

Compactifications of manifolds with boundary

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This paper is concerned with compactifications of high-dimensional manifolds. Siebenmann’s iconic 1965 dissertation [L. C. Siebenmann, The obstruction to finding a boundary for an open manifold of dimension greater than five, Ph.D. thesis, Princeton Univ. (1965), MR 2615648] provided necessary and sufficient conditions for an open manifold M^m ($m \geq 6$) to be compactifiable by addition of a manifold boundary. His theorem extends easily to cases where M^m is noncompact with compact boundary; however, when ∂M^m is noncompact, the situation is more complicated. The goal becomes a “completion” of M^m , i.e. a compact manifold \widehat{M}^m containing a compactum $A \subseteq \partial M^m$ such that $\widehat{M}^m \setminus A \approx M^m$. Siebenmann did some initial work on this topic, and O’Brien [G. O’Brien, The missing boundary problem for smooth manifolds of dimension greater than or equal to six, *Topology Appl.* **16** (1983) 303–324, MR 722123] extended that work to an important special case. But, until now, a complete characterization had yet to emerge. Here, we provide such a characterization. Our second main theorem involves \mathcal{Z} -compactifications. An important open question asks whether a well-known set of conditions laid out by Chapman and Siebenmann [T. A. Chapman and L. C. Siebenmann, Finding a boundary for a Hilbert cube manifold, *Acta Math.* **137** (1976) 171–208, MR 0425973] guarantee \mathcal{Z} -compactifiability for a manifold M^m . We cannot answer that question, but we do show that those conditions are satisfied if and only if $M^m \times [0, 1]$ is \mathcal{Z} -compactifiable. A key ingredient in our proof is the above Manifold Completion Theorem — an application that partly explains our current interest in that topic, and also illustrates the utility of the π_1 -condition found in that theorem.

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1. Introduction

This paper is about “nice” compactifications of high-dimensional manifolds. The simplest of these compactification is the addition of a boundary to an open manifold. That was the topic of Siebenmann’s famous 1965 dissertation [30], the main result of which can easily be extended to include noncompact manifolds with compact boundaries. When M^m has noncompact boundary, one may ask for a compactification \widehat{M}^m that “completes” ∂M^m . That is a more delicate problem. Siebenmann addressed a very special case in his dissertation, before O’Brien [26] characterized completable n -manifolds in the case where M^m and ∂M^m are both 1-ended. Since completable manifolds can have infinitely many (non-isolated) ends, O’Brien’s theorem does not imply a full characterization of completable n -manifolds. We obtain such a characterization here, thereby completing an unfinished chapter in the study of noncompact manifolds.

A second type of compactification considered here is the \mathcal{Z} -compactification. These are similar to the compactifications discussed above — in fact, those are special cases — but \mathcal{Z} -compactifications are more flexible. For example, a \mathcal{Z} -boundary for an open manifold need not be a manifold, and a manifold that admits no completion can admit a \mathcal{Z} -compactification. These compactifications have proven to be useful in both geometric group theory and manifold topology, for example, in attacks on the Borel and Novikov Conjectures. A major open problem (in our minds) is a characterization of \mathcal{Z} -compactifiable manifolds. A set of necessary conditions was identified by Chapman and Siebenmann [6], and it is hoped that those conditions are sufficient. We prove what might be viewed the next best thing: If M^m satisfies the Chapman–Siebenmann conditions (and $m \neq 4$), then $M^m \times [0, 1]$ is \mathcal{Z} -compactifiable. We do this by proving that $M^m \times [0, 1]$ is completable — an application that partly explains the renewed interest in manifold completions, and also illustrates the usefulness of the conditions found in the Manifold Completion Theorem.

1.1. The manifold completion theorem

An m -manifold M^m with (possibly empty) boundary is *completable* if there exists a compact manifold \widehat{M}^m and a compactum $C \subseteq \partial \widehat{M}^m$ such that $\widehat{M}^m \setminus C$ is homeomorphic to M^m . In this case \widehat{M}^m is called a (*manifold*) *completion* of M^m . A primary goal of this paper is the following characterization theorem for $m \geq 6$. Definitions will be provided subsequently.

Theorem 1.1. (Manifold Completion Theorem) *An m -manifold M^m ($m \geq 6$) is completable if and only if*

- (a) M^m is inward tame,
- (b) M^m is peripherally π_1 -stable at infinity,
- (c) $\sigma_\infty(M^m) \in \varprojlim \{ \widetilde{K}_0(\pi_1(N)) \mid N \text{ a clean neighborhood of infinity} \}$ is zero, and
- (d) $\tau_\infty(M^m) \in \varprojlim^1 \{ \text{Wh}(\pi_1(N)) \mid N \text{ a clean neighborhood of infinity} \}$ is zero.

Together, conditions (a) and (c) ensure that (nice) neighborhoods of infinity have finite homotopy type, while Condition (d) allows one to upgrade certain, naturally arising, homotopy equivalences to simple homotopy equivalences. These conditions have arisen in other contexts, such as [6, 30].

Condition (b) can be thought of as “ π_1 -stability rel boundary”; it seems unique to the situation at hand. In the special case where M^m is 1-ended and $N_0 \supseteq N_1 \supseteq \dots$ is a cofinal sequence of (nice) connected neighborhoods of infinity, it demands that each sequence

$$\pi_1(\partial_M N_i \cup N_{i+1}) \leftarrow \pi_1(\partial_M N_i \cup N_{i+2}) \leftarrow \pi_1(\partial_M N_i \cup N_{i+3}) \leftarrow \dots$$

be stable where $\partial_M N_i$ denotes $\partial M^m \cap N_i$. This reduces to ordinary π_1 -stability when ∂M^m is compact. A complete discussion of this condition can be found in Sec. 4.

Remark 1. Several comments are in order:

- (1) Dimensions ≤ 5 are discussed briefly in Sec. 2; our main focus is $m \geq 6$.
- (2) If ∂M^m is compact and M^m is inward tame then M^m has finitely many ends (see Sec. 5), so the ends are isolated and disjoint from ∂M^m . In that case Theorem 1.1 reduces to Siebenmann’s dissertation [30]. As such, Theorem 1.1 can be viewed as a generalization of [30].
- (3) The special case of the Manifold Completion Theorem, where M^m and ∂M^m are 1-ended, was proved by O’Brien [26]; that is where “peripheral π_1 -stability” was first defined. But since candidates for completion can be infinite-ended (e.g. let $C \subseteq S^{m-1}$ be a Cantor set and $M^m = B^m \setminus C$), the general theorem is not a corollary. In the process of generalizing [26], we simplify the proof presented there and correct an error in the formulation of Condition (c). We also exhibit some interesting examples which answer a question posed by O’Brien about a possible weakening condition (b).
- (4) If Condition (b) is removed from Theorem 1.1, one arrives at Chapman and Siebenmann’s conditions for characterizing \mathcal{Z} -compactifiable Hilbert cube manifolds [6]. A \mathcal{Z} -compactification theorem for finite-dimensional manifolds is the subject of the second main result of this paper. We will describe that theorem and the necessary definitions now.

1.2. The stable \mathcal{Z} -compactification theorem for manifolds

To extend the idea of a completion to Hilbert cube manifolds Chapman and Siebenmann introduced the notion of a “ \mathcal{Z} -compactification”. A compactification $\widehat{X} = X \sqcup Z$ of a space X is a \mathcal{Z} -compactification if there is a homotopy $H : \widehat{X} \times [0, 1] \rightarrow \widehat{X}$ such that $H_0 = \text{id}_{\widehat{X}}$ and $H_t(\widehat{X}) \subseteq X$ for all $t > 0$. Subsequently, this notion has been fruitfully applied to more general spaces — notably, finite-dimensional manifolds and complexes; see, for example, [2, 3, 9, 5, 11]. A completion of a finite-dimensional manifold is a \mathcal{Z} -compactification, but a \mathcal{Z} -compactification

need not be a completion. In fact, a manifold that allows no completion can still admit a \mathcal{Z} -compactification; the exotic universal covers constructed by Mike Davis are some of the most striking examples (just apply [1]). Such manifolds must satisfy Conditions (a), (c) and (d), but the converse remains open.

Question. Does every finite-dimensional manifold that satisfies conditions (a), (c) and (d) of Theorem 1.1 admit a \mathcal{Z} -compactification?

This question was posed more generally in [6] for locally compact ANRs, but in [17] a 2-dimensional polyhedral counterexample was constructed. The manifold version remains open. In this paper, we prove a best possible “stabilization theorem” for manifolds.

Theorem 1.2. (Stable \mathcal{Z} -compactification Theorem for Manifolds) *An m -manifold M^m ($m \geq 5$) satisfies Conditions (a), (c) and (d) of Theorem 1.1, if and only if $M^m \times [0, 1]$ admits a \mathcal{Z} -compactification. In fact, $M^m \times [0, 1]$ is completable if and only if M^m satisfies those conditions.*

Remark 2. In [10], Ferry showed that if a locally finite k -dimensional polyhedron X satisfies conditions (a), (c) and (d), then $X \times [0, 1]^{2k+5}$ is \mathcal{Z} -compactifiable. Theorem 1.1 can be viewed as a sharpening of Ferry’s theorem in cases where X is a manifold.

1.3. Outline of this paper

The remainder of this paper is organized as follows. In Sec. 2, we review the status of Theorem 1.1 in dimensions < 6 . In Sec. 3, we fix some terminology and notation; then in Secs. 4–7, we carefully discuss each of the four conditions present in Theorem 1.1. In Secs. 9–10, we prove Theorem 1.1, and in Sec. 11, we prove Theorem 1.2. In Sec. 12, we provide a counterexample to a question posed in [26] about a possible relaxation of Condition (b), and in Sec. 13, we provide the proof of a technical lemma that was postponed until the end of the paper.

2. Manifold Completions in Dimensions < 6

The Manifold Completion Theorem is true in dimensions ≤ 3 , but much simpler versions are possible in those dimensions. For example, Tucker [32] showed that a 3-manifold can be completed if and only if each component of each clean neighborhood of infinity has finitely generated fundamental group — a condition that is implied by inward tameness alone.

Since we have been unable to find the optimal 2-dimensional completion theorem in the literature, we take this opportunity to provide such a theorem. If M^2 has finitely generated first homology (e.g. if M^2 is inward tame), then by classical work (see [24, 28]) $\text{int}(M^2) \approx \Sigma^2 - P$, where Σ^2 is a closed surface and P is a finite set of points. Therefore, M^2 contains a compact codimension 1 submanifold C such

that each of the components $\{N_i\}_{i=1}^k$ of $\overline{M^2 \setminus C}$ is a noncompact manifold whose frontier is a circle onto which it deformation retracts. Complete the N_i individually as follows:

- (i) If N_i contains no portion of ∂M^2 , add a circle at infinity; and
- (ii) If N_i contains components of ∂M^2 , perform the Kerékjártó-Freudenthal end-point compactification to N_i .

Classification 9.26 of [4], applied to each N_i of type (ii), ensures that the result is a manifold completion of M^2 . As a consequence, we have the following theorem.

Theorem 2.1. *A connected 2-manifold M^2 is completable if and only if $H_1(M^2)$ is finitely generated; in particular, Theorem 1.1 is valid when $n = 2$.*

In dimension 5 our proof of Theorem 1.1 goes through verbatim, provided it is always possible to work in neighborhoods of infinity with boundaries in which Freedman’s 4-dimensional Disk Embedding Theorem holds. That issue is discussed in [27]; [13, Sec. 11.9] in the less general setting of Siebenmann’s thesis, but the issues here are the same. In the language of [13]: Theorem 1.1 holds provided Condition (b) is strengthened to require the existence of arbitrarily small neighborhoods of infinity with stable peripheral pro- π_1 groups that are “good”. A caveat is that whenever [12] is applied, conclusions are topological, rather than piecewise-linear (PL) or smooth.

Remarkably, Siebenmann’s thesis fails in dimension 4 (see [25, 34]). Counterexamples to his theorem are, of course, counterexamples to Theorem 1.1 as well.

As for low-dimensional versions of Theorem 1.2: if $m \leq 3$ and M^m satisfies Condition (a) then M^m is completable (hence \mathcal{Z} -compactifiable), so $M^m \times [0, 1]$ is completable and \mathcal{Z} -compactifiable. If $m = 4$, then $M^4 \times [0, 1]$ is a 5-manifold, which (see Sec. 11) satisfies the conditions of Theorem 1.1. Whether that leads to a completion depends on 4-dimensional issues, in particular the “goodness” of the (stable) peripheral fundamental groups of the ends of $M^4 \times [0, 1]$. Those groups are determined by, but are not the same as, the fundamental groups at the ends of M^4 . If desired, a precise group-theoretic condition can be formulated from [18, Proposition 11.1].

3. Conventions, Notation, and Terminology

For convenience, all manifolds are assumed to be PL. That assumption is particularly useful for the topic at hand, since numerous instances of “smoothing corners” would be required in the smooth category (an issue that is covered nicely in [26]). With proper attention to such details, analogous theorems can be obtained in the smooth or topological category. Unless stated otherwise, an m -manifold M^m is permitted to have a boundary, denoted ∂M^m . We denote the *manifold interior* by $\text{int } M^m$. For $A \subseteq M^m$, the *point-set interior* will be denoted $\text{Int}_{M^m} A$ and the *frontier* by $\text{Fr}_{M^m} A$ (or for conciseness, $\text{Int}_M A$ and the *frontier* by $\text{Fr}_M A$). A *closed*

manifold is a compact boundaryless manifold, while an *open manifold* is a non-compact boundaryless manifold.

For $q < m$, a q -dimensional submanifold $Q^q \subseteq M^m$ is *properly embedded* if it is a closed subset of M^m and $Q^q \cap \partial M^m = \partial Q^q$; it is *locally flat* if each $p \in \text{int } Q^q$ has a neighborhood pair homeomorphic to $(\mathbb{R}^m, \mathbb{R}^q)$ and each $p \in \partial Q^q$ has a neighborhood pair homeomorphic to $(\mathbb{R}_+^m, \mathbb{R}_+^q)$. By this definition, the only properly embedded codimension 0 submanifolds of M^m are unions of its connected components; a more useful type of codimension 0 submanifold is the following: a codimension 0 submanifold $Q^m \subseteq M^m$ is *clean* if it is a closed subset of M^m and $\text{Fr}_M Q^m$ is a properly embedded locally flat (hence, bicollared) $(m - 1)$ -submanifold of M^m . In that case, $\overline{M^m \setminus Q^m}$ is also clean, and $\text{Fr}_M Q^m$ is a clean codimension 0 submanifold of both ∂Q^m and $\partial(\overline{M^m \setminus Q^m})$.

When the dimension of a manifold or submanifold is clear, we sometimes omit the superscript; for example, denoting a clean codimension 0 submanifold by Q . Similarly, when the ambient space is clear, we denote (point-set) interiors and frontiers by $\text{Int } A$ and $\text{Fr } A$.

For any codimension 0 clean submanifold $Q \subseteq M^m$, let $\partial_M Q$ denote $Q \cap \partial M^m$; alternatively $\partial_M Q = \partial Q \setminus \text{int}(\text{Fr } Q)$. Similarly, we will let $\text{int}_M Q$ denote $Q \cap \text{int } M^m$; alternatively $\text{int}_M Q = Q \setminus \partial M^m$.

4. Ends, Pro- π_1 , and the Peripheral π_1 -Stability Condition

4.1. Neighborhoods of infinity, partial neighborhoods of infinity, and ends

Let M^m be a connected manifold. A *clean neighborhood of infinity* in M^m is a clean codimension 0 submanifold $N \subseteq M^m$ for which $\overline{M^m \setminus N}$ is compact. Equivalently, a clean neighborhood of infinity is a set of the form $\overline{M^m \setminus C}$ where C is a compact clean codimension 0 submanifold of M^m . A *clean compact exhaustion* of M^m is a sequence $\{C_i\}_{i=1}^\infty$ of clean compact connected codimension 0 submanifolds with $C_i \subseteq \text{Int}_M C_{i+1}$ and $\cup C_i = M^m$. By letting $N_i = \overline{M^m \setminus C_i}$ we obtain the corresponding *cofinal sequence of clean neighborhoods of infinity*. Each such N_i has finitely many components $\{N_i^j\}_{j=1}^{k_i}$. By enlarging C_i to include all of the compact components of N_i , we can arrange that each N_i^j is noncompact; then, by drilling out regular neighborhoods of arcs connecting the various components of each $\text{Fr}_M N_i^j$ (further enlarging C_i), we can also arrange that each $\text{Fr}_M N_i^j$ is connected. A clean N_i with these latter two properties is called a *0-neighborhood of infinity*. Most constructions in this paper will begin with a clean compact exhaustion of M^m with a corresponding cofinal sequence of clean 0-neighborhoods of infinity.

Assuming the above arrangement, an *end* ε of M^m is determined by a nested sequence $(N_i^{k_i})_{i=1}^\infty$ of components of the N_i ; each component is called a *neighborhood of ε* . More generally, any subset of M^m that contains one of the $N_i^{k_i}$ is a neighborhood of ε , and any nested sequence $(W_j)_{j=1}^\infty$ of connected neighborhoods of ε , for which $\cap W_j = \emptyset$, also determines the end ε . A more thorough discussion of ends

can be found in [19]. Here, we will abuse notation slightly by writing $\varepsilon = (N_i^{k_i})_{i=1}^\infty$, keeping in mind that a sequence representing ε is not unique.

At times, we will have need to discuss components $\{N^j\}$ of a neighborhood of infinity N without reference to a specific end of M^m . In that situation, we will refer to the N^j as a *partial neighborhoods of infinity for M^m* (*partial 0-neighborhoods* if N is a 0-neighborhood of infinity). Clearly, every noncompact clean connected codimension 0 submanifold of M^m with compact frontier is a partial neighborhood of infinity with respect to an appropriately chosen compact C ; if its frontier is connected it is a partial 0-neighborhood of infinity.

4.2. The fundamental group of an end

For each end ε of M^m , we will define the *fundamental group at ε* by using inverse sequences. Two inverse sequences of groups $A_0 \xleftarrow{\alpha_1} A_1 \xleftarrow{\alpha_2} A_3 \xleftarrow{\alpha_3} \dots$ and $B_0 \xleftarrow{\beta_1} B_1 \xleftarrow{\beta_2} B_3 \xleftarrow{\beta_3} \dots$ are *pro-isomorphic* if they contain subsequences that fit into a commutative diagram of the form

$$\begin{array}{ccccccc}
 G_{i_0} & \xleftarrow{\lambda_{i_0+1,i_1}} & G_{i_1} & \xleftarrow{\lambda_{i_1+1,i_2}} & G_{i_2} & \xleftarrow{\lambda_{i_2+1,i_3}} & G_{i_3} & \dots \\
 & \searrow & \swarrow & \searrow & \swarrow & \searrow & \swarrow & \\
 & & H_{j_0} & \xleftarrow{\mu_{j_0+1,j_1}} & H_{j_1} & \xleftarrow{\mu_{j_1+1,j_2}} & H_{j_2} & \xleftarrow{\mu_{j_2+1,j_3}} \dots
 \end{array} \tag{4.1}$$

where the connecting homomorphisms in the subsequences are (as always) compositions of the original maps. An inverse sequence is *stable* if it is pro-isomorphic to a constant sequence $C \xleftarrow{\text{id}} C \xleftarrow{\text{id}} C \xleftarrow{\text{id}} \dots$. Clearly, an inverse sequence is pro-isomorphic to each of its subsequences; it is stable if and only if it contains a subsequence for which the images stabilize in the following manner:

$$\begin{array}{ccccccc}
 G_0 & \xleftarrow{\lambda_1} & G_1 & \xleftarrow{\lambda_2} & G_2 & \xleftarrow{\lambda_3} & G_3 & \dots \\
 & \searrow & \swarrow & \searrow & \swarrow & \searrow & \swarrow & \\
 & & \text{Im}(\lambda_1) & \xleftarrow{\cong} & \text{Im}(\lambda_2) & \xleftarrow{\cong} & \text{Im}(\lambda_3) & \xleftarrow{\cong} \dots
 \end{array} \tag{4.2}$$

where all unlabeled homomorphisms are restrictions or inclusions. (Here, we have simplified notation by relabeling the entries in the subsequence with integer subscripts.)

Given an end $\varepsilon = (N_i^{k_i})_{i=1}^\infty$, choose a ray $r : [1, \infty) \rightarrow M^m$ such that $r([i, \infty)) \subseteq N_i^{k_i}$ for each integer $i > 0$ and form the inverse sequence

$$\pi_1(N_1^{k_1}, r(1)) \xleftarrow{\lambda_2} \pi_1(N_2^{k_2}, r(2)) \xleftarrow{\lambda_3} \pi_1(N_3^{k_3}, r(3)) \xleftarrow{\lambda_4} \dots, \tag{4.3}$$

where each λ_i is an inclusion induced homomorphism composed with the change-of-basepoint isomorphism induced by the path $r|_{[i-1,i]}$. We refer to r as the *base ray* and the sequence (4.3) as a representative of the “fundamental group at ε based at r ” — denoted $\text{pro-}\pi_1(\varepsilon, r)$. Any similarly obtained representation (e.g. by choosing

a different sequence of neighborhoods of ε) using the same base ray can be seen to be pro-isomorphic. We say *the fundamental group at ε is stable* if (4.3) is a stable sequence. A key observation from the theory of ends is that stability of $\text{pro-}\pi_1(\varepsilon, r)$ depends on *neither* the choice of neighborhoods nor that of the base ray. See [19] or [14].

4.3. Relative connectedness, relative π_1 -stability, and the peripheral π_1 -stability condition

Let Q be a manifold and $A \subseteq \partial Q$. We say that Q is *A-connected at infinity* if Q contains arbitrarily small neighborhoods of infinity V for which $A \cup V$ is connected.

Example 1. If P is a compact manifold with connected boundary, $X \subseteq \partial P$ is a closed set, and $Q = P \setminus X$, then Q has one end for each component of X but Q is ∂Q -connected at infinity. More generally, if B is a clean connected codimension 0 manifold neighborhood of X in ∂P and $A = B \setminus X$, then Q is A -connected at infinity.

The following lemma is straightforward.

Lemma 4.1. *Let Q be a noncompact manifold and A a clean codimension 0 submanifold of ∂Q . Then Q is A -connected at infinity if and only if $Q \setminus A$ is 1-ended.*

If $A \subseteq \partial Q$ and Q is A -connected at infinity: let $\{V_i\}$ be a cofinal sequence of clean neighborhoods of infinity for which each $A \cup V_i$ is connected; choose a ray $r : [1, \infty) \rightarrow \text{int } Q$ such that $r([i, \infty)) \subseteq V_i$ for each $i > 0$; and form the inverse sequence

$$\pi_1(A \cup V_1, r(1)) \xleftarrow{\mu_2} \pi_1(A \cup V_2, r(2)) \xleftarrow{\mu_3} \pi_1(A \cup V_3, r(3)) \xleftarrow{\mu_4} \dots, \tag{4.4}$$

where bonding homomorphisms are obtained as in (4.3). We say Q is *A- π_1 -stable at infinity* if (4.4) is stable. Independence of this property from the choices of $\{V_i\}$ and r follows from the traditional theory of ends by applying Lemmas 4.1 and 4.2.

Lemma 4.2. *Let Q be a noncompact manifold and A a clean codimension 0 submanifold of ∂Q for which Q is A -connected at infinity. Then, for any cofinal sequence of clean neighborhoods of infinity $\{V_i\}$ and ray $r : [1, \infty) \rightarrow Q$ as described above, the sequence (4.4) is pro-isomorphic to any sequence representing $\text{pro-}\pi_1(Q \setminus A, r)$.*

Proof. It suffices to find a single cofinal sequence of connected neighborhoods of infinity $\{N_i\}$ in $Q \setminus A$ for which the corresponding representation of $\text{pro-}\pi_1(Q \setminus A, r)$ is pro-isomorphic to (4.4). Toward that end, for each i let $C_1 \supseteq C_2 \supseteq \dots$ be a nested sequence of relative regular neighborhoods of A in Q such that $\cap C_i = A$. By “cleanness” of the V_i , each C_i can be chosen so that $C_i \cup V_i$ is a clean codimension 0 submanifold of Q which deformation retracts onto $A \cup V_i$. Then $N_i = (C_i \cup V_i) \setminus A$

is a clean neighborhood of infinity in $Q \setminus A$ and $N_i \hookrightarrow C_i \cup V_i$ is a homotopy equivalence. For each i there is a canonical isomorphism $\alpha_i : \pi_1(A \cup V_i, r(i)) \rightarrow \pi_1(N_i, r(i))$ which is the composition

$$\pi_1(A \cup V_i, r(i)) \xrightarrow{\cong} \pi_1(C_i \cup V_i, r(i)) \xrightarrow{\cong} \pi_1(N_i, r(i)).$$

These isomorphisms fit into a commuting diagram

$$\begin{array}{ccccccc} \pi_1(A \cup V_1, r(1)) & \xleftarrow{\mu_2} & \pi_1(A \cup V_2, r(2)) & \xleftarrow{\mu_3} & \pi_1(A \cup V_3, r(3)) & \xleftarrow{\mu_4} & \dots \\ \alpha_1 \downarrow \cong & & \alpha_2 \downarrow \cong & & \alpha_3 \downarrow \cong & & \\ \pi_1(N_1, r(1)) & \xleftarrow{\lambda_2} & \pi_1(N_2, r(2)) & \xleftarrow{\lambda_3} & \pi_1(N_3, r(3)) & \xleftarrow{\lambda_4} & \dots \end{array}$$

completing the proof. □

Remark 3. In the above discussion, we allow for the possibility that $A = \emptyset$. In that case, A -connectedness at infinity reduces to 1-endedness and $A\text{-}\pi_1$ -stability to ordinary π_1 -stability at that end.

Definition 4.3. Let M^m be a manifold and ε an end of M^m .

- (1) M^m is *peripherally locally connected at infinity* if it contains arbitrarily small 0-neighborhoods of infinity N with the property that each component N^j is $\partial_M N^j$ -connected at infinity.
- (2) M^m is *peripherally locally connected at ε* if ε has arbitrarily small 0-neighborhoods P that are $\partial_M P$ -connected at infinity.

An N with the property described in condition (1) will be called a *strong 0-neighborhood of infinity* for M^m , and a P with the property described in condition (2) will be called a *strong 0-neighborhood of ε* . More generally, any connected partial 0-neighborhood of infinity Q that is $\partial_M Q$ -connected at infinity will be called a *strong partial 0-neighborhood of infinity*.

Lemma 4.4. M^m is *peripherally locally connected at infinity* iff M^m is *peripherally locally connected at each of its ends*.

Proof. Clearly, the initial condition implies the latter. For the converse, let N' be an arbitrary neighborhood of infinity in M^m and for each end ε , let P_ε be a 0-neighborhoods of ε , contained in N' , which is $\partial_M P_\varepsilon$ -connected at infinity. By compactness of the Freudenthal boundary of M^m , there is a finite subcollection $\{P_{\varepsilon_k}\}_{k=1}^n$ that covers the end of M^m ; in other words, $C = M^m - \bigcup_{k=1}^n P_{\varepsilon_k}$ is compact. If the P_{ε_k} are pairwise disjoint, we are finished; just let $N = \bigcup_{k=1}^n P_{\varepsilon_k}$. If not, adjust the P_{ε_k} within N' so they are in general position with respect to one another, then let $\{Q_j\}_{j=1}^s$ be the set of components of $\bigcup_{k=1}^n P_{\varepsilon_k}$ and note that each Q_j is a $\partial_M Q_j$ -connected partial 0-neighborhood of infinity. □

Remark 4. In the next section, we show that *every* inward tame manifold M^m is peripherally locally connected at infinity. As a consequence, that condition plays less prominent role than the next definition.

Definition 4.5. Let M^m be a manifold and ε an end of M^m .

- (1) M^m is *peripherally π_1 -stable at infinity* if contains arbitrarily small strong 0-neighborhoods of infinity N with the property that each component N^j is $\partial_M N^j$ - π_1 -stable at infinity.
- (2) M^m is *peripherally π_1 -stable at ε* if ε has arbitrarily small strong 0-neighborhoods P that are $\partial_M P$ - π_1 -stable at infinity.

It is easy to see that peripheral π_1 -stability at infinity implies peripheral π_1 -stability at each end; and when M^m is finite-ended, peripheral π_1 -stability at each end implies peripheral π_1 -stability at infinity. An argument could be made for defining peripheral π_1 -stability at infinity to mean “peripherally π_1 -stability at each end”. For us, that point is moot; in the presence of inward tameness the two alternatives are equivalent.

Lemma 4.6. *An inward tame manifold M^m is peripherally π_1 -stable at infinity if and only if it is peripherally π_1 -stable at each of its ends.*

Proof of this lemma is technical, and not central to the main argument. For that reason, we save the proof for later (see Sec. 13). Although it is not needed here, it would be interesting to know whether Lemma 4.6 holds without the assumption of inward tameness.

5. Finite Domination and Inward Tameness

A topological space P is *finitely dominated* if there exists a finite polyhedron K and maps $u : P \rightarrow K$ and $d : K \rightarrow P$ such that $d \circ u \simeq \text{id}_P$. If choices can be made so both $d \circ u \simeq \text{id}_P$ and $u \circ d \simeq \text{id}_K$, i.e. $P \simeq K$, we say P has *finite homotopy type*. For simplicity, we will restrict our attention to cases where P is a locally finite polyhedron — a class that contains the PL manifolds, submanifolds, and subspaces considered here.

Lemma 5.1. *Let M^m be a manifold and $A \subseteq \partial M$. Then M^m is finitely dominated [respectively, has finite homotopy type] if and only if $M^m \setminus A$ is finitely dominated [respectively, has finite homotopy type].*

Proof. $M^m \setminus A \hookrightarrow M^m$ is a homotopy equivalence, and these properties are homotopy invariants. □

Lemma 5.2. *A locally finite polyhedron P is finitely dominated if and only if there exists a homotopy $H : P \times [0, 1] \rightarrow P$ such that $H_0 = \text{id}_P$ and $\overline{H_1(P)}$ is compact.*

Proof. Assuming a finite domination, as described above, the homotopy between id_P and $d \circ u$ has the desired property. For the converse, let K be a compact polyhedral neighborhood of $\overline{H_1(P)}$, $u : K \hookrightarrow P$, and $d = H_1 : P \rightarrow K$. □

A locally finite polyhedron P is *inward tame* if it contains arbitrarily small polyhedral neighborhoods of infinity that are finitely dominated. Equivalently, P contains a cofinal sequence $\{N_i\}$ of closed polyhedral neighborhoods of infinity each admitting a “taming homotopy” $H : N_i \times [0, 1] \rightarrow N_i$ that pulls N_i into a compact subset of itself. By an application of the Homotopy Extension Property (similar to [20, Lemma 3.4]) we can require taming homotopies to be fixed on $\text{Fr } N_i$. From there, it is easy to see that, in an inward tame polyhedron, *every* closed neighborhood of infinity admits a taming homotopy.^a

Lemma 5.3. *Let M^m be a manifold and A a clean codimension 0 submanifold of ∂M^m . If M^m is inward tame then so is $M^m \setminus A$.*

Proof. For an arbitrarily small clean neighborhood of infinity N in M^m , let H be a taming homotopy that fixes $\text{Fr } N$. Then H extends via the identity to a homotopy that pulls $A \cup N$ into a compact subset of itself, so $A \cup N$ is finitely dominated. Arguing as in Lemma 4.2, $M^m \setminus A$ has arbitrarily small clean neighborhoods of infinity homotopy equivalent to such an $A \cup N$. □

Remark 5. Important cases of Lemma 5.3 are when $A = \partial M^m$ and when V is a clean neighborhood of infinity (or a component of one) and $A = \partial_M V$. Notice that Lemma 5.3 is valid when M^m is compact and H is the “empty map”.

A finitely dominated space has finitely generated homology, from which it can be shown that an inward tame manifold with compact boundary is finite-ended (see [21, Proposition 3.1]). That conclusion fails for manifolds with noncompact boundary; see item (3) of Remark 1. The following variation is crucial to this paper.

Proposition 5.4. *If a noncompact connected manifold M^m and its boundary each have finitely generated homology, then M^m has finitely many ends. More specifically, the number of ends of M^m is bounded above by $\dim H_{m-1}(M^m, \partial M^m; \mathbb{Z}_2) + 1$.*

Proof. Let C be a clean connected compact codimension 0 submanifold of M^m , with the property that $N = \overline{M^m \setminus C}$ is a 0-neighborhood of infinity, and let $\{N^j\}_{j=1}^k$ be the collection of connected components of N^n . It suffices to show that $k \leq \dim H_{m-1}(M^m, \partial M^m; \mathbb{Z}_2) + 1$. For the remainder of this proof (and only this proof), all homology is with \mathbb{Z}_2 -coefficients.

Note that ∂C is the union of clean codimension 0 submanifolds $\partial_M C$ and $\text{Fr } C$, which intersect in their common boundary $\partial(\text{Fr } C)$. So by a generalized version of Poincaré duality [23, Theorem 3.43] and the Universal Coefficients Theorem, for all i , we have

$$H_i(C, \partial_M C) \cong H_{m-i}(C, \text{Fr } C). \tag{5.1}$$

Claim 1. $\dim H_{m-1}(C, \partial_M C) \geq k - 1$.

^aFor a discussion of “tameness” terminology and its variants, see [19, Sec. 3.5.5].

By the long exact sequence for the pair $(C, \text{Fr } C)$, we have

$$\begin{array}{ccccccc} \dots & \rightarrow & H_1(C, \text{Fr } C) & \rightarrow & \tilde{H}_0(\text{Fr } C) & \rightarrow & \tilde{H}_0(C) \\ & & & & \parallel & & \parallel \\ & & & & (\mathbb{Z}_2)^{k-1} & & 0. \end{array}$$

So the claim follows from identity (5.1).

Claim 2. $\text{rank } H_{m-1}(N, \partial_M N) \geq k$.

This claim follows from the long exact sequence for the triple $(N, \partial N, \partial_M N)$:

$$\begin{array}{ccccccc} \rightarrow & H_m(N, \partial N) & \rightarrow & H_{m-1}(\partial N, \partial_M N) & \rightarrow & H_{m-1}(N, \partial_M N) & \rightarrow \\ & \parallel & & \parallel & & & \\ & 0 & & (\mathbb{Z}_2)^k & & & \end{array}$$

where triviality of $H_m(N, \partial N)$ is due to the noncompactness of all components of N , and the middle equality is from excision.

The relative Mayer–Vietoris Theorem for pairs [23, Sec. 2.2], applied to $(M^m, \partial M^m)$ expressed as $(C \cup N, \partial_M C \cup \partial_M N)$, contains

$$H_{m-1}(\text{Fr } C, \partial \text{Fr } C) \rightarrow H_{m-1}(C, \partial_M C) \oplus H_{m-1}(N, \partial_M N) \rightarrow H_{m-1}(M^m, \partial M^m) \tag{5.2}$$

from which we can deduce

$$\begin{aligned} & \dim(H_{m-1}(C, \partial_M C) \oplus H_{m-1}(N, \partial_M N)) \\ & \leq \dim H_{m-1}(\text{Fr } C, \partial \text{Fr } C) + \dim H_{m-1}(M^m, \partial M^m). \end{aligned}$$

Since $H_{m-1}(\text{Fr } C, \partial \text{Fr } C) \cong (\mathbb{Z}_2)^k$ (from excision), then by Claims 1 and 2, we have

$$(k - 1) + k \leq k + \dim H_{m-1}(M^m, \partial M^m).$$

So $k \leq \dim H_{m-1}(M^m, \partial M^m) + 1$. □

Corollary 5.5. *If M^m is inward tame, then M^m is peripherally locally connected at infinity.*

Proof. By Lemma 4.1, it suffices to show that each compact codimension 0 clean submanifold $D \subseteq M^m$ is contained in a compact codimension 0 clean submanifold $C \subseteq M^m$ so that if $N = \overline{M^m \setminus C}$, then each component N^j of N has the property that $N^j \setminus \partial M^m$ is 1-ended.

Since M^m is inward tame, each of its clean neighborhoods of infinity is finitely dominated, so $\overline{M^m \setminus D}$ has finitely many components, each of which is finitely dominated. Let P^l be one of those components. Then, $\text{Fr } P^l$ is a compact clean codimension 0 submanifold of ∂D , whose interior is the boundary of $P^l \setminus \partial M^m$. Since $\text{int}(\text{Fr } P^l)$ and $P^l \setminus \partial M^m$ each have finitely generated homology ($P^l \setminus \partial M^m$ is finitely dominated), then by Proposition 5.4, $P^l \setminus \partial M^m$ has finitely many ends. Choose a compact clean codimension 0 submanifold K_l of $P^l \setminus \partial M^m$ that intersects

$\text{int}(\text{Fr } P^l)$ non-trivially and has exactly one (unbounded) complementary component in $P^l \setminus \partial M^m$ for each of those ends. After doing this for each of the component P^l of $\overline{M^m \setminus D}$, let $C = D \cup (\cup K_l)$. □

6. Finite Homotopy Type and the σ_∞ -Obstruction

Finitely generated projective left Λ -modules S and T are *stably equivalent* if there exist finitely generated free Λ -modules F_1 and F_2 such that $S \oplus F_1 \cong T \oplus F_2$. Under the operation of direct sum, the stable equivalence classes of finitely generated projective modules form a group $\tilde{K}_0(\Lambda)$, the *reduced projective class group* of Λ . In [33], Wall associated to each path connected finitely dominated space P a well-defined $\sigma(P) \in \tilde{K}_0(\mathbb{Z}[\pi_1(P)])$ which is trivial if and only if P has finite homotopy type. (Here $\mathbb{Z}[\pi_1(P)]$ denotes the integral group ring corresponding to $\pi_1(P)$). In the literature, $\tilde{K}_0(\mathbb{Z}[G])$ is sometimes abbreviated to $\tilde{K}_0(G)$.) As one of the necessary and sufficient conditions for completability of a 1-ended inward tame open manifold M^m ($m > 5$) with stable pro- π_1 , Siebenmann defined the *end obstruction* $\sigma_\infty(M^m)$, to be (up to sign) the finiteness obstruction $\sigma(N)$ of an arbitrary clean neighborhood of infinity N whose fundamental group “matches” the stable pro- $\pi_1(\varepsilon(M^m))$.^b

In cases where M^m is multi-ended or has non-stable pro- π_1 (or both), a more general definition of $\sigma_\infty(M^m)$, introduced in [6], is required. Its definition employs several ideas from [30, Sec. 6]. First note that there is a covariant functor \tilde{K}_0 from groups to abelian groups taking G to $\tilde{K}_0(\mathbb{Z}[G])$, which may be composed with the π_1 -functor to get a functor from path connected spaces to abelian groups; here we use an observation by Siebenmann allowing base points to be ignored. Next extend the functor and the finiteness obstruction to non-path-connected P (abusing notation slightly) by letting

$$\tilde{K}_0(\mathbb{Z}[\pi_1(P)]) = \bigoplus \tilde{K}_0(\mathbb{Z}[\pi_1(P^j)]),$$

where $\{P^j\}$ is the set of path components of P , and letting

$$\sigma(P) = (\sigma(P^1), \dots, \sigma(P^k))$$

recalling that P is finitely dominated and, hence, has finitely many components — each finitely dominated.

Now, for an inward tame locally finite polyhedron P (or more generally locally compact ANR), let $\{N_j\}$ be a nested cofinal sequence of closed polyhedral

^bThe main theorem of [26] incorrectly uses $\sigma(M^m)$ — the finiteness obstruction of the entire manifold M^m — in place of $\sigma_\infty(M^m)$. The mistake is an erroneous application of Siebenmann’s Sum Theorem to conclude that triviality of $\sigma(M^m)$ implies triviality of $\sigma(N)$ for each clean neighborhood of infinity N . Siebenmann [30] (correctly) used the Sum Theorem to show that, in the case of *stable* pro- π_1 , it is enough to check the obstruction once — for a well-chosen clean neighborhood of infinity. He denoted that obstruction $\sigma(\varepsilon)$. In our situation (and O’Brien’s) such a simplification is not possible. We use the subscripted “ ∞ ” to help distinguish the general situation from Siebenmann’s special case.

neighborhoods of infinity and define

$$\sigma_\infty(P) = (\sigma(N_1), \sigma(N_2), \sigma(N_3), \dots) \in \varprojlim \{ \tilde{K}_0[\mathbb{Z}[\pi_1(N_i)]] \}.$$

The bonding maps of the target inverse sequence

$$\tilde{K}_0[\mathbb{Z}[\pi_1(N_1)]] \leftarrow \tilde{K}_0[\mathbb{Z}[\pi_1(N_2)]] \leftarrow \tilde{K}_0[\mathbb{Z}[\pi_1(N_3)]] \leftarrow \dots$$

are induced by inclusion, with the Sum Theorem for finiteness obstructions [30, Theorem 6.5] assuring consistency. Clearly, $\sigma_\infty(P)$ vanishes if and only if each N_i has finite homotopy type; by another application of the Sum Theorem, this happens if and only if every closed polyhedral neighborhood of infinity has finite homotopy type.

Remark 6. Alternatively, we could define $\sigma_\infty(P)$ to lie in the inverse limit of the *inverse system* corresponding to all closed polyhedral neighborhoods of infinity, partially ordered by inclusion. These inverse limits are isomorphic, and in either case, the combination of Conditions (a) and (c) of Theorem 1.1 is equivalent to the requirement that all clean neighborhoods of infinity have finite homotopy type — a property referred to as absolute inward tameness in [19].

We close this section with an observation that builds upon Lemma 5.3. Both play key roles in the proof of Theorem 1.1.

Lemma 6.1. *Let M^m be a manifold and A a clean codimension 0 submanifold of ∂M^m . If M^m is inward tame and $\sigma_\infty(M^m)$ vanishes, then $M^m \setminus A$ is inward tame and $\sigma_\infty(M^m \setminus A)$ also vanishes.*

Proof. Lemma 5.3 assures us that if M^m is inward tame, then so too is $M^m \setminus A$. The latter ensures that $\sigma_\infty(M^m \setminus A)$ is defined. Arguing as we did in the proof of Lemma 5.3, $M^m \setminus A$ contains arbitrarily small neighborhoods of infinity which are homotopy equivalent to $A \cup N$, where N is a clean neighborhood of infinity in M^m . If $\sigma_\infty(M^m) = 0$, then N has finite homotopy type; and since $A \cup N = \overline{A \setminus N} \cup N$, where $\overline{A \setminus N}$ is a compact $(m - 1)$ -manifold, then $A \cup N$ has finite homotopy type (by a direct argument or easy application of the Sum Theorem for the finiteness obstruction). The vanishing of $\sigma_\infty(M^m \setminus A)$ then follows from the above discussion. □

7. The τ_∞ -Obstruction

The τ_∞ obstruction in Condition (d) of Theorem 1.1 was first defined in [6] and applied to Hilbert cube manifolds; the role it plays here is similar. It lies in the derived limit of an inverse sequence of Whitehead groups. For a more detailed discussion, the reader should see [6].

The *derived limit* of an inverse sequence

$$G_0 \xleftarrow{\lambda_1} G_1 \xleftarrow{\lambda_2} G_2 \xleftarrow{\lambda_3} \dots$$

of abelian groups is the quotient group:

$$\varprojlim^1 \{G_i, \lambda_i\} = \left(\prod_{i=0}^{\infty} G_i \right) / \{(g_0 - \lambda_1 g_1, g_1 - \lambda_2 g_2, g_2 - \lambda_3 g_3, \dots) \mid g_i \in G_i\}.$$

It is a standard fact that pro-isomorphic inverse sequences of abelian groups have isomorphic derived limits.

Suppose a manifold M^m contains a cofinal sequence $\{N_i\}$ of clean neighborhoods of infinity with the property that each inclusion $\text{Fr } N_i \hookrightarrow N_i$ is a homotopy equivalence.^c Let $W_i = N_i \setminus N_{i+1}$ and note that $\text{Fr } N_i \hookrightarrow W_i$ is a homotopy equivalence. See Fig. 1.

Since $\text{Fr } N_i$ and W_i are finite polyhedra, the inclusion determines a Whitehead torsion $\tau(W_i, \text{Fr } N_i) \in \text{Wh}(\pi_1(\text{Fr } N_i))$ (see [7]). As in the previous section, we must

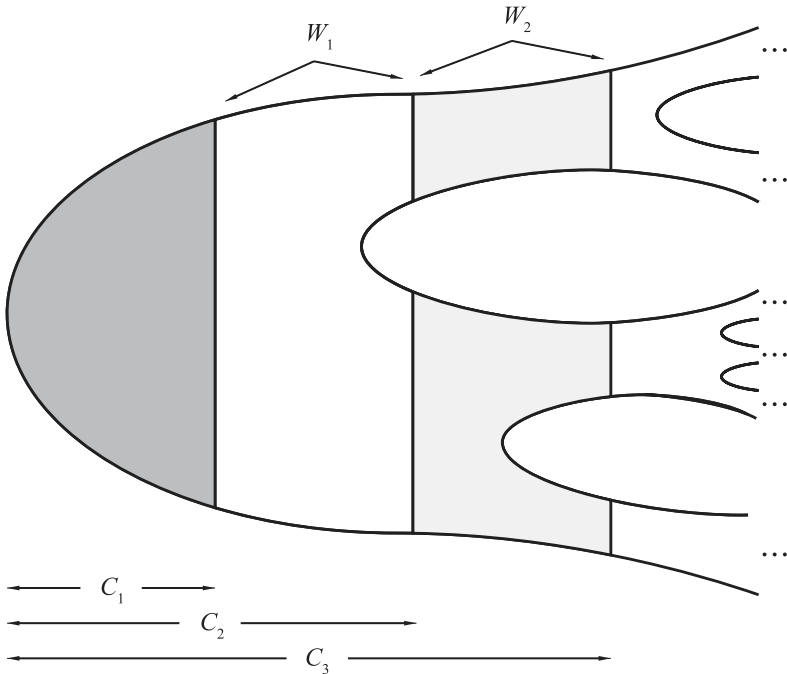


Fig. 1. Decomposition of M^m into $\{W_i\}_{i=1}^{\infty}$.

^cA manifold admitting such sequence of neighborhoods of infinity is called *pseudo-collarable*. See [16, 21, 22] for discussion of that topic.

allow for non-connected $\text{Fr } N_i$ so we define

$$\text{Wh}(\pi_1(\text{Fr } N_i)) = \bigoplus \text{Wh}(\pi_1(\text{Fr } N_i^j)),$$

where $\{\text{Fr } N_i^j\}$ is the (finite) set of components of $\text{Fr } N_i$ and

$$\tau(W_i, \text{Fr } N_i) = (\tau(W_i^1, \text{Fr } N_i^1), \dots, \tau(W_i^k, \text{Fr } N_i^k)).$$

These groups fit into and inverse sequence of abelian groups

$$\text{Wh}(\pi_1(N_1)) \leftarrow \text{Wh}(\pi_1(N_2)) \leftarrow \text{Wh}(\pi_1(N_3)) \leftarrow \dots$$

where the bonding homomorphisms are induced by inclusions. (To match [6], we have substituted $\pi_1(N_i)$ for the canonically equivalent $\pi_1(\text{Fr } N_i)$.) Let $\tau_i = \tau(W_i, \text{Fr } N_i) \in \text{Wh}(\pi_1(N_i))$. Then

$$\tau_\infty(M^m) = [(\tau_1, \tau_2, \tau_3, \dots)] \in \varprojlim^1 \{\text{Wh}(\pi_1(N_i))\},$$

where $[(\tau_1, \tau_2, \tau_3, \dots)]$ is the coset containing $(\tau_1, \tau_2, \tau_3, \dots)$.

If $\tau_\infty(M^m)$ is trivial, it is possible to adjust the choices of the N_i so that each inclusion $\text{Fr } N_i \hookrightarrow W_i$ has trivial torsion, and hence is a simple homotopy equivalence. Roughly speaking, the adjustment involves “lending and borrowing torsion to and from immediate neighbors of the W_i ”. The procedure is as described in [6, Sec. 6], except that a Splitting Theorem for finite-dimensional manifolds (see [26, pp. 318]) replaces [6, Lemma 6.1]. The reader is warned that the procedure described in [26, Sec. 4] is flawed; we recommend [6].

8. Geometric Characterization of Completable Manifolds and a Review of h- and s-Cobordisms

The following geometric characterization of completable manifolds, which has analogs in [26, 32], paves the way for the proof of Theorem 1.1. It leads naturally to the consideration of h- and s-cobordisms, which we will briefly review for later use.

Lemma 8.1. (Geometric characterization of completable manifolds) *A non-compact manifold with boundary M^m is completable iff $M^m = \bigcup_{i=1}^\infty C_i$ where, for all i :*

- (i) C_i is a compact clean codimension 0 submanifold of M^m ,
- (ii) $C_i \subset \text{Int } C_{i+1}$, and
- (iii) if W_i denotes $\overline{C_{i+1} \setminus C_i}$, then $(W_i, \text{Fr } C_i) \approx (\text{Fr } C_i \times [0, 1], \text{Fr } C_i \times \{0\})$.

Proof. For the forward implication, suppose \widehat{M}^m is a compact manifold, A is closed subset set of $\partial \widehat{M}^m$, and $M^m = \widehat{M}^m \setminus A$. Write A as $\bigcap_i F_i$, where $\{F_i\}_{i=1}^\infty$ is a sequence of compact clean codimension 0 submanifolds of $\partial \widehat{M}^m$ with $F_{i+1} \subseteq \text{Int } F_i$. Let $c : \partial \widehat{M}^m \times [0, 1] \rightarrow \widehat{M}^m$ be a collar on $\partial \widehat{M}^m$ with $c(\partial \widehat{M}^m \times \{0\}) = \partial \widehat{M}^m$ and, for

each i , let $C_i = \widehat{M}^m \setminus c(\text{Int}(F_i) \times [0, 1/i])$. Assertions (i) and (ii) are clear. Moreover,

$$\begin{aligned} W_i &\approx F_i \times [0, 1/i] \setminus (\text{Int } F_{i+1} \times [0, 1/(i+1)]) \\ &\approx F_i \times [0, 1/i] \end{aligned}$$

via a homeomorphism taking $c(F_i \times \{1/i\})$ onto $F_i \times \{1/i\}$. Then, since $\text{Fr } C_i = c(F_i \times \{1/i\}) \cup \partial F_i \times [0, 1/i] \approx F_i$, an application of relative regular neighborhood theory allows an adjustment of that homeomorphism so that $\text{Fr } C_i$ is taken onto $F_i \times \{1/i\}$. A reparametrization of the closed interval completes the proof of assertion (iii). (Note that this works even when the F_i have multiple and varying numbers of components. See Fig. 2.)

For the converse, we reverse the above procedure to embed M^m in a copy of C_1 . Details can be found in [32, Lemma 1]. □

The above lemma shows that a strategy for completing a manifold is to fill up a neighborhood of infinity in M^m with a sequence of cobordisms, then modify those cobordisms (when possible) so they become products.

Recall that an (absolute) *cobordism* is a triple (W, A, B) , where W is a manifold with boundary and A and B are disjoint manifolds without boundary for which $A \cup B = \partial W$. The triple (W, A, B) is a *relative cobordism* if A and B are disjoint codimension 0 clean submanifolds of ∂W . In that case, there is an associated absolute cobordism $(V, \partial A, \partial B)$, where $V = \partial W \setminus (\text{int } A \cup \text{int } B)$. We view absolute cobordisms as special cases of relative cobordisms where $V = \emptyset$. A relative cobordism is an *h-cobordism* if each of the inclusions $A \hookrightarrow W$, $B \hookrightarrow W$, $\partial A \hookrightarrow V$, and

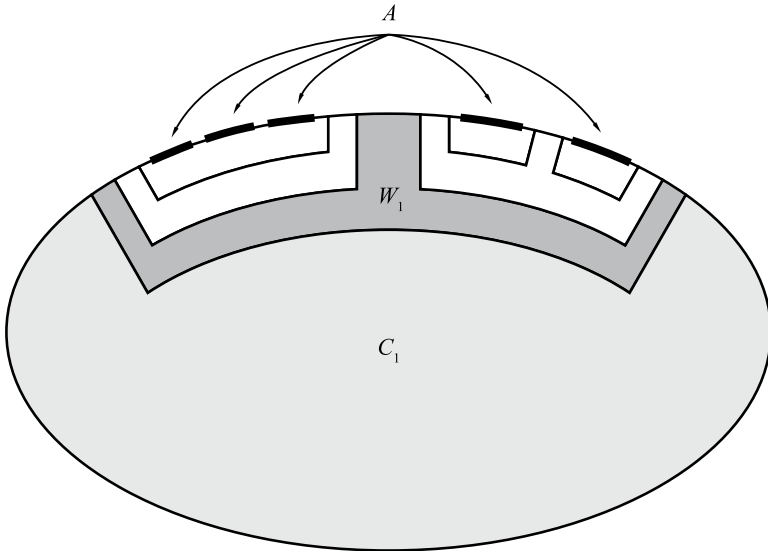


Fig. 2. Decomposing completed M^m into product cobordisms.

$\partial B \hookrightarrow V$ is a homotopy equivalence; it is an *s-cobordism* if each of these inclusions is a simple homotopy equivalence. (For convenience, $\emptyset \hookrightarrow \emptyset$ is considered a simple homotopy equivalence.) A relative cobordism is *nice* if it is absolute or if $(V, \partial A, \partial B) \approx (\partial A \times [0, 1], \partial A \times \{0\}, \partial A \times \{1\})$. The crucial result, proof (and additional discussion) of which may be found in [29], is the following theorem.

Theorem 8.2. (Relative s-cobordism Theorem) *A compact nice relative cobordism (W, A, B) with $\dim W \geq 6$ is a product, i.e. $(W, A, B) \approx (A \times [0, 1], A \times \{0\}, A \times \{1\})$, if and only if it is an s-cobordism.*

Remark 7. A situation similar to a nice relative cobordism occurs when $\partial W = A \cup B'$, where A and B' are codimension 0 clean submanifolds of ∂W with a common non-empty boundary $\partial A = \partial B'$. By choosing a clean codimension 0 submanifold $B \subseteq B'$ with the property that $B' \setminus \text{Int } B \approx \partial B \times [0, 1]$ we arrive at a nice relative cobordism (W, A, B) . When this procedure is applied, we will refer to (W, A, B) as a *corresponding nice relative cobordism*. For notational consistency, we will always adjust the term B' on the far right of the triple (W, A, B') , leaving A alone.

For our purposes, the following lemma will be crucial.

Lemma 8.3. *Let W be a compact manifold with $\partial W = A \cup B'$, where A and B' are codimension 0 clean submanifolds of ∂W with a common boundary. Suppose $A \hookrightarrow W$ is a homotopy equivalence and that there is a homotopy $J : W \times [0, 1] \rightarrow W$ such that $J_0 = \text{id}_W$, J is fixed on $\partial B'$, and $J_1(W) \subseteq B'$. Then $B' \hookrightarrow W$ is a homotopy equivalence, so the corresponding nice relative cobordism (W, A, B) is an h-cobordism.*

Proof. Choose $p \in \partial A = \partial B'$, to be used as the basepoint for A , B' and W . Let $i : A \hookrightarrow W$ and $\iota : B' \hookrightarrow W$ denote inclusions and define $f : A \rightarrow B'$ by $f(x) = J_1(x)$. Then

$$\iota \circ f = J_1 \circ i. \tag{8.1}$$

Clearly, $J_1 : W \rightarrow W$ induces the identity isomorphism on $\pi_1(W, p)$, and since i is a homotopy equivalence, it induces a π_1 -isomorphism. So, from (8.1), we may deduce that $f_* : \pi_1(A, p) \rightarrow \pi_1(B', p)$ is injective. Moreover, since f restricts to the identity function mapping ∂A onto $\partial B'$, [8] allows us to conclude that f_* is an isomorphism. From there it follows that $\iota_* : \pi_1(B', p) \rightarrow \pi_1(W, p)$ is also an isomorphism.

Let $p : \widetilde{W} \rightarrow W$ be the universal covering projection, $\widetilde{A} = p^{-1}(A)$, and $\widetilde{B}' = p^{-1}(B')$. Since i_* and ι_* are both π_1 -isomorphisms these are the universal covers of A and B' , respectively. By generalized Poincaré duality for non-compact manifolds,

$$H_k(\widetilde{W}, \widetilde{B}'; \mathbb{Z}) \cong H_c^{n-k}(\widetilde{W}, \widetilde{A}; \mathbb{Z}),$$

where cohomology is with compact supports. Since $\widetilde{A} \hookrightarrow \widetilde{W}$ is a proper homotopy equivalence, all of these relative cohomology groups vanish, so $H_k(\widetilde{W}, \widetilde{B}'; \mathbb{Z}) = 0$ for all k . By the relative Hurewicz theorem, $\pi_k(\widetilde{W}, \widetilde{B}') = 0$ for all k , so the same is

true for $\pi_k(W, B')$. An application of Whitehead’s theorem allows us to conclude that $B' \hookrightarrow W$ is a homotopy equivalence. \square

9. Proof of the Manifold Completion Theorem: Necessity

We will prove necessity of the conditions in Theorem 1.1 by a straightforward application of Lemma 8.1.

Proof of Theorem 1.1. (necessity) Suppose \widehat{M}^m is a compact manifold and A is closed subset set of $\partial\widehat{M}^m$ such that $M^m = \widehat{M}^m \setminus A$. As in the proof of Lemma 8.1 write $A = \bigcap_i F_i$, where $\{F_i\}$ is a sequence of compact clean codimension 0 submanifolds of $\partial\widehat{M}^m$ with $F_{i+1} \subseteq \text{Int } F_i$, and let $c : \partial\widehat{M}^m \times [0, 1] \rightarrow \widehat{M}^m$ be a collar on $\partial\widehat{M}^m$ with $c(\partial\widehat{M}^m \times \{0\}) = \partial\widehat{M}^m$. For each i , let $\widehat{N}_i = c(F_i \times [0, 1/i])$ and $N_i = \widehat{N}_i \setminus A$. Then $\{N_i\}$ is cofinal sequence of clean neighborhoods of infinity in M^m with $\text{Fr } N_i = c(F_i \times \{1/i\} \cup \partial F_i \times [0, 1/i])$. Since $F_i \times \{1/i\} \cup \partial F_i \times [0, 1/i] \hookrightarrow F_i \times [0, 1/i]$ and $N_i \hookrightarrow \widehat{N}_i$ are both homotopy equivalences, then so is $\text{Fr } N_i \hookrightarrow N_i$; and since each N_i has finite homotopy type, conditions (a) and (c) of Theorem 1.1 both hold (by the discussion in Secs. 5 and 6).

If we let $W_i = \overline{N_i \setminus N_{i+1}}$, then $\tau_\infty(M^m)$ is determined by the Whitehead torsions of inclusions $\text{Fr } N_i \hookrightarrow W_i$ (see Sec. 7). Associate W_i with $F_i \times [0, 1/i]$ and $\text{Fr } N_i$ with $F_i \times \{1/i\} \cup \partial F_i \times [0, 1/i]$, as in the proof of Lemma 8.1. Then, the fact that both $F_i \times \{1/i\} \hookrightarrow F_i \times [0, 1/i]$ and $F_i \times \{1/i\} \hookrightarrow F_i \times \{1/i\} \cup \partial F_i \times [0, 1/i]$ are simple homotopy equivalences ensures that $\tau(W_i, \text{Fr } N_i) = 0$. So condition (d) is satisfied.

It remains to verify the peripheral π_1 -stability condition. Fix $i \geq 1$ and let F_i^j be one component of F_i , $\widehat{N}_i^j = c(F_i^j \times [0, 1/i])$ and $N_i^j = \widehat{N}_i^j \setminus A$. Then $\partial_M N_i^j = c(F_i^j \times \{0\}) \setminus A$ and N_i^j is clearly $\partial_M N_i^j$ -connected at infinity. For each $k > i$, let F_k' be the union of all components of F_k contained in F_i^j , $\widehat{N}_k' = c(F_k' \times [0, 1/k])$ and $N_k' = \widehat{N}_k' \setminus A$. By definition, we may consider the sequence

$$\pi_1(\partial_M N_i^j \cup N_{i+1}') \xleftarrow{\mu_2} \pi_1(\partial_M N_i^j \cup N_{i+2}') \xleftarrow{\mu_3} \pi_1(\partial_M N_i^j \cup N_{i+3}') \xleftarrow{\mu_4} \dots, \tag{9.1}$$

where basepoints are suppressed and bonding homomorphisms are compositions of maps induced by inclusions and change-of-basepoint isomorphisms. Each of those inclusions is the top row of a commutative diagram

$$\begin{array}{ccc} \partial_M N_i^j \cup N_k' & \hookrightarrow & \partial_M N_i^j \cup N_{k+1}' \\ \downarrow \text{incl} & & \downarrow \text{incl} \\ \partial_M N_i^j \cup \widehat{N}_k' & & \partial_M N_i^j \cup \widehat{N}_{k+1}' \\ \downarrow \approx & & \downarrow \approx \\ (F_i^j \times \{0\}) \cup (F_k' \times [0, 1/k]) & \hookrightarrow & (F_i^j \times \{0\}) \cup (F_{k+1}' \times [0, 1/k + 1]), \end{array}$$

where the bottom row is an obvious homotopy equivalence, as are all vertical maps. It follows that the initial inclusion is a homotopy equivalence as well. As a result, all bonding homomorphisms in (9.1) are isomorphisms, so the sequence is stable. \square

10. Proof of the Manifold Completion Theorem: Sufficiency

Throughout this section $\{C_i\}_{i=1}^\infty$ will denote a clean compact exhaustion of M^m with a corresponding cofinal sequence of clean 0-neighborhoods of infinity $\{N_i\}_{i=1}^\infty$, each of which has a finite set of connected components $\{N_i^j\}_{j=1}^{k_i}$. For each i we will let $W_i = \overline{N_i \setminus N_{i+1}}$, a compact clean codimension 0 submanifold of M^m . Note that ∂W_i may be expressed as $\text{Fr } N_i \cup (\partial_M W_i \cup \text{Fr } N_{i+1})$, a union of two clean codimension 0 submanifolds of ∂W_i intersecting in a common boundary $\partial(\text{Fr } N_i)$. (Figs. 1 and 2 contain useful schematics.) The proof of Theorem 1.1, will be accomplished by gradually improving the exhaustion of M^m so that ultimately, conditions (i)–(iii) of Lemma 8.1 are all satisfied.

Lemma 10.1. *If M^m is inward tame and $\sigma_\infty(M^m)$ vanishes, then for each i , $\sigma(N_i)$ and $\sigma(N_i \setminus \partial M^m)$ are both zero.*

Proof. By our discussion in Sec. 6, if M^m is inward tame and $\sigma_\infty(M^m) = 0$, then each N_i has finite homotopy type. Since $N_i \hookrightarrow N_i \setminus \partial M^m$ is a homotopy equivalence, so does $N_i \setminus \partial M^m$. □

Proposition 10.2. *If M^m satisfies Conditions (a)–(c) of Theorem 1.1 then the $\{C_i\}$ and the corresponding $\{N_i\}$ can be chosen so that, for each i ,*

- (1) $\text{Fr } N_i \hookrightarrow N_i$ is a homotopy equivalence, and
- (2) $\partial_M W_i \cup \text{Fr } N_{i+1} \hookrightarrow N_i$ is a homotopy equivalence; therefore,
- (3) the nice relative cobordisms corresponding to $(W_i, \text{Fr } N_i, \partial_M W_i \cup \text{Fr } N_{i+1})$ are h -cobordisms.

Proof. By Lemma 10.1 and the definition of peripheral π_1 -stability at infinity, we can begin with a clean compact exhaustion $\{C_i\}_{i=1}^\infty$ of M^m and a corresponding sequence of neighborhoods of infinity $\{N_i\}_{i=1}^\infty$, each with a finite set of connected components $\{N_i^j\}_{j=1}^{k_i}$, so that for all $i \geq 1$ and $1 \leq j \leq k_i$,

- (i) N_i^j is inward tame,
- (ii) N_i^j is $(\partial_M N_i^j)$ -connected and $(\partial_M N_i^j)$ - π_1 -stable at infinity, and
- (iii) $\sigma_\infty(N_i^j) = 0$.

By Lemmas 5.3, 4.2, and 6.1, this implies that

- (i') $N_i^j \setminus \partial_M N_i^j$ is inward tame,
- (ii') $N_i^j \setminus \partial M^m$ is 1-ended and has stable fundamental group at infinity, and
- (iii') $\sigma_\infty(N_i^j \setminus \partial M^m) = 0$.

These are precisely the hypotheses of Siebenmann’s Relativized Main Theorem [30, Theorem 10.1], so $N_i^j \setminus \partial M^m$ contains an open collar neighborhood of infinity $V_i^j \approx \partial V_i^j \times [0, \infty)$. Following the proof in [30] (similar to what is done in [26, Theorem 3.2]), this can be done so that $\partial N_i^j \setminus \partial M^m (= \text{int}(\text{Fr } N_i^j))$ and ∂V_i^j

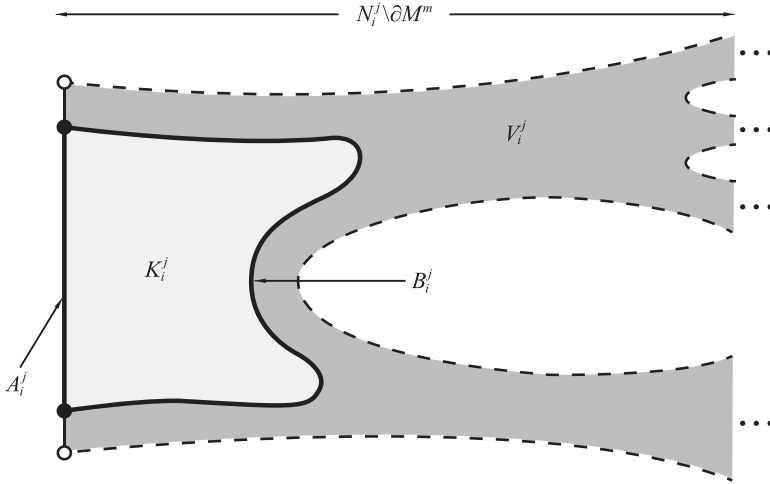


Fig. 3. $V_i^j \approx \partial V_i^j \times [0, 1)$ contained in $N_i^j \setminus \partial M^m$.

contain clean compact codimension 0 submanifolds A_i^j and B_i^j , respectively, so that $(\partial N_i^j \setminus \partial M^m) \setminus \text{int } A_i^j = \partial V_i^j \setminus \text{int } B_i^j \approx \partial A_i^j \times [0, 1)$. See Fig. 3.

Then $K_i^j = \overline{N_i^j \setminus V_i^j}$ is a clean codimension 0 submanifold of M^m which intersects C_i in A_i^j . To save on notation, replace C_i with $C_i \cup (\cup K_i^j)$, which is still a clean compact codimension 0 submanifold of M^m , but with the added property that

$$N_i \setminus \partial M^m \approx \text{int}(\text{Fr } N_i) \times [0, \infty). \tag{10.1}$$

Since adding $\partial_M N_i$ back in does not affect homotopy types, we also have that

$$\text{Fr } N_i \hookrightarrow N_i \text{ is a homotopy equivalence.} \tag{10.2}$$

Having enlarged the C_i , pass to a subsequence if necessary to regain the property that $C_i \subseteq \text{Int } C_{i+1}$ for all i .

Letting $N_i = \overline{M^m \setminus C_i}$ gives a nested cofinal sequence of clean neighborhoods of infinity $\{N_i\}$ with the property that each inclusion $\text{Fr } N_i \hookrightarrow N_i$ is a homotopy equivalence; in other words, we have obtained a pseudo-collar structure on M^m . For each $i \geq 1$, let $W_i = \overline{N_i \setminus N_{i+1}}$, a clean compact codimension 0 submanifold of M^m with $\partial W_i = \text{Fr } N_i \cup (\partial_M W_i \cup \text{Fr } N_{i+1})$.

Claim 1. $\text{Fr } N_i \hookrightarrow W_i$ is a homotopy equivalence.

Condition (10.2) applied to N_i ensures the existence a strong deformation retraction H_t of N_i onto $\text{Fr } N_i$. That same condition applied to N_{i+1} ensures the existence of a retraction $r : N_{i+1} \rightarrow \text{Fr } N_{i+1}$, which extends to a retraction $\hat{r} : N_i \rightarrow W_i$. The composition $\hat{r}H_t$, restricted to W_i , gives a deformation retraction of W_i onto $\text{Fr } N_i$.

Claim 2. $\partial_M W_i \cup \text{Fr } N_{i+1} \hookrightarrow W_i$ is a homotopy equivalence.

By applying Lemma 8.3, it is enough to show that there exists a homotopy $H : W_i \times [0, 1] \rightarrow W_i$, fixed on $\partial(\text{Fr } N_i)$, with the property that $H_1(W_i) \subseteq \partial_M W_i \cup \text{Fr } N_{i+1}$. Toward that end, let B be a collar neighborhood of $\partial_M W_i$ in W_i and let $D = \overline{W_i} \setminus \overline{B}$. Use the collar structure on $N_i \setminus \partial M^m$ to obtain a homotopy $K : N_i \times [0, 1] \rightarrow N_i$, fixed on $\partial(\text{Fr } N_i)$, which pushes N_i into the complement of D ; in other words $K_1(N_i) \subseteq B \cup N_{i+1}$. Compose this homotopy with the retraction $\widehat{r} : N_i \rightarrow W_i$ used in the previous claim to get a homotopy $\widehat{r}K_t$ of W_i (still fixed on $\partial(\text{Fr } N_i)$) with $\widehat{r}K_1(W_i) \subseteq B \cup \text{Fr } N_{i+1}$. Follow this with a homotopy that deformation retracts B onto $\partial_M W_i$ while sending $\text{Fr } N_{i+1}$ into itself to complete the desired homotopy and prove Claim 2.

We can now write $M^m = C_1 \cup W_1 \cup W_2 \cup W_3 \cup \dots$ where, for each i ,

- W_i is a compact clean codimension 0 submanifold of M^m ,
- $\partial W_i = \text{Fr } N_i \cup (\partial_M W_i \cup \text{Fr } N_{i+1})$, and
- both $\text{Fr } N_i \hookrightarrow W_i$ and $\partial_M W_i \cup \text{Fr } N_{i+1} \hookrightarrow W_i$ are homotopy equivalences.

As such, the corresponding nice relative cobordisms (as described in Remark 7) are h-cobordisms. □

Proposition 10.3. *If M^m satisfies conditions (b)–(d) of Theorem 1.1 the conclusion of Proposition 10.2 can be improved so that, for each i , the nice relative cobordisms corresponding to $(W_i, \text{Fr } N_i, \partial_M W_i \cup \text{Fr } N_{i+1})$ are s-cobordisms. In that case, $(W_i, \text{Fr } N_i) \approx (\text{Fr } N_i \times [0, 1], \text{Fr } N_i \times \{0\})$ for all i , and M^m is completable.*

Proof. By the triviality of $\tau_\infty(M^m)$, it is possible to adjust the choices of the N_i so that each inclusion $\text{Fr } N_i \hookrightarrow W_i$ has trivial Whitehead torsion, i.e. $\tau(W_i, \text{Fr } N_i) = 0$, and hence is a simple homotopy equivalence. As was discussed in Sec. 7, the adjustment involves “lending and borrowing torsion to and from immediate neighbors of the W_i ” as described in [6, Sec. 6], except that a Splitting Theorem for finite-dimensional manifolds (see [26, p. 318]) replaces [6, Lemma 6.1].

To complete the proof, apply the Relative s-cobordism Theorem to each W_i then apply Lemma 8.1. □

11. \mathcal{Z} -Compactifications and the Proof of Theorem 1.2

In this section, we prove Theorem 1.2. Since $M^m \times [0, 1]$ satisfies conditions (a), (c) and (d) of Theorem 1.1 if and only if M^m satisfies those same conditions (see [6]), it suffices to prove the following proposition which is based on work found in [18].

Proposition 11.1. *If a manifold M^m is inward tame at infinity, then $M^m \times [0, 1]$ is peripherally π_1 -stable at infinity.*

Proof. Apply Corollary 5.5 to obtain a cofinal sequence $\{N_i\}$ of clean neighborhoods of infinity for M^m with the property that, for all i , each component N_i^j of N_i is $\partial_M N_i^j$ -connected at infinity. Since $\{N_i \times [0, 1]\}$ is a cofinal sequence of clean

neighborhoods of infinity for $M^m \times [0, 1]$ it suffices to show that the corresponding connected components, $N_i^j \times [0, 1]$, are all $\partial_{M \times [0, 1]}(N_i^j \times [0, 1])$ -connected and $(\partial_{M \times [0, 1]}(N_i^j \times [0, 1]))$ - π_1 -stable at infinity. By Lemmas 4.1 and 4.2, that is equivalent to showing that, for each N_i^j , $\text{int}_M(N_i^j) \times (0, 1)$ is 1-ended and has stable pro- π_1 at that end. Every connected topological space becomes 1-ended upon crossing with $(0, 1)$, so that condition is immediate. The π_1 -stability property is proved with a small variation on the main technical argument from [18]; in particular, Corollary 3.6 from that paper. The “small variation” is necessary because the earlier argument assumed the product of an *open* manifold with $(0, 1)$. That issue is easily overcome by arranging that the analog of homotopy K_t used in [18, Proposition 3.3] sends the manifold interior of $\text{Int}_M(N_i^j)$ into itself and sends $\text{Fr } N_i^j$ into itself for all t . That is easily accomplished since $\text{Fr } N_i^j$ has an open collar neighborhood at infinity. □

12. A Counterexample to a Question of O’Brien

We now give a negative answer to a question posed by O’Brien [26, pp. 308].

Question. (For a 1-ended manifold M^m with 1-ended boundary) Let $\{V_i\}$ be a cofinal sequence of clean 0-neighborhoods of infinity. If $\{\pi_1(\partial M^m \cup V_i)\}_{i \geq 1}$ is stable, does it follow that M^m is peripherally π_1 -stable at infinity?

The key ingredient in our counterexamples is a collection of contractible open n -manifolds W^n (one for each $n \geq 3$), constructed by Sternfeld in his dissertation [31] (see also [15]). Each W^n has the property that it cannot be embedded in any compact n -manifold. Although these W^n have finite homotopy type, they are not inward tame, since they contain arbitrarily small clean connected neighborhoods of infinity with non-finitely generated fundamental groups. Our counterexamples will be the $(n + 1)$ -manifolds $W^n \times [0, 1]$. First a general observation.

Proposition 12.1. *Let W^n be a connected open n -manifold. If W^n has finite homotopy type, then $W^n \times [0, 1]$ is 1-ended and inward tame, with $\sigma_\infty(W^n \times [0, 1]) = 0$.*

Proof. It suffices to exhibit arbitrarily small connected clean neighborhood of infinity in W^n with finite homotopy type. Let $N \subseteq W^n$ be a clean neighborhood of infinity and $a \in (0, 1)$. By choosing N small and a close to 1, we can obtain arbitrarily small neighborhoods of infinity in $W^n \times [0, 1]$ of the form

$$V(N, a) = (N \times [0, 1]) \cup (W^n \times [a, 1]).$$

Since $V(N, a)$ deformation retracts onto $W^n \times \{a\}$, it is connected and has finite homotopy type. □

Example 2. Consider the $(n + 1)$ -manifold $M^{n+1} = W^n \times [0, 1]$, where W^n is the Sternfeld n -manifold ($n \geq 3$) described above. Then $\partial M^{n+1} = W^n \times \{0\}$. A standard duality argument shows that *every* contractible open manifold of dimension

≥ 2 is 1-ended. Let $\{N_i\}$ be a cofinal sequence of clean connected neighborhoods of infinity in W^n , and for each $i \geq 1$, let $V_i = V(N_i, \frac{i}{i+1})$, as defined in the previous proof. By Seifert-van Kampen, each $V_i \cup \partial M^{n+1}$ is simply connected, so the inverse sequence $\{\pi_1(\partial M^{n+1} \cup V_i)\}_{i \geq 1}$ is pro-trivial, hence, stable.

To see that M^{n+1} is not peripherally π_1 -stable at infinity, first assume that $n \geq 5$. Then, if M^{n+1} were peripherally π_1 -stable at infinity, it would be completable by Theorem 1.1. (The triviality of $\tau_\infty(M^{n+1})$ is immediate since M^{n+1} is simply connected at infinity, which follows from the simple connectivity of the V_i .) But, if \widehat{M}^{n+1} were a completion, then $W^n \times \{0\} \hookrightarrow \partial \widehat{M}^{n+1}$ would be an embedding into a closed n -manifold, contradicting Sternfeld’s theorem.

To obtain analogous examples when $n = 3$ or $n = 4$, we cannot rely on the Manifold Completion Theorem. But a direct analysis of the fundamental group calculations in Sternfeld’s proof reveals that the peripheral pro- π_1 -systems arising in $W^n \times [0, 1)$ are nonstable in those dimensions as well.

13. Proof of Lemma 4.6

We now return to Lemma 4.6, which asserts that the two natural candidates for the definition of “peripherally π_1 -stable at infinity” (the global versus the local approach) are equivalent for inward tame manifolds. The intuition behind the lemma is fairly simple. If M^m contains arbitrarily small 0-neighborhoods of infinity N with the property that each component N^j is $\partial_M N^j$ - π_1 -stable at infinity, then those components provide arbitrarily small neighborhoods of the ends satisfying the necessary π_1 -stability condition. Conversely, if each end ε has arbitrarily small strong 0-neighborhoods P that are $\partial_M P$ - π_1 -stable at infinity, we can use the compactness of the set of ends (in the Freudenthal compactification) to find, within any neighborhood of infinity, a finite collection $\{P_1, \dots, P_k\}$ of such neighborhoods which cover the end of M^m . If we can do this so the P_i are pairwise disjoint, we are finished — just let $N = \cup P_i$. That is not as easy as one might hope, but we are able to attain the desired conclusion by proving the following proposition.

Proposition 13.1. *Suppose M^m is inward tame and each end ε has arbitrarily small strong 0-neighborhoods P_ε that are $\partial_M P_\varepsilon$ - π_1 -stable at infinity. Then every strong partial 0-neighborhood of infinity $Q \subseteq M^m$ is $\partial_M Q$ - π_1 -stable at infinity.*

Our proof requires that we break the stability condition into a pair of weaker conditions. An inverse sequence of groups is:

- *semistable* (sometimes called *pro-epimorphic*) if it is pro-isomorphic to an inverse sequence of surjective homomorphisms;
- *pro-monomorphic* if it is pro-isomorphic to an inverse sequence of injective homomorphisms.

It is an elementary fact that an inverse sequence is stable if and only if it is both semistable and pro-monomorphic.

We will make use of the following topological characterizations of the above properties, when applied to $\text{pro-}\pi_1$. In these theorems, a “space” should be locally compact, locally connected, and metrizable.

Proposition 13.2. *Let X be a 1-ended space and $r : [0, \infty) \rightarrow X$ a proper ray. Then $\text{pro-}\pi_1(X, r)$ is*

- (1) *semistable if and only if, for every compact set $C \subseteq X$, there exists a larger compact set $D \subseteq X$ such that for any compact set E with $D \subseteq E \subseteq X$, every loop in $X \setminus D$ with base point on r can be pushed into $X \setminus E$ by a homotopy with image in $X \setminus C$ keeping the base point on r , and*
- (2) *pro-monomorphic if and only if X contains a compact set C with the property that, for every compact set D with $C \subseteq D \subseteq X$, there exists a compact set $E \supseteq D$ with the property that every loop in $X \setminus E$ that contracts in $X \setminus C$ also contracts in $X \setminus D$.*

These are standard. See, for example [14] or [19]. In the case that $\text{pro-}\pi_1(X, r)$ is pro-monomorphic, the compact set C in the above proposition is called a π_1 -core for X . Notice that by Proposition 13.2, the property of (1-ended) X having pro-monomorphic $\text{pro-}\pi_1(X, r)$ is independent of the choice of r .

It is a non-obvious (but standard) fact that having semistable $\text{pro-}\pi_1(X, r)$ is also independent of the choice of r . As for the characterization of semistable $\text{pro-}\pi_1(X, r)$, we are mostly interested in the following easy corollary.

Corollary 13.3. *If X is a 1-ended space and $\text{pro-}\pi_1(X, r)$ is semistable for some (hence every) proper ray r , then for each compact set $C \subseteq X$, there is a larger compact set $D \subseteq X$ such that, for every compact set $E \subseteq X$ and every path $\lambda : [0, 1] \rightarrow X \setminus D$ with $\lambda(\{0, 1\}) \subseteq E$, there is a path homotopy in $X \setminus C$ taking λ to a path λ' in $X \setminus E$.*

We are now ready for our primary task.

Proof of Proposition 13.1. Let Q be a strong partial 0-neighborhood of infinity in M^m . By Lemma 4.2, proving that Q is $\partial_M Q$ - π_1 -stable at infinity is equivalent to proving that the 1-ended space $Q \setminus \partial M^m$ has stable $\text{pro-}\pi_1$. We will take the latter approach.

By Lemma 5.3 $Q \setminus \partial M^m$ is inward tame, so a modification of the argument in [21, Proposition 3.2] ensures that $\text{pro-}\pi_1(Q \setminus \partial M^m, r)$ is semistable. It is therefore enough to show that $\text{pro-}\pi_1(Q \setminus \partial M^m, r)$ is pro-monomorphic. We will do that by verifying the condition described in Proposition 13.2, i.e. we will show that $Q \setminus \partial M^m$ contains a π_1 -core.

By hypothesis, each end ε of Q has a strong 0-neighborhood P_ε which is $\partial_M P_\varepsilon$ - π_1 -stable at infinity and lies in $\text{Int}_M Q$. Since the set of ends of Q is compact in the Freudenthal compactification, there is a finite subcollection $\{P_{\varepsilon_i}\}_{i=1}^k$ whose union

is a neighborhood of infinity in Q . Place the collection of submanifolds $\{P_{\varepsilon_i}\}_{i=1}^k$ in general position.

Claim 1. *For each $\Omega \subseteq \{1, \dots, k\}$ the set $\bigcap_{j \in \Omega} P_{\varepsilon_j}$ has finitely many components, each of which is a clean codimension 0 submanifold of M^m .*

General position ensures that each component is a clean codimension 0 submanifold of M^m . Since each P_{ε_j} is a closed subset of M^m each component T of $\bigcap_{j \in \Omega} P_{\varepsilon_j}$ is closed in M^m , and since T cannot also be open in M^m it must have non-empty frontier. Since $\{P_{\varepsilon_j}\}_{j \in \Omega}$ is in general position, so also is the collection of (compact) frontiers, $\{\text{Fr } P_{\varepsilon_j}\}_{j \in \Omega}$. So, for each $i \neq j$ in Ω , $\Delta_{i,j} = \text{Fr } P_{\varepsilon_i} \cap \text{Fr } P_{\varepsilon_j}$ is a clean codimension 1 submanifold of $\text{Fr } P_{\varepsilon_i}$ and $\text{Fr } P_{\varepsilon_j}$. The union of these $\Delta_{i,j}$ separate $\bigcup_{j=1}^k \text{Fr } P_{\varepsilon_j}$ into finitely many pieces, and since the frontier of each T is a union of these pieces, there can only be finitely many such T .

Choose an embedding $b : \partial M^m \times [0, 1] \rightarrow M^m$ with $b(x, 0) = x$ for all $x \in \partial M^m$ and whose image B is a regular neighborhood of ∂M^m in M^m . With some additional care, arrange that B intersects: Q in $b(\partial_M Q \times [0, 1])$; each P_{ε_i} in $b(\partial_M P_{\varepsilon_i} \times [0, 1])$; and (more specifically) each component T of each finite intersection $\bigcap_{j \in \Omega} P_{\varepsilon_j}$ in $b(\partial_M T \times [0, 1])$. For each $0 \leq s < t \leq 1$, let $B^{[s,t]} = b(\partial M^m \times [s, t])$, $B^{(s,t)} = b(\partial M^m \times (s, t))$, etc. For $A \subseteq \partial M^m$, let $B_A = b(A \times [0, 1])$ and define $B_A^{[s,t]}$, $B_A^{(s,t)}$, etc. analogously.

By hypothesis and Proposition 13.2, we can choose a clean codimension 0 compact π_1 -core C_i for each $P_{\varepsilon_i} \setminus \partial M^m$. Then choose t so small that $B^{[0,t]} \cap (\bigcup_{i=1}^k C_i) = \emptyset$. Let $C'_0 \equiv \overline{Q \setminus \bigcup_{i=1}^k P_{\varepsilon_i}}$, then let $C_0 = C'_0 \setminus B^{[0,t]}$ so that C_0 is a compact clean codimension 0 submanifold of $Q \setminus \partial M^m$. Let $C = \bigcup_{i=0}^k C_i \subseteq Q \setminus \partial M^m$. Notice that the collection $\{B^{[0,t]}_{\partial M^m Q}, P_{\varepsilon_1}, \dots, P_{\varepsilon_k}\}$ covers $Q \setminus \text{Int}_Q C$.

Choose a clean codimension 0 compact submanifold of $D' \subseteq Q \setminus \partial M^m$ so large that

- (i) $\text{Int}_Q D' \supseteq C$,
- (ii) D' contains every compact component of $\bigcap_{j \in \Omega} P_{\varepsilon_j}$ for all $\Omega \subseteq \{1, \dots, k\}$, and
- (iii) for any compact set $E \subseteq Q \setminus \partial M^m$ such that $D' \subseteq E$, if λ is a path in $T \setminus \partial M^m$, where T is an unbounded component of $P_{\varepsilon_i} \cap P_{\varepsilon_j}$ for some $i, j \in \{1, \dots, k\}$, and λ lies outside D' with endpoints outside E , then there is a path homotopy of λ in $(T \setminus \partial M^m) \setminus C$ pushing λ outside E . (This uses Corollary 13.3 and the fact that each T , being a clean partial neighborhood of infinity in M^m , has the property that $T \setminus \partial M^m$ has finitely many ends, each with semistable $\text{pro-}\pi_1$.)

Now, choose a compact set $D \subseteq Q \setminus \partial M^m$ such that

- (i') $D \supseteq D'$.
- (ii') for every $\Omega \subseteq \{1, \dots, k\}$ and every unbounded component T of $\bigcap_{j \in \Omega} P_{\varepsilon_j}$, each $x \in (T \setminus \partial M^m) \setminus D$ can be pushed to infinity in $(T \setminus \partial M^m) \setminus D'$. (This is possible since there are only finitely such T .)
- (iii') if $x = b(y, t_0) \in B \setminus D$, then $b(y \times [0, t_0]) \cap D' = \emptyset$.

Claim 2. D is a π_1 -core for $Q \setminus \partial M^m$.

Toward that end, let F be a compact subset of $Q \setminus \partial M^m$ containing D , then choose $G \subseteq Q \setminus \partial M^m$ to be an even larger compact set with the following property:

(†) for each $i \in \{1, \dots, k\}$, loops in $P_{\varepsilon_i} \setminus \partial M^m$ lying outside G which contract in $(P_{\varepsilon_i} \setminus \partial M^m) \setminus C$, also contract in $(P_{\varepsilon_i} \setminus \partial M^m) \setminus F$.

Let $\alpha : [0, 1] \times [0, 1] \rightarrow (Q \setminus \partial M^m) \setminus D$. The interiors of sets $\{B_{\partial M^m Q}^{[0,t]}, P_{\varepsilon_1}, \dots, P_{\varepsilon_k}\}$ cover $(Q \setminus \partial M^m) \setminus D$, so we can subdivide $[0, 1]^2$ into subsquares $\{R_t\}$ so small that the image of each R_t lies in $B^{(0,t)}$ or one of the $P_{\varepsilon_i} \setminus \partial M^m$ and hence, in $B^{(0,t)} \setminus D$ or one of the $(P_{\varepsilon_i} \setminus \partial M^m) \setminus D$. Since each vertex of this subdivision is sent to a point x in $B^{(0,t)} \setminus D$ and/or $T \setminus D$, where T is an unbounded component of the intersection of the P_{ε_i} which contain the images of the subsquares containing that vertex, then by the choice of D we can push x into $(Q \setminus \partial M^m) \setminus G$ along a path that does not leave T and does not intersect D' . In those cases where $x = b(y, t_0) \in B^{(0,t)} \setminus D$, push x out of G along $b(y \times (0, 1))$, so that the track also stays in $B^{(0,t)} \setminus D'$, by property (iii').

Doing the above for each vertex adjusts α up to homotopy in $(Q \setminus \partial M^m) \setminus D'$ so that each vertex of the subdivision is taken into $(Q \setminus \partial M^m) \setminus G$ and each R_t is still taken into the same P_{ε_i} (or $B^{(0,t)}$) as before.

Next, we move to the 1-skeleton of our subdivision of $[0, 1]^2$. If an edge e is the intersection $R_t \cap R_{t'}$ of two squares, i.e. e is not in $\partial([0, 1]^2)$, we use property (iii) to adjust α up to homotopy so e is mapped into $(Q \setminus \partial M^m) \setminus G$, noting that this homotopy may cause the “new” α to drift into $(Q \setminus \partial M^m) \setminus C$. (If e is sent into $B^{(0,t)}$, we can use (iii') to ensure that the push stays in $B^{(0,t)} \setminus D'$ as well.)

Do the above for each edge until the entire 1-skeleton of the subdivision of $[0, 1]^2$ is mapped into $(Q \setminus \partial M^m) \setminus G$. The image of α now lies in $(Q \setminus \partial M^m) \setminus C$. Notice that the restriction of α to each R_t is a map of a disk into a single P_{ε_i} (or $B^{(0,t)}$) missing C_i with boundary being mapped into $P_{\varepsilon_i} \setminus G$. So by the choice of G , we may redefine α on R_t to be the same on its boundary, but to take R_t into $P_{\varepsilon_i} \setminus F$ or $B^{(0,t)} \setminus F$. Assembling the $\alpha|_{R_t}$ we get a map $\alpha' : [0, 1] \times [0, 1] \rightarrow (Q \setminus \partial M^m) \setminus F$ that agrees with α on $\partial([0, 1]^2)$. □

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