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Non-cocompact Group Actions and π_1-Semistability at Infinity

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Abstract. A finitely presented 1-ended group G has *semistable fundamental group at infinity* if G acts geometrically on a simply connected and locally compact ANR Y having the property that any two proper rays in Y are properly homotopic. This property of Y captures a notion of connectivity at infinity stronger than "1-ended", and is in fact a feature of G, being independent of choices. It is a fundamental property in the homotopical study of finitely presented groups. While many important classes of groups have been shown to have semistable fundamental group at infinity, the question of whether every G has this property has been a recognized open question for nearly forty years. In this paper we attack the problem by considering a proper *but non-cocompact* action of a group J on such an Y. This J would typically be a subgroup of infinite index in the geometrically acting over-group G; for example J might be infinite cyclic or some other subgroup whose semistability properties are known. We divide the semistability property of G into a J-part and a "perpendicular to J" part, and we analyze how these two parts fit together. Among other things, this analysis leads to a proof (in a companion paper) that a class of groups previously considered to be likely counter examples do in fact have the semistability property.

1 Introduction

In this paper we consider a new approach to the semistability problem for finitely presented groups. This is a problem at the intersection of group theory and topology. It has been solved for many classes of finitely presented groups (see, for example, [BM91, Bow04, GG12, GM96, LR75, Mih83, Mih86, Mih87, MT92b, MT92a, Mih16]), but not in general. We begin by stating the problem.

The Problem. Consider a finitely presented infinite group G acting cocompactly by cell-permuting covering transformations on a 1-ended, simply connected, locally finite CW complex Y. Pick an expanding sequence $\{C_n\}$ of compact subsets with $\operatorname{int} C_n \subseteq C_{n+1}$ and $\bigcup C_n = Y$; then choose a proper "base ray" $\omega\colon [0,\infty) \to Y$ with the property that $\omega([n, n+1])$ lies in $Y - C_n$. Consider the inverse sequence

$$(1.1)\qquad \pi_1\big(Y - C_0, \omega(0)\big) \xleftarrow{\lambda_1} \pi_1\big(Y - C_1, \omega(1)\big) \xleftarrow{\lambda_2} \pi_1\big(Y - C_2, \omega(3)\big) \xleftarrow{\lambda_2} \cdots,$$

where the λ_i are defined using subsegments of ω.

The Problem is

EITHER to prove that this inverse sequence is always semistable, *i.e.,* is pro-isomorphic to a sequence with epimorphic bonding maps,

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OR to find a group G for which that statement is false.

This problem is known to be independent of the choice of Y, $\{C_n\}$, and ω, and it is equivalent to some more geometrical versions of semistability, which we now recall.

A 1-ended, locally finite CW complex Y, with proper base ray ω, *has semistable fundamental group at* ∞ if any of the following equivalent conditions holds:

(1) Sequence (1.1) is pro-isomorphic to an inverse sequence of surjections.
(2) Given n there exists m such that, for any q, any loop in $Y - C_m$ based at a point $\omega(t)$ can be homotoped in $Y - C_n$, with base point traveling along ω, to a loop in $Y - C_q$.
(3) Any two proper rays in Y are properly homotopic.

Just as a basepoint is needed to define the fundamental group of a space, a base ray is needed to define the fundamental pro-group at ∞. And just as a path between two basepoints defines an isomorphism between the two fundamental groups, a proper homotopy between two base rays defines a pro-isomorphism between the two fundamental pro-groups at ∞. In the absence of such a proper homotopy, it can happen that the two pro-groups are not pro-isomorphic (see [Geo08, Example 16.2.4]). Thus, in the case of G acting cocompactly by covering transformations as above, semistability is necessary and sufficient for the "fundamental pro-group at infinity of G" to be well-defined up to pro-isomorphism.

The approach presented here. In its simplest form, our approach is to restrict attention to the sub-action on Y of an infinite finitely generated subgroup J having infinite index in G. We separate the topology of Y at infinity into "the J-directions" and "the directions in Y orthogonal to J", with the main result being that, having appropriate analogs of semistability in the two directions, implies that Y has semistable fundamental group at ∞.

For the purposes of an introduction, we first describe a special case of the Main Theorem and give a few examples. A more far-reaching, but more technical, version of the Main Theorem is given in Section 3.

Suppose J is a finitely generated group acting by cell-permuting covering transformations on a 1-ended locally finite and simply connected CW complex Y. Let $\Gamma(J, J^0)$ be the Cayley graph of J with respect to a finite generating set J^0 and let $m\colon \Gamma \to Y$ be a J-equivariant map.

(a) J is *semistable at infinity in* Y if for any compact set $C \subseteq Y$, there is a compact set $D \subseteq Y$ such that if r and s are two proper rays (based at the same point) in $\Gamma(J, J^0) - m^{-1}(D)$, then mr and ms are properly homotopic in $Y - C$ relative to $mr(0) = ms(0)$.

Standard methods show that this property does not depend on the choice of finite generating set J^0.

(b) J is *co-semistable at infinity in* Y if for any compact set $C \subseteq Y$ there is a compact set $D \subseteq Y$ such that for any proper ray r in $Y - J \cdot D$ and any loop α based at $r(0)$ whose image lies in $Y - D$, α can be pushed to infinity in $Y - C$ by a proper homotopy with the base point tracking r.

Theorem 1.1 (Main Theorem—a special case) *If J is both semistable at infinity in Y and co-semistable at infinity in Y, then Y has semistable fundamental group at infinity.*

Remark 1.2 (i) To our knowledge, the theorems proved here are the first non-obvious results that imply a semistable fundamental group at ∞ for a space Y that might not admit a cocompact action by covering transformations.

(ii) In the special case where J is an infinite cyclic group, condition (a) above is always satisfied, since $\Gamma(J, J^0)$ can be chosen to be homeomorphic to $\mathbb{R}$; any two proper rays in $\mathbb{R}$ which begin at the same point and lie outside a nonempty compact subset of $\mathbb{R}$ are properly homotopic in their own images. Moreover, since condition (b) is implied by the main hypothesis of [GG12] (via [Wri92, Lemma 3.1] or [Geo08, Th.16.3.4]), Theorem 1.1 implies the main theorem of [GG12].

(iii) The converse of Theorem 1.1 is trivial. If Y is semistable at infinity and J is any finitely generated group acting as covering transformations on Y, it follows directly from the definitions that J is both semistable at infinity in Y and co-semistable at infinity in Y. So, our theorem effectively reduces checking the semistability of the fundamental group at infinity of a space to separately checking two strictly weaker conditions.

(iv) In our more general version of Theorem 1.1 (not yet stated), the group J will be permitted to vary for different choices of compact set C. No over-group containing these various groups is needed unless we want to extend our results to locally compact ANRs. That issue is discussed in Corollary 9.1.

Some examples. We now give four illuminating examples. Admittedly, the conclusion of Theorem 1.1 is known by previous methods in the first three examples, but they are included because they nicely illustrate how the semistability and co-semistability hypotheses lead to the semistability conclusion of the theorem. Moreover, an understanding of these examples helps to motivate later proofs. In the case of the fourth example, the conclusion was not previously known.

Example 1.3 Let G be the Baumslag–Solitar group

$$B(1,2) = \langle a, t \mid t^{-1}at = a^2 \rangle$$

acting by covering transformations on $Y = T \times \mathbb{R}$, where T is the Bass–Serre tree corresponding to the standard graph of groups representation of G, and let $J = \langle a \rangle \cong \mathbb{Z}$. Then J is semistable at infinity in Y for the reasons described in Remark 1.2(ii). To see that J is co-semistable at infinity in Y, choose $D \subseteq Y$ to be of the form $T_0 \times [-n, n]$, where $n \geq 1$ and T_0 is a finite subtree containing the "origin" 0 of T. Then each component of $Y - J \cdot D$ is simply connected (it is a subtree crossed with $\mathbb{R}$). So pushing α to infinity along r can be accomplished by first contracting α to its basepoint, then sliding that basepoint along r to infinity.

Example 1.4 Let $J = \langle a, b \mid \rangle$ be the fundamental group of a punctured torus of constant curvature -1, and consider the corresponding action of J on $Y = \mathbb{H}^2$. Figure 1 shows $\mathbb{H}^2$ with an embedded tree representing the image of a well-chosen $m\colon \Gamma(J, \{a, b\}) \to \mathbb{H}^2$. The shaded region represents a typical $J \cdot D$ for a carefully chosen compact $D \subseteq \mathbb{H}^2$, which is represented by the darker shading. The components

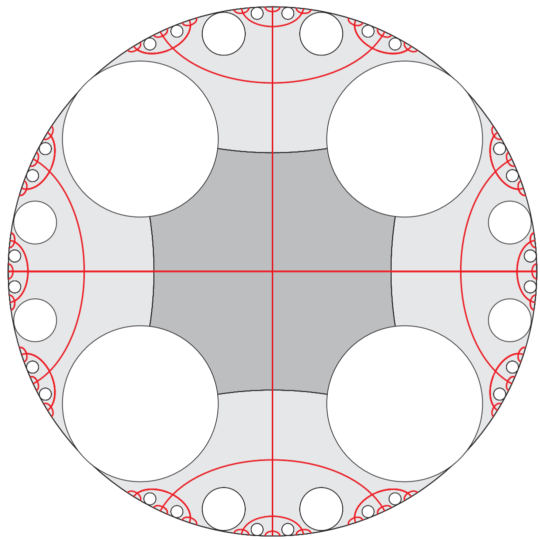

Figure 1.

of $\mathbb{H}^2 - J \cdot D$ are open horoballs. Notice that two proper rays in $\Gamma(J, \{a, b\}) - m^{-1}(D)$ that begin at the same point are not necessarily properly homotopic in $\Gamma(J, \{a, b\}) - m^{-1}(D)$; but their images are properly homotopic in $\mathbb{H}^2 - D$, so J is semistable at infinity in $\mathbb{H}^2$. Moreover, since each component of $\mathbb{H}^2 - J \cdot D$ is simply connected, J is co-semistable at infinity in $\mathbb{H}^2$ for the same reason as in Example 1.3.

Example 1.5 Let $K \subseteq S^3$ be a figure-eight knot; endow $S^3 - K$ with a hyperbolic metric and consider the corresponding proper action of the knot group J on $\widetilde{S^3 - K} = \mathbb{H}^3$. Much like the previous example, there exists a nice geometric embedding of a Cayley graph of J into $\mathbb{H}^3$ and choices of compact $D \subseteq \mathbb{H}^3$ so that $\mathbb{H}^3 - J \cdot D$ is an infinite collection of (3-dimensional) open horoballs. Since J itself is known to be 1-ended with semistable fundamental group at infinity (a useful case to keep in mind), the first condition of Theorem 1.1 is immediate. And again, co-semistability at infinity follows from the simple connectivity of the horoballs.

Example 1.6 For many years an outstanding class of finitely presented groups not known to be semistable at ∞ has been the class of finitely presented ascending HNN extensions whose base groups are finitely generated but not finitely presented.[1] While Theorem 3.1 does not establish semistability for this whole class, it does so for a significant subclass: those of "finite depth". This new result is established in [Mih], a paper that makes use of the more technical Main Theorem 3.1 proved here. In particular, allowing the group J to vary (see Remark 1.2(iv)) is important in this example.

[1]The case of finitely presented base group was settled long ago in [Mih85].

Outline of the paper. The paper is organized as follows. We consider 1-ended simply connected locally finite CW complexes Y, and groups J that act on Y as covering transformations. In Section 2 we review a number of equivalent definitions for a space and group to have semistable fundamental group at ∞. In Section 3 we state our Main Theorem 3.1 in full generality and formally introduce the two somewhat orthogonal notions in the hypotheses of Theorem 3.1. The first is that of a finitely generated group J being semistable at ∞ in Y with respect to a compact set C, and the second defines what it means for J to be co-semistable at ∞ in Y with respect to C. In Section 4 we give a geometrical outline and overview of the proof of the main theorem. In Section 5 we prove a number of foundational results. Suppose C is a compact subset of Y and J is a finitely generated group acting as covering transformations on Y. Define $J \cdot C$ to be $\bigcup_{j\in J} j(C)$. We consider components U of $Y - J \cdot C$ such that the image of U in $J\backslash Y$ is not contained in a compact set. Such U are called J-unbounded. We show there are only finitely many J-unbounded components of $Y - J \cdot C$ up to translation in J, and the J-stabilizer of a J-unbounded component is an infinite group. In Section 6 we use van Kampen's Theorem to show that for a finite subcomplex C of Y, the J-stabilizer of a J-unbounded component of $Y - J \cdot C$ is a finitely generated group. A bijection between the ends of the stabilizer of a J-unbounded component of $Y - J \cdot C$ and "J-bounded ends" of that component is produced in Section 7. The constants that arise in our bijection are shown to be J-equivariant. In Section 8, we prove our main theorem. A generalization of our main theorem from CW complexes to absolute neighborhood retracts is proved in Section 9.

2 Equivalent Definitions of Semistability

Some equivalent forms of semistability have been stated in the Introduction. It will be convenient to have the following theorem.

Theorem 2.1 (see [CM14, Theorem 3.2]) *With Y as before, the following are equivalent:*

(i) *Y has semistable fundamental group at ∞.*

(ii) *Let $r\colon [0,\infty) \to Y$ be a proper base ray. Then for any compact set C, there is a compact set D such that for any third compact set E and loop α based at $r(0)$ whose image lies in $Y - D$, α is homotopic to a loop in $Y - E$ by a homotopy with image in $Y - C$, where α tracks r.*

(iii) *For any compact set C, there is a compact set D such that if r and s are proper rays based at v and with image in $Y - D$, then r and s are properly homotopic* $\mathrm{rel}\{v\}$ *by a proper homotopy supported in $Y - C$.*

(iv) *If C is compact in Y, there is a compact set D in Y such that for any third compact set E and proper rays r and s based at a vertex v and with image in $Y - D$, there is a path α in $Y - E$ connecting points of r and s such that the loop determined by α and the initial segments of r and s is homotopically trivial in $Y - C$.*

Proof That the first three conditions are equivalent is shown in [CM14, Theorem 3.2]. Condition (iv) is clearly equivalent to the more standard condition (iii). ∎

3 The Main Theorem and its Definitions

We are now ready to state our main theorem in its general form. After doing so, we will provide a detailed discussion of the definitions that go into that theorem. Both the theorem and the definitions generalize those found in the introduction.

Theorem 3.1 (Main Theorem) *Let Y be a* 1*-ended simply connected locally finite CW complex. Assume that for each compact subset C_0 of Y there is a finitely generated group J acting as cell preserving covering transformations on Y so that* (a) *J is semistable at ∞ in Y with respect to C_0 and* (b) *J is co-semistable at ∞ in Y with respect to C_0. Then Y has semistable fundamental group at ∞.*

Remark 3.2 If there is a group G (not necessarily finitely generated) acting as covering transformations on Y such that each of the groups J of Theorem 3.1 is isomorphic to a subgroup of G, then the condition that Y is a locally finite CW complex can be relaxed to: Y is a locally compact absolute neighborhood retract (ANR) (see Corollary 9.1).

The *distance* between vertices of a CW complex will always be the number of edges in a shortest edge path connecting them. The space Y is a 1-ended simply connected locally finite CW complex, and for each compact subset C_0 of Y, $J(C_0)$ is an infinite finitely generated group acting as covering transformations on Y and preserving some locally finite cell structure on Y. Fix $*$, a base vertex in Y. Let J^0 be a finite generating set for J and let $\Lambda(J, J^0)$ be the Cayley graph of J with respect to J^0. Let $z_{(J,J^0)}: (\Lambda(J, J^0), 1) \to (Y, *)$ be a J-equivariant map so that each edge of Λ is mapped to an edge path of length at most $K(J^0)$. If r is an edge path in Λ, then $z(r)$ is called a Λ-*path* in Y. The vertices $J*$ are called J*-vertices*.

If C_0 is a compact subset of Y, then the group J is *semistable at ∞ in Y with respect to C_0* if there exists a compact set C in Y and some (equivalently any) finite generating set J^0 for J such that for any third compact set D and proper edge path rays r and s in $\Lambda(J, J^0)$ that are based at the same vertex v and are such that $z(r)$ and $z(s)$ have image in $Y - C$, there is a path δ in $Y - D$ connecting $z(r)$ and $z(s)$ such that the loop determined by δ and the initial segments of $z(r)$ and $z(s)$ is homotopically trivial in $Y - C_0$ (compare to Theorem 2.1(iv)).

Note that this definition requires less than one requiring $z(r)$ and $z(s)$ be properly homotopic rel$\{z(v)\}$ in $Y - C_0$ (compare to Theorem 2.1(iii)). It may be that the path δ is not homotopic to a path in the image of z by a homotopy in $Y - C_0$. By a standard argument, this definition is independent of generating set J^0 and base point $*$, although C may change as J^0, $*$, and z do. When J is semistable at infinity in Y with respect to C_0, we can say J is *semistable at ∞ in Y with respect to J^0, C_0, C, and z.* Observe that if $\hat{C}$ is compact containing C, then J is also semistable at ∞ in Y with respect to J^0, C_0, $\hat{C}$, and z.

If J is 1-ended and semistable at ∞ or 2-ended, then J is always semistable at ∞ in Y with respect to any compact subset C_0 of Y. The semistability of the fundamental group at ∞ of a locally finite CW complex only depends on the 2-skeleton of the complex (see, for example, [LR75, Lemma 3]). Similarly, the semistability at ∞ of a group in a CW complex only depends on the 2-skeleton of the complex.

The notion of J being co-semistable at infinity in a space Y is a bit technical, but has its roots in a simple idea that is fundamental to the main theorems of [GG12, Wri92]. In both of these papers J is an infinite cyclic group acting as covering transformations on a 1-ended simply connected space Y with pro-monomorphic fundamental group at ∞. Wright [Wri92] showed that under these conditions the following could be proved:

($*$) Given any compact set $C_0 \subset Y$ there is a compact set $C \subset Y$ such that any loop in $Y - J \cdot C$ is homotopically trivial in $Y - C_0$.

Condition ($*$) is all that is needed in [GG12, Wri92] in order to prove the main theorems. In [GGM] condition ($*$) is used to show that Y is proper 2-equivalent to $T \times \mathbb{R}$ (where T is a tree). Interestingly, there are many examples of finitely presented groups G (and spaces) with infinite cyclic subgroups satisfying ($*$) but where the fundamental group at ∞ of G is not pro-monomorphic (see [GGM]). In fact, if G has pro-monomorphic fundamental group at ∞, then either G is simply connected at ∞ or (by a result of B. Bowditch [Bow04]) G is virtually a closed surface group and $\pi_1^\infty(G) = \mathbb{Z}$.

Our co-semistability definition generalizes the conditions of ($*$) in two fundamental ways, and our main theorem still concludes that Y has a semistable fundamental group at ∞ (just as in the main theorem of [GG12]).

(1) We expand J from an infinite cyclic group to an arbitrary finitely generated group and we allow J to change as compact subsets of Y become larger.

(2) We weaken the requirement that loops in $Y - J \cdot C$ be trivial in $Y - C_0$ to only requiring that loops in $Y - J \cdot C$ can be "pushed" arbitrarily far out in $Y - C_0$.

We are now ready to set up our co-semistability definition. A subset S of Y is *bounded* in Y if S is contained in a compact subset of Y. Otherwise, S is *unbounded* in Y. Fix an infinite finitely generated group J acting as covering transformations on Y and a finite generating set J^0 of J. Assume that J respects a cell structure on Y. Let $p\colon Y \to J\backslash Y$ be the quotient map. If K is a subset of Y, and there is a compact subset D of Y such that $K \subset J \cdot D$ (equivalently $p(K)$ has image in a compact set), then K is a *J-bounded* subset of Y. Otherwise, K is a *J-unbounded* subset of Y. If $r\colon [0,\infty) \to Y$ is proper and pr has image in a compact subset of $J\backslash Y$, then r is said to be *J-bounded*. Equivalently, r is a J-bounded proper edge path in Y if and only if r has image in $J \cdot D$ for some compact set $D \subset Y$. In this case, there is an integer M (depending only on D) such that each vertex of r is within (edge path distance) M of a vertex of $J*$. Hence, r "determines" a unique end of the Cayley graph $\Lambda(J, J^0)$.

For a non-empty compact set $C_0 \subset Y$ and finite subcomplex C containing C_0 in Y, let U be a J-unbounded component of $Y - J \cdot C$ and let r be a J-bounded proper ray with image in U. We say that J is co-*semistable at ∞ in U with respect to r and C_0* if for any compact set D and loop $\alpha\colon [0,1] \to U$ with $\alpha(0) = \alpha(1) = r(0)$ there is a homotopy $H\colon [0,1] \times [0,n] \to Y - C_0$ such that $H(t,0) = \alpha(t)$ for all $t \in [0,1]$ and $H(0,s) = H(1,s) = r(s)$ for all $s \in [0,n]$ and $H(t,n) \subset Y - D$ for all $t \in [0,1]$. This means that α can be pushed along r by a homotopy in $Y - C_0$ to a loop in $Y - D$. We say J is *co-semistable at ∞ in Y with respect to C_0* (and C) if J is co-semistable

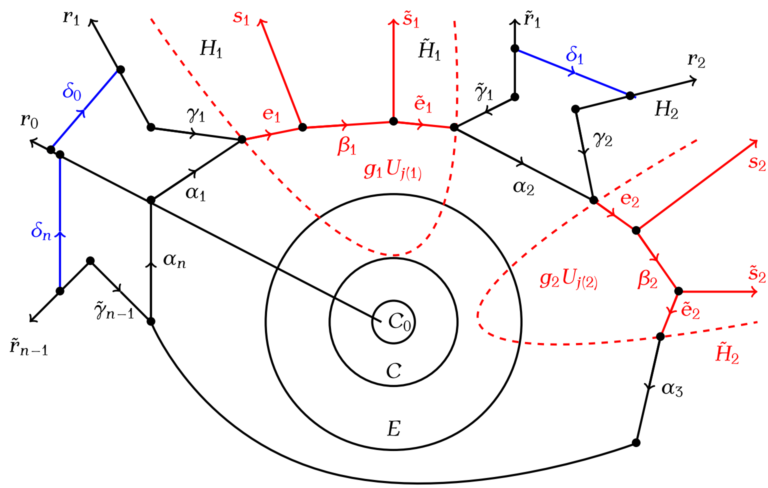

Figure 2.

at ∞ in U with respect to r and C_0 for each J-unbounded component U of $Y - J \cdot C$, and any proper J-bounded ray r in U. Note that if $\hat{C}$ is a finite complex containing C, then J is also co-semistable at ∞ in Y with respect to C_0 and $\hat{C}$.

It is important to notice that our definition only requires that loops in U can be pushed arbitrarily far out in $Y - C_0$ along proper J-bounded rays in U (as opposed to all proper rays in U).

4 An Outline of the Proof of the Main Theorem

A number of technical results are necessary to prove the main theorem. The outline in this section is intended to give the geometric intuition behind these results and describe how they connect to prove the main theorem. Figure 2 will be referenced throughout this section. Here, C_0 is an arbitrary compact subset of Y, J^0 is a finite generating set for the group J that respects a locally finite cell structure on Y and acts as covering transformations on Y. The finite subcomplex C of Y is such that J is co-semistable at ∞ in Y with respect to C_0 and C, and J is semistable at ∞ in Y with respect to J^0, C_0 and C.

The proper base ray is r_0, E is a finite union of specially selected compact sets, and α is a loop based on r_0 with image in $Y - E$. The path α is broken into subpaths $\alpha = (\alpha_1, e_1, \beta_1, \widetilde{e}_1, \alpha_2, \ldots, \alpha_n)$ where the α_i lie in $J \cdot C$, the β_i lie in $Y - J \cdot C$, and the edges e_i and $\widetilde{e}_i$ serve as "transition edges". We let F be an arbitrary large compact set, and we must show that α can be pushed along r_0 to a loop outside of F by a homotopy avoiding C_0 (see Theorem 2.1(ii)).

In Sections 5 and 6 we show that $Y - J \cdot C$ has only finitely many J-unbounded components (up to translation in J) and that the stabilizer of any one of these components is infinite and finitely generated. We pick a finite collection of J-unbounded components of $Y - J \cdot C$ such that no two are J-translates of one another, and any

J-unbounded component of $Y - J \cdot C$ is a translate of one of these finitely many. Each $g_i U_{f(i)}$ in Figure 2 is such that $g_i \in J$ and $U_{f(i)}$ is one of these finitely many components. The edges e_i have initial vertex in $J \cdot C$ and terminal vertex in $g_i U_{f(i)}$. Similarly for $\widetilde{e}_i$. The fact that the stabilizer of a J-unbounded component of $Y - J \cdot C$ is finitely generated and infinite allows us to construct the proper edge path rays $r_i, \widetilde{r}_i, s_i$ and $\widetilde{s}_i$ in Figure 2. Let S_i be the (finitely generated infinite) J-stabilizer of $g_i U_{f(i)}$. Lemma 7.6 allows us to construct proper edge path rays r_i in $J \cdot C$ (far from C_0) that are "S_i-edge paths" and proper rays s_i in $g_i U_{f(i)}$ so that s_i and r_i are (uniformly over all i) "close" to one another. Hence, r_i is properly homotopic rel$\{r_i(0)\}$ to (γ_i, e_i, s_i) by a homotopy in $Y - C_0$. This means e_i can be "pushed" between s_i and (γ_i^{-1}, r_i) into $Y - F$ by a homotopy avoiding C_0, and we have the first step in moving α into $Y - F$ by a homotopy avoiding C_0, which is similar for $\widetilde{r}_i, \widetilde{s}_i$ and $\widetilde{e}_i$.

Since all of the paths/rays α_i, γ_i, r_i, $\widetilde{\gamma}_i$, and $\widetilde{r}_i$ have image in $J \cdot C$, they are uniformly (only depending on the size of the compact set C) close to J-paths/rays. But the semistability at ∞ of J in Y with respect to C_0 then implies that there is a path δ_i connecting $(\widetilde{\gamma}_{i-1}^{-1}, \widetilde{r}_{i-1})$ and $(\alpha_i, \gamma_i^{-1}, r_i)$ in $Y - F$ such that the loop determined by δ_i and the initial segments of $(\widetilde{\gamma}_{i-1}^{-1}, \widetilde{r}_{i-1})$ and $(\alpha_i, \gamma_i^{-1}, r_i)$ is homotopically trivial by a homotopy avoiding C_0. Geometrically, that means α_i can be pushed outside of F by a homotopy between $(\widetilde{\gamma}_{i-1}^{-1}, \widetilde{r}_{i-1})$ and (γ_i^{-1}, r_i), and with image in $Y - C_0$.

All that remains is to push the β_i into $Y - F$ by a homotopy between s_i and $\widetilde{s}_i$. A serious technical issue occurs here. If we knew that s_i and $\widetilde{s}_i$ converged to the same end of $g_i U_{f(i)}$, then we could find a path in $g_i U_{f(i)} - F$ connecting s_i and $\widetilde{s}_i$ and Lemma 8.6 explains how to use the assumption that J is co-semistable at ∞ in Y with respect to C_0, to slide β_i between s_i and $\widetilde{s}_i$ to a path in $Y - F$, finishing the proof of the main theorem. But at this point there is no reason to believe s_i and $\widetilde{s}_i$ determine the same end of $g_i U_{f(i)}$. This is where two of the main lemmas (and two of the most important ideas) of the paper, Lemmas 8.4 and 8.5 come in. All but finitely many of the components gU_i of $Y - J \cdot C$ avoid a certain compact subset of E. If $g_i U_{f(i)}$ is one of these components, then Lemma 8.4 explains how to select the proper ray $\widetilde{r}_i$ and a path ψ in $Y - F$ connecting r_i and $\widetilde{r}_i$ so that the loop determined by ψ, initial segments of r_i and $\widetilde{r}_i$ and the path $(\gamma_i, e_i, \beta_i, \widetilde{e}_i, \widetilde{\gamma}_i^{-1})$ is homotopically trivial in $Y - C_0$ (so that the section of α defined by $(e_i, \beta_i, \widetilde{e}_i)$ can be pushed into $Y - F$ by a homotopy between (γ_i^{-1}, r_i) and $(\widetilde{\gamma}_i^{-1}, \widetilde{r}_i)$). Lemma 8.5 tells us how to select the compact set E so that if $g_i U_{f(i)}$ is one of the finitely many remaining components of $Y - J \cdot U$, then the proper rays s_i and $\widetilde{s}_i$ can be selected so that s_i and $\widetilde{s}_i$ converge to the same end of $g_i U_{f(i)}$. In either case, α is homotopic rel$\{r_0\}$ to a loop in $Y - F$ by a homotopy in $Y - C_0$.

5 Stabilizers of J-unbounded Components

Throughout this section, J is a finitely generated group acting as cell preserving covering transformations on a simply connected locally finite 1-ended CW complex Y, and $p: Y \to J \backslash Y$ is the quotient map. Suppose C, is a large (see Theorem 6.1) finite subcomplex of Y and U is a J-unbounded component of $Y - J \cdot C$. Lemma 5.10 and Theorem 6.1 show the J-stabilizer of U is finitely generated and infinite. Lemma 7.4

shows that there is a finite subcomplex $D(C) \subset Y$ such that for any compact E containing D and any J-unbounded component U of $Y - J \cdot C$, there is a special bijection $\mathcal{M}$ between the set of ends of the J-stabilizer of U and the ends of $U \cap (J \cdot E)$. For C compact in Y, Lemma 5.7 shows there are only finitely many J-unbounded components of $Y - J \cdot C$ up to translation in J.

Suppose that J is semistable at ∞ in Y with respect to C_0 and C, U is a J-unbounded component of $Y - J \cdot C$ and J is co-semistable at ∞ in U with respect to the proper J-bounded ray r and C_0. Once again, co-semistability at ∞ only depends on the 2-skeleton of Y, and from this point on we may assume that Y is 2-dimensional. The next two lemmas reduce complexity again by showing that in certain instances we need only consider locally finite 2-complexes with edge path loop attaching maps on 2-cells. Such complexes are in fact simplicial, and this is important for our arguments in Section 6.

Lemma 5.1 *Suppose Y is a locally finite 2-complex, and the finitely generated group J acts as cell preserving covering transformations on Y; then there is a J-equivariant subdivision of the 1-skeleton of Y and a locally finite 2-complex X also admitting a cell preserving J-action such that the following hold:*

(i) *The image of a 2-cell attaching map for X is a finite subcomplex of X.*

(ii) *The space X has the same 1-skeleton as Y, there is a J-equivariant bijection between the cells of Y and X that is the identity on vertices and edges, and if a is a 2-cell attaching map for Y and a' is the corresponding 2-cell attaching map for X, then a and a' are homotopic in the image of a, and a' is an edge path loop with the same image as a.*

(iii) *The action of J on X is the obvious action induced by the action of J on Y.*

(iv) *If K_1 is a finite subcomplex of Y and K_2 is the corresponding finite subcomplex of X, then there is a bijective correspondence between the J-unbounded components of $Y - J \cdot K_1$ and $X - J \cdot K_2$ so that if U_1 is a J-unbounded component of $Y - J \cdot K_1$ and U_2 is the corresponding component of $X - J \cdot K_2$, then U_1 and U_2 are both a union of open cells and the bijection of cells between Y and X induces a bijection between the open cells of U_1 and U_2. In particular, the J-stabilizer of U_1 is equal to that of U_2.*

Proof Suppose D is a 2-cell of Y and the attaching map on S^1 for D is a_D. Then the image of a_D is a compact connected subset of the 1-skeleton of Y. If e is an edge of Y, then $\mathrm{im}(a_D) \cap e$ is either $\emptyset$, a single closed interval or a pair of closed intervals (we consider a single point to be an interval). In any case, add vertices when necessary to make the end points of these intervals vertices. This process is automatically J-equivariant and locally finite. The map a_D is homotopic (in the image of a_D) to an edge path loop b_D with image the same as that of a_D. Let Z be the 1-skeleton of Y. Attach a 2-cell D' to Z with attaching map b_D. For $j \in J$ the attaching map for jD is ja_D, and we automatically have an attach map for X (corresponding to the cell jD) defined by jb_D. This construction is J-equivariant. Call the resulting locally finite 2-complex X and define the action of J on X in the obvious way.

It remains to prove part (iv). Suppose K_1 and K_2 are corresponding finite subcomplexes of Y and X, respectively. Recall that vertices are open (and closed) cells of a

CW complex, and every point of a CW complex belongs to a unique open cell. If A is an open cell of Y, then either A is a cell of $J \cdot K_1$ or A is a subset of $Y - J \cdot K_1$.

Claim 5.1.1. Suppose U is a component of $Y - J \cdot K_1$. If p and q are distinct points of U, then there is a sequence of open cells $A_0, \ldots, A_n$ of U such that $p \in A_0$, $q \in A_n$ and either $A_i \cap \bar{A}_{i+1} \neq \varnothing$ or $\bar{A}_i \cap A_{i+1} \neq \varnothing$. (Here $\bar{A}$ is the closure of A in Y, equivalently the closed cell corresponding to A.)

Proof Let α be a path in U from p to q. By local finiteness, there are only finitely many closed cells $B_0, \ldots, B_n$ that intersect the compact set $\mathrm{im}(\alpha)$. Note that $B_i \notin K$ so that the open cell A_i for B_i is a subset of U. In particular, $\mathrm{im}(\alpha) \subset A_0 \cup \cdots \cup A_n$. Let $0 = x_0$ and assume that $\alpha(x_0) = p \in A_0$. Let x_1 be the last point of $\alpha^{-1}(B_0)$ in $[0,1]$ (it may be that $x_1 = x_0$). If $\alpha(x_1) \notin A_0$, then $\alpha(x_1) \in A_1 \cup \cdots \cup A_n$, and we assume that $\alpha(x_1) \in A_1$. In this case, $\alpha(x_1) \in \bar{A}_0 \cap A_1 (= B_0 \cap A_1)$.

If $\alpha(x_1) \in A_0$, then take a sequence of points $\{t_i\}$ in $(x_1, 1]$ converging to x_1. Infinitely many $\alpha(t_i)$ belong to some A_j for $j \geq 1$ (say $j = 1$). Then $\alpha(x_1) \in A_0 \cap \bar{A}_1$.

Let x_2 be the last point of $\alpha^{-1}(B_1)$ in $[0,1]$. Continue inductively. ■

Claim 5.1.2. If $A_1 \neq A_2$ are open cells of Y such that $A_1 \cap \bar{A}_2 \neq \varnothing$ and A_i corresponds to the open cell Q_i of X for $i \in \{1,2\}$, then $Q_i \cap \bar{Q}_{i+1} \neq \varnothing$.

Proof We only need check this when A_1 or A_2 is a 2-cell (otherwise $Q_i = A_i$). Note that A_1 is not a 2-cell, since otherwise $A_1 \cap \bar{A}_2 = \varnothing$. If A_2 is a 2-cell, and $A_1 \cap \bar{A}_2 \neq \varnothing$, then by construction, $A_1 \subset \bar{A}_2$ and $Q_1 \subset \bar{Q}_2$. ■

Write U as a union $\bigcup_{i \in I} A_i$ of the open cells in U. Let Q_i be the open cell of X corresponding to A_i. By Claims 5.1.1 and 5.1.2, $\bigcup_{i \in I} Q_i$ is a connected subset of $X - J \cdot K_2$. The roles of X and Y can be reversed in Claims 5.1.1 and 5.1.2. Then writing a component of $X - J \cdot K_2$ as a union of its open cells $\bigcup_{l \in L} Q_l$ (and letting A_l be the open cell of Y corresponding to Q_l), we have that $\bigcup_{l \in L} A_l$ is a connected subset of $Y - J \cdot K_1$. ■

Remark 5.2 There are maps $g\colon X \to Y$ and $f\colon Y \to X$ that are the identity on 1-skeletons and such that fg and gf are properly homotopic to the identity maps relative to the 1-skeleton. In particular, X and Y are proper homotopy equivalent. This basically follows from the proof of [Geo08, Theorem 4.1.8]. These facts are not used in this paper.

The remainder of this section is a collection of elementary (but useful) lemmas. The *boundary* of a subset S of Y (denoted ∂S) is the closure of S (denoted $\bar{S}$) delete the interior of S. If K is a subcomplex of a 2-complex Y, then ∂K is a union of vertices and edges.

Lemma 5.3 *If $A \subset Y$, then $p(A) = p(J \cdot A)$ and $p^{-1}(p(A)) = J \cdot A$. If C is compact in Y and B is compact in $J \backslash Y$ such that $p(C) \subset B$, then there is a compact set $A \subset Y$ such that $C \subset A$ and $p(A) = B$.*

Proof The first part of the lemma follows directly from the definition of $J \cdot A$. Cover $B \subset J\backslash Y$ by finitely many evenly covered open sets U_i for $i \in \{1, \dots, n\}$ such that $\bar{U}_i$ is compact and evenly covered. Pick a finite number of sheets over the $\bar{U}_i$ that cover C and so that there is at least one sheet over each $\bar{U}_i$. Call these sheets $K_1, \dots, K_m$. Let $A = (\bigcup_{i=1}^m K_i) \cap p^{-1}(B)$. Then $C \subset A$, and A is compact, since $(\bigcup_{i=1}^m K_i)$ is compact and $p^{-1}(B)$ is closed. We claim that $p(A) = B$. Clearly, $p(A) \subset B$. If $b \in B$, then there is $j \in \{1, \dots, n\}$ such that $b \in \bar{U}_j$. Then there is $k_b \in K_{j'}$ such that $p(k_b) = b$, and so $k_b \in p^{-1}(B) \cap (\bigcup_{i=1}^m K_i)$ and p maps A onto B. ∎

Remark 5.4 If C is a compact subset of Y, j is an element of J and U is a component of $Y - J \cdot C$, then $j(U)$ is a component of $Y - J \cdot C$, and $p(U)$ is a component of $J\backslash Y - p(C)$.

Lemma 5.5 *Suppose C is a non-empty compact subset of Y and U is an unbounded component of $Y - J \cdot C$. Then ∂U is an unbounded subset of $J \cdot C$.*

Proof Otherwise ∂U is closed and bounded in Y and therefore compact. But ∂U separates U from $J \cdot C$, contradicting the fact that Y is 1-ended. ∎

The next remark establishes a minimal set of topological conditions on a topological space X in order to define the number of ends of X.

Remark 5.6 If X is a connected, locally compact, locally connected Hausdorff space and C is compact in X, then C union all bounded components of $X - C$ is compact, any neighborhood of C contains all but finitely many components of $X - C$, and $X - C$ has only finitely many unbounded components.

Lemma 5.7 *Suppose C is a compact subset of Y and U is a component of $Y - J \cdot C$. Then U is J-unbounded if and only if $p(U)$ is an unbounded component of $J\backslash Y - p(C)$. Hence up to translation by J there are only finitely many J-unbounded components of $Y - J \cdot C$.*

Proof First observe that $p(C) \cap p(U) = \emptyset$. Suppose $p(U)$ is unbounded. Choose a ray $r\colon [0, \infty) \to p(U)$ such that r is proper in $J\backslash Y$. Select $u \in U$ such that $p(u) = r(0)$. Lift r to $\widetilde{r}$ at u. Then $\widetilde{r}$ has image in U, and there is no compact set $D \subset Y$ such that $\mathrm{im}(\widetilde{r}) \subset J \cdot D$. Hence, U is J-unbounded. If U is J-unbounded, then by definition, $p(U)$ is not a subset of a compact subset of Y. ∎

Lemma 5.8 *Suppose C is a compact subset of Y. Then there is a compact subset $D \subset Y$ such that $C \subset D$, every J-bounded component of $Y - J \cdot C$ is a subset of $J \cdot D$ and each component of $Y - J \cdot D$ is J-unbounded.*

Proof Let U be a J-bounded component of $Y - J \cdot C$. Then $p(U)$ is a bounded component of $J\backslash Y - p(C)$. Let B be the union of $p(C)$ and all bounded components of $J\backslash Y - p(C)$. Then B is compact (Remark 5.6). By Lemma 5.3, there is a compact set D containing C such that $p(D) = B$. ∎

Lemma 5.9 *Suppose C and D are finite subcomplexes of Y. Then only finitely many J-unbounded components of $Y - J \cdot C$ intersect D.*

Proof Note that $J \cdot C$ is a subcomplex of Y. If the lemma is false, then for each $i \in \mathbb{Z}^+$ there are distinct unbounded components U_i of $Y - J \cdot C$ such that $U_i \cap D \neq \emptyset$. Choose $u_i \in U_i \cap D$. Let E_i be an (open) cell containing u_i. Then $E_i \subset U_i$ and the E_i are distinct. Then infinitely many cells of Y intersect D, contrary to the local finiteness of Y. ∎

Lemma 5.10 *Suppose C is a finite subcomplex of Y and U is a J-unbounded component of $Y - J \cdot C$. Then there are infinitely many $j \in J$ such that $j(U) = U$. In particular, the J-stabilizer of U is an infinite subgroup of J.*

Proof If $x \in \partial U \subset \partial(J \cdot C)$ then any neighborhood of x intersects U. Let $x_1, x_2, \ldots$ be a sequence in U converging to x. By local finiteness infinitely many x_i belong to some open cell D of U, and so $x \in \bar{D}$. By Lemma 5.5, there are infinitely many open cells D of U and distinct $j_D \in J$ such that $j_D(C) \cap \bar{D} \neq \emptyset$. For all such D, $j_D^{-1}(\bar{D}) \cap C \neq \emptyset$ and by the local finiteness of Y, there are infinitely many such D with $j_D^{-1}(D)$ all the same. If $j_{D_1}^{-1}(D_1) = j_{D_2}^{-1}(D_2)$, then $j_{D_2} j_{D_1}^{-1}(D_1) = D_2$, so $j_{D_2} j_{D_1}^{-1}$ stabilizes U. ∎

Lemma 5.11 *Suppose C is a finite subcomplex of Y, U is a J-unbounded component of $Y - J \cdot C$ and $S < J$ is the subgroup of J that stabilizes U. Then for any $g \in J$, the stabilizer of gU is gSg^{-1}.*

Proof Simply observe that $hgU = gU$ if and only if $g^{-1}hgU = U$ if and only if $g^{-1}hg \in S$ if and only if $h \in gSg^{-1}$. ∎

Lemma 5.12 *Suppose $C \subset Y$ is compact and R_1 is a J-unbounded component of $Y - J \cdot C$. If $D \subset Y$ is compact, and $C \subset D$, then there is a J-unbounded component R_2 of $Y - J \cdot D$ such that $R_2 \subset R_1$.*

Proof Choose an unbounded component V_2 of $J \backslash Y - p(D)$ such that $V_2 \subset p(R_1)$. By Lemma 5.7, there is a component R_2' of $Y - J \cdot D$ such that $p(R_2') = V_2$, and so R_2' is J-unbounded. Choose points $x \in R_1$ and $y \in R_2'$ such that $p(x) = p(y) \in V_2$. Then the covering transformation taking y to x takes R_2' to a J-unbounded component R_2 of $Y - J \cdot D$. As $x \in R_2 \cap R_1$, we have $R_2 \subset R_1$. ∎

6 Finite Generation of Stabilizers

The following principal result of this section allows us to construct proper rays in J-unbounded components of $Y - J \cdot D$ that track corresponding proper rays in a copy of a Cayley graph of the corresponding stabilizer of that component. These geometric constructions are critical to the proof of our main theorem.

Theorem 6.1 *Suppose J is a finitely generated group acting as cell preserving covering transformations on the simply connected,* 1*-ended,* 2*-dimensional, locally finite CW* *complex Y. Let $p\colon Y \to J\backslash Y$ be the quotient map. Suppose D is a connected finite subcomplex of Y such that the image of $\pi_1(p(D))$ in $\pi_1(J\backslash Y)$ (under the map induced by inclusion of $p(D)$ into $J\backslash Y$) generates $\pi_1(J\backslash Y)$. Then for any J-unbounded component V of $Y - J\cdot D$, the stabilizer of V under the action of J is finitely generated.*

By Lemma 5.1 and Remark 5.2, we can assume that Y is simplicial. Theorem 6.2.11 [Geo08] is a cellular version of van Kampen's theorem. The following is an application of that theorem.

Theorem 6.2 *Suppose X_1 and X_2 are path connected subcomplexes of a path connected CW complex X such that $X_1 \cup X_2 = X$ and $X_1 \cap X_2 = X_0$ is non-empty and path connected. Let $x_0 \in X_0$. For $i = 0, 1, 2$ let A_i be the image of $\pi_1(X_i, x_0)$ in $\pi_1(X, x_0)$ under the map induced by inclusion of X_i into X. Then $\pi_1(X, x_0)$ is isomorphic to the amalgamated product $A_1 *_{A_0} A_2$.*

Theorem 6.3 *Suppose that X is a connected, locally finite, 2-dimensional simplicial complex. If K is a finite subcomplex of X such that the inclusion map $i\colon K \hookrightarrow X$ induces an epimorphism on fundamental group and U is an unbounded component of $X - K$, then the image of $\pi_1(U)$ in $\pi_1(X)$ under the map induced by the inclusion of U into X is a finitely generated group.*

Proof If V is a bounded component of $X - K$, then $V \cup K$ is a finite subcomplex of X. So without loss, assume that each component of $X - K$ is unbounded. If e is edge in $X - K$ and both vertices of e belong to K, then by barycentric subdivision, we can assume that each open edge in $X - K$ has at least one vertex in $X - K$. Equivalently, if both vertices of an edge belong to K, then the edge belongs to K. If T is a triangle of X and each vertex of T belongs to K, then each edge belongs to K, and T belongs to K (otherwise, the open triangle of T is a bounded component of $X - K$).

The *largest* subcomplex Z of X contained in a component U of $X - K$ contains all vertices of X that are in U, all edges each of whose vertices are in U, and all triangles each of whose vertices are in U.

Lemma 6.4 *Suppose that U is a component of $X - K$ and Z is the largest subcomplex of X contained in U. Then Z is a strong deformation retract of U. In particular, Z is connected.*

Proof If e (resp. T) is an open edge (resp. triangle) of X that is a subset of U but not of Z, then some vertex of e (respectively T) belongs to K and some vertex of e (resp. T) belongs to Z. Say e has vertices v and w and $v \in Z$ and $w \in K$; then clearly $[v, w)$ linearly strong deformation retracts to v. If T is a triangle of X with vertices $v, w \in Z$ and $u \in K$, then for each point $p \in [v, w]$ the linear strong deformation retraction from of $(u, p]$ to p agrees with those defined for $(u, v]$ and $(u, w]$ and defines a strong deformation for the triangle $[v, w, u] - \{u\}$ to the edge $[v, w]$, and similarly if $v \in Z$ and $u, w \in K$. Combining these deformation retractions gives a strong deformation retraction of U to Z. ∎

Suppose that U is a component of $X - K$ and Z is the largest subcomplex of X contained in U. Let Q_1 be the (finite) subcomplex of X consisting of all edges and triangles that intersect both U and K (and hence intersect both Z and K). By Lemma 6.4, we can add finitely many edges in Z to Q_1 so that the resulting complexes Q_2 and $Q_2 \cap Z$ are connected. The complex $Q_3 = Q_2 \cup (X - U)$ is a connected subcomplex of X.

The subcomplexes Q_3 and Z are connected and cover X, and $Q_3 \cap Z = Q_2 \cap Z$ is a non-empty connected finite subcomplex of X. Let A_0, A_1, and A_2 be the image of $\pi_1(Q_3 \cap Z)$, $\pi_1(Q_3)$ and $\pi_1(Z)$, respectively in $\pi_1(X)$ under the homomorphism induced by inclusion. By Theorem 6.2, $\pi_1(X)$ is isomorphic to the amalgamated product $A_1 *_{A_0} A_2$. Now as $K \subset Q_3$, $A_1 = \pi_1(X)$. But then normal forms in amalgamated products imply that $A_2 = A_0$. As $Q_3 \cap Z$ is a finite complex, A_0, and hence A_2, is finitely generated. This completes the proof of Theorem 6.3. ∎

Suppose J is a finitely generated group acting on a simply connected 2-dimensional simplicial complex Y and let K be a finite subcomplex of $J\backslash Y$ such that the image of $\pi_1(K)$ under the homomorphism induced by the inclusion map of K into $J\backslash Y$, generates $\pi_1(J\backslash Y)$. Let D be a finite subcomplex of Y that projects onto K so that $p^{-1}(K) = J \cdot D$. Let X_1 be an unbounded component of $J\backslash Y - K$. The number of J-unbounded components of $Y - J \cdot D$ that project to X_1 is the index of the image of $\pi_1(X_1)$ in $\pi_1(J\backslash Y) = J$ under the homomorphism induced by inclusion, and the stabilizer of such a J-unbounded component is isomorphic to the image of $\pi_1(X_1)$ in $\pi_1(J\backslash Y) = J$ under the homomorphism induced by inclusion. Hence, Theorem 6.1 is a direct corollary of Theorem 6.3.

7 A Bijection Between J-bounded Ends and Stabilizers

As usual, J^0 is a finite generating set for an infinite group J that acts as covering transformations on a 1-ended simply connected locally finite 2-dimensional CW complex Y. Assume that C is a finite subcomplex of Y and U is a J-unbounded component of $Y - J \cdot C$. The main result of this section connects the ends of the J-stabilizer of U to the J-bounded ends of U (and allows us to construct the r and s rays in Figure 2). Recall that $z\colon (\Lambda(J, J^0), 1) \to (Y, *)$ and K is an integer such that for each edge e of Λ, $z(e)$ is an edge path of length at most K.

Lemma 7.1 *Suppose C and D are finite subcomplexes of Y, U is a J-unbounded component of $Y - J \cdot C$, and some vertex of $J \cdot D$ belongs to U. Let S be the J-stabilizer of U. Then there is an integer $N_{7.1}(U, C, D)$ such that for each vertex $v \in U \cap (J \cdot D)$ there is an edge path of length at most N from v to $S*$, and for each element $s \in S$ there is an edge path of length at most N from $s*$ to a vertex of $U \cap (J \cdot D)$.*

Proof Without loss of generality, assume that $* \in D$ and D is connected. Let A be an integer such that any two vertices in D can be connected by an edge path of length at most A. For each vertex v of $U \cap (J \cdot D)$, let α_v be a path of length at most A from v to a vertex w_v* of $J*$. The covering transformation w_v^{-1} takes α_v to an edge path ending at $*$ and of length at most A. The vertices of $U \cap (J \cdot D)$ are partitioned into a finite collection of equivalence classes, where v and u are related if $w_v^{-1}(\alpha_v)$ and

$w_u^{-1}(\alpha_u)$ have the same initial point. Equivalently, $w_v w_u^{-1} u = v$. In particular, $u \sim v$ implies $w_v w_u^{-1} \in S$. Let d_Λ denote edge path distance in the Cayley graph $\Lambda(J, J^0)$ and $|g|_\Lambda = d_\Lambda(1, g)$. Note that, as vertices of Λ,

$$d_\Lambda(w_v w_u^{-1}, w_v) = |w_u|_\Lambda.$$

For each (out of the finitely many) equivalence class of vertices in $U \cap (J \cdot D)$, distinguish u in that class. Let N_1 be the largest of the numbers $|w_u|_\Lambda$ (over the distinguished u). If u is distinguished and $v \sim u$, then let β be an edge path in Λ of length at most N_1 from w_v to $w_v w_u^{-1}$. Then $z\beta$ (from $w_v *$ to $w_v w_u^{-1} * \in S*$) has length at most KN_1. The path $(\alpha_v, z\beta)$ (from v to $w_v w_u^{-1} * \in S*$) has length at most $N_1 K + A$.

Let α be an edge path from $*$ to a vertex of $U \cap (J \cdot D)$. Then for each $s \in S$, $s(\alpha)$ is an edge path from $s*$ to a vertex of $U \cap (J \cdot D)$. Let $N_2 = |\alpha|$; then let N be the largest of the integers $N_1 K + A$ and N_2. ∎

Remark 7.2 Assume we are in the setup of Lemma 7.1. Suppose $g \in J$. Then each vertex of $(gU) \cap (J \cdot D)$ is within N of a vertex of $gS*$ and within $N + |g|K$ of $gSg^{-1}*$ (as $d_\Lambda(gs, gsg^{-1}) = |g^{-1}|$), where by Lemma 5.11, gSg^{-1} stabilizes gU. Also, each vertex of $gS*$ is within N of a vertex of $(gU) \cap (J \cdot D)$ and each vertex of $gSg^{-1}*$ is within $N + |g|K$ of a vertex of $(gU) \cap (J \cdot D)$. By Lemma 5.7, there are only finitely many J-unbounded components of $Y - J \cdot C$ up to translation in J. Hence, finitely many integers N cover all cases.

If $C \subset E$ are compact subsets of Y and U is a J-unbounded component of $Y - J \cdot C$, let $\mathcal{E}(U, E)$ be the set of equivalence classes of J-bounded proper edge path rays of $U \cap (J \cdot E)$, where two such rays r and s are equivalent if for any compact set F in Y, there is an edge path from a vertex of r to a vertex of s with image in $(U \cap (J \cdot E)) - F$. If X is a connected locally finite CW complex, let $\mathcal{E}(X)$ be the set of ends of X. In the next lemma it is not necessary to factor the map m through $z\colon \Lambda(J, J^0) \to Y$ in order for it to be true, but for our purposes, it is more applicable this way. For a 2-dimensional CW complex X and subcomplex A of X, let A_1 be the subcomplex comprised of $A \cup \{$all vertices connected by an edge to a vertex of $A\} \cup \{$all edges with at least one vertex in $A\}$. Let

$$St(A) = A_1 \cup \{\text{all 2-cells whose attaching maps have image in } A_1\}.$$

Inductively define $St^n(A) = St(St^{n-1}(A))$ for all $n > 1$. The next lemma is a standard result that we will employ a number of times.

Lemma 7.3 *Suppose L is a positive integer; then there is an integer $M(L)$ such that if α is an edge path loop in Y of length at most L and α contains a vertex of $J*$, then α is homotopically trivial in $St^{M(L)}(v)$ for any vertex v of α.*

Proof Since Y is simply connected, each of the (finitely many) edge path loops at $*$ that have length at most L is homotopically trivial in $St^{M_1}(*)$ for some integer M_1. If α is a loop at $*$ of length L and v is a vertex of α, then $St^{M_1}(*) \subset St^{M_1+L}(v)$, and so α is homotopically trivial in $St^M(v)$ where $M = M_1 + L$. The lemma follows by translation in J. ∎

Lemma 7.4 *Suppose C is a finite subcomplex of Y and U is a J-unbounded component of $Y - J \cdot C$. Let S^0 be a finite generating set for S (the J-stabilizer of U), and let $\Lambda(S, S^0)$ be the Cayley graph of S with respect to S^0. Let $m_1\colon \Lambda(S, S^0) \to \Lambda(J, J^0)$ be an S-equivariant map where $m_1(v) = v$ for each vertex v of $\Lambda(S, S^0)$, and each edge of $\Lambda(S, S^0)$ is mapped to an edge path in $\Lambda(J, J^0)$. Let $m = zm_1\colon \Lambda(S, S^0) \to Y$. Then there is a compact set $D_{7.4}(C, U, S^0) \subset Y$ such that for any compact subset E of Y containing D, there is a bijection*

$$\mathcal{M}_U\colon \mathcal{E}\big(\Lambda(S, S^0)\big) \rightarrowtail\!\!\!\twoheadrightarrow \mathcal{E}(U, E) = \mathcal{E}(U, D)$$

and an integer $I_{7.4}(U, C, D)$ such that if q is a proper edge path ray in $\Lambda(S, S^0)$ and $\mathcal{M}([q]) = [t]$, then there is a $t' \in [t]$ such that for each vertex v of $m(q)$ there is an edge path of length at most I from v to a vertex of t', and if w is a vertex of t', then there is an edge path of length at most I from w to a vertex of $m(q)$.

Proof Throughout this proof, $\Lambda = \Lambda(S, S^0)$. We call the points $m(S)(= S*) \subset Y$, the S-vertices of Y. There is an integer $\mathbf{B}(\mathbf{S^0})$ such that if e is an edge of Λ, then the edge path $m(e)$ has length at most B. Fix α_0 an edge path in Y from $*$ to a vertex of $u \in U$. If $[v, w]$ is an edge of Λ, then $(v\alpha_0^{-1}, m(e), w\alpha_0)$ is an edge path of length at most $B + 2|\alpha_0|$ in Y connecting vu and wu (the terminal points of $v(\alpha_0)$ and $w(\alpha_0)$). Hence, there is an integer $\mathbf{A}$ (depending only on the integer $B + 2|\alpha_0|$) and an edge path of length at most A in U from the terminal point of $v(\alpha_0)$ to the terminal point of $w(\alpha_0)$. Let $\mathbf{I} = |\alpha_0| + \max\{A, B\}$. Let $\mathbf{D_1}$ be a finite subcomplex of Y containing $St^{A+B}(*) \cup St(C)$. By Lemma 7.1, there is an integer $\mathbf{N}$ such that each vertex of $(J \cdot D_1) \cap U$ is connected by an edge path of length at most N to a vertex of $S*$. There is an integer $\mathbf{Z}$ such that if a and b are vertices of U that belong to an edge path in Y of length at most $N + |\alpha_0|$, and this path contains a point of $J*$, then there is an edge path of length at most Z in U connecting a and b. Let $\mathbf{D}$ contain $D_1 \cup St^{Z+N}(*)$.

Let q be a proper edge path ray in Λ with $q(0) = 1$. Let the consecutive S-vertices of $m(q)$ be $v_0 = *, v_1, v_2, \ldots$. (So the edge path distance in Y between v_i and v_{i+1} is at most B.) For simplicity, assume that v_i is the element of S that maps $*$ to v_i. Then $v_i(\alpha_0)$ is an edge path that ends in U. By the definition of D_1, there is an edge path β_i in $U \cap (J \cdot D)$ from the end point of $v_i(\alpha_0)$ to the end point of $v_{i+1}(\alpha_0)$ of length at most A (see the left-hand side of Figure 3). For each vertex v of the proper edge path ray $\beta_q = (\beta_0, \beta_1, \ldots)$ (in $U \cap (J \cdot D)$), there is an edge path of length at most $A + |\alpha_0| \le I$ from v to a vertex of $m(q)$. For each vertex w of $m(q)$ there is an edge path of length at most $B + |\alpha_0| \le I$ from w to a vertex of β_q. In particular, β_q is a proper J-bounded ray in U. If $p \in [q] \in \mathcal{E}(\Lambda(S, S^0))$ (with $p(0) = 1$), then $m(p)$ is of bounded distance from β_p. If δ_i is a sequence of edge paths in Λ each beginning at a vertex of q and ending at a vertex of p, such that any compact subset intersects only finitely many δ_i, then the paths $m(\delta_i)$ connect $m(q)$ to $m(p)$ and (since m is a proper map) any compact subset of Y intersects only finitely many $m(\delta_i)$. The $m(\delta_i)$ determine (using translates of α_0 as above) edge paths in $U \cap (J \cdot D)$ connecting β_q and β_p so that $[\beta_p] = [\beta_q]$ in $\mathcal{E}(U, E)$ for any finite subcomplex E of Y that contains D. This defines a map $\mathcal{M}\colon \mathcal{E}(\Lambda) \to \mathcal{E}(U, E)$ that satisfies the last condition of our lemma, and it remains to show that $\mathcal{M}$ is bijective.

Let r be a proper edge path J-bounded ray in U. Then r has image in $J \cdot E$ for some finite subcomplex E containing D. Let $v_1, v_2, \ldots$ be the consecutive vertices of r. By Lemma 7.1 there is an integer N_E such that each v_i is within N_E of $S*$. Let τ_i be a shortest edge path from v_i to $S*$, so that $|\tau_i| \leq N_E$. We can assume without loss that the image of τ_i is in $J \cdot E$. Let $w_i \in S*$ be the terminal point of τ_i. Let z_i be the first vertex of τ_i in $J \cdot D_1$. Then the segment of τ_i from z_i to w_i has length at most N. For each i there is an edge path in Y of length at most $2N_E + 1$ connecting w_i to w_{i+1}. Hence, there is a proper edge path ray $q(r)$ in Λ such that $m(q(r))$ contains each w_i. The proper edge path ray $\beta_{q(r)}$ has image in $U \cap (J \cdot D_1)$, and there is an edge path of length at most Z in $U \cap (J \cdot D)$ from z_i to a vertex of $\beta_{q(r)}$. Hence, there is an edge path in $U \cap (J \cdot E)$ of length at most $Z + N_E$ from v_i to a vertex of $\beta_{q(r)}$ so that $[r] = [\beta_{q(r)}]$ in $\mathcal{E}(U, E)$. In particular, $\mathcal{M}$ is onto.

Finally, we show $\mathcal{M}$ is injective. Suppose a and b are distinct proper edge path rays in Λ with initial point 1, such that $[\beta_a] = [\beta_b]$ in $\mathcal{E}(U, E)$ for some E containing D. Let τ_i be a sequence of edge paths in $U \cap (J \cdot E)$ where each begins at a vertex of β_a, ends at a vertex of β_b, and so that only finitely many intersect any given compact set (a cofinal sequence). By the construction of β_a and β_b, we can assume the initial point of τ_i is the end point of $v_i \alpha_0$ for v_i a vertex of a in Λ and the terminal point of τ_i is the end point of $w_i \alpha_0$ for w_i a vertex of b. By Lemma 7.1, there is an integer $N_E (\geq |\alpha_0|)$ such that each vertex of τ_i is within N_E of $S*$. For each i, this defines a finite sequence A_i of points in $S*$ beginning with $v_i *$ on $m(a)$, ending with $w_i *$ on $m(b)$, each within N_E of a point of τ_i and adjacent points of A_i are within $2N_E + 1$ of one another. Since the τ_i are cofinal, so are the A_i. Since the distance between adjacent points of A_i is bounded, if u and v are vertices of $\Lambda(S, S^0)$ such that $m(u)$ and $m(v)$ are adjacent in A_i, then there is a bound on the distance between u and v in $\Lambda(S, S^0)$. This implies a and b determine the same end of $\Lambda(S, S^0)$. ∎

Remark 7.5 Consider Lemma 7.4 for components gU of $Y - J \cdot C$ for $g \in J$. The stabilizer of gU is gSg^{-1}, and there can be no bound on the integers $I(gU, C, D)$ or the size of $D(C, gU)$. For gU, one can consider instead $m_g \colon \Lambda(S, S^0) \to Y$ by $m_g(x) = gm(x)$ (so $m_g(1) = g*$). Lemma 7.6 is a generalization of Lemma 7.4 that applies to all J-translates of U. Since there are only finitely many J-unbounded components of $Y - J \cdot C$ up to J-translation, the dependency of I and D on U can be eliminated, and in the next lemma $I_{7.6}$ and $D_{7.6}$ are taken to only depend on C.

For C compact in Y, let $\mathcal{U} = \{U_1, \ldots, U_l\}$ be a set of J-unbounded components of $Y - J \cdot C$ such that if U is any J-unbounded component of $Y - J \cdot C$, then $U = gU_i$ for some $g \in J$ and some $i \in \{1, \ldots, l\}$. Also assume that $U_i \neq gU_j$ for any $i \neq j$ and any $g \in J$. Call $\mathcal{U}$ a *component transversal* for $Y - J \cdot C$. Let S_i^0 be a finite generating set for S_i, the J-stabilizer of U_i and $\Lambda_i = \Lambda(S_i, S_i^0)$ the Cayley graph of S_i with respect to S_i^0. For $g \in J$, let $m_{(g,i)} \colon \Lambda_i \to Y$ be defined by $m_{(g,i)}(x) = gm_i(x)$ (where $m_i \colon \Lambda_i \to Y$ is defined by Lemma 7.4). In particular, $m_{(g,i)}(S_i) = gS_i *$.

Lemma 7.6 *For $i \in \{1, \ldots, l\}$, let $D_i = D_{7.4}(C, U_i, S_i^0)$, $D_{7.6}(C) = \bigcup_{i=1}^{l} D_i \subset Y$, $I_{7.6}(C) = \max\{I_{7.4}(U_i, C, D_i)\}_{i=1}^{l}$, and $\mathcal{M}_i \colon \mathcal{E}(\Lambda_i) \rightarrowtail\!\!\!\!\rightarrow \mathcal{E}(U_i, E)$ (Lemma 7.4). For E*

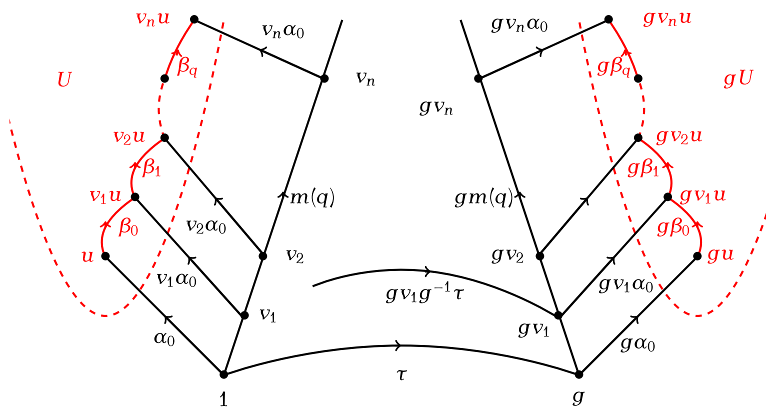

Figure 3.

compact containing $D_{7.6}(C)$ and $g \in J$, there is a bijection

$$\mathcal{M}_{(g,i)} \colon \mathcal{E}(\Lambda_i) \rightarrowtail\!\!\twoheadrightarrow \mathcal{E}(gU_i, E), \text{ where } \mathcal{M}_{(g,i)}([q]) = g\mathcal{M}_i([q])$$

such that if q is a proper edge path ray in Λ_i and $\mathcal{M}_{(g,i)}([q]) = [t]$, then there is $t' \in [t]$ such that for each vertex v of $m_{(g,i)}(q)$, there is an edge path of length at most $I_{7.6}(C)$ from v to a vertex of t', and if w is a vertex of t', then there is an edge path of length at most $I_{7.6}(C)$ from w to a vertex of $m_{(g,i)}(q) = gm_i(q)$.

8 Proof of the Main Theorem

We now set notation for the proof of our main theorem. Let C_0 be compact in Y, and let J^0 be a finite generating set for the infinite group J that acts as cell preserving covering transformations on Y. Let C be a finite subcomplex of Y such that J is co-semistable at ∞ in Y with respect to C_0 and C and J is semistable at ∞ in Y with respect to J^0, C_0, and C. As in the setup for Lemma 7.6, we let $\mathcal{U} = \{U_1, \dots, U_l\}$ be a component transversal for $Y - J \cdot C$, S_i^0 be a finite generating set for S_i, the J-stabilizer of U_i, and $\Lambda_i = \Lambda(S_i, S_i^0)$ be the Cayley graph of S_i with respect to S_i^0. For $g \in J$, let $m_{(g,i)} \colon \Lambda_i \to Y$ be defined by $m_{(g,i)}(x) = gm_i(x)$ (where $m_i \colon \Lambda_i \to Y$ is defined by Lemma 7.4). In particular, $m_{(g,i)}(S_i) = gS_i*$.

The next lemma is a direct consequence of Lemma 7.1.

Lemma 8.1 *Let N_i be $N_{7.1}(U_i, C, St(C))$ and $N_{8.1} = \max\{N_1, \dots, N_l\}$. If $g \in J$ and $[v, w]$ is an edge of Y with $v \in gU_i$ and $w \in J \cdot C$, then there are edge paths of length at most $N_{8.1}$ from v and w to gS_i* and for each $q \in S_i*$, an edge path of length at most $N_{8.1}$ from gq to a vertex of $St(J \cdot C) \cap gU_i$.*

Lemma 8.2 *There is an integer $M_{8.2}(C)$ and compact set $D_{8.2}(C)$ in Y containing $St^{M_{8.2}}(C)$ such that for any $U_i \in \{U_1, \dots, U_l\}$, $g \in J$ and edge $[v, w]$ of Y with $v \in gU_i - D_{8.2}$ and $w \in J \cdot C$ (see Figure 4), we have the following:*

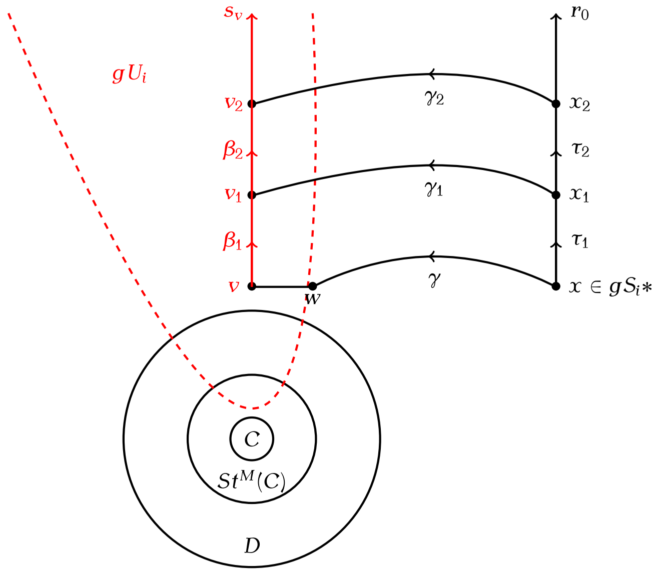

Figure 4.

(i) *There is an edge path γ of length at most $N_{8.1}$ from a vertex $x = gx'* \in gS_i*$ to w, where x' is a vertex in an unbounded component Q of $\Lambda(S_i, S_i^0) - m^{-1}_{(g,i)}(St^{M_{8.2}}(C))$.*

(ii) *If γ is as in part 1, and r_0' is any proper edge path ray in Q beginning at x' (so $r_0 = m_{(g,i)}(r_0')$ is a proper edge path ray beginning at x), then there is a proper J-bounded ray s_v beginning at v such that s_v has image in gU_i and is properly homotopic* rel$\{v\}$ *to $([v,w], \gamma^{-1}, r_0)$ by a proper homotopy with image in $St^{M_{8.2}}(\mathrm{im}(r_0)) \subset Y - C$. So (by hypothesis) J is co-semistable at ∞ in gU_i with respect to s_v and C_0.*

Proof Let A' be an integer such that if $s \in \bigcup_{i=1}^{l} S_i^0$, then there is an edge path of length at most A' in $\Lambda(J, J^0)$ from 1 to s. The image of this path under $z\colon (\Lambda, 1) \to (Y, *)$ is a path in Y of length at most $KA' = A$. Let $N = N_{8.1}$. Select B an integer such that if a and b are vertices of $St(J \cdot C) \cap gU_i$ (for any $g \in J$ and $i \in \{1, \dots, l\}$) of distance at most $2N + A + 1$ in Y, then they can be joined by an edge path of length at most B in gU. By Lemma 7.3, there is an integer $M_{8.2}$ such that if β is a loop in Y of length at most $A + B + 2N + 1$ and containing a vertex of $J*$, then β is homotopically trivial in $St^M(b)$ for any vertex b of β.

There are only finitely many pairs (g, i) with $g \in J$ and $i \in \{1, \dots, l\}$ such that $gS_i* \cap St^M(C) \neq \varnothing$. If $gS_i \cap St^M(C) = \varnothing$, then $m^{-1}_{(g,i)}(St^M(C)) = \varnothing$. Lemma 8.1 implies there is an edge path γ of length at most $N_{8.1}$ from a vertex $x = gx'* \in gS_i*$ to w. Now let $r_0' = (e_0, e_1, \dots)$ be any proper edge path ray at $x' \in \Lambda(S_i, S_i^0)$. Let τ_i be the edge path $m_{(g,i)}(e_i)$ so that τ_i is an edge path in Y of length at most A and $r_0 = m_{(g,i)}(r_0') = (\tau_1, \tau_2, \dots)$ is a proper edge path at x (see Figure 4).

Let $x_0' = x'$ and x_j' be the end point of e_j so that $x_j = gx_j'*$ is the end point of τ_j. Let $\gamma_0 = (\gamma, [w, v])$ (of length at most $N+1$). For $j \geq 1$, let γ_j be an edge path of length at most $N_{8.1}$ from x_j to $v_j \in gU_i \cap St(J \cdot C)$ (by Lemma 8.1). By the definition of B, there is an edge path β_j in gU_i from v_j to v_{j+1} of length at most B. Let s_v be the proper edge path $(\beta_1, \beta_2, \dots)$, with initial vertex v. The loop $(\gamma_{j-1}, \beta_j, \gamma_j^{-1}, \tau_j^{-1})$ has length at most $A + B + 2N + 1$ and contains the J-vertex x_j, and so is homotopically trivial in $St^M(x_j) \subset Y - C$. Combining these homotopies shows that s_v is properly homotopic rel$\{v\}$ to $([v, w], \gamma^{-1}, r_0)$ by a proper homotopy with image in $St^M(\text{im}(r_0)) \subset Y - C$. As long as $D_{8.2}$ contains $St^M(C)$, the conclusion of our lemma is satisfied for all such pairs (g, i).

If (g, i) is one of the finitely many pairs such that $gS_i \cap St^M(C) \neq \emptyset$, then we need only find a compact $D_{(g,i)}$ so that the lemma is valid for the pair (g, i) and $D_{(g,i)}$, since we can let D be compact containing $St^M(C)$ and the union of these finitely many $D_{(g,i)}$.

Fix (g, i) and let E be compact in $\Lambda(S_i, S_i^0) = \Lambda_i$ containing the compact set $m_{(g,i)}^{-1}(St^M(C))$ and all bounded components of $\Lambda_i - m_{(g,i)}^{-1}(St^M(C))$. Let $D_{(g,i)}$ be compact in Y containing $m_{(g,i)}(E)$. Select γ exactly as in the first case. Since x' is a vertex of Λ_i in an unbounded component Q of $\Lambda_i - m_{(g,i)}^{-1}(St^M(C))$, there is a proper edge path ray r_0' at x' with image in Q. Then $r_0 = m_{(g,i)}(r_0')$ is a proper edge path ray at x and the vertices of r_0' are mapped to vertices $x_0 = x, x_1, \dots$ of $(gS_i*) - St^M(C)$. Select paths τ_i and β_i as in the first case and the same argument shows that $s_v = (\beta_1, \beta_2, \dots)$ is properly homotopic rel$\{v\}$ to $([v, w], \gamma^{-1}, r_0)$ by a proper homotopy with image in $St^M(\text{im}(r_0)) \subset Y - C$. ∎

Remark 8.3 The homotopy of Lemma 8.2 (pictured in Figure 4) of s_v to $([v, w], \gamma^{-1}, r_0)$ is sometimes called a *ladder homotopy*. The *rungs* of the ladder are the γ_i and the *sides* of the ladder are s_v and r_0. The loops determined by two consecutive rungs and the segments of the two sides connecting these rungs have bounded length and contain a vertex of $J*$. Lemma 7.3 implies there is an integer M such that each such loop is homotopically trivial by a homotopy in $St^M(v)$ for v any vertex of that loop. Combining these homotopies gives a ladder homotopy.

We briefly recall the outline of Section 4. We determine a compact set $E(C_0, C)$ such that for any compact set F, loops outside of E and based on a proper base ray r_0 can be pushed outside F relative to r_0 and by a homotopy avoiding C_0. A loop outside E is written in the form

$$\alpha = (\alpha_1, e_1, \beta_1, \widetilde{e}_1, \alpha_2, e_2\beta_2, \widetilde{e}_2 \dots, \alpha_{n-1}, e_{n-1}, \beta_{n-1}, \widetilde{e}_{n-1}, \alpha_n)$$

where α_i is an edge path in $J \cdot C$, e_i (resp. $\widetilde{e}_i$) is an edge with terminal (resp. initial) vertex in $Y - J \cdot C$ and β_i is an edge path in $Y - J \cdot C$ (see Figure 2).

We can push the α_j subpaths of α arbitrarily far out between $(\widetilde{\gamma}_{j-1}^{-1}, \widetilde{r}_{j-1})$ and (γ_j^{-1}, r_j) using the semistability of J in Y with respect to C. Lemmas 8.4 and 8.6 consider subpaths of the form $(e, \beta, \widetilde{e})$ in α. The edges e and $\widetilde{e}$ are properly pushed off to infinity using ladder homotopies given by Lemma 8.2. The β paths present difficulties and two cases are considered. If β lies in gU_i and gS_i* does not intersect $St^{M_{8.2}}(C)$, then Lemma 8.4 provides a proper homotopy to compatibly push $(e, \beta, \widetilde{e})$ arbitrarily far out. In Lemma 8.6, we consider paths $(e, \beta, \widetilde{e})$ not considered

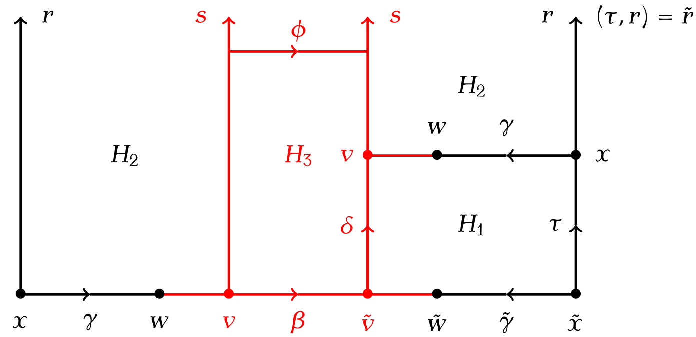

Figure 5.

in Lemma 8.4. For $g \in J$ and $i \in \{1, \dots, l\}$, there are only finitely many cosets gS_i such that $(gS_i*) \cap St^{M_{8.2}}(C) \neq \varnothing$ and we are reduced to considering paths $(e, \beta, \widetilde{e})$ with β in gU_i for these gS_i.

Lemma 8.4 *Suppose that $g \in J$, $i \in \{1, \dots, l\}$ and $([w, v], \beta, [\widetilde{v}, \widetilde{w}])$ is an edge path in $Y - D_{8.2}$. Suppose further that*

(i) *$w, \widetilde{w} \in J \cdot C$ and $v, \widetilde{v} \in gU_i$;*
(ii) *β is an edge path in gU_i;*
(iii) *γ (resp. $\widetilde{\gamma}$) is an edge path of length at most $N_{8.1}$ from $x = gx' \in gS_i*$ (resp. $\widetilde{x} = g\widetilde{x}' \in gS_i$) to w (resp. $\widetilde{w}$), (such paths exist by Lemma 8.1);*
(iv) *x' and $\widetilde{x}'$ belong to the **same** unbounded component Q of $\Lambda(S_i, S_i^0) - m_{(g,i)}^{-1}(St^{M_{8.2}}(C))$ (in particular, when $m_{(g,i)}^{-1}(St^{M_{8.2}}(C)) = \varnothing$).*

Then there are proper Λ_i-edge path rays r' at x' and $\widetilde{r}'$ at $\widetilde{x}'$ such that r' and $\widetilde{r}'$ have image in Q and if $r = m_{(g,i)}(r')$ and $\widetilde{r} = m_{(g,i)}(\widetilde{r}')$, then for any compact set $F \subset Y$, there is an integer $d \geq 0$ and edge path ψ in $Y - F$ from $r(d)$ to $\widetilde{r}(d)$ such that the loop

$$\left(r|_{[0,d]}^{-1}, \gamma, [w, v], \beta, [\widetilde{v}, \widetilde{w}], \widetilde{\gamma}^{-1}, \widetilde{r}|_{[0,d]}, \psi^{-1}\right)$$

is homotopically trivial by a homotopy in $Y - C_0$. (So $([w, v], \beta, [\widetilde{v}, \widetilde{w}])$ can be pushed between (γ^{-1}, r) and $(\widetilde{\gamma}^{-1}, \widetilde{r})$ to a path in $Y - F$ by a homotopy in $Y - C_0$.)

Proof Let r' be any proper edge path in Q with initial point x'. Let $\tau' = (e', \dots, e'_k)$ be an edge path in Q from $\widetilde{x}'$ to x' with consecutive vertices $(\widetilde{x}' = t'_0, t', \dots, t'_k = x')$. Let $\widetilde{r}' = (\tau', r')$. Let $t_j = m_{(g,i)}(t'_j)$ for all $j \in \{0, 1, \dots, k\}$, $r = m_{(g,i)}(r')$, $\widetilde{r} = m_{(g,i)}(\widetilde{r}')$, and $\tau = m_{(g,i)}(\tau')$ (an edge path from $\widetilde{x}$ to x with image in $Y - St^{M_{8.2}}(C)$).

By Lemma 8.1 and the definition of $M_{8.2}$, there is an edge path δ in gU_i from $\widetilde{v}$ to v such that the loop $([\widetilde{v}, \widetilde{w}], \widetilde{\gamma}^{-1}, \tau, \gamma, [w, v], \delta^{-1})$ is homotopically trivial by a ladder homotopy H_1 (with rungs connecting the two sides τ and δ and) with image in $St^{M_{8.2}}(\{t_0, t, \dots, t_k\}) \subset Y - C$.

By Lemma 8.2, there is a proper edge path ray s at v and with image in gU_i such that r is properly homotopic rel$\{x\}$ to $(\gamma, [w, v], s)$ by a ladder homotopy H_2 in $Y - C$. Since J is co-semistable at ∞ in Y with respect to C_0 and C (and s is J-bounded), the loop (β, δ) can be pushed along s by a homotopy H_3 (with image in $Y - C_0$) to a loop ϕ in $Y - F$, where if ϕ is based at $s(k)$, then $s([k, \infty))$ avoids F. Combine these homotopies as in Figure 5 to obtain ψ. ■

If U is a J-unbounded component of $Y - J \cdot C$, and s and $\widetilde{s}$ are proper edge path rays in Y and with image in U, then we say s and $\widetilde{s}$ *converge to the same end of U* (in Y) if for any compact set F in Y, there are edge paths in $U - F$ connecting s and $\widetilde{s}$. Figure 2 can serve as a visual aid for Lemma 8.5.

Lemma 8.5 *There is a compact set $D_{8.5}(C, U_1, \ldots, U_l)$ such that the following hold:*
If $g \in J$, $i \in \{1, \ldots, l\}$, and $([w, v], \beta, [\widetilde{v}, \widetilde{w}])$ is an edge path in $Y - D_{8.5}$ with $w, \widetilde{w} \in J \cdot C$ and β a path in gU_i, then there are edge paths γ and $\widetilde{\gamma}$ of length at most $N_{8.1}$ from $x = gx'* \in gS_i*$ to w and $\widetilde{x} = g\widetilde{x}'* \in gS_i*$ to $\widetilde{w}$ respectively, and proper edge path rays r' at x' and $\widetilde{r}'$ at $\widetilde{x}'$ with image in $\Lambda(S_i, S_i^0) - m_{(g,i)}^{-1}(D_{8.2})$, such that for $r = m_{(g,i)}(r')$ and $\widetilde{r} = m_{(g,i)}(\widetilde{r}')$, one of the following two statements is true.

(i) *For any compact set F in Y, there is an integer $d \in [0, \infty)$ and edge path ψ in $Y - F$ from $r(d)$ to $\widetilde{r}(d)$ such that the loop*

$$\left(r|_{[0,d]}^{-1}, \gamma, [w, v], \beta, [\widetilde{v}, \widetilde{w}], \widetilde{\gamma}^{-1}, \widetilde{r}|_{[0,d]}, \psi^{-1}\right)$$

is homotopically trivial by a homotopy in $Y - C_0$.

(ii) *There are proper J-bounded edge path rays s at v and $\widetilde{s}$ at $\widetilde{v}$ with image in gU_i such that, the ray s (resp. $\widetilde{s}$) is properly homotopic* rel$\{v\}$ *to $([v, w], \gamma^{-1}, r)$ (resp.* rel$\{\widetilde{v}\}$ *to $([\widetilde{v}, \widetilde{w}], \widetilde{\gamma}^{-1}, \widetilde{r})$) by a (ladder) homotopy in $Y - C$ (just as in Lemma 8.2), and s and $\widetilde{s}$ converge to the same end of gU_i.*

Proof We define $D_{8.5}$ to be the union of a finite collection of compact sets. The first is $D = D_{8.2}(C)$ (which contains $St^{M_{8.2}}(C)$). If $\Lambda(S_i, S_i^0) - m_{(g,i)}^{-1}(St^{M_{8.2}}(C))$ has only one unbounded component (in particular when $m_{g,i}^{-1}(St^{M_{8.2}}(C)) = \emptyset$), then conclusion (i) is satisfied (by Lemma 8.4). There are only finitely many pairs (g, i) with $g \in J$ and $i \in \{1, \ldots, l\}$ such that $\Lambda(S_i, S_i^0) - m_{g,i}^{-1}(St^{M_{8.2}}(C))$ has more than one unbounded component. List these pairs as $(g(1), \iota(1)), \ldots, (g(t), \iota(t))$. Now assume that $gU_i = g(q)U_{\iota(q)}$ for some $q \in \{1, \ldots, t\}$. There are finitely many unbounded components of $\Lambda(S_i, S_i^0) - m_{(g,i)}^{-1}(St^{M_{8.2}}(C))$. List them as $K_1, \ldots, K_a$. Consider pairs (K_j, K_k) with $j \neq k$.

If for every compact set F in Y, there are vertices $y_j' \in K_j$ and $y_k' \in K_k$, edge paths τ_j and τ_k of length at most $N_{8.1}$ from $m_{(g,i)}(y_j')$ to gU_i and $m_{(g,i)}(y_k')$ to gU_i respectively, and an edge path in $gU_i - F$ connecting the terminal point of τ_j and the terminal point of τ_k, then we call the pair (K_j, K_k) *inseparable* and let $F_{(j,k)} = \emptyset$. Otherwise, we call the pair *separable* and let $F_{(j,k)}$ be the compact subset of Y for which this condition fails. Let $E_{(g,i)} = \bigcup_{j \neq k} F_{(j,k)}$. As $gU_i = g_q U_{\iota(q)}$, define $E^q = E_{(g,i)}$.

We now define $D_{8.5} = D_{8.2}(C) \cup E^1 \cup \cdots \cup E^t$. As noted above, we need only consider the case where β has image in $g(q)U_{\iota(q)}$ for some $q \in \{1, \ldots, t\}$. Simplifying notation again let $g = g(q)$ and $U_i = U_{\iota(q)}$. Lemma 8.1 implies there are edge paths γ and $\widetilde{\gamma}$ of length at most $N_{8.1}$ from $x = gx'* \in gS_i*$ to w and $\widetilde{x} = g\widetilde{x}'* \in gS_i*$ to $\widetilde{w}$, respectively.

Again, let $K_1, \ldots, K_a$ be the unbounded components of $\Lambda(S_i, S_i^0) - m_{(g,i)}^{-1}(St^{M_{8.2}}(C))$. Assume that x' belongs to K_1. If $\widetilde{x}'$ also belongs to K_1, then conclusion (i) of our lemma follows directly from Lemma 8.4.

So, we can assume $\widetilde{x}'$ belongs to $K_2 \neq K_1$. Notice that the existence of β (in $Y - D_{8.5}$) implies that the pair (K_1, K_2) is inseparable. This implies that there is a sequence of pairs of vertices $(y'_{1(j)}, y'_{2(j)})$ for $j \in \{1, 2, \ldots\}$ with $y'_{1(j)} \in K_1$, $y'_{2(j)} \in K_2$ and edge paths $\tau_{1(j)}$ and $\tau_{2(j)}$ of length at most $N_{8.1}$ from $m_{(g,i)}(y'_{1(j)})$ to gU_i and $m_{(g,i)}(y'_{2(j)})$ to gU_i, respectively, and an edge path β_j in gU_i from the terminal point of $\tau_{1(j)}$ to the terminal point of $\tau_{2(j)}$ and such that only finitely may β_j intersect any compact set. Pick proper edge path rays r' in K_1 at x' and $\widetilde{r}'$ in K_2 at $\widetilde{x}'$ so that for infinitely many pairs $(y'_{1(j)}, y'_{2(j)})$, r' passes through $y'_{1(j)}$ and $\widetilde{r}'$ passes through $y'_{2(j)}$. Let $r = m_{(g,i)}(r')$ and $\widetilde{r} = m_{(g,i)}(r')$. Choose s and $\widetilde{s}$ for r and $\widetilde{r}$, respectively as in Lemma 8.2 where γ and $\widetilde{\gamma}$ for r and $\widetilde{r}$ are chosen to be $\tau_{1(j)}$ and $\tau_{2(j)}$ when ever possible. Lemma 8.2 implies the ray s is properly homotopic rel$\{v\}$ to $([v, w], \gamma_w^{-1}, r)$ and $\widetilde{s}$ is properly homotopic rel$\{\widetilde{v}\}$ to $([\widetilde{v}, \widetilde{w}], \widetilde{\gamma}^{-1}, \widetilde{r})$ by ladder homotopies in $Y - C$. The paths β_j show that s and $\widetilde{s}$ converge to the same end of gU_i, so that conclusion (ii) of our lemma is satisfied. ∎

Lemma 8.6 *Suppose U is a J-unbounded component of $Y - J \cdot C$, F is any compact subset of Y and s_1 and s_2 are J-bounded proper edge path rays in U determining the same end of U, and with $s_1(0) = s_2(0)$, then there is an integer n and a path β from the vertex $s_1(n)$ to the vertex $s_2(n)$ such that the image of β is in $Y - F$ and $(s_1|_{[0,n]}, \beta, s_2|_{[0,n]}^{-1})$ is homotopically trivial in $Y - C_0$.*

Proof Choose an integer n such that $s_1([n, \infty))$ and $s_2([n, \infty))$ avoid F. Since s_1 and s_2 determine the same end of U, there is an edge path α in $U - F$ from $s_1(n)$ to $s_2(n)$. Consider the loop $(s_1|_{[0,n]}^{-1}, s_2|_{[0,n]}, \alpha^{-1})$ based at $s_1|_{[n,\infty)}$. By co-semistability, there is a homotopy $H\colon [0,1] \times [0, l] \to Y - C_0$ (see Figure 6) such that

$$H(0,t) = H(1,t) = s_1(n+t) \text{ for } t \in [0,l],\ H(t,l) \in Y - F \text{ for } t \in [0,1],$$
$$H|_{[0,1]\times\{0\}} = (s_1|_{[0,n]}^{-1}, s_2|_{[0,n]}, \alpha^{-1})$$

Define $\tau(t) = H(t, l)$ for $t \in [0,1]$ (so that $\tau(0) = \tau(1) = s_1(l+n)$). Now define

$$\beta = \left(s_1|_{[n,n+l]}, \tau, s_1|_{[n,n+l]}^{-1}, \alpha\right)$$

to finish the proof. ∎

Lemma 8.7 *Suppose r_1' and r_2' are proper edge path rays in $\Lambda(J, J^0)$ such that $m_{(g,i)}(r_1') = r_1$ and $m_{(g,i)}(r_2') = r_2$ have image in $Y - C$. There is a compact set $D_{8.7}(C)$ in Y such that if α is an edge path in $(J \cdot C) \cap (Y - D_{8.7})$ from $r_1(0)$ to $r_2(0)$ and F is any compact set in Y, then there is an edge path ψ in $Y - F$ from r_1 to r_2 such that the loop determined by ψ, α and the initial segments of r_1 and r_2 is homotopically trivial in $Y - C_0$.*

Proof There is an integer $N_{8.7}(C)$ such that for each vertex v of C there is an edge path in Y from v to $*$ of length at most $N_{8.7}$. Then for each vertex v of $J \cdot C$ there is

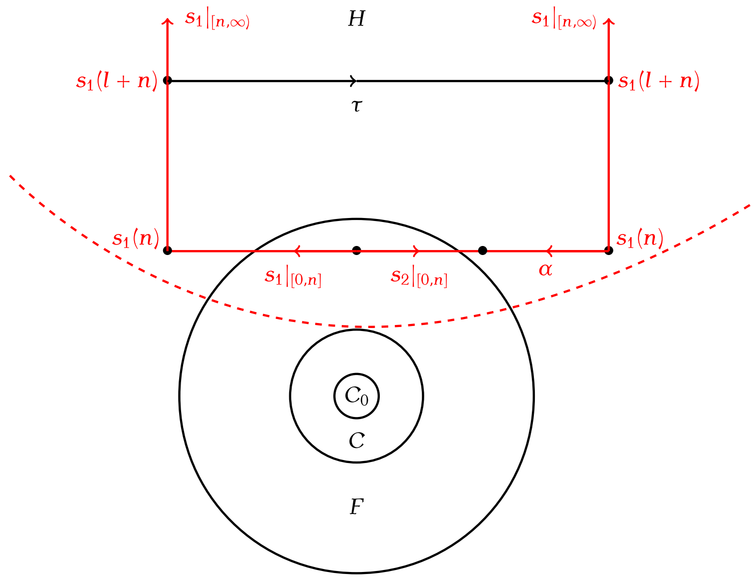

Figure 6.

an edge path of length at most $N_{8.7}$ from v to $J*$. Choose an integer P such that if v' and w' are vertices of $\Lambda(J, J^0)$ and $z(v') = v$ and $z(w') = w$ are connected by an edge path of length at most $2N_{8.7} + 1$ in Y, then v' and w' are connected by an edge path of length at most P in $\Lambda(J, J^0)$. Recall that if e is an edge of $\Lambda(J, J^0)$ then $z(e)$ is an edge path of length at most K. By Lemma 7.3, there is an integer $M_{8.7}$ such that any loop containing a vertex of $J*$ and of length at most $KP + 2N_{8.7} + 1$ is homotopically trivial in $St^{M_{8.7}}(v)$ for any vertex v of this loop.

Let $D_{8.7} = St^{M_{8.7}}(C)$. Write α as the edge path $(e_1, \dots, e_p)$ with consecutive vertices $v_0, v_1, \dots, v_p$. Let β_0 and β_p be trivial and for $i \in \{1, \dots, p-1\}$, let β_i be an edge path of length at most $N_{8.7}$ from v_i to some vertex g_i* for $g_i \in J$. Let $g_0 = r_1'(0)$ and $g_p = r_2'(0)$ (so $g_0* = v_0$ and $g_p* = v_p$). For $i \in \{0, \dots, p-1\}$, there is an edge path τ_i' in $\Lambda(J, J^0)$ from g_{i-1} to g_i of length at most P. Let $\tau_i = z(\tau_i')$ (an edge path of length at most PK). Then the loop $(\beta_i, \tau_{i+2}, \beta_{i+1}^{-1}, e_i^{-1})$ has length at most $KP + 2N_{8.7} + 1$ and so is homotopically trivial in $St^{M_{8.7}}(v)$ for any vertex v of the loop. Let $\tau' = (\tau_1', \dots, \tau_p')$; then α is homotopic rel$\{v_0, v_p\}$ to $z(\tau') = \tau$ by a (ladder) homotopy in $Y - C$. Since J is semistable at ∞ in Y with respect to J^0, C_0, and C, there is an edge path ψ in $Y - F$ from r_1 to (τ, r_2) such that the loop determined by ψ, τ and the initial segments of r_1 and r_2 is homotopically trivial in $Y - C_0$. Now combine this homotopy with the homotopy of α and τ. ∎

Proof of Theorem 3.1 Let C_0 be a finite subcomplex of Y and let J_0 be a finite generating set for an infinite finitely generated group J, where J acts as cell preserving covering transformations on Y, J is semistable at ∞ in Y with respect to J_0, C_0 and C (a finite subcomplex of Y) and J is co-semistable at ∞ in Y with respect to C_0 and C.

Also assume that $Y - J \cdot C$ is a union of J-unbounded components. Let $U_1, \dots, U_l$ be J-unbounded components of $Y - J \cdot C$ forming a component transversal for $Y - J \cdot C$ and let S_i be the J-stabilizer of U_i for $i \in \{1, \dots, l\}$. Let $N_{8.1}$ be defined for C and $U_1, \dots, U_l$ as in Lemma 8.1. Let r_0' be a proper edge path ray in $\Lambda(J, J^0)$ at 1 and $r_0 = zr_0'$.

Let E be compact containing $St^{N_{8.1}}(D_{8.7}) \cup D_{8.5}(C, U_1, \dots, U_l)$ and such that once r_0 leaves E it never returns to $D_{8.5}(C)$. Suppose α is an edge path loop based on r_0 with image in $Y - E$ (see Figure 2). Let F be any compact subset of Y. Our goal is to find a proper homotopy $H\colon [0,1] \times [0,1] \to Y - C_0$ such that $H(0,t) = H(1,t)$ is a subpath of r_0, $H(t,0) = \alpha$ and $H(t,1)$ has image in $Y - F$ (so that Y has semistable fundamental group at ∞ by Theorem 2.1(ii)). Write α as

$$\alpha = (\alpha_1, e_1, \beta_1, \widetilde{e}_1, \alpha_2, e_2\beta_2, \widetilde{e}_2 \dots, \alpha_{n-1}, e_{n-1}, \beta_{n-1}, \widetilde{e}_{n-1}, \alpha_n),$$

where α_i is an edge path in $J \cdot C$, e_i (resp. $\widetilde{e}_i$) is an edge with terminal (resp. initial) vertex in $Y - J \cdot C$ and β_i is an edge path in the J-unbounded component $g_i U_{f(i)}$ of $Y - J \cdot C$ where $f(i) \in \{1, \dots, l\}$.

By Lemmas 8.1 and 8.2 and the definition of $D_{8.5}(C)$, there is an edge path γ_i of length at most $N_{8.1}$, from a vertex $x_i = gx_i' *$ of $g_i S_{f(i)} *$ to the initial vertex of e_i, and there are proper edge path rays r_i' at x_i' in $\Lambda(S_{f(i)}, S'_{f(i)})$ and s_i at the end point of e_i such that s_i has image in $g_i U_{f(i)}$, and r_i is properly homotopic to (γ_i, e_i, s_i) (where $r_i = m_{(g, f(i))}(r_i')$), by a proper (ladder) homotopy H_i with image in $Y - C$. Similarly, there is an edge path $\widetilde{\gamma}_i$ of length at most $N_{8.1}$ from $\widetilde{x}_i$, a vertex of $g_i S_{j(i)} *$, to the terminal vertex of $\widetilde{e}_i$, and there are J-bounded proper edge path rays $\widetilde{r}_i$ at $\widetilde{\gamma}_j(0)$ and $\widetilde{s}_i$ at the initial point of $\widetilde{e}_i$, such that $\widetilde{r}_i = m_{(g_i, f(i))}(\widetilde{r}_i')$ for some proper ray $\widetilde{r}_i'$ in $\Lambda(S_{f(i)}, S'_{f(i)})$, $\widetilde{s}_i$ has image in $g_i U_{f(i)}$ and $\widetilde{s}_i$ is properly homotopic to $(\widetilde{e}_i, \widetilde{\gamma}_i^{-1}, \widetilde{r}_i)$ by a proper (ladder) homotopy $\widetilde{H}_i$ with image in $Y - C$. In particular, the r_i, and $\widetilde{r}_i$-rays have image in $Y - C$.

By Lemma 8.5, either r_i is properly homotopic rel$\{r_i(0)\}$ to the ray $(\gamma_i, e_i, \beta_i, \widetilde{e}_i, \widetilde{\gamma}_i^{-1}, \widetilde{r}_i)$ by a homotopy in $Y - C_0$ or the rays s_i and $\widetilde{s}_i$ converge to the same end of $g_i U_{f(i)}$. In the former case, the path $(\gamma_i, e_i, \beta_i, \widetilde{e}_i, \widetilde{\gamma}_i^{-1})$ can be moved by a homotopy along r_i and $\widetilde{r}_i$ to a path outside F where the homotopy has image in $Y - C_0$.

In the later case, Lemma 8.6 implies there is an integer n_i and edge path $\widetilde{\beta}_i$ from $s_i(n_i)$ to $\widetilde{s}_i(n_i)$ and with image in $Y - F$ such that β_i can be moved by a homotopy along s_i and $\widetilde{s}_i$ to $\widetilde{\beta}_i$, such that this homotopy has image in $Y - C_0$. In any case, the (ladder) homotopy H_i (of r_i to (γ_i, e_i, s_i)) tells us that (γ_i, e_i) can be moved (by a homotopy in $Y - C_0$) along r_i and s_i to a path in $Y - F$ and similarly for $(\widetilde{\gamma}_i, \widetilde{e}_i)$ using $\widetilde{H}_i$. Combining these three homotopies, we have in the latter case (as in the former),

($*$) The path $(\gamma_i, e_i, \beta_i, \widetilde{e}_i, \widetilde{\gamma}_i^{-1})$ can be moved by a homotopy along r_i and $\widetilde{r}_i$ to a path outside F by a homotopy with image in $Y - C_0$.

For consistent notation, let $\widetilde{r}_0 = r_n$ be the tail of r_0 beginning at $\alpha_1(0)$, and let $\widetilde{\gamma}_0$ and γ_n be the trivial paths at the initial point of α_1. It remains to show that for $0 \le i \le n$, there is a path δ_i in $Y - F$ from $\widetilde{r}_i$ to r_{i+1} such that the loop determined by δ_i, the path $(\widetilde{\gamma}_i, \alpha_{i+1}, \gamma_{i+1}^{-1})$, and the initial segments of $\widetilde{r}_i$ and r_{i+1} is homotopically trivial in $Y - C_0$. These homotopies are given by Lemma 8.7, since the paths γ_i and $\widetilde{\gamma}_i$

all have length at most $N_{8.1}$, and so by the definition of E they have image in $Y - D_{8.7}$ (as do the α_i), and since the rays r_i and $\widetilde{r}_i$ have image in $Y - C$. ■

9 Generalizations to Absolute Neighborhood Retracts

There is no need for a space X to be a CW complex in order to define what it means for a finitely generated group J to be semistable at ∞ in X with respect to a compact subset C_0 of X, or for J to be co-semistable at ∞ in X with respect to C_0.

Corollary 9.1 *Suppose X is a 1-ended simply connected locally compact absolute neighborhood retract (ANR), and G is a group (not necessarily finitely generated) acting as covering transformations on X. Assume that for each compact subset C_0 of X, there is a finitely generated subgroup J of G so that* (a) *J is semistable at ∞ in X with respect to C_0, and* (b) *J is co-semistable at ∞ in X with respect to C_0. Then X has semistable fundamental group at ∞.*

Proof By a theorem of J. West [Wes77] the locally compact ANR $G\backslash X$ is proper homotopy equivalent to a locally finite polyhedron Y_1. A simplicial structure on Y_1 lifts to a simplicial structure on its universal cover Y, and G acts as cell preserving covering transformations on Y. A proper homotopy equivalence from $G\backslash X$ to Y_1 lifts to a G-equivariant proper homotopy equivalence $h: X \to Y$. Let $f: Y \to X$ be a (G-equivariant) proper homotopy inverse of h. Since the semistability of the fundamental group at ∞ of a space is invariant under proper homotopy equivalence, it suffices to show that Y satisfies the hypothesis of Theorem 3.1.

First we show that if C_0 is compact in Y, then there is a finitely generated subgroup J of G such that J is semistable at ∞ in Y with respect to C_0. There is a finitely generated subgroup J of G, with finite generating set J^0 and compact set $C \subset X$ such that J is semistable at ∞ with respect to J^0, $h^{-1}(C_0)$, C and z_1, where $z_1: \Lambda(J, J_0) \to X$ is J-equivariant. Note that $z = hz_1$ is J-equivariant. Let r' and s' be proper edge path rays in Λ such that $r'(0) = s'(0)$ and both $r = z_1(r')$ and $s = z_1(s')$ have image in $X - C$. Then given any compact set D in X there is path δ_D in $X - D$ from r to s such that the loop determined by δ_D and the initial segments of r and s is homotopically trivial in $X - h^{-1}(C_0)$.

Now, let D be compact in Y. Suppose that r' and s' are proper edge path rays in Λ such that $r'(0) = s'(0)$ and both $r = hz_1(r')$ and $s = hz_1(s')$ have image in $X - h(C)$ (in particular, $z_1(r')$ and $z_1(s')$ have image in $X - C$). Let δ be a path from $z_1(r')$ to $z_1(s')$ in $X - h^{-1}(D)$ (so that $h(\delta)$ is a path from r to s in $Y - D$) such that the loop determined by δ and the initial segments of $z_1(r')$ and $z_1(s')$ is homotopically trivial by a homotopy H_0 with image in $X - h^{-1}(C_0)$. Then the loop determined by $h(\delta)$ and the initial segments of r and s is homotopically trivial in $Y - C_0$ by the homotopy hH_0.

Finally, we show that if C_0 is compact in Y, there is a finitely generated subgroup J of G such that J is co-semistable at ∞ in Y with respect to C_0. Consider the compact set $h^{-1}(C_0) \subset X$. Choose C compact in X such that J is co-semistable at ∞ in X with respect to $h^{-1}(C_0)$ and C.

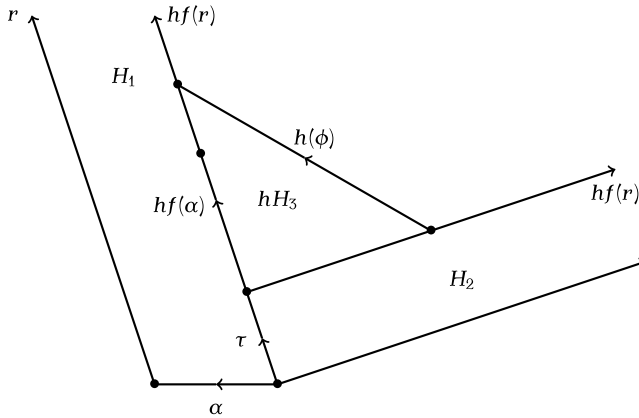

Figure 7.

Let $H\colon Y \times [0,1] \to Y$ be a proper homotopy such that $H(y,0) = y$ and $H(y,1) = hf(y)$ for all $y \in Y$. Let D_1 be compact in Y so that if s is a proper ray in $Y - D_1$, then the proper homotopy of s to $hf(s)$ (induced by H) has image in $Y - C_0$. Let $D_2 = D_1 \cup f^{-1}(C)$. It suffices to show that if r is a J-bounded proper ray in $Y - J \cdot D_2$ and α is a loop in $Y - J \cdot D_2$ with initial point $r(0)$, then for any compact set F in Y, α can be pushed along r to a loop in $Y - F$ by a homotopy in $Y - C_0$. Define $\tau(t) = H(r(0), t)$ for $t \in [0,1]$.

Let $H_1\colon [0,\infty) \times [0,1] \to Y - C_0$ be the proper homotopy (induced by H) of the proper ray (α, r) to $(hf(\alpha), hf(r))$ so that $H_1(t,0) = (\alpha, r)(t)$, $H_1(t,1) = (hf(\alpha), hf(r))(t)$ for $t \in [0,\infty)$, and $H_1(0,t) = \tau(t)$ (see Figure 7). Let $H_2\colon [0,\infty) \times [0,1] \to Y - C_0$ be the proper homotopy (induced by H) of r to $hf(r)$ so that $H_2(t,0) = r(t)$, $H_2(t,1) = hf(r)(t)$ for $t \in [0,\infty)$ and $H_2(0,t) = \tau(t)$ for $t \in [0,1]$.

Recall that f is J-equivariant. Since r and α have an image in $Y - J \cdot D_2$ (and $f^{-1}(C) \subset D_2$), $f(r)$ and $f(\alpha)$ have an image in $X - J \cdot C$. Also, $f(r)$ is J-bounded in X. There is a homotopy H_3 with an image in $X - h^{-1}(C_0)$ that moves $f(\alpha)$ along $f(r)$ to a loop ϕ in $X - h^{-1}(F)$, where if $fr(q)$ is the initial point of ϕ, then $fr([q,\infty)) \subset X - h^{-1}(F)$. The homotopy hH_3 has image in $Y - C_0$ and moves $hf(\alpha)$ along $hf(r)$ to the loop $h(\phi)$ in $Y - F$. Combine the homotopies H_1, H_2, and H_3 as in Figure 7 to see that α can be moved along r into $Y - F$ by a homotopy in $Y - C_0$. ■

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