$, there exists an ε-map of $\widehat{X}$ into X. Since X is a 2-dimensional polyhedron, a standard result from dimension theory [Mi, Th. 4.5.13] implies that $\dim \widehat{X} = 2$. Then [BM, Prop. 2.6] guarantees that $\dim(Z) \le 1$.

Claim (b). *There exists a homotopy equivalence $G : \widehat{X} \to \mathrm{int}(S^1 \times B^2)$ so that $G|_Z$ embeds Z in $\mathcal{N}_1^3$.*

Begin with a homotopy equivalence $F : \widehat{X} \to \mathrm{int}(S^1 \times B^2)$. By the dimension theoretic result mentioned earlier, we may approximate $f = F|_Z$ arbitrarily closely by an embedding $g : Z \to \mathcal{N}_1^3$. Since $\mathrm{int}(S^1 \times B^2)$ is an ANR and since Z has arbitrarily small neighborhoods in $\widehat{X}$ of the form $\widehat{X}_{p,\infty}$, it follows that g may be extended to a map $\overline{g} : \widehat{X}_{p,\infty} \to \mathrm{int}(S^1 \times B^2)$. By precomposing with a strong deformation retraction $r_p : \widehat{X} \to \widehat{X}_{p,\infty}$ we may extend $\overline{g}$ to $G : \widehat{X} \to \mathrm{int}(S^1 \times B^2)$.

By choosing g sufficiently close to f and p sufficiently large in the above paragraph, we can make $\overline{g}$ arbitrarily close to $\overline{f} = F|_{\widehat{X}_{p,\infty}}$. Since sufficiently close maps of a compactum into $\mathrm{int}(S^1 \times B^2)$ are homotopic (by the straight-line homotopy), we may assume that $\overline{g}$ is homotopic to $\overline{f}$. Now F and $\widehat{X}_{p,\infty} \hookrightarrow \widehat{X}$ are both homotopy equivalences, hence, $\overline{f}$ is a homotopy equivalence. It follows that $\overline{g}$ is a homotopy equivalence, and since r_p is a homotopy equivalence, so is $G = \overline{g} \circ r_p$. This completes the proof of Claim (b).

Next, recall that (by the convention established in Section 6), $S^1 \times B^4 = (S^1 \times B^2) \times [-1,1]^2$. Choose a small irrational number $r_0 \in (0,1)$ and define $\psi : \widehat{X} \to S^1 \times B^4 = (S^1 \times B^2) \times [-1,1]^2$ by $\psi(x) = (G(x),(r_0,r_0))$. Clearly, ψ is a homotopy equivalence and, since $\psi(Z) \subset \mathcal{N}_1^3 \times \{(r_0,r_0)\} \subset \mathbb{R}^3 \times \mathbb{R}^2$, at most one $\mathbb{R}^5$-coordinate of each point in $\psi(Z)$ is rational. Thus, $\psi(Z) \cap \Omega_{5,k}^{(3)} = \emptyset$ for all k.

To complete Step 1, we use basic general position to adjust ψ to the desired map. Begin by choosing a cover $\mathcal{U}$ of $\psi(\widehat{X})$ by round open n-balls each contained in $\mathrm{int}(S^1 \times B^4)$. Then choose a triangulation $\mathbf{T}$ of X so that each K_i (and hence, each $X_{p,\infty}$) corresponds to a subcomplex, and for each simplex $\sigma \in \mathbf{T}$, there exists $U_\sigma \in \mathcal{U}$ such that $\psi(\sigma) \subset U_\sigma$. In addition, choose $\mathbf{T}$ so that $\mathrm{mesh}(\mathbf{T}|_{X_{p,\infty}}) \to 0$ as $p \to \infty$ (here the mesh is measured in $\widehat{X}$).

Begin by defining ϕ on the vertex set $\mathbf{T}^{(0)}$ so that the following properties are satisfied:

(i) for each $\sigma \in \mathbf{T}$, $\phi(\sigma^{(0)}) \subset U_\sigma$,
(ii) if $v \in \mathbf{T}^{(0)}|_{X_{p,\infty}}$, then $\mathrm{dist}(\phi(v),\psi(v)) < 1/p$,
(iii) $\{\phi(v) \mid v \in \mathbf{T}^{(0)}\}$ is in general position in $\mathbb{R}^5$,
(iv) $\{\phi(v) \mid v \in \mathbf{T}^{(0)}\} \cap (B^- \cup B^+) = \emptyset$, and
(v) if $v \in \mathbf{T}^{(0)}$ and $\phi(v) = (x_1,\ldots,x_5) \in \mathbb{R}^5$, then $x_5 > r_0$.

Extend ϕ linearly over each simplex of $\mathbf{T}$, then extend to all of $\widehat{X}$ by letting $\phi|_Z = \psi|_Z$. Condition (i) ensures that ϕ sends $\widehat{X}$ into $\operatorname{int}(S^1 \times B^4)$ and that ϕ is homotopic to ψ. Condition (ii) guarantees that ϕ is continuous, while conditions (iii) and (iv) ensure that $\phi|_X$ is a p.l. embedding transverse to $B^- \cup B^+$. Condition (v) makes ϕ an embedding by arranging that $\phi(X) \cap \phi(Z) = \emptyset$.

Step 2: *Deriving a contradiction.* Let $A^-_{1,\infty}$ denote the 1-dimensional polyhedron $(\phi^{-1}(B^-)) \cap X$, and for integers $1 \le p \le q$, let $A^-_{p,q} = A^-_{1,\infty} \cap X_{p,q}$ and $A^-_{p,\infty} = A^-_{1,\infty} \cap X_{p,\infty}$. Define $A^+_{1,\infty}$, $A^+_{p,q}$ and $A^+_{p,\infty}$ similarly. Also, let $E_{1,\infty} = \{\overline{e}_i \mid i = 1, 2, \ldots\}$ and $E_{p,q} = \{\overline{e}_i \mid p \le i \le q\}$. Then define $E^+_{1,\infty} = \{\overline{e}_i \in E_\infty \mid \phi(\overline{e}_i) \in \mathbb{H}^+\}$, $E^-_{1,\infty} = E_{1,\infty} \setminus E^+_{1,\infty}$, $E^+_{p,q} = E_{p,q} \cap E^+_{1,\infty}$ and $E^-_{p,q} = E_{p,q} \cap E^-_{1,\infty}$.

Since $\phi(\widehat{X})$ is a compact ANR, there exists $\varepsilon > 0$ so that loops in $\phi(\widehat{X})$ of diameter less than ε bound singular disks in $\phi(\widehat{X})$ having diameter less than 1. Since $\phi(\widehat{X})$ may be pushed into $\phi(X)$ by arbitrarily small homotopies, it follows that loops in $\phi(X)$ having diameter less than ε bound singular disks in $\phi(X)$ having diameter less than 1.

Choose k sufficiently large that $\operatorname{mesh}(\Omega_{5,k}) < \varepsilon$. Since $\phi(Z) \cap \Omega^{(3)}_{5,k} = \emptyset$, sufficiently small neighborhoods of $\phi(Z)$ miss $\Omega^{(3)}_{5,k}$. Hence, there exists $p_0 \in \mathbb{Z}$ so that $\phi(\widehat{X}_{p_0,\infty}) \cap \Omega^{(3)}_{5,k} = \emptyset$. Since $B^- \setminus \Omega^{(3)}_{5,k}$ and $B^+ \setminus \Omega^{(3)}_{5,k}$ consist of finite collections of pairwise disjoint open 4-cubes, each having diameter less than ε, loops in $\phi(X_{p_0,\infty}) \cap B^-$ and $\phi(X_{p_0,\infty}) \cap B^+$ bound singular disks in $\phi(X)$ having diameters less than 1.

Pulling the above information back into X, and noting that

$$\operatorname{dist}(B^-, \phi(E^+_\infty)) > 1 \quad \text{and} \quad \operatorname{dist}(B^+, \phi(E^-_\infty)) > 1,$$

we deduce that loops in $A^-_{p_0,\infty}$ contract in $X \setminus E^+_\infty$ and loops in $A^+_{p_0,\infty}$ contract in $X \setminus E^-_\infty$. By Lemma 4.3, if $q \ge p_0$, then loops in $A^-_{p_0,q}$ contract in $X_{p_0,q} \setminus E^+_{p_0,q}$ and loops in $A^+_{p_0,q}$ contract in $X_{p_0,q} \setminus E^-_{p_0,q}$.

Now, for each $q \ge p_0$, the map $\phi|_{X_{p_0,q}} : X_{p_0,q} \to \operatorname{int}(S^1 \times B^4)$ satisfies all hypotheses of Proposition 7.3. We may conclude that

$$|\phi(L_{p_0}) \cap B^-| \ge 2^{|E^+_{p_0,q}|} \quad \text{and} \quad |\phi(L_{p_0}) \cap B^+| \ge 2^{|E^-_{p_0,q}|}.$$

But as $q \to \infty$, at least one of $|E^+_{p_0,q}|$ or $|E^-_{p_0,q}|$ approaches infinity, and since $\phi(L_{p_0})$ is a p.l. embedded circle which intersects $B^- \cup B^+$ transversely at finitely many points, this gives a contradiction. ∎

9. Questions. Along with the open question at the conclusion of Section 1, some interesting questions remain.

Question. *Does there exist a finite n such that $X \times [-1,1]^n$ is $\mathcal{Z}$-compactifiable?*

Question. *Does there exist an open n-manifold M^n for which $M^n \times Q$ is $\mathcal{Z}$-compactifiable but M^n is not?*

Question (motivated by [Be, p. 135]). *If Y is a finite $K(G,1)$, does its universal cover admit a $\mathcal{Z}$-compactification?*

Note. The universal cover of the finite aspherical 2-complex discussed near the end of Section 1 is $\mathcal{Z}$-compactifiable.

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