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ON THE FUNDAMENTAL GROUPS OF TREES OF MANIFOLDS

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We consider limits of inverse sequences of closed manifolds, whose consecutive terms are obtained by connect summing with closed manifolds, which are in turn trivialized by the bonding maps. Such spaces, which we refer to as *trees of manifolds*, need not be semilocally simply connected at any point and can have complicated fundamental groups.

Trees of manifolds occur naturally as visual boundaries of standard nonpositively curved geodesic spaces, which are acted upon by right-angled Coxeter groups whose nerves are closed PL-manifolds. This includes, for example, those Coxeter groups that act on Davis' exotic open contractible manifolds. Also, all of the homogeneous cohomology manifolds constructed by Jakobsche are trees of manifolds. In fact, trees of manifolds of this type, when constructed from PL-homology spheres of common dimension at least 4, are boundaries of negatively curved geodesic spaces.

We prove that if Z is a tree of manifolds, the natural homomorphism $\varphi : \pi_1(Z) \to \check{\pi}_1(Z)$ from its fundamental group to its first shape homotopy group is injective. If $Z = \operatorname{bdy} X$ is the visual boundary of a nonpositively curved geodesic space X, or more generally, if Z is a Z-set boundary of any ANR X, then the first shape homotopy group of Z coincides with the fundamental group at infinity of X: $\check{\pi}_1(Z) = \pi_1^\infty(X)$. We therefore obtain an injective homomorphism $\psi : \pi_1(Z) \to \pi_1^\infty(X)$, which allows us to study the relationship between these groups. In particular, if $Z = \operatorname{bdy} \Gamma$ is the boundary of one of the Coxeter groups Γ mentioned above, we get an injective homomorphism $\psi : \pi_1(\operatorname{bdy} \Gamma) \to \pi_1^\infty(\Gamma)$.

1. Trees of manifolds

Definition 1.1. We shall call a topological space Z a *tree of manifolds* if there is an inverse sequence

$$M_1 \xleftarrow{f_{2,1}} M_2 \xleftarrow{f_{3,2}} M_3 \xleftarrow{f_{4,3}} \cdots ,$$

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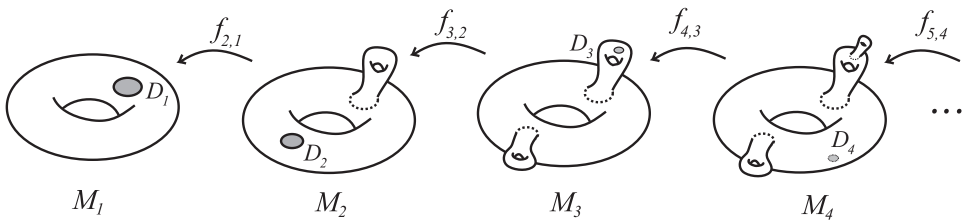

Figure 1. A tree of manifolds.

called a *defining sequence* for Z, of distinct closed PL-manifolds M_n with collared disks $D_n \subseteq M_n$, and continuous functions $f_{n+1,n} : M_{n+1} \to M_n$ that have the following properties:

(P1) $$Z = \varprojlim \bigl(M_1 \xleftarrow{f_{2,1}} M_2 \xleftarrow{f_{3,2}} M_3 \xleftarrow{f_{4,3}} \cdots\bigr).$$

(P2) For each n, the restriction of $f_{n+1,n}$ to the set $f_{n+1,n}^{-1}(M_n \setminus \operatorname{int} D_n)$, call it $h_{n+1,n}$, is a homeomorphism onto $M_n \setminus \operatorname{int} D_n$, and $h_{n+1,n}^{-1}(\partial D_n)$ is bicollared in M_{n+1}.

(P3) For each n, we have $\lim_{m\to\infty} \operatorname{diam} f_{m,n}(D_m) = 0$, where $f_{m,n} = f_{n+1,n} \circ f_{n+2,n+1} \circ \cdots \circ f_{m,m-1} : M_m \to M_n$ and $f_{n,n} = \mathrm{id}_{M_n}$.

(P4) For each pair $n < m$, the sets $f_{m,n}(D_m)$ and ∂D_n are disjoint.

Remark 1.2. It follows that, for $m \geq n+2$, the set

$$E_{m,n} = \operatorname{int} D_n \cup f_{n+1,n}(\operatorname{int} D_{n+1}) \cup f_{n+2,n}(\operatorname{int} D_{n+2}) \cup \cdots \cup f_{m-1,n}(\operatorname{int} D_{m-1})$$

can be written as the union of $m-n$, or fewer, open disks in M_n and that $f_{m,n}$ restricted to $f_{m,n}^{-1}(M_n \setminus E_{m,n})$ is a homeomorphism onto $M_n \setminus E_{m,n}$, which we will denote by $h_{m,n}$. Moreover, if for $n < m$ we define the spheres $S_{m,n} = h_{m,n}^{-1}(\partial D_n) \subseteq M_m$, we see that the collection $\mathcal{S}_n = \{S_{n,1}, S_{n,2}, \ldots, S_{n,n-1}\}$ decomposes M_n into a connected sum

$$M_n = (N_{n,1} \# N_{n,2} \# \cdots \# N_{n,n-1}) \# N_{n,n} \approx M_{n-1} \# N_{n,n}.$$

Hence, Z can be thought of as the limit of a growing tree of connected sums of closed manifolds. In particular, in dimensions greater than two, we have

$$\pi_1(M_n) = \pi_1(N_{n,1}) * \pi_1(N_{n,2}) * \cdots * \pi_1(N_{n,n-1}) * \pi_1(N_{n,n});$$

and in dimension two, we have

$$\pi_1(M_n) = F_{n,1} *_{\pi_1(S_{n,1})} F_{n,2} *_{\pi_1(S_{n,2})} \cdots *_{\pi_1(S_{n,n-2})} F_{n,n-1} *_{\pi_1(S_{n,n-1})} F_{n,n},$$

where $F_{n,i}$ denotes the free fundamental group of the appropriately punctured $N_{n,i}$.

Note also that each $S_{n,i} \approx \partial D_i$ naturally embeds in Z.

Definition 1.3. We will call a defining sequence $M_1 \xleftarrow{f_{2,1}} M_2 \xleftarrow{f_{3,2}} M_3 \xleftarrow{f_{4,3}} \cdots$ *well-balanced* if the set $\bigcup_{m\geq 3} E_{m,1}$ either has finitely many components or is dense in M_1, and if for each $n \geq 2$, the set $h^{-1}_{n,n-1}(M_{n-1} \setminus D_{n-1}) \cup \big(\bigcup_{m\geq n+2} E_{m,n}\big)$ either has finitely many components or is dense in M_n.

Remark 1.4. Whether Z has a well-balanced defining sequence or not, will play a role only in the case when the manifolds M_n are 2-dimensional closed surfaces.

2. The first shape homotopy group

We briefly recall the definition of the first shape homotopy group of a pointed compact metric space (Z, z_0). More details can be found in [Mardešić and Segal 1982].

Definition 2.1. Let (Z, z_0) be a pointed compact metric space. Choose an inverse sequence

$$(Z_1, z_1) \xleftarrow{f_{2,1}} (Z_2, z_2) \xleftarrow{f_{3,2}} (Z_3, z_3) \xleftarrow{f_{4,3}} \cdots$$

of pointed compact polyhedra such that

$$(Z, z_0) = \varprojlim \big((Z_i, z_i),\, f_{i+1,i}\big).$$

The *first shape homotopy group* of Z, based at z_0, is then given by

$$\check{\pi}_1(Z, z_0) = \varprojlim \big(\pi_1(Z_1, z_1) \xleftarrow{f_{2,1\#}} \pi_1(Z_2, z_2) \xleftarrow{f_{3,2\#}} \pi_1(Z_3, z_3) \xleftarrow{f_{4,3\#}} \cdots\big).$$

This definition of $\check{\pi}_1(Z, z_0)$ does not depend on the choice of the inverse sequence.

Remark 2.2. Let $p_i : (Z, z_0) \to (Z_i, z_i)$ denote the projections of the limit (Z, z_0) into its inverse sequence $\big((Z_i, z_i),\, f_{i+1,i}\big)$ such that $p_i = f_{i+1,i} \circ p_{i+1}$ for all i. Since the maps p_i induce homomorphisms $p_{i\#} : \pi_1(Z, z_0) \to \pi_1(Z_i, z_i)$ such that $p_{i\#} = f_{i+1,i\#} \circ p_{i+1\#}$ for all i, we obtain an induced homomorphism $\varphi : \pi_1(Z, z_0) \to \check{\pi}_1(Z, z_0)$, which is given by $\varphi([\alpha]) = ([\alpha_1], [\alpha_2], [\alpha_3], \ldots)$, where $\alpha_i = p_i \circ \alpha$.

The following examples illustrate that $\varphi : \pi_1(Z, z_0) \to \check{\pi}_1(Z, z_0)$ need not be injective and is typically not surjective.

Example 2.3. Consider the "topologist's sine curve"

$$Y = \big\{(x, y, z) \in \mathbb{R}^3 \mid z = 0, 0 < x \leq 1, y = \sin 1/x\big\} \cup \big(\{0\} \times [-1, 1] \times \{0\}\big).$$

Define $Y_i = Y \cup \big([0, 1/i] \times [-1, 1] \times \{0\}\big)$. Let Z and Z_i be the subsets of $\mathbb{R}^3$ obtained by revolving Y and Y_i about the y-axis, respectively, and let $f_{i+1,i} : Z_{i+1} \hookrightarrow Z_i$ be inclusion. Then Z is the limit of the inverse sequence $(Z_i,\, f_{i+1,i})$. If we take $z_0 = (1, \sin 1, 0)$, then $\pi_1(Z, z_0)$ is infinite cyclic, while $\check{\pi}_1(Z, z_0)$ is trivial.

Example 2.4. We can make the space Z of the previous example path connected, by taking any arc $a \subseteq \mathbb{R}^3$, such that $a \cap Z = \partial a = \{z_0, (0, 1, 0)\}$, and then considering $Z^+ = Z \cup a$. Notice that both $\pi_1(Z^+, z_0)$ and $\check{\pi}_1(Z^+, z_0)$ are infinite cyclic. However, the homomorphism $\varphi : \pi_1(Z^+, z_0) \to \check{\pi}_1(Z^+, z_0)$ is trivial.

Example 2.5. Consider the Hawaiian Earring — the union $Z = \bigcup_{k=1}^{\infty} C_k$ of the circles $C_k = \{(x, y) \in \mathbb{R}^2 \mid x^2 + (y - 1/k)^2 = (1/k)^2\}$. Put $Z_i = C_1 \cup C_2 \cup \cdots \cup C_i$ and let $z_0 = z_i = (0, 0)$. Define $f_{i+1,i} : Z_{i+1} \to Z_i$ by $f_{i+1,i}(t) = (0, 0)$ for $t \in C_{i+1}$ and $f_{i+1,i}(t) = t$ for $t \in Z_{i+1} \setminus C_{i+1}$. Then (Z, z_0) is the limit of the inverse sequence $((Z_i, z_i), f_{i+1,i})$. While this time $\varphi : \pi_1(Z, z_0) \to \check{\pi}_1(Z, z_0)$ is injective (see Remark 3.2(i) below), it is not surjective: let $l_i : (S^1, *) \to (C_i, z_0)$ be a fixed homeomorphism and, following an idea of Griffiths', consider for each i the element

$$g_i = [l_1][l_1][l_1]^{-1}[l_1]^{-1}[l_1][l_2][l_1]^{-1}[l_2]^{-1}[l_1][l_3][l_1]^{-1}[l_3]^{-1} \dots [l_1][l_i][l_1]^{-1}[l_i]^{-1}$$

of $\pi_1(Z_i, z_i)$. Then the sequence $(g_i)_i$ is an element of the group $\check{\pi}_1(Z, z_0)$ which is clearly not in the image of φ. (Indeed, combining this observation with the appendix of [Zdravkovska 1981], where the higher dimensional analogue is being discussed, we see that the homomorphism $\varphi : \pi_1(Z') \to \check{\pi}_1(Z')$ is not surjective for *any* metric compactum Z' which is shape equivalent to Z.)

Example 2.6. The canonical homomorphism from the fundamental group of a tree of manifolds to its first shape homotopy group is not surjective if, using the notation of Remark 1.2, $\pi_1(N_{n,n}) \neq 1$ for infinitely many n. Indeed, in Example 2.5 it is irrelevant, for the nonsurjectivity argument, just how the circles C_k are joined. We may as well consider the limit (Z, z_0) of any inverse sequence $((Z_i, z_i), f_{i+1,i})$, where $Z_1 = C_1$ is a circle, Z_{i+1} is equal to Z_i joined with some circle C_{i+1} at some point $p_i \in Z_i$, $f_{i+1,i}(t) = p_i$ for $t \in C_{i+1}$ and $f_{i+1,i}(t) = t$ otherwise. If we change the definition of the loops l_i to include a possible change of base point, the argument of Example 2.5 goes through to show that

$$\varphi : \pi_1(Z, z_0) \to \check{\pi}_1(Z, z_0)$$

is not surjective. Therefore, this homomorphism is also not surjective for trees of manifolds, provided $\pi_1(N_{n,n}) \neq 1$ for infinitely any n. In the case of a tree of manifolds, we instead run loops l_n through the punctured attachments $N_{n,n}$, where they can pick up appropriate nontrivial fundamental group elements, whenever available, that is, whenever $\pi_1(N_{n,n}) \neq 1$. For the corresponding element $(g_n)_n$ of the shape group not to be in the image of the homomorphism, it suffices if this occurs infinitely often. (Moreover, as was the case in Example 2.5, $\varphi : \pi_1(Z') \to \check{\pi}_1(Z')$ is not surjective for any metric compactum Z' which is shape equivalent to such Z.)

Remark 2.7. In contrast to the above examples, it is known that if the compact metric space Z is connected, locally path connected and semilocally simply connected, then $\varphi : \pi_1(Z, z_0) \to \check{\pi}_1(Z, z_0)$ is an isomorphism [Kuperberg 1975].

3. Statement of the main theorem

Theorem 3.1. *Suppose Z is a tree of manifolds, and $z_0 \in Z$. In case Z is 2-dimensional, suppose further that Z admits a well-balanced defining sequence. Then the canonical homomorphism $\varphi : \pi_1(Z, z_0) \to \check{\pi}_1(Z, z_0)$ is injective.*

We will prove this theorem in Section 6.

Remark 3.2. Other classes of spaces Z for which $\varphi : \pi_1(Z, z_0) \to \check{\pi}_1(Z, z_0)$ has been shown to be injective, include (i) one-dimensional compacta [Curtis and Fort 1959; Eda and Kawamura 1998; Cannon and Conner 1998] and (ii) subsets of closed surfaces [Fischer and Zastrow 2005].

Our result can be viewed in the spirit of earlier work [Edwards and Hastings 1976; Ferry 1980; Geoghegan and Krasinkiewicz 1991] regarding the "improvability" of a compactum within its shape class when comparing its homotopy theory, strong shape theory and shape theory. Specifically, the homomorphism $\varphi : \pi_1(Z, z_0) \to \check{\pi}_1(Z, z_0)$ factors through the *first strong shape homotopy group* $\pi_1^s(Z, z_0)$:

$$\pi_1(Z, z_0) \xrightarrow{\varphi_1} \pi_1^s(Z, z_0) \xrightarrow{\varphi_2} \check{\pi}_1(Z, z_0),$$

where φ_2 is always surjective and has kernel

$$\underleftarrow{\lim}^1 \big(\pi_2(Z_1, z_1) \xleftarrow{f_{2,1\#}} \pi_2(Z_2, z_2) \xleftarrow{f_{3,2\#}} \pi_2(Z_3, z_3) \xleftarrow{f_{4,3\#}} \cdots\big).$$

(See [Mardešić and Segal 1982, §III.9] for details.) The question of improvability, in this context, then becomes: Which of the homomorphisms φ_1, φ_2, or φ can be made injective, surjective, or bijective upon replacing Z by a metric compactum Z', which is shape equivalent to Z?

Let Z be a tree of manifolds. Then the above π_2-system is Mittag-Leffler, so that its $\lim^1$ is trivial. Consequently, $\varphi_2 : \pi_1^s(Z, z_0) \to \check{\pi}_1(Z, z_0)$ is an isomorphism. Example 2.6 then shows that $\varphi_1 : \pi_1(Z, z_0) \to \pi_1^s(Z, z_0)$ can almost never be surjectively improved, while Theorem 3.1 states that it is already injectively improved. It follows that these trees of manifolds are all-around "1-improved".

4. Coxeter group boundaries

We now present an application of Theorem 3.1 to boundaries of certain nonpositively curved geodesic spaces. Recall that a metric space is *proper* if all of its closed metric balls are compact. A *geodesic space* is a metric space in which any two points lie in a geodesic, i.e. a subset that is isometric to an interval of the real line in its usual metric. A proper geodesic space X is said to be *nonpositively curved* if any two points on the sides of a geodesic triangle in X are no further apart than their corresponding points on a reference triangle in Euclidean 2-space. The *visual boundary* of a nonpositively curved geodesic space X, denoted by bdy X, is defined to be the set of all geodesic rays emanating from an arbitrary but fixed point x_0 endowed with the compact open topology. (See [Bridson and Haefliger 1999] for more details.) Let some geodesic base ray $\omega : [0, \infty) \to X$ with $\omega(0) = x_0$ be given and let us denote the concentric metric spheres and closed concentric metric balls by $S_{x_0}(i) = \{x \in X \mid d(x, x_0) = i\}$ and $B_{x_0}(i) = \{x \in X \mid d(x, x_0) \leq i\}$, respectively. Under the relatively mild assumption that the pointed concentric metric spheres $(S_{x_0}(i), \omega(i))$ have the pointed homotopy type of ANRs, it is shown in [Conner and Fischer 2003] that

$$\check{\pi}_1(\text{bdy}\, X, \omega) = \pi_1^\infty(X, \omega).$$

Here, $\pi_1^\infty(X, \omega)$ is the *fundamental group at infinity* of X, that is, the limit of the sequence

$$\pi_1\big(X \setminus B_{x_0}(1), \omega(2)\big) \leftarrow \pi_1\big(X \setminus B_{x_0}(2), \omega(3)\big) \leftarrow \pi_1\big(X \setminus B_{x_0}(3), \omega(4)\big) \leftarrow \cdots,$$

whose bonds are induced by inclusion followed by a base point slide along ω. In fact, this relationship holds in more generality. (See Remark 4.2 below.)

A class of visual boundaries to which Theorem 3.1 applies, arises from nonpositively curved simplicial complexes, which are acted upon by certain Coxeter groups, whose definition we now briefly recall: let V be a finite set and $m : V \times V \to \{\infty\} \cup \{1, 2, 3, \ldots\}$ a function with the property that $m(u, v) = 1$ if and only if $u = v$, and $m(u, v) = m(v, u)$ for all $u, v \in V$. Then the group

$$\Gamma = \big\langle V \mid (uv)^{m(u,v)} = 1 \text{ for all } u, v \in V \big\rangle$$

defined in terms of generators and relations is called a *Coxeter group*. If moreover $m(u, v) \in \{\infty, 1, 2\}$ for all $u, v \in V$, then Γ is called *right-angled*. The abstract simplicial complex

$$N(\Gamma, V) = \{\varnothing \neq S \subseteq V \mid S \text{ generates a finite subgroup of } \Gamma\}$$

is called the *nerve* of the group Γ. For a right-angled Coxeter group, the isomorphism type of the nerve $N(\Gamma, V) = N(\Gamma)$ does not depend on the Coxeter system (Γ, V) but only on the group Γ [Radcliffe 2001].

Conversely, given any finite simplicial complex M, there is exactly one right-angled Coxeter group Γ whose nerve $N(\Gamma)$ is isomorphic to the first barycentric subdivision M' of M; namely, the Coxeter group Γ which is generated by the vertex set of M' and whose only relations are of the form $(uv)^2 = 1$ whenever $\{u, v\} \in M'$ [Davis 2002].

For the rest of this section, Γ will be a right-angled Coxeter group whose nerve $N(\Gamma)$ is a closed PL-manifold. This includes, for example, the Coxeter groups generated by the reflections of any one of Davis' exotic open contractible manifolds of dimension 4 and higher, for which the nerves are PL-homology spheres; see [Davis 1983].

As described, for example, in [Davis 2002], Γ acts properly discontinuously on a nonpositively curved (and hence contractible) simplicial complex $X(\Gamma)$, its so-called *Davis–Vinberg complex*, by isometry and with compact quotient. In [Fischer 2003] it is shown that the visual boundary of $X(\Gamma)$ is a (well-balanced) tree of manifolds. (The proof given there also applies to the nonorientable case, by virtue of [Stallings 1995].) The visual boundary of $X(\Gamma)$ is usually referred to as the *boundary* of Γ and is denoted by $\operatorname{bdy}\Gamma$. Since Coxeter groups are semi-stable at infinity [Mihalik 1996] and since Γ is one-ended, $\pi_1^\infty(X(\Gamma), \omega) = \pi_1^\infty(\Gamma)$ is actually an invariant of the group Γ [Geoghegan and Mihalik 1996].

In summary, we obtain:

Corollary 4.1. *Let Γ be a right-angled Coxeter group whose nerve $N(\Gamma)$ is a closed PL-manifold. Then the canonical homomorphism $\psi : \pi_1(\operatorname{bdy}\Gamma) \to \pi_1^\infty(\Gamma)$ is injective.*

Remark 4.2. The coincidence of $\check{\pi}_1(\operatorname{bdy} X, *)$ with $\pi_1^\infty(X, *)$ holds in a context larger than that of nonpositively curved geodesic spaces and their boundaries. Identifying each point x of a nonpositively curved geodesic space X with the unique geodesic segment from x_0 to x, we obtain a natural compactification $\bar{X} = X \cup \operatorname{bdy} X$. Since open metric balls in X are convex and since they can be used to refine any open cover of X, it follows from [Hu 1965, Theorem IV.4.1] that X is an ANR. Moreover, geodesic retraction towards the base point x_0 gives rise to a homotopy $H : \bar{X} \times [0, 1] \to \bar{X}$ such that $H_0 = \mathrm{id}$, $H_1(\bar{X}) = \{x_0\}$ and $H_t(\bar{X}) \cap \operatorname{bdy} X = \varnothing$ for all $t > 0$. In particular, $\bar{X}$ is contractible and ϵ-dominated by X for every $\epsilon > 0$. Hence, $\bar{X}$ is an AR [Hanner 1951] and $\operatorname{bdy} X$ is a Z-set boundary of X. (Recall that a closed subset Z of a compact ANR $\bar{X}$ is called a *Z-set* in $\bar{X}$ if there is a homotopy $H : \bar{X} \times [0, 1] \to \bar{X}$ such that $H_0 = \mathrm{id}$ and $H_t(\bar{X}) \cap Z = \varnothing$ for all $t > 0$. In this situation, $\bar{X}$ is called a *Z-compactification* of $X = \bar{X} \setminus Z$ and Z is called a *Z-set boundary of* X.) We claim that if Z is a Z-set boundary of any ANR X, then the first shape homotopy group of Z always coincides with the fundamental group at infinity of X. Indeed, if Z happens to have compact polyhedral neighborhoods $N_1 \supseteq N_2 \supseteq N_3 \supseteq \cdots$ in $\bar{X}$ with $Z = \bigcap_{i=1}^\infty N_i$, then $\check{\pi}_1(Z, *)$ coincides with $\pi_1^\infty(X, *)$ by definition, so long as the base points are chosen consistently. However, even if such polyhedral neighborhoods are not available, we can argue via a detour through Hilbert cube manifold theory, the basic facts of which are reviewed in [van Mill 1989, Chapter 7]: since $\bar{X}$ is a compact ANR, then $\bar{X} \times [-1, 1]^{\mathbb{N}}$ is a compact Hilbert cube manifold [Edwards 1980], or equivalently, a compact ANR with the disjoint-cells property [Toruńczyk 1980]. Choose a map $f : \bar{X} \to [0, 1]$ with $f^{-1}(\{0\}) = Z$ and define $g : \bar{X} \times [-1, 1]^{\mathbb{N}} \to \bar{X} \times [-1, 1]^{\mathbb{N}}$ by $g(s, (t_i)_{i\in\mathbb{N}}) = (s, (f(s) \cdot t_i)_{i\in\mathbb{N}})$. Let Y denote the image of g. Then Y is a retract of $\bar{X} \times [-1, 1]^{\mathbb{N}}$ and therefore an ANR. Also, Y is homeomorphic to a Z-compactification of $X \times [-1, 1]^{\mathbb{N}}$ by Z, where we identify Z with $Z \times \{\mathbf{0}\}$. Since $X \times [-1, 1]^{\mathbb{N}}$ has the disjoint-cells property, so does $Y \approx (X \times [0, 1]^{\mathbb{N}}) \cup Z$. Consequently, Y is a Hilbert cube manifold. (In fact, since $g : \bar{X} \times [-1, 1]^{\mathbb{N}} \to Y$ is a cell-like map between compact Hilbert cube manifolds, it is a near-homeomorphism.) Hilbert cube manifolds are triangulable and the fundamental group at infinity of X is isomorphic to that of $X \times [-1, 1]^{\mathbb{N}}$. Therefore we have recovered our claim.

5. Trees of homology spheres as boundaries of negatively curved geodesic spaces

In this section we present a general procedure for building trees of manifolds from sequences of homology spheres (of dimension at least 4) in such a way that the resulting tree of manifolds becomes the visual boundary of a negatively curved geodesic space in a natural way. This procedure is flexible enough to match a variety of given trees of manifolds. For example, at the end of the section, we will use it to reproduce Jakobsche's homogeneous cohomology manifolds in this way. It follows that their fundamental groups are subgroups of the fundamental groups at infinity of the underlying negatively curved geodesic spaces.

For $\kappa < 0$, let $\mathbb{H}^d(\kappa)$ denote the simply connected complete Riemannian manifold of dimension d with constant negative sectional curvature κ. If we change the comparison space in the definition of *nonpositively curved geodesic space* from Euclidean 2-space to $\mathbb{H}^2(\kappa)$, we arrive at the concept of a *negatively curved geodesic space* (with curvature bound κ). Since a negatively curved geodesic space X is, in particular, nonpositively curved, its visual boundary bdy X can be defined as in Section 4 and the results mentioned there apply.

Consider now a sequence $N_1, N_2, N_3, \ldots$ of PL-homology spheres of common dimension $d \geq 4$. Then each N_n bounds a unique $(d + 1)$-dimensional compact contractible PL-manifold L_n. In [Ancel and Guilbault 1997] it is shown that, given any $\kappa < 0$, the interior of every compact contractible PL-manifold of dimension at least 5 admits a geodesic metric of negative curvature with curvature bound κ. If we equip the interior $\mathring{L}_n$ of each L_n with this geodesic metric of negative curvature, using some arbitrary but fixed common curvature bound κ, then the visual boundary of $\mathring{L}_n$ is homeomorphic to N_n. Moreover, by the very nature of the construction in [Ancel and Guilbault 1997], we can find in each $\mathring{L}_n$ various geodesic subsets A, which are isometric to half-spaces of $\mathbb{H}^{d+1}(\kappa)$ and whose boundaries in $\mathring{L}_n$ are isometrically embedded copies of $\mathbb{H}^d(\kappa)$.

In order to see where such a region A might be located, we recall that the construction of [Ancel and Guilbault 1997] represents each $\mathring{L}_n$ as the union of three pieces: two open cones $\mathbb{O}(Q_0)$ and $\mathbb{O}(Q_1)$ on PL-manifolds Q_0 and Q_1, which are homology d-cells, and the product $\mathbb{O}(\Sigma) \times [0, 1]$ of the open cone on their common boundary $\Sigma = \partial Q_0 \approx \partial Q_1$ with the unit interval. Each of the cones $\mathbb{O}(Q_i)$ are equipped with the κ-cone metric, as described in [Bridson and Haefliger 1999, p. 59], and $\mathbb{O}(\Sigma) \times [0, 1]$ is given a metric which can be described as follows: let $\mathbb{H}^d_0(\kappa)$ be a copy of $\mathbb{H}^d(\kappa)$ in $\mathbb{H}^{d+1}(\kappa)$ passing through the origin and let $\mathbb{H}^d_1(\kappa)$ be a parallel copy of $\mathbb{H}^d_0(\kappa)$, translated by one unit along the perpendicular through the origin. The closure of the region between $\mathbb{H}^d_0(\kappa)$ and $\mathbb{H}^d_1(\kappa)$ is homeomorphic to $\mathbb{H}^d(\kappa) \times [0, 1]$ in a natural way, and we metrize $\mathbb{H}^d(\kappa) \times [0, 1]$ by this correspondence. The metric on $\mathbb{O}(\Sigma) \times [0, 1]$ is now obtained by identifying each $\mathbb{O}(\sigma^{d-1}) \times [0, 1]$, where σ^{d-1} is a top-dimensional simplex of Σ, with a natural copy in $\mathbb{H}^d(\kappa) \times [0, 1]$ endowed with this new metric. The three pieces $\mathbb{O}(Q_0)$, $\mathbb{O}(Q_1)$, and $\mathbb{O}(\Sigma) \times [0, 1]$ are then glued together along the strongly geodesic subsets $\mathbb{O}(\partial Q_i) \approx \mathbb{O}(\Sigma) \times \{i\}$ $(i = 0, 1)$ to form $\mathring{L}_n$. Therefore, we can find a subset A of $\mathring{L}_n$, as described above, in any $\mathbb{O}(\sigma^d)$ with $\sigma^d \in Q_i$ $(i = 0, 1)$ as well as in any $\mathbb{O}(\sigma^{d-1}) \times [0, 1]$ with $\sigma^{d-1} \in \Sigma$. Notice that A determines a disk D at infinity whose interior can be arranged to include any given interior point at infinity of either $\mathbb{O}(\sigma^d)$ or $\mathbb{O}(\sigma^{d-1}) \times [0, 1]$. We also can arrange for D to be as small at infinity as we like. See Figure 2.

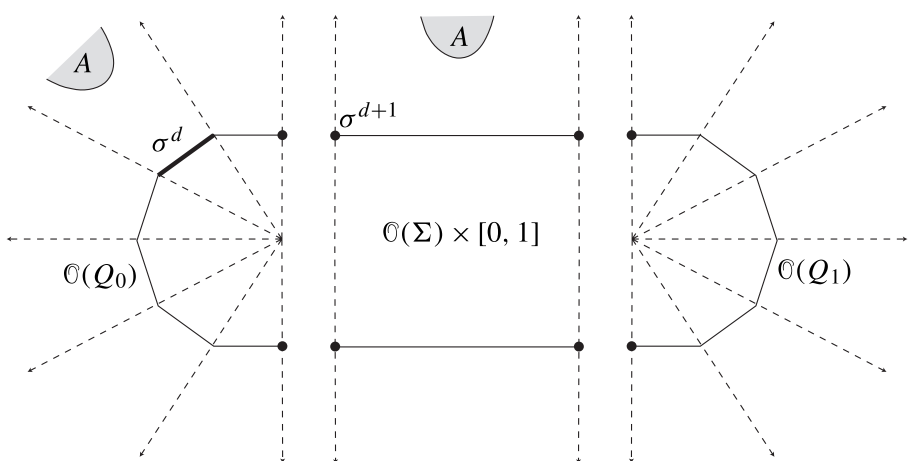

Figure 2. $\mathring{L}_n$ is made up of $\mathbb{O}(Q_0)$, $\mathbb{O}(\Sigma) \times [0, 1]$ and $\mathbb{O}(Q_1)$.

Now, from each of $\mathring{L}_1$ and $\mathring{L}_2$, remove the interior of such a half-space A and glue the remainders along the embedded copies of $\mathbb{H}^d(\kappa)$ described above. This yields a negatively curved geodesic space (with curvature bound κ) whose visual boundary is the connected sum $N_1 \# N_2$. Continuing in this fashion, we obtain a negatively curved geodesic space X (with curvature bound κ) whose visual boundary bdy X is the inverse limit Z of the homology spheres $M_n = N_1 \# N_2 \# \cdots \# N_n$, which, when regarded as an inverse system under geodesic retraction, satisfies the definition of a tree of manifolds.

We have a lot of flexibility in how to choose the subsets A of $\mathring{L}_n$. In particular, we are in great control of location and size of D in N_n. We therefore can produce a variety of trees of manifolds in this way. We could, for example, have the terms $N_2, N_3, \ldots$ accumulate to exactly one "bad" point of Z. Going the other extreme, we could distribute the terms so as to make Z non-semilocally simply connected at every point, assuming that none of the N_i is simply connected. Such is the case with Jakobsche's homogeneous cohomology manifolds, for example, whose definition we now briefly recall.

In [Jakobsche 1991] Jakobsche describes how to construct a d-dimensional homogeneous cohomology manifold Z from an orientable d-dimensional closed PL-manifold N and a countable collection $\mathcal{N}$ of (distinct) d-dimensional PL-homology spheres. The construction renders Z as a tree of manifolds, with a defining sequence whose first term is $M_1 = N$ and whose general term M_n is a connected sum of N and various members of the collection $\mathcal{N}$. Jakobsche imposes an axiom system on the defining sequence, which ensures that the resulting space Z only depends on the pair $(N, \mathcal{N})$. This is achieved by requiring that each of the elements of $\mathcal{N}$ is attached infinitely often and in an increasingly dense fashion.

Combining this discussion with Theorem 3.1 and the remarks of Section 4, we record:

Proposition 5.1. *Let N be a d-dimensional PL-homology sphere with $d \geq 4$ and let $\mathcal{N}$ be a countable collection of distinct d-dimensional PL-homology spheres. Consider the Jakobsche homogeneous cohomology manifold Z based on the pair $(N, \mathcal{N})$. Then for any negative real number $\kappa < 0$, Z is the visual boundary of some negatively curved geodesic space X with curvature bound κ. Moreover, for every $\omega \in Z = \mathrm{bdy}\, X$, the natural homomorphism $\psi : \pi_1(Z, \omega) \to \pi_1^\infty(X, \omega)$ is injective.*

6. Proof of the main theorem

Let Z be a tree of manifolds with defining sequence $M_1 \xleftarrow{f_{2,1}} M_2 \xleftarrow{f_{3,2}} \cdots$. Let D_n, $h_{m,n}$, $E_{m,n}$, and $\mathcal{S}_n = \{S_{n,1}, S_{n,2}, \ldots, S_{n,n-1}\}$ be as in Section 1. Again, we denote projection by $p_n : Z \to M_n$. Since the result is known for one-dimensional spaces (Remark 3.2), we may assume that the (common) dimension of the manifolds M_n

is at least 2. Let $\alpha : S^1 \to Z$ be a loop such that $\alpha_n = p_n \circ \alpha : S^1 \to M_n$ is null-homotopic for each n. We wish to show that $\alpha : S^1 \to Z$ is null-homotopic. We will do this by constructing a map $\beta : D^2 \to Z$ with $\beta|_{S^1} = \alpha$.

By assumption, we may choose maps $\beta_n : D^2 \to M_n$ with $\beta_n|_{S^1} = \alpha_n$. The difficulty of the proof, of course, is that in general $\beta_n \neq f_{n+1,n} \circ \beta_{n+1}$, so that the sequence $(\beta_n)_n$ does not even constitute a function $D^2 \to Z$ into the inverse limit, much less a map extending α.

An outline of our strategy. Although we might not be in a position to move the maps α_n the slightest bit, we can place β_n in general position with respect to the spheres of the collection $\mathcal{S}_n$ while having $\beta_n|_{S^1}$ approximate α_n with increasing accuracy as n increases. Indeed, we will arrange for each *cancellation pattern* $\beta_n^{-1}(\bigcup \mathcal{S}_n)$, to consist of finitely many pairwise disjoint straight line segments in D^2 having their endpoints in S^1. Ideally, we would like to paste together our map β from appropriate pieces belonging to the maps of the sequence $(\beta_n)_n$, namely those pieces that cancel the elements of $\pi_1(N_{n,n})$. However, these cancellation patterns will in general not be compatible. For example, in dimensions greater than two, the cancellation pattern for an element

$$[\alpha_{n+1}] = h_1 * k_1 * h_2 * k_2 * \cdots * h_5 * k_5 = 1 \in \pi_1(M_{n+1}) = \pi_1(M_n) * \pi_1(N_{n+1,n+1})$$

might be witnessed by β_{n+1} as

$$h_1(k_1(h_2(k_2)h_3)k_3(h_4)k_4)h_5(k_5) = 1.$$

The induced cancellation pattern for

$$[\alpha_n] = f_{n+1,n\#}([\alpha_{n+1}]) = h_1 * 1 * h_2 * 1 * \cdots * 1 * h_5 * 1 = 1 \in \pi_1(M_n) * \{1\}$$

as obtained from $f_{n+1,n} \circ \beta_{n+1}$ would then be given by

$$h_1((h_2h_3)(h_4))h_5 = 1.$$

On the other hand, the map β_n might cancel $[\alpha_n]$ as $(h_1h_2)(h_3(h_4)h_5) = 1$. This is illustrated in Figure 3, which depicts the sets $\beta_n^{-1}(\partial D_n)$, $(f_{n+1,n} \circ \beta_{n+1})^{-1}(\partial D_n)$, and $\beta_{n+1}^{-1}(S_{n+1,n})$ (after general position) as dashed lines. If k_1 is not trivial and if k_3 does not cancel k_4 in $\pi_1(N_{n+1,n+1})$, then we cannot use any of the pieces of the map β_n to construct β.

As a remedy, we will repeatedly select subsequences until, at least approximately, all cancellation patterns are coherent. That is, until the sets $\beta_n^{-1}\big(\bigcup \mathcal{S}_n\big)$ are approximately nested with increasing n. Once this is achieved, the union of these cancellation patterns will produce a limiting pattern $\mathcal{P}$ of possibly infinitely many straight line segments in D^2 whose interiors are pairwise disjoint and whose endpoints lie in S^1. Each segment of $\mathcal{P}$, at least approximately, will then map under

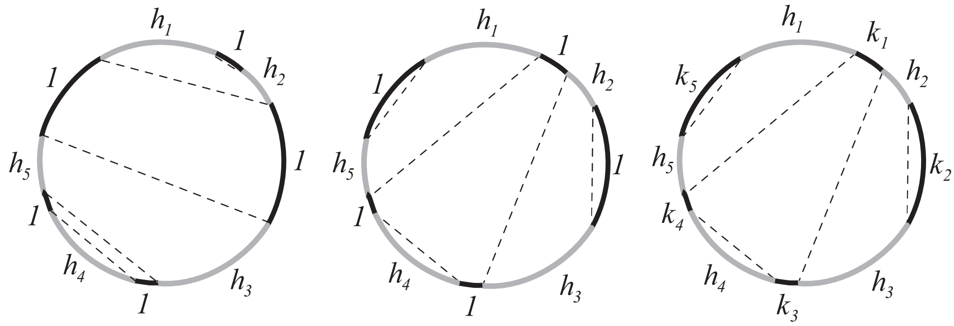

Figure 3. Incompatible cancellations.

some β_n into some $S_{n,i}$. Note that we must accommodate the possibility that the image of α_n meets some $S_{n,i}$ in infinitely many points. This effect is accounted for by a possible increase of segments $c \subseteq \beta_m^{-1}(\bigcup \mathcal{S}_m)$ for which $\beta_m(c) \subseteq S_{m,i}$, as m increases. We will then define the map $\beta : D^2 \to Z$ in two stages.

First, we will extend $\alpha : S^1 \to Z$ to a map $\beta : S^1 \cup \mathcal{P} \to Z$. If $\dim Z = 2$, this can be done so that each segment of $\mathcal{P}$ maps to a local geodesic of that simple closed curve of Z which corresponds to the appropriate ∂D_i. If $\dim Z \geq 3$, any coherent extension into the spheres of Z corresponding to ∂D_i will do, so long as the extension to a segment does not deviate too much from the image of its endpoints. All this must be done with sufficient care, so as to make the map $\beta : S^1 \cup \mathcal{P} \to Z$ uniformly continuous, which will allow us to extend it to the closure of its domain.

Next, we will focus on the components of the subset of D^2 on which the map β is not yet defined. We shall call these components *holes*. The boundary of a hole H, denoted by $\operatorname{bdy} H$, is a simple closed curve, which either maps to a singleton under β, in which case we extend β trivially over the closure $\operatorname{cl} H$, or $p_n \circ \beta(\operatorname{bdy} H) \subseteq N_n^*$ for some n, where

$$N_1^* = M_1 \setminus \left(\bigcup_{m\geq 3} E_{m,1}\right)$$

and

$$N_n^* = M_n \setminus \Big(h_{n,n-1}^{-1}(M_{n-1} \setminus D_{n-1}) \cup \left(\bigcup_{m\geq n+2} E_{m,n}\right)\Big) \text{ for } n \geq 2.$$

The map $p_n \circ \beta : \operatorname{bdy} H \to N_n^* \subseteq M_n$ can be extended to a map $p_n \circ \beta : \operatorname{cl} H \to M_n$ so long as the hole H is sufficiently "thin", because M_n is an ANR. For the moment, assume that $\dim Z \geq 3$. The map $p_n \circ \beta : \operatorname{cl} H \to M_n$ can then be cut off at $S_{n,n-1} = h_{n,n-1}^{-1}(\partial D_{n-1})$ and pushed off $\bigcup_{m\geq n+2} E_{m,n}$. This allows us to extend the map $p_n \circ \beta : \operatorname{bdy} H \to N_n^*$ to a map $p_n \circ \beta : \operatorname{cl} H \to N_n^*$. Since N_n^* naturally embeds in Z, we have an extension of $\beta : \operatorname{bdy} H \to Z$ to $\beta : \operatorname{cl} H \to Z$. For each n, there will be *finitely* many maps $p_n \circ \beta : \operatorname{bdy} H \to N_n^* \subseteq M_n$ for which the hole H is not thin enough to make this argument. In those cases, some $f_{m,n} \circ \beta_m : D^2 \to M_n$, with sufficiently large m, will be witness to the fact that $p_n \circ \beta : \operatorname{bdy} H \to M_n$ is null-homotopic after all. This is due to the approximate nestedness of the cancellation patterns $\beta_n^{-1}\big(\bigcup \mathscr{S}_n\big)$. Since for sufficiently large n the subset of Z which is homeomorphic to N_n^* is arbitrarily small, this procedure guarantees continuity of the resulting map $\beta : D^2 \to Z$.

If $\dim Z = 2$, the above process requires a little bit more care and is helped by the assumption that the defining tree is well-balanced. Specifically, the sets N_n^* will either be ANRs or one-dimensional. In the former case, we can adapt the argument we just made, and in the latter case, we make use of Remark 3.2(i).

The remainder of Section 6 contains the necessary details.

General position and other approximations. We choose subsets $S^+_{m,n} \subseteq M_m$ such that

(1) $S^+_{m,n}$ is the image of an embedding of $S_{m,n} \times [-1, 1]$ into $M_m \setminus D_m$, under which $S_{m,n} \times \{0\}$ is mapped onto $S_{m,n}$;
(2) the collections $\mathscr{S}^+_m = \{S^+_{m,1}, S^+_{m,2}, \ldots, S^+_{m,m-1}\}$ consist of pairwise disjoint sets;
(3) $h_{m+1,m}(S^+_{m+1,n}) \subseteq \operatorname{int} S^+_{m,n}$;
(4) $\bigcap_{m>n} h_{m,n}(S^+_{m,n}) = \partial D_n$.

For every $n \geq 2$, we then choose finite collections $\mathscr{A}_n$ and $\mathscr{G}_n$ of arcs in S^1 such that

(5) the elements of $\mathscr{A}_n$ are pairwise disjoint;
(6) the elements of $\mathscr{G}_n$ are pairwise disjoint;
(7) $S^1 = \big(\bigcup \mathscr{A}_n\big) \cup \big(\bigcup \mathscr{G}_n\big)$;
(8) $\big(\bigcup \mathscr{A}_n\big) \cap \big(\bigcup \mathscr{G}_n\big)$ is finite;
(9) $\alpha_n\big(\bigcup \mathscr{A}_n\big) \subseteq \bigcup \{\operatorname{int} S^+ \mid S^+ \in \mathscr{S}^+_n\}$;
(10) $\alpha_n\big(\bigcup \mathscr{G}_n\big) \subseteq M_n \setminus h_{n+1,n}\big(\bigcup \mathscr{S}^+_{n+1} \setminus \{S^+_{n+1,n}\}\big)$.

Remark 6.1. The maps $\alpha_n|_{\mathscr{G}_n}$ and $\alpha_n|_{\mathscr{A}_n}$ represent $[\alpha_n] \in \pi_1(M_n)$ as an element of the decomposition given in Remark 1.2.

On each M_n we fix a metric d_n. Put $\epsilon_1 = 1$. Inductively, we choose $\epsilon_n > 0$ so that $d_n\left(f_{m,n}(x), f_{m,n}(y)\right) < \min\left\{\frac{1}{m}, \epsilon_n\right\}$ whenever $n < m$ and $d_m(x, y) < \epsilon_m$.

Notation. Suppose X is a topological space which is connected, locally path connected, and semilocally simply connected. Fix a cover $\mathscr{U}$ of X consisting of open path connected subsets U of X for which the inclusion induced homomorphism $\pi_1(U) \to \pi_1(X)$ is trivial. Consider the following category $\Upsilon(\mathscr{U})$. As objects we take the elements of $\mathcal{U}$. For the set $\mathrm{Hom}(U, V)$ of morphisms $U \xrightarrow{[\tau]} V$ we take equivalence classes $[\tau]$ of paths $\tau : [0, 1] \to X$ with $\tau(0) \in U$ and $\tau(1) \in V$, where $\tau \sim \mu$ if and only if there is a homotopy $H : [0, 1] \times [0, 1] \to X$ such that $H(s, 0) = \tau(s)$, $H(s, 1) = \mu(s)$, $H(0, t) \in U$ and $H(1, t) \in V$ for all $s, t \in [0, 1]$. If $\tau : [a, b] \to X$ is a path whose domain is an arbitrary compact interval, then $[\tau]$ will of course denote the equivalence class of the path $\tau' : [0, 1] \to X$ given by $\tau'(t) = \tau(a + t(b - a))$. We compose morphisms

$$U \xrightarrow{[\tau]} V \quad \text{and} \quad V \xrightarrow{[\mu]} W$$

of this category as follows. Choose an arbitrary path $\gamma : [0, 1] \to V$ with $\gamma(0) = \tau(1)$ and $\gamma(1) = \mu(0)$ and put $[\tau][\mu] = [\tau \cdot \gamma \cdot \mu]$, where $\tau \cdot \gamma \cdot \mu$ denotes the usual concatenation of the three paths. Well-definition and associativity is checked easily. The equivalence class containing a constant path in U yields an identity morphism

$$U \xrightarrow{\mathbf{1}_U} U.$$

Also, for every morphism $U \xrightarrow{[\tau]} V$ we have a morphism $V \xrightarrow{[\overline{\tau}]} U$, given by $\overline{\tau}(t) = \tau(1-t)$, such that $[\tau][\overline{\tau}] = \mathbf{1}_U$ and $[\overline{\tau}][\tau] = \mathbf{1}_V$. Hence, the category $\Upsilon(\mathcal{U})$ is a groupoid. For a fixed $U_0 \in \mathcal{U}$, we obtain a group $\Pi_1(\mathcal{U}, U_0) = \mathrm{Hom}(U_0, U_0)$.

We leave the straightforward proof of the next lemma to the reader.

Lemma 6.2. *Let (X, d) be a connected, locally path connected, semilocally simply connected, and compact metric space. Then there is a finite cover $\mathcal{U}$ of X consisting of open path connected subsets such that every loop, which lies in the union of two elements of $\mathcal{U}$ contracts in X. Let $\epsilon > 0$ be a Lebesgue number for any such cover $\mathcal{U}$. If*

$$U \xrightarrow{[\tau]} V \quad \textit{and} \quad U \xrightarrow{[\mu]} V$$

are morphisms of $\Upsilon(\mathcal{U})$ such that $d(\tau(t), \mu(t)) < \epsilon$ for all $t \in [0, 1]$, then these two morphisms $[\tau]$ and $[\mu]$ agree. Moreover, for every $x_0 \in U_0 \in \mathcal{U}$, the function $\zeta : \pi_1(X, x_0) \to \Pi_1(\mathcal{U}, U_0)$ given by $\zeta([\tau]) = [\tau]$, is an isomorphism. □

Convention. We want to choose ϵ_n sufficiently small so that $3\epsilon_n$ is a Lebesgue number for some covers $\mathcal{U}_n$, $\mathcal{V}_{n,i}$, $\mathcal{V}^+_{n,i}$ of M_n, $S_{n,i}$, $S^+_{n,i}$, respectively, which are chosen as in Lemma 6.2. Whenever convenient, and without further notice, we will perform computations in $\Pi_1(\mathcal{U}_n, *)$, $\Pi_1(\mathcal{V}_{n,i}, *)$, $\Pi_1(\mathcal{V}^+_{n,i}, *)$ instead of $\pi_1(M_n, *)$, $\pi_1(S_{n,i}, *)$, $\pi_1(S^+_{n,i}, *)$, respectively. That is, we might perform groupoid computations in $\Upsilon(\mathcal{U}_n)$, $\Upsilon(\mathcal{V}_{n,i})$, $\Upsilon(\mathcal{V}^+_{n,i})$, respectively.

We choose maps $\beta'_n : D^2 \to M_n$ such that $\beta'_1 = \beta_1$ and, denoting $\alpha'_n = \beta'_n|_{S^1}$, such that for $n \geq 2$:

(11) β'_n is in general position with respect to $\mathcal{S}_n = \{S_{n,1}, S_{n,2}, \cdots, S_{n,n-1}\}$;

(12) α_n and α'_n differ only over $\bigcup \{\operatorname{int} h_{n+1,n}(S^+_{n+1,i}) \mid i = 1, 2, \ldots, n-1\}$;

(13) $d_n(\alpha_n, \alpha'_n) < \epsilon_n$.

Hence,

(14) $\alpha'_n\big(\bigcup \mathcal{A}_n\big) \subseteq \bigcup \{\operatorname{int} S^+ \mid S^+ \in \mathcal{S}^+_n\}$;

(15) $\alpha'_n\big(\bigcup \mathcal{G}_n\big) \subseteq M_n \setminus h_{n+1,n}\big(\bigcup \mathcal{S}^+_{n+1} \setminus \{S^+_{n+1,n}\}\big)$.

The limiting cancellation pattern. (This procedure was inspired by [Cannon and Conner 1998].) We may assume that the set $\bigcup\{\beta'^{-1}_n(S) \mid S \in \mathcal{S}_n\}$, for $n \geq 2$, consists of finitely many disjoint arcs, whose collection we will denote by $\mathcal{B}_n$. (We can eliminate possible simple closed curves, one innermost circle at a time, because the image of such a circle lies in some $S_{n,i}$, where it must be null-homotopic. This is also true in dimension 2, because $M_1, M_2, \ldots$ is a sequence of *distinct* closed surfaces. Therefore, as usual, we can cut off the map at this circle, cap it within $S_{n,i}$, and move this portion of the map off of $S_{n,i}$.) The endpoints of the elements of $\mathcal{B}_n$ lie in S^1. In fact, we may arrange that $\mathcal{B}_n$ is a finite collection of disjoint straight line segments in D^2, whose endpoints lie in $\bigcup \mathcal{A}_n$. In the case of surfaces, we also arrange that for each $c \in \mathcal{B}_n$ with $\beta'_n(c) \subseteq S_{n,i}$, the map $f_{n,i} \circ \beta'_n|_c : c \to \partial D_i$ is a local geodesic, as measured by some fixed homeomorphism $\rho_i : \partial D_i \xrightarrow{\sim} S^1$.

Lemma 6.3. *Suppose $i < n < k \leq m$. Let $l \in \mathcal{A}_k$ and $c \in \mathcal{B}_m$ be such that $l \cap \partial c \neq \varnothing$ and $\beta'_m(c) \subseteq S_{m,i}$. Then $\alpha_k(l) \subseteq \operatorname{int} S^+_{k,i}$ and there is exactly one $q \in \mathcal{A}_n$ with $l \subseteq q$ and $\alpha_n(q) \subseteq \operatorname{int} S^+_{n,i}$.*

Proof. Since $\alpha'_m(\partial c) = \beta'_m(\partial c) \subseteq S_{m,i} \subseteq \operatorname{int} h_{m+1,m}(S^+_{m+1,i})$, property (12) implies that $\alpha_m(\partial c) \subseteq \operatorname{int} h_{m+1,m}(S^+_{m+1,i})$. Applying $h_{m,k}$ to this inclusion and using (3), we obtain $\alpha_k(\partial c) \subseteq \operatorname{int} h_{k+1,k}(S^+_{k+1,i}) \subseteq \operatorname{int} S^+_{k,i}$. Since $l \cap \partial c \neq \varnothing$, it follows from (2) and (9) that $\alpha_k(l) \subseteq \operatorname{int} S^+_{k,i}$. Applying $h_{k,n}$, this yields $\alpha_n(l) \subseteq \operatorname{int} h_{n+1,n}(S^+_{n+1,i})$. Consequently, because of (5)–(10), there is $q \in \mathcal{A}_n$ with $l \subseteq q$ and $\alpha_n(q) \subseteq \operatorname{int} S^+_{n,i}$. □

Using Lemma 6.3 as a selection principle, we now single out cancellation patterns for each $[\alpha_n] \in \pi_1(M_n)$. For every pair $m \geq n \geq 2$ we select a subset $\mathcal{B}_{m,n} \subseteq \mathcal{B}_m$ with the following properties:

(16) For every $d \in \mathcal{B}_{m,n}$ there is exactly one $i < n$ such that $\beta'_m(d) \subseteq S_{m,i}$.

(17) If there is a $c \in \mathcal{B}_m$ whose endpoints lie in two distinct elements $q_1, q_2 \in \mathcal{A}_n$, then there is exactly one $d \in \mathcal{B}_{m,n}$ having one endpoint in each of q_1 and q_2; moreover, $\beta'_m(d) \subseteq S_{m,i}$ if and only if $\alpha_n(q_1) \subseteq \operatorname{int} S^+_{n,i}$ and $\alpha_n(q_2) \subseteq \operatorname{int} S^+_{n,i}$.

(18) There is no $d \in \mathcal{B}_{m,n}$ having both endpoints in the same element of $\mathcal{A}_n$.

(19) If $n \leq k \leq m$, then $\mathcal{B}_{m,n} \subseteq \mathcal{B}_{m,k}$.

Remark 6.4. $\mathcal{B}_{m,n}$ is one possible cancellation pattern for $[\alpha_n] = 1 \in \pi_1(M_n)$ (as represented by $\alpha_n|_{\mathcal{G}_n}$ and $\alpha_n|_{\mathcal{A}_n}$), namely the one induced by the cancellation pattern $\mathcal{B}_{m,m}$ of $[\alpha_m] = 1 \in \pi_1(M_m)$ (as represented by $\alpha_m|_{\mathcal{G}_m}$ and $\alpha_m|_{\mathcal{A}_m}$). However, the cancellation patterns $\mathcal{B}_{m,n}$ and $\mathcal{B}_{k,n}$ for $[\alpha_n]$ need not be compatible.

We shall call $\mathcal{B}_{m,n}$ and $\mathcal{B}_{k,n}$ *equivalent* if one can be obtained from the other by moving the endpoints of their segments within the arcs of $\mathcal{A}_n$. Clearly, for every $n \geq 2$, infinitely many of $(\mathcal{B}_{m,n})_{m \geq n}$ are equivalent. So, we can select an increasing sequence $(m_k)_k$ of natural numbers such that for all $n \geq 2$, all of $(\mathcal{B}_{m_k,n})_{m_k \geq n}$ are equivalent. We can go even further, and assume that each sequence $(\mathcal{B}_{m_k,n})_{m_k \geq n}$ converges to a finite collection $\mathcal{P}_n$ of straight line segments in D^2, whose endpoints lie on S^1 and whose interiors are pairwise disjoint. Finally, we define $\mathcal{P} = \bigcup_{n \geq 2} \mathcal{P}_n$.

From (19) we get:

Lemma 6.5. *If $n \leq k$, then $\mathcal{P}_n \subseteq \mathcal{P}_k$. In particular, $\mathcal{P}$ is a (possibly infinite) collection of straight line segments, whose endpoints lie on S^1 and whose interiors are pairwise disjoint.* □

The limiting pattern $\mathcal{P}$ has the following separation property:

Lemma 6.6. *If $x, y \in S^1$ are such that $\alpha_n(x)$ and $\alpha_n(y)$ are separated by an element of $\mathcal{S}_n$ in M_n, for some n, then there is a $c \in \mathcal{P}$ such that c separates x from y in D^2.*

Proof. Say, $S_{n,i}$ $(i < n)$ separates $\alpha_n(x)$ from $\alpha_n(y)$ in M_n. Then ∂D_i separates $\alpha_i(x)$ from $\alpha_i(y)$ in M_i. So, by (3) and (4), there is an $N > n$ such that if $m \geq N$, then $h_{m,i}(S^+_{m,i})$ separates $\alpha_i(x)$ from $\alpha_i(y)$ in M_i. Hence, if $m \geq N$, then $S^+_{m,i}$ separates $\alpha_m(x)$ from $\alpha_m(y)$ in M_m, and consequently $S^+_{m,i}$ separates $\alpha'_m(x)$ from $\alpha'_m(y)$ in M_m, by (3) and (12). Now suppose $m_k > N$. Then $S_{m_k,i}$ separates $\alpha'_{m_k}(x)$ from $\alpha'_{m_k}(y)$ in M_{m_k}. So, if q is any arc in D^2 from x to y, we must have $\beta'_{m_k}(q) \cap S_{m_k,i} \neq \varnothing$. Consequently, there is a $c \in \mathcal{B}_{m_k}$ such that $\beta'_{m_k}(c) \subseteq S_{m_k,i}$ and such that c separates x from y in D^2. Let the endpoints of c be in the elements $l_1, l_2 \in \mathcal{A}_{m_k}$. By Lemma 6.3, there are $q_1, q_2 \in \mathcal{A}_N$ with $l_1 \subseteq q_1$, $l_2 \subseteq q_2$, $\alpha_N(q_1) \subseteq \operatorname{int} S^+_{N,i}$, and $\alpha_N(q_2) \subseteq \operatorname{int} S^+_{N,i}$. Hence, $\alpha'_N(q_1) \subseteq \operatorname{int} S^+_{N,i}$ and $\alpha'_N(q_2) \subseteq \operatorname{int} S^+_{N,i}$, so that x and y are neither in q_1 nor in q_2, since $S^+_{N,i}$ separates $\alpha'_N(x)$ from $\alpha'_N(y)$. In particular, $q_1 \neq q_2$ and each of the two components of $S^1 \setminus (q_1 \cup q_2)$ contains one of x or y. Therefore, by (17), for every $m_k > N$, there is exactly one segment $d \in \mathcal{B}_{m_k,N}$ having one endpoint in each of q_1 and q_2, and any such segment must separate x from y in D^2. Now, the sequence $(\mathcal{B}_{m_k,N})_{m_k > N}$, which consists of equivalent patterns, converges to $\mathcal{P}_N$. We conclude that there is a $c \in \mathcal{P}_N \subseteq \mathcal{P}$, which separates x from y in D^2. □

In order to simplify subscripts later on, we now replace β'_m by $f_{m_{k+1},m} \circ \beta'_{m_{k+1}}$ and $\mathcal{B}_{m,n}$ by $\mathcal{B}_{m_{k+1},n}$, respectively, whenever $m_k < m < m_{k+1}$. Consequently, we may assume that every $(\mathcal{B}_{m,n})_{m \geq n}$ is a sequence of equivalent patterns converging to $\mathcal{P}_n$. Clearly, this substitution does not affect Properties (16)–(19), Lemma 6.5, or Lemma 6.6. Notice also that, because of our choice of $\epsilon_{m_{k+1}}$ on page 61, Properties (11)–(13) are still valid. Since we will no longer need the collections $\mathcal{A}_n$ and $\mathcal{G}_n$, we dispose of them at this point. Finally, we verify

Lemma 6.7. *For every $c \in \mathcal{P}_n$ there is exactly one $i \in \{1, 2, \ldots, n-1\}$ such that for each $m \geq n$ there is a $c_m \in \mathcal{B}_{m,n}$ such that the following properties hold*:

(i) $\beta'_m(c_m) \subseteq S_{m,i}$.

(ii) $f_{m,n} \circ \beta'_m(c_m) \subseteq S_{n,i}$.

(iii) $c_m \to c$ *as* $m \to \infty$.

(iv) $\alpha_n(\partial c) \subseteq S_{n,i}$.

(v) $\alpha_i(\partial c) \subseteq \partial D_i$.

Proof. By construction, there is exactly one $i < n$ such that there are $c_m \in \mathcal{B}_{m,n}$ with $\beta'_m(c_m) \subseteq S_{m,i}$ and $c_m \to c$. Item (ii) follows from (i) and (v) follows from (iv). So, we only have to prove (v). To this end, write $\partial c_m = \{x_m, y_m\}$ and $\partial c = \{x, y\}$ such that $x_m \to x$ and $y_m \to y$. For all $m \geq n$, we have $f_{m,i} \circ \alpha'_m(\partial c_m) = f_{m,i} \circ \beta'_m(\partial c_m) \subseteq \partial D_i$. Let $\epsilon > 0$ be given. Choose $m \geq n$ such that $d_i(\alpha_i(x_m), \alpha_i(x)) < \epsilon/2$ and $1/m < \epsilon/2$. Then, since $d_m(\alpha'_m, \alpha_m) < \epsilon_m$, we get

$$\begin{aligned}
&d_i\bigl(f_{m,i} \circ \alpha'_m(x_m), \alpha_i(x)\bigr) \\
&\qquad = d_i\bigl(f_{m,i} \circ \alpha'_m(x_m),\, f_{m,i} \circ \alpha_m(x)\bigr) \\
&\qquad \leq d_i\bigl(f_{m,i} \circ \alpha'_m(x_m),\, f_{m,i} \circ \alpha_m(x_m)\bigr) + d_i\bigl(f_{m,i} \circ \alpha_m(x_m),\, f_{m,i} \circ \alpha_m(x)\bigr) \\
&\qquad \leq 1/m + d_i\bigl(\alpha_i(x_m), \alpha_i(x)\bigr) \\
&\qquad \leq \epsilon/2 + \epsilon/2 = \epsilon.
\end{aligned}$$

Hence $\alpha_i(x) \in \partial D_i$. Similarly, $\alpha_i(y) \in \partial D_i$. $\square$

Extending the map over the limiting cancellation pattern.

Lemma 6.8. *Suppose dim $Z \geq 3$. Then for each $i \in \mathbb{N}$, there is a map $\gamma_i : D^2 \to Z$ such that*

(a) *$\gamma_i(x) = \alpha(x)$ for all $x \in S^1$ with $\alpha_i(x) \in \partial D_i$*;

(b) *$p_i \circ \gamma_i(x) \in \partial D_i$ for all $x \in D^2$*;

(c) *if $(c_k)_k$ is a sequence of segments in $\mathcal{P}$ converging to a straight line segment $c \subseteq D^2$ with $\partial c = \{x, y\} \subseteq S^1$ and $\alpha_i(x) = \alpha_i(y) \in \partial D_i$, then $\gamma_i(c)$ is a singleton, namely $\gamma_i(c) = \alpha(\partial c)$.*

Proof. Fix $i \in \mathbb{N}$. Consider the set $\mathcal{P}^*$ of all straight line segments of $c \subseteq D^2$ with $\partial c = \{x, y\} \subseteq S^1$ such that $\alpha_i(x) = \alpha_i(y) \in \partial D_i$ and such that there is a sequence $(c_k)_k$ of segments in $\mathcal{P}$ with $c_k \to c$. Since the interiors of the elements of $\mathcal{P}$ are pairwise disjoint, the same is true for $\mathcal{P}^*$. The union $\bigcup \mathcal{P}^*$ is compact and $\alpha(\partial c)$ is a singleton for every $c \in \mathcal{P}^*$. We only have to show that there is a continuous extension $\gamma_{i,i} : D^2 \to \partial D_i$ of the restriction $\alpha_i|_{\alpha_i^{-1}(\partial D_i)} : \alpha_i^{-1}(\partial D_i) \to \partial D_i$ such that $\gamma_{i,i}(c) = \alpha_i(\partial c)$ for all $c \in \mathcal{P}^*$. Because then we can define $\gamma_{n,i} = h_{n,i}^{-1} \circ \gamma_{i,i}$ if $n > i$ and $\gamma_{n,i} = f_{i,n} \circ \gamma_{i,i}$ if $n < i$, so that $\gamma_i = (\gamma_{n,i})_n$ is the desired map. We put $\gamma_{i,i}(x) = \alpha_i(x)$ if $x \in S^1$ with $\alpha_i(x) \in \partial D_i$ and $\gamma_{i,i}(c) = \alpha_i(\partial c)$ if $c \in \mathcal{P}^*$. Clearly, $\gamma_{i,i} : \alpha_i^{-1}(\partial D_i) \cup \left(\bigcup \mathcal{P}^*\right) \to \partial D_i$ is a continuous map. Since $\alpha_i^{-1}(\partial D_i) \cup \left(\bigcup \mathcal{P}^*\right)$ is a closed subset of D^2 and $\dim D^2 = 2 \leq \dim \partial D_i$, it follows from the mapping-into-spheres definition of dimension [van Mill 1989, Theorem 4.6.4] that we can extend $\gamma_{i,i}$ continuously to all of D^2. □

Remark 6.9. If we denote the boundary of an arc I as $\partial I = \{x, y\}$ we shall implicitly assume that the arc is directed from x to y.

Lemma 6.10. *Suppose* $\dim Z = 2$ *and* $1 \leq i < k \leq m$. *Let* $c_1 \in \mathcal{B}_k$ *and* $c_2 \in \mathcal{B}_m$ *be such that* $\beta'_k(c_1) \subseteq S_{k,i}$ *and* $\beta'_m(c_2) \subseteq S_{m,i}$. *Denote* $\partial c_i = \{x_i, y_i\}$. *Let* $I_i \subseteq S^1$ *be counterclockwise arcs with* $\partial I_i = \{y_i, x_i\}$, *respectively. Suppose that* $I_1 \cap I_2$ *is an arc, and that each* (≤ 2) *component of* $(I_2 \setminus I_1) \cup (I_1 \setminus I_2)$ *maps under* α_{i+1} *to a subset of* M_{i+1} *of diameter less than* ϵ_{i+1}. *Then* $[f_{k,i} \circ \beta'_k|_{c_1}]\overline{[f_{m,i} \circ \beta'_m|_{c_2}]} = 1 \in \pi_1(\partial D_i)$.

Proof. The following computations are in accordance with the convention on page 62. In particular, since all relevant paths are within the required tolerance, we can use Lemma 6.2. We have

$$\begin{aligned}
[f_{k,i+1} \circ \beta'_k|_{c_1}][\alpha_{i+1}|_{I_1}] &= [f_{k,i+1} \circ \beta'_k|_{c_1}][f_{k,i+1} \circ \alpha_k|_{I_1}] \\
&= [f_{k,i+1} \circ \beta'_k|_{c_1}][f_{k,i+1} \circ \alpha'_k|_{I_1}] \\
&= [f_{k,i+1} \circ \beta'_k|_{c_1}][f_{k,i+1} \circ \beta'_k|_{I_1}] \\
&= [f_{k,i+1} \circ \beta'_k|_{c_1 \cup I_1}] \\
&= 1 \in \pi_1(M_{i+1}),
\end{aligned}$$

since $f_{k,i+1} \circ \beta'_k(D^2) \subseteq M_{i+1}$. Similarly, $[f_{m,i+1} \circ \beta'_m|_{c_2}][\alpha_{i+1}|_{I_2}] = 1 \in \pi_1(M_{i+1})$. Hence,

$$\begin{aligned}
[f_{k,i+1} \circ \beta'_k|_{c_1}]\overline{[f_{m,i+1} \circ \beta'_m|_{c_2}]} &= [f_{k,i+1} \circ \beta'_k|_{c_1}][\alpha_{i+1}|_{I_1}]\overline{[\alpha_{i+1}|_{I_2}]}\,\overline{[f_{m,i+1} \circ \beta'_m|_{c_2}]} \\
&= 1 \in \pi_1(M_{i+1}).
\end{aligned}$$

Therefore, the injective homomorphism $\mathrm{incl}_\# : \pi_1(S_{i+1,i}) \to \pi_1(M_{i+1})$ takes

$$[f_{k,i+1} \circ \beta'_k|_{c_1}]\overline{[f_{m,i+1} \circ \beta'_m|_{c_2}]}$$

to 1. This completes the proof, because $f_{i+1,i}|_{S_{i+1,i}}$ induces an isomorphism

$$\pi_1(S_{i+1,i}) \to \pi_1(\partial D_i).$$ □

Lemma 6.11. *Suppose* $\dim Z = 2$. *Let* $c \in \mathcal{P}_n$, $i \in \{1, 2, \ldots, n-1\}$, $c_m \in \mathcal{B}_{m,n}$ $(m \geq n)$ *with* $\beta'_m(c_m) \subseteq S_{m,i}$ *and* $c_m \to c$ *as* $m \to \infty$. *Then* $f_{m,i} \circ \beta'_m|_{c_m}$ *converges to a local geodesic* $\beta_c : c \to \partial D_i$ (*which only depends on* c) *as* $m \to \infty$ *with* $\beta_c|_{\partial c} = \alpha_i|_{\partial c}$.

Proof. Each $f_{m,i} \circ \beta'_m|_{c_m}$ is a local geodesic in ∂D_i. Say $\partial c_m = \{x_m, y_m\}$, $\partial c = \{x, y\}$, with $x_m \to x$ and $y_m \to y$. Then, as in the proof of Lemma 6.7, we get

$$d_i(f_{m,i} \circ \beta'_m(x_m), \alpha_i(x)) = d_i(f_{m,i} \circ \alpha'_m(x_m), \alpha_i(x)) \to 0$$

as $m \to \infty$. The same holds when x_m is replaced by y_m and x is replaced by y, respectively. So, by Lemma 6.10, $f_{m,i} \circ \beta'_m|_{c_m}$ converges to a unique local geodesic $\beta_c : c \to \partial D_i$ with $\beta_c|_{\partial c} = \alpha_i|_{\partial c}$. □

We now begin to define the desired map $\beta : D^2 \to Z$.

Definition 6.12. We first define a function $\beta : S^1 \cup \left(\bigcup \mathcal{P}\right) \to Z$. For $x \in S^1$, we define $\beta(x) = \alpha(x)$. For $x \in c \in \mathcal{P}_n$, say $\alpha_n(\partial c) \subseteq S_{n,i}$ with $i \in \{1, 2, \ldots, n-1\}$, we consider two cases:

(A) If $\dim Z \geq 3$, we define $\beta(x) = \gamma_i(x)$, where γ_i is the map of Lemma 6.8.

(B) If $\dim Z = 2$, we define $\beta|_c$ via its projections $(p_k \circ \beta|_c : c \to M_k)_k$. Specifically, we define $p_i \circ \beta|_c$ to be the local geodesic β_c of Lemma 6.11; for $k > i$ we put $p_k \circ \beta|_c = h_{k,i}^{-1} \circ \beta|_c$; and for $k < i$ we let $p_k \circ \beta|_c = f_{i,k} \circ \beta|_c$.

In either case, we have $p_n \circ \beta(c) \subseteq S_{n,i}$ and $\beta|_{\partial c} = \alpha|_{\partial c}$.

Remark 6.13. In the 2-dimensional case (B), we make the following observation, based on Lemma 6.10. For each $i \in \mathbb{N}$ there is a $\delta_i > 0$ such that if $d_1, d_2 \in \mathcal{P}$ with $\alpha_i(\partial d_1) \subseteq \partial D_i$ and $\alpha_i(\partial d_2) \subseteq \partial D_i$ and $d(d_1, d_2) < \delta_i$, then $[p_i \circ \beta|_{d_1}][\overline{p_i \circ \beta|_{d_2}}] = 1 \in \pi_1(\partial D_i)$.

Lemma 6.14. *The function* $\beta : S^1 \cup \left(\bigcup \mathcal{P}\right) \to Z$ *is uniformly continuous.*

Proof. We need to show that $p_n \circ \beta : S^1 \cup (\bigcup \mathcal{P}) \to M_n$ is uniformly continuous for every n. Fix n and assume, to the contrary, that there is an $\epsilon > 0$ and two sequences $(x_k)_k$ and $(\tilde{x}_k)_k$ in $S^1 \cup \left(\bigcup \mathcal{P}\right)$ such that $d(x_k, \tilde{x}_k) \to 0$ as $k \to \infty$ but $d_n(p_n \circ \beta(x_k), p_n \circ \beta(\tilde{x}_k)) \geq \epsilon$ for all k. Since $p_n \circ \beta|_{S^1} = \alpha_n$ is uniformly continuous, and since we always can change to subsequences, we may assume without loss of generality that, say, $x_k \notin S^1$ for all k. For each k choose a $d_k \in \mathcal{P}$ with $x_k \in d_k$. Say, $\partial d_k = \{y_k, z_k\} \subseteq S^1$.

(A) First assume that $\dim Z \geq 3$. Then, by definition, $\beta(t) = \gamma_i(t)$ whenever $t \in c \in \mathcal{P}$ with $\alpha_i(\partial c) \subseteq \partial D_i$. Since we may change to subsequences, in order to arrive at the desired contradiction, we only have to consider two cases: either $\tilde{x}_k \in S^1$ for all k, or $\tilde{x}_k \notin S^1$ for all k.

Case 1: $\tilde{x}_k \in S^1$ for all k. Taking subsequences, we may assume that $(\tilde{x}_k)_k$ converges to some $x \in S^1$. Then also $x_k \to x$ as $k \to \infty$. Interchanging y_k and z_k so as to arrange for $d(z_k, x) \leq d(y_k, x)$ for all k, we may assume that $z_k \to x$. Choose $N > n$ such that

(a) $\operatorname{diam} f_{m,n}(\partial D_m) < \epsilon/2$ for all $m \geq N$.

Choose $\delta > 0$ such that

(b) $d_n(p_n \circ \gamma_m(s), p_n \circ \gamma_m(t)) < \epsilon/2$ for all $m \in \{1, 2, \ldots, N-1\}$ and $s, t \in D^2$ with $d(s,t) < \delta$; and

(c) $d_n(\alpha_n(s), \alpha_n(t)) < \epsilon/2$ for all $s, t \in S^1$ with $d(s,t) < \delta$.

Choose $K \in \mathbb{N}$ such that for all $k \geq K$, we have $d(x_k, \tilde{x}_k) < \delta/2$ and $d(z_k, \tilde{x}_k) < \delta/2$. Now, fix any $k \geq K$. We claim that $d_n(p_n \circ \beta(x_k), p_n \circ \beta(z_k)) < \epsilon/2$. That will yield our contradiction in Case 1, because then, using (c),

$$\begin{aligned} d_n\big(p_n\circ\beta(x_k), p_n\circ\beta(\tilde{x}_k)\big) &\leq d_n\big(p_n\circ\beta(x_k), p_n\circ\beta(z_k)\big)+d_n\big(p_n\circ\beta(z_k), p_n\circ\beta(\tilde{x}_k)\big) \\ &= d_n\big(p_n\circ\beta(x_k), p_n\circ\beta(z_k)\big)+d_n\big(\alpha_n(z_k), \alpha_n(\tilde{x}_k)\big) \\ &< \epsilon/2+\epsilon/2 = \epsilon. \end{aligned}$$

In order to prove this claim, choose $m \in \mathbb{N}$ such that $\alpha_m(\partial d_k) \subseteq \partial D_m$. If $m \geq N$, then the claim follows from (a), since for every $t \in d_k$, we have $p_n \circ \beta(t) = p_n \circ \gamma_m(t) = f_{m,n}(p_m \circ \gamma_m(t))$ and $p_m \circ \gamma_m(t) \in \partial D_m$. If $m < N$, then the claim follows from (b), because $d(x_k, z_k) < \delta$ and $p_n \circ \beta|_{d_k} = p_n \circ \gamma_m|_{d_k}$.

Case 2: $\tilde{x}_k \notin S^1$ for all k. For each k choose $\tilde{d}_k \in \mathcal{P}$ such that $\tilde{x}_k \in \tilde{d}_k$. Say, $\partial\tilde{d}_k = \{\tilde{y}_k, \tilde{z}_k\} \subseteq S^1$. Without loss of generality, we may assume that $x_k \to x$, $\tilde{x}_k \to x$, $y_k \to y$, $\tilde{y}_k \to \tilde{y}$, $z_k \to z$, and $\tilde{z}_k \to \tilde{z}$ as $k \to \infty$, for some $x \in D^2$ and $y, \tilde{y}, z, \tilde{z} \in S^1$. There is no loss in generality to assume further that $z = \tilde{z}$. In fact, if $x \in S^1$, we may assume that $x = z$ and if $x \notin S^1$, we may assume that $y = \tilde{y}$. Say $\alpha_{m_k}(\partial d_k) \subseteq \partial D_{m_k}$ and $\alpha_{\tilde{m}_k}(\partial\tilde{d}_k) \subseteq \partial D_{\tilde{m}_k}$. Switching to further subsequences, we need to address only three subcases:

(i) $m_k \to \infty$ and $\tilde{m}_k \to \infty$ as $k \to \infty$;

(ii) $m_k \to \infty$ as $k \to \infty$ and $(\tilde{m}_k)_k$ is constant;

(iii) both $(m_k)_k$ and $(\tilde{m}_k)_k$ are constant.

First we look at subcase (i): Notice that

$$\begin{aligned} \operatorname{diam} p_n \circ \beta(d_k) &= \operatorname{diam} p_n \circ \gamma_{m_k}(d_k) \\ &= \operatorname{diam} f_{m_k,n}(\underbrace{p_{m_k} \circ \gamma_{m_k}(d_k)}_{\in\partial D_{m_k}}) \leq \operatorname{diam} f_{m_k,n}(\partial D_{m_k}), \end{aligned}$$

which approaches 0 as $k \to \infty$.

Similarly, diam $p_n \circ \beta(\tilde{d}_k) \to 0$. Therefore,

$$\begin{aligned}
&d_n\big(p_n \circ \beta(x_k),\, p_n \circ \beta(\tilde{x}_k)\big) \\
&\quad \le d_n\big(p_n \circ \beta(x_k),\, p_n \circ \beta(z_k)\big) + d_n\big(p_n \circ \beta(z_k),\, p_n \circ \beta(\tilde{z}_k)\big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + d_n\big(p_n \circ \beta(\tilde{z}_k),\, p_n \circ \beta(\tilde{x}_k)\big) \\
&\quad = d_n\big(p_n \circ \beta(x_k),\, p_n \circ \beta(z_k)\big) + d_n\big(\alpha_n(z_k), \alpha_n(\tilde{z}_k)\big) + d_n\big(p_n \circ \beta(\tilde{z}_k),\, p_n \circ \beta(\tilde{x}_k)\big),
\end{aligned}$$

which goes to 0 as $k \to \infty$. Contradiction.

Next, we turn to subcase (ii): Say $\tilde{m}_k = i$ for all k. If $x = z$, the final estimate of subcase (i) goes through and leads to the same contradiction, because

$$p_n \circ \beta|_{\tilde{d}_k} = p_n \circ \gamma_i|_{\tilde{d}_k}.$$

So, we assume that $y = \tilde{y}$. Let c be the straight line segment in D^2 with endpoints y and z. Then $\alpha_i(y) = \lim_{k\to\infty} \alpha_i(\tilde{y}_k) \in \partial D_i$ and $\alpha_i(z) = \lim_{k\to\infty} \alpha_i(\tilde{z}_k) \in \partial D_i$. Since $m_k \to \infty$, then diam $\alpha_i(\partial d_k) =$ diam $p_i \circ \beta(\partial d_k) \le$ diam $p_i \circ \beta(d_k) \to 0$ as $k \to 0$, as in Case 1. Hence $\alpha_i(y) = \lim_{k\to\infty} \alpha_i(y_k) = \lim_{k\to\infty} \alpha_i(z_k) = \alpha_i(z)$. Therefore, applying Lemma 6.8(c) to the sequence $\tilde{d}_k \to c$, we conclude that $\gamma_i(c) = \alpha(\partial c)$. Consequently, $p_i \circ \beta(\tilde{x}_k) = p_i \circ \gamma_i(\tilde{x}_k) \to p_i \circ \gamma_i(x) = \alpha_i(y)$ as $k \to \infty$. Since $p_i \circ \beta(\tilde{x}_k) \in \partial D_i$ for all k, we get that $p_n \circ \beta(\tilde{x}_k) \to \alpha_n(y)$ as $k \to \infty$. But also, diam $p_n \circ \beta(d_k) \to 0$, so that $d_n(p_n \circ \beta(x_k), p_n \circ \beta(y_k)) \to 0$ as $k \to \infty$. Since $p_n \circ \beta(y_k) = \alpha_n(y_k) \to \alpha_n(y)$ we obtain the contradictory statement that the sequences $(p_n \circ \beta(\tilde{x}_k))_k$ and $(p_n \circ \beta(x_k))_k$ have the same limit, namely $\alpha_n(y)$.

Now to subcase (iii): Since $\alpha_{m_k}(z_k) \in \partial D_{m_k}$ and $\alpha_{\tilde{m}_k}(\tilde{z}_k) \in \partial D_{\tilde{m}_k}$ for all k, and $\lim_{k\to\infty} \alpha(z_k) = \lim_{k\to\infty} \alpha(\tilde{z}_k) = \alpha(z)$, then there is an $i \in \mathbb{N}$ such that $m_k = \tilde{m}_k = i$ for all k. Hence, $\lim_{k\to\infty} p_i \circ \beta(x_k) = \lim_{k\to\infty} p_i \circ \gamma_i(x_k) = p_i \circ \gamma_i(x) = \lim_{k\to\infty} p_i \circ \gamma_i(\tilde{x}_k) = \lim_{k\to\infty} p_i \circ \beta(\tilde{x}_k)$. Yet another contradiction.

(B) Now we consider the case dim $Z = 2$. We break the analysis into the same cases.

Case 1: $\tilde{x}_k \in S^1$ for all k. As above we assume that $(\tilde{x}_k)_k$, $(x_k)_k$, and $(z_k)_k$ converge to the same point $x \in S^1$. We also assume that $y_k \to y$ for some $y \in S^1$.

If $m_k \to \infty$, we choose $N > n$ and $\delta > 0$ such that (a) and (c) above hold. We can establish the same claim as before and arrive at the same contradiction, because $p_n \circ \beta(d_k) \subseteq f_{m_k,n}(\partial D_{m_k})$.

Now suppose $(m_k)_k$ is constant, say $m_k = i$ for all k. Since $\alpha_i(\partial d_k) = \alpha_{m_k}(\partial d_k) \subseteq \partial D_{m_k} = \partial D_i$ for all k, then $\alpha_i(\{x, y\}) \subseteq \partial D_i$. By Remark 6.13, $\big(p_i \circ \beta|_{d_k}\big)_k$ converges to a local geodesic of ∂D_i with endpoints $\alpha_i(x)$ and $\alpha_i(y)$. Hence $p_i \circ \beta(x_k) \to \alpha_i(x)$ as $k \to \infty$. Since $\alpha_i(x)$ and all $p_i \circ \beta(x_k)$ are in ∂D_i, then $p_n \circ \beta(x_k) \to \alpha_n(x)$ as $k \to \infty$. This is a contradiction, since also $p_n \circ \beta(\tilde{x}_k) = \alpha_n(\tilde{x}_k) \to \alpha_n(x)$ as $k \to \infty$.

Case 2: $\tilde{x}_k \notin S^1$ for all k. One handles subcase (i) exactly as above: note that $p_n \circ \beta(d_k) \subseteq f_{m_k,n}(\partial D_{m_k})$. Subcases (ii) and (iii) are similar to, but simpler than, the above: say $\tilde{m}_k = i$ for all k. Again, by Remark 6.13, $\big(p_i \circ \beta|_{\tilde{d}_k}\big)_k$ converges to a well-defined local geodesic in ∂D_i whose endpoints are $\alpha_i(y)$ and $\alpha_i(z)$. $\square$

Definition 6.15. Let F denote the closure of $S^1 \cup \big(\bigcup \mathcal{P}\big)$ in D^2.

Corollary 6.16. *$\beta : S^1 \cup \big(\bigcup \mathcal{P}\big) \to Z$ extends continuously to a map $\beta : F \to Z$.* $\square$

Extending the map over the remaining holes. We will call a component of $D^2 \setminus F$ a *hole* and denote the collection of all holes by $\mathcal{H}$. If $\mathcal{P} = \varnothing$, then $\mathcal{H} = \{\mathring{D}^2\}$. Without loss of generality we may assume that this is not the case. The closure (in D^2) of each hole $H \in \mathcal{H}$ is a compact convex subset of the plane with nonempty interior and hence homeomorphic to a disk whose boundary bdy H is the union of a countable collection $\mathcal{C}_H$ of disjoint arcs in S^1 and a countable collection $\mathcal{L}_H$ of straight line segments in D^2 whose endpoints lie in S^1. Currently, β is defined on the boundary of each hole. Definition 6.24 below (for dim $Z \geq 3$) and Definition 6.27 (for dim $Z = 2$) will extend β over the closure of each hole, so that we at last obtain a function $\beta : D^2 \to Z$.

It will be convenient to have some measure of the size of a hole:

Definition 6.17. Let $H \in \mathcal{H}$. For every segment $c \in \mathcal{L}_H$ we consider the complementary arc $d = \mathrm{cl}\big((\mathrm{bdy}\ H) \setminus c\big)$ and fix unit-speed homeomorphisms $l_{H,c} : [0, 1] \to c$ and $r_{H,c} : [0, 1] \to d$ of opposite orientation in bdy H. Let $s \in \mathbb{N} \cup \{1/2\}$ be the largest s with the property that there is a $c \in \mathcal{L}_H$ with $d(l_{H,c}(x), r_{H,c}(x)) < 1/s$ for all $x \in [0, 1]$. We will call $1/s$ the *size* of H, denoted by size H, and c the *base* of H. If there is more than one possible base for a hole, then we fix one of them arbitrarily.

The proof of the next lemma is an elementary exercise.

Lemma 6.18. *Given any $\delta > 0$, there are only finitely many holes $H \in \mathcal{H}$ with* size $H \geq \delta$. $\square$

We will now sort our holes according to which parts of the tree their boundary is mapped.

Definition 6.19. We define the *location* of a hole $H \in \mathcal{H}$ as follows:

(i) loc $H = 1$ if there is no $n \in \mathbb{N}$ with $p_n \circ \beta(\mathrm{bdy}\ H) \subseteq D_n$;

(ii) loc $H = n+1$ if n is the largest positive integer such that $p_n \circ \beta(\mathrm{bdy}\ H) \subseteq D_n$;

(iii) loc $H = \infty$ if $p_n \circ \beta(\mathrm{bdy}\ H) \subseteq D_n$ for infinitely many positive integers n.

Recall the definition of the sets N_n^* from page 60.

Lemma 6.20. *Let $H \in \mathcal{H}$ and $n = \operatorname{loc} H$. If $n = \infty$, then $\beta(\operatorname{bdy} H)$ is a singleton. If n is finite then $p_n \circ \beta(\operatorname{bdy} H) \subseteq N_n^*$.*

Proof. Suppose $n = \infty$. Then there is an increasing sequence $(m_k)_k$ of positive integers such that $p_{m_k} \circ \beta(\operatorname{bdy} H) \subseteq D_{m_k}$ for all k. So, for each $i \in \mathbb{N}$ we have $\operatorname{diam} p_i \circ \beta(\operatorname{bdy} H) = \operatorname{diam} f_{m_k,i} \circ p_{m_k} \circ \beta(\operatorname{bdy} H) \leq \operatorname{diam} f_{m_k,i}(D_{m_k}) \to 0$ as $k \to \infty$. Therefore $p_i \circ \beta(\operatorname{bdy} H)$ is a singleton for every i and hence so is $\beta(\operatorname{bdy} H)$.

Now suppose n is finite. By construction, for each $c \in \mathcal{L}_H$, either there is an $i \in \mathbb{N}$ such that $p_i \circ \beta(c) \subseteq \partial D_i$, or $\beta(c)$ is a singleton. Hence, by Lemma 6.6, for each $i \in \mathbb{N}$, either $p_i \circ \beta(\operatorname{bdy} H) \subseteq D_i$ or $p_i \circ \beta(\operatorname{bdy} H) \subseteq M_i \setminus \mathring{D}_i$. Since $p_i \circ \beta(\operatorname{bdy} H) \not\subseteq D_i$ for all $i \geq n$, and $p_{n-1} \circ \beta(\operatorname{bdy} H) \subseteq D_{n-1}$ if $n \geq 2$, then $p_n \circ \beta(\operatorname{bdy} H) \subseteq N_n^*$. □

Lemma 6.21 [van Mill 1989, Theorem 5.1.1]. *Let Y be a compact (separable metric) ANR and $k \in \mathbb{N}$. There is a real number $\xi_{Y,k} > 0$ such that for every (separable metric) space X and any two maps $f, g : X \to Y$ with $d(f(x), g(x)) < \xi_{Y,k}$ for all $x \in X$, there is a homotopy $T : X \times [0, 1] \to Y$ such that $T(x, 0) = f(x)$, $T(x, 1) = g(x)$, and $\operatorname{diam} T(\{x\} \times [0, 1]) < 1/k$ for all $x \in X$.*

Addendum: Since Y is locally contractible, it can also be arranged that whenever $X = [0, 1]$, $f(0) = g(0)$, and $f(1) = g(1)$, that $T(\{0\} \times [0, 1])$ and $T(\{1\} \times [0, 1])$ be singletons.

Note that each N_n^* is compact and embeds into Z by way of the sequence

$$\iota_n = (N_n^* \to M_i)_i : N_n^* \to Z$$

given by the restrictions of $f_{n,1}, f_{n,2}, \ldots, f_{n,n-1}, \mathrm{id}, h_{n+1,n}^{-1}, h_{n+2,n}^{-1}, \ldots$.

Lemma 6.22. *Let $H \in \mathcal{H}$ and suppose that $n = \operatorname{loc} H$ is finite. Then the map $p_n \circ \beta|_{\operatorname{bdy} H} : \operatorname{bdy} H \to N_n^*$ is null-homotopic.*

Proof. First assume that $\dim Z \geq 3$. Since $\mathrm{incl}_\# : \pi_1(N_n^*) \to \pi_1(M_n)$ is injective, we only need to show that the loop $p_n \circ \beta|_{\operatorname{bdy} H} : \operatorname{bdy} H \to N_n^*$ contracts in M_n. For each $c \in \mathcal{L}_H$ let c' be the arc of S^1 with $\partial c' \cap (\operatorname{bdy} H) = \partial c$ and let $\kappa_c : c \to c'$ be the unit-speed homeomorphism with $\kappa_c|_{\partial c} = \mathrm{id}\,|_{\partial c}$. List the elements of $\mathcal{L}_H = \{c_1, c_2, c_3, \ldots\}$. For each $j \in \mathbb{N}$ choose $\zeta_{n,j} > 0$ such that if $x, y \in F$ with $d(x, y) < \zeta_{n,j}$, then $d_n\big(p_n \circ \beta(x), p_n \circ \beta(y)\big) < \xi_{M_n,j}$. Choose $s_1 \in \mathbb{N}$ such that $d(x, \kappa_{c_i}(x)) < \zeta_{n,1}$ for all $i > s_1$ and all $x \in c_i$. Inductively, choose $s_j > s_{j-1}$ such that $d(x, \kappa_{c_i}(x)) < \zeta_{n,j}$ for all $i > s_j$ and all $x \in c_i$. Put $s = s_1$ and let $\mathcal{L}'_H = \{c_1, c_2, \ldots, c_s\}$. Put

$$\operatorname{bdy}' H = \big(\bigcup \mathcal{C}_H\big) \cup \big(\bigcup \{c' \mid c \in \mathcal{L}_H \setminus \mathcal{L}'_H\}\big) \cup \big(\bigcup \mathcal{L}'_H\big).$$

By Lemma 6.21 it suffices to show that $p_n \circ \beta|_{\operatorname{bdy}' H} : \operatorname{bdy}' H \to M_n$ is null-homotopic, because of the way we chose our $\zeta_{n,j}$. Moreover, since each c_i can be approximated arbitrarily closely by some element of $\mathcal{P}$, we will assume, without loss of generality, that $c_1, c_2, \ldots, c_s \in \mathcal{P}$.

Choose $k > n$ such that $c_1, c_2, \ldots, c_s \in \mathcal{P}_k$. Choose $\delta > 0$ so that for every $x, y \in S^1$ with $d(x, y) < \delta$ we have $d_n(\alpha_n(x), \alpha_n(y)) < \epsilon_n$ and, moreover, $\alpha_n(x)$ and $\alpha_n(y)$ are in the same element of $\mathcal{S}_n^+$ provided either of $\alpha_n(x)$ or $\alpha_n(y)$ is in an element of $\mathcal{S}_n$. By Lemma 6.7, there is an $m > k$ and $c'_1, c'_2, \ldots, c'_s \in \mathcal{B}_{m,k}$ whose endpoints lie within δ of the endpoints of $c_1, c_2, \ldots, c_s$ and are such that $f_{m,k} \circ \beta'_m(c'_i)$ and $\alpha_k(\partial c_i)$, and hence also $p_k \circ \beta(c_i)$, lie in the same element of $\mathcal{S}_k$. Applying $f_{k,n}$, we see that $f_{m,n} \circ \beta'_m(c'_i)$ and $p_n \circ \beta(c_i)$ lie in the same element of $\mathcal{S}_n$. Also, each element of $\mathcal{S}_n^+$ is simply connected. The map $f_{m,n} \circ \beta'_m : D^2 \to M_n$ is now witness to the fact that $p_n \circ \beta : \mathrm{bdy}' H \to M_n$ is null-homotopic, upon verification that $d_n(f_{m,n} \circ \beta'_m(x), p_n \circ \beta(y)) < 2\epsilon_n$ whenever x is an endpoint of some c'_i and y the corresponding endpoint of c_i, which is within δ. (Recall Lemma 6.2 and our convention on page 62.) We estimate:

$$\begin{aligned} d_n(f_{m,n} \circ \beta'_m(x), p_n \circ \beta(y)) \\ &= d_n(f_{m,n} \circ \alpha'_m(x), \alpha_n(y)) \\ &\leq d_n(f_{m,n} \circ \alpha'_m(x), f_{m,n} \circ \alpha_m(x)) + d_n(f_{m,n} \circ \alpha_m(x), \alpha_n(y)) \\ &\leq d_n(f_{m,n} \circ \alpha'_m(x), f_{m,n} \circ \alpha_m(x)) + d_n(\alpha_n(x), \alpha_n(y)) \\ &\leq \epsilon_n + \epsilon_n, \end{aligned}$$

because $d_m(\alpha'_m, \alpha_m) < \epsilon_m$.

Now suppose $\dim Z = 2$. Then $\mathrm{incl}_\# : \pi_1(N_n^*) \to \pi_1(M_n)$ may not be injective. However, since the defining sequence is assumed to be well-balanced we know that the natural homomorphism $\pi_1(N_n^*) \to \check{\pi}_1(N_n^*)$ is injective, because N_n^* is either an ANR (see Remark 2.7) or one-dimensional (see Remark 3.2(i)). To exploit this fact, we put $N_1^0 = M_1$ and for $i \geq 1$ we set $N_1^i = M_1 \setminus \bigcup\{f_{m,1}(\mathring{D}_m) \mid 1 \leq m \leq i\}$. Similarly, if $n \geq 2$, we put $N_n^{n-1} = M_n \setminus h_{n,n-1}^{-1}(M_{n-1} \setminus D_{n-1})$ and for $i \geq n$, we define $N_n^i = M_n \setminus \big(h_{n,n-1}^{-1}(M_{n-1} \setminus D_{n-1}) \cup \big(\bigcup\{f_{m,n}(\mathring{D}_m) \mid n \leq m \leq i\}\big)\big)$. The intersection $N_n^* = \bigcap_{i=n-1}^{\infty} N_n^i$ can be interpreted as the limit of the inverse sequence

$$N_n^{n-1} \hookleftarrow N_n^n \hookleftarrow N_n^{n+1} \hookleftarrow N_n^{n+2} \hookleftarrow \cdots,$$

whose bonding maps consist of inclusions. (The terms are not necessarily all distinct.) Hence, it suffices to show that $p_n \circ \beta|_{\mathrm{bdy}\, H}$ contracts in each N_n^i. However, the homomorphism $(h_{i+1,n}^{-1})_\# : \pi_1(N_n^i) \to \pi_1(M_{i+1})$ is injective, so that we only need to show that $p_{i+1} \circ \beta|_{\mathrm{bdy}\, H}$ contracts in M_{i+1}. This is done exactly as above. The only necessary adjustment is to choose m sufficiently large so that $[f_{m,n} \circ \beta'_m|_{c_i}][\overline{p_n \circ \beta|_{c_i}}] = 1$ in the fundamental group of the element of $\mathcal{S}_n$ in which they lie. $\square$

While the spaces N_n^* may not be ANRs, we still have:

Corollary 6.23. *Let* $n, k \in \mathbb{N}$. *There is a real number* $\eta_{n,k} > 0$ *such that for every polyhedron* K *with dim* $K \leq$ *dim* $Z - 2$ *and any two maps* $f, g : K \to N_n^*$ *with* $d(f(x), g(x)) < \eta_{n,k}$ *for all* $x \in K$, *there is a homotopy* $T : K \times [0, 1] \to N_n^*$ *such that* $T(x, 0) = f(x)$, $T(x, 1) = g(x)$, *and* $\operatorname{diam} T(\{x\} \times [0, 1]) < 1/k$ *for all* $x \in K$. *Moreover, if* $K = [0, 1]$, $f(0) = g(0)$, *and* $f(1) = g(1)$, *then* $T(\{0\} \times [0, 1])$ *and* $T(\{1\} \times [0, 1])$ *are singletons.*

Proof. Choose $m > n+1$ such that $\operatorname{diam} f_{j,n}(D_j) < 1/(2k)$ for all $j \geq m$. Consider the ANR $Y = M_n \setminus \left(h_{n,n-1}^{-1}(M_{n-1} \setminus D_{n-1}) \cup E_{m,n}\right)$. (If $n = 1$, consider instead $Y = M_1 \setminus E_{m,1}$.) Put $\eta_{n,k} = \xi_{Y,2k}$ and let $T : K \times [0, 1] \to Y$ be the homotopy of Lemma 6.21 from f to g with $\operatorname{diam} T(\{x\} \times [0, 1]) \leq 1/(2k)$ for all $x \in K$. Since dim $(K \times [0, 1]) \leq$ dim $Z - 1$, we can swipe this homotopy into N_n^*, moving points less than $1/(2k)$. □

For each $n, k \in \mathbb{N}$ choose a $\delta_{n,k} > 0$ such that if $x, y \in F$ with $d(x, y) < \delta_{n,k}$, then $d_n(p_n \circ \beta(x), p_n \circ \beta(y)) < \eta_{n,k}$. We want to arrange that for every $n \in \mathbb{N}$, both sequences $(\eta_{n,k})_k$ and $(\delta_{n,k})_k$ are decreasing and have their limit equal to zero.

We now extend $\beta : F \to Z$ to a function $\beta : D^2 \to Z$. We start with dimension 3 and higher:

Definition 6.24. Suppose dim $Z \geq 3$. We wish to extend β to the holes $\mathcal{H}$. Let $H \in \mathcal{H}$, $n = \operatorname{loc} H$ and let $1/s$ be the size of H. Currently, β is defined on bdy H. We will now extend β over cl H. In doing so, we consider three cases:

(i) If $n = \infty$, we define $\beta(\operatorname{cl} H)$ to be the singleton $\beta(\operatorname{bdy} H)$.

(ii) Suppose n is finite and that there is a $k \in \mathbb{N}$ such that $\delta_{n,k-1} \leq 1/s < \delta_{n,k}$. Let $c \in \mathcal{L}_H$ be the base of H. Then $d_n(p_n \circ \beta \circ l_{H,c}(x), p_n \circ \beta \circ r_{H,c}(x)) < \eta_{n,k}$ for all x in $[0, 1]$. Also, $p_n \circ \beta \circ l_{H,c} : [0, 1] \to N_n^*$ and $p_n \circ \beta \circ r_{H,c} : [0, 1] \to N_n^*$, by Lemma 6.20. By Corollary 6.23 there is a homotopy $T : [0, 1] \times [0, 1] \to N_n^*$ with

$$T(x, 0) = p_n \circ \beta \circ l_{H,c}(x), \quad T(x, 1) = p_n \circ \beta \circ r_{H,c}(x),$$

and $\operatorname{diam} T(\{x\} \times [0, 1]) < 1/k$ for all $x \in [0, 1]$. We also arrange for $T(\{0\} \times [0, 1])$ and $T(\{1\} \times [0, 1])$ to be singletons. Define the quotient map $\varphi_{H,c} : [0, 1] \times [0, 1] \to \operatorname{cl} H$ by $\varphi_{H,c}(x, t) = t \cdot r_{H,c}(x) + (1 - t) \cdot l_{H,c}(x)$. The only nontrivial fibers of $\varphi_{H,c}$ are $\{0\} \times [0, 1]$ and $\{1\} \times [0, 1]$. Hence, there is a unique map $T' : \operatorname{cl} H \to N_n^*$ with $T' \circ \varphi_{H,c} = T$. We define $\beta|_{\operatorname{cl} H} = \iota_n \circ T'$.

(iii) Finally, suppose n is finite and $\delta_{n,1} \leq 1/s$. In this case we extend the map $p_n \circ \beta|_{\operatorname{bdy} H} : \operatorname{bdy} H \to N_n^*$ to any map $p_n \circ \beta|_{\operatorname{cl} H} : \operatorname{cl} H \to N_n^*$; it will not matter how. By Lemma 6.22 we may extend $p_n \circ \beta|_{\operatorname{bdy} H} : \operatorname{bdy} H \to N_n^*$ to some map $T' : \operatorname{cl} H \to N_n^*$. As before, we put $\beta|_{\operatorname{cl} H} = \iota_n \circ T'$.

Doing this for every hole H, completes the definition of a function $\beta : D^2 \to Z$, which extends the map $\beta : F \to Z$, in case $\dim Z \geq 3$.

If $\dim Z = 2$ and if N_n^* is not an ANR, then we might not be able to carry out part (ii) of Definition 6.24. We therefore need one more result.

Lemma 6.25 [Cannon and Conner 1998]. *Let Y be a compact one-dimensional connected metric space and $f : S^1 \to Y$ a null-homotopic map. Then there is a collection $\aleph$ of straight line segments in D^2 whose endpoints lie in S^1 and with disjoint interiors, and a map $g : D^2 \to Y$ with $g|_{S^1} = f$ such that g is constant on every element of $\aleph$ and on every component of the complement of the closure of $\bigcup \aleph$ in D^2.*

Corollary 6.26. *Suppose $\dim Z = 2$. Let $n \in \mathbb{N}$ and suppose N_n^* is one-dimensional. Then for each $H \in \mathcal{H}$ with $\operatorname{loc} H = n$ there is a map $\chi_H : \operatorname{cl} H \to N_n^*$ such that $\chi_H|_{\operatorname{bdy} H} = p_n \circ \beta|_{\operatorname{bdy} H}$ and such that the following holds: for every $\delta > 0$ and every $\epsilon > 0$, there is a finite subset $\mathcal{H}' \subseteq \mathcal{H}$ such that for every $H \in \mathcal{H} \setminus \mathcal{H}'$ with $\operatorname{loc} H = n$ and every $x \in \operatorname{cl} H$, there is an $x' \in \operatorname{bdy} H$ with $d(x, x') < \delta$ and $d_n(\chi_H(x), \chi_H(x')) < \epsilon$.*

Proof. Let an $H \in \mathcal{H}$ with $\operatorname{loc} H = n$ be given. Using Lemmas 6.22 and 6.25, we can construct a collection $\aleph_H$ of straight line segments in $\operatorname{cl} H$ whose endpoints lie in $\operatorname{bdy} H$ and whose interiors are pairwise disjoint and disjoint from $\operatorname{bdy} H$, and a map $\chi_H : \operatorname{cl} H \to N_n^*$ such that $\chi_H|_{\operatorname{bdy} H} = p_n \circ \beta|_{\operatorname{bdy} H}$ and χ_H is constant on every element of $\aleph_H$ and on every component of the complement of the closure of $\bigcup \aleph_H$ in $\operatorname{cl} H$. This works, because we can arrange that the line segments of Lemma 6.25 do not have both endpoints in one and the same $c \in \mathcal{L}_H$. (Recall that for each $c \in \mathcal{L}_H$ either there is an $i \in \mathbb{N}$ such that $p_i \circ \beta|_c : c \to \partial D_i$ is a local geodesic or $\beta(c)$ is a singleton.)

Now let all $(\chi_H)_{H \in \mathcal{H}, \operatorname{loc} H = n}$ be defined and $\delta, \epsilon > 0$ be given. Choose $\delta_1 \in (0, \delta/4)$ such that if $x, y \in F$ and $d(x, y) < \delta_1$, then $d_n(p_n \circ \beta(x), p_n \circ \beta(y)) < \epsilon/2$. Let $C_1, C_2, \ldots$ be the distinct nonseparating simple closed curves of N_n^*. Since for each i there is a unique $m_i \geq n$ with $C_i = h_{m_i,n}(\partial D_{m_i})$, there is an N such that $\operatorname{diam} C_i < \epsilon/2$ for all $i > N$. Choose pairwise disjoint regular neighborhoods $V(C_1, M_n), V(C_2, M_n), \ldots, V(C_N, M_n)$ of $C_1, C_2, \ldots, C_N$ in M_n, respectively. For each $i \in \{1, 2, \ldots, N\}$, choose a finite cover $\mathcal{W}_i$ of $V(C_i, M_n)$ as in Lemma 6.2 and choose $\Lambda_1 > 0$ sufficiently small so that it is a Lebesgue number for each of the covers $\mathcal{W}_i$, with respect to the metric d_n. This allows us to perform computations in $\Pi_1(\mathcal{W}_i, *)$ rather than $\pi_1(V(C_i, M_n), *)$. We also want to choose Λ_1 sufficiently small so that arcs in any $C_1, C_2, \ldots, C_N$ of arclength less than Λ_1, as measured in S^1 after applying $\rho_{m_i} \circ h_{m_i,n}^{-1}$, have d_n-diameters that are less than a Lebesgue number for the respective cover $\mathcal{W}_i$; where ρ_{m_i} is as defined on page 63. Choose $\delta_2 \in (0, 16\Lambda_1/\delta)$ such that if τ is an arc in C_i with $i \in \{1, 2, \ldots, N\}$

whose arclength, as measured in S^1 after applying $\rho_{m_i} \circ h^{-1}_{m_i,n}$, is less than δ_2, then the diameter of τ, as measured in the metric d_n of M_n, is less than $\epsilon/2$. Choose $\Lambda_2 > 0$ such that if $a, b \in C_i$ with $i \in \{1, 2, \ldots, N\}$ and $0 < d_n(a, b) < \Lambda_2$ then the closure of one the of components of $C_i \setminus \{a, b\}$ has arclength less than $\delta \cdot \delta_2/16$, as measured in S^1 after applying $\rho_{m_i} \circ h^{-1}_{m_i,n}$. Choose $\delta_3 \in (0, \delta_1)$ such that if $x, y \in F$ and $d(x, y) < \delta_3$, then $d_n(p_n \circ \beta(x), p_n \circ \beta(y)) < \Lambda_2$. Choose $\delta_4 \in (0, \delta_3)$ such that if $H \in \mathcal{H}$ whose base c has length at least δ_1 and whose size $H < \delta_4$, then the arclength of $r_{H,c}$ is less than twice the length of c; where we use the notation $r_{H,c}$ and $l_{H,c}$ of Definition 6.17. Choose $\delta_5 \in (0, \delta_4)$ such that if $x, y \in F$ with $d(x, y) < \delta_5$, then $d_n(p_n \circ \beta(x), p_n \circ \beta(y)) < \Lambda_1$; and such that if moreover $p_n \circ \beta(x) \in C_i$ for some $i \in \{1, 2, \ldots, N\}$, then $p_n \circ \beta(y) \in V(C_i, M_n)$.

Now let $\mathcal{H}'$ be the set of all $H \in \mathcal{H}$ such that size $H \geq \delta_5$. By Lemma 6.18, this is a finite set. Let $H \in \mathcal{H} \setminus \mathcal{H}'$ with loc $H = n$ and $x \in \operatorname{cl} H$. We may assume that $x \in H$. Let c be the base of H.

First suppose that diam $p_n \circ \beta(c) < \epsilon/2$. Choose $y \in \operatorname{bdy} H \subseteq F$ such that $\chi_H(x) = \chi_H(y)$. Choose $x', y' \in c$ such that each of $d(x, x')$ and $d(y, y')$ is less than size $H < \delta_5 < \delta_1 < \delta$. Then

$$\begin{aligned} d_n(\chi_H(x), \chi_H(x')) &= d_n(\chi_H(y), \chi_H(x')) \\ &= d_n(p_n \circ \beta(y), p_n \circ \beta(x')) \\ &\leq d_n(p_n \circ \beta(y), p_n \circ \beta(y')) + d_n(p_n \circ \beta(y'), p_n \circ \beta(x')) \\ &< \epsilon/2 + \epsilon/2 = \epsilon, \end{aligned}$$

and we are done.

Now suppose diam $p_n \circ \beta(c) \geq \epsilon/2$. Since $\beta(c)$ is not a singleton, we can choose $i \in \mathbb{N}$ such that $p_n \circ \beta(c) \subseteq C_i$. Then $i \in \{1, 2, \ldots, N\}$, by choice of N, and $p_n \circ \beta(\operatorname{bdy} H) \subseteq V(C_i, M_n)$, by choice of δ_5. Also, the length of $p_n \circ \beta(c)$, as measured in S^1 after applying $\rho_{m_i} \circ h^{-1}_{m_i,n}$, is at least δ_2, and the length of c is at least δ_1.

Claim: Suppose $d \in \aleph_H$ with $\partial d = \{a, b\}$ and $a \in c$. Say $a = l_{H,c}(t_1)$ and $b = r_{H,c}(t_2)$. Put $a' = r_{H,c}(t_1)$ and $b' = l_{H,c}(t_2)$. We claim that the arc σ in the image of $r_{H,c}$ from a' to b has length less than $\delta/4$.

Reason: Suppose, to the contrary, that σ has length at least $\delta/4$. We have $d(a, a') <$ size $H < \delta_5 < \delta_4 < \delta_1$. Also, $d(b, b') <$ size $H < \delta_5 < \delta_3$. Since the arclength of $r_{H,c}$ is less than twice the length of c, the corresponding segment τ on c with endpoints a and b' has length at least $\delta/8$. Since the length of c is not more than 2, τ covers at least $(\delta/8)/2 = \delta/16$ of the length of c. Since the length of $p_n \circ \beta(c)$ is at least δ_2, then $p_n \circ \beta(\tau)$ has arclength at least $(\delta/16) \cdot \delta_2$ and so does $\mu = \operatorname{cl}[C_i \setminus p_n \circ \beta(\tau)]$. (Notice that if μ had arclength less than $\delta\delta_2/16 < \Lambda_1$, then $[\chi_H|_\tau] = [p_n \circ \beta|_\tau] \neq 1 \in \Pi_1(\mathcal{W}_i, *)$. However, if l is the simple closed curve formed by concatenating τ, d and the remaining portion of bdy H, then $[\chi_H|_l] = 1 \in \Pi_1(\mathcal{W}_i, *)$. This is a contradiction, because χ_H is constant on d and $p_n \circ \beta \circ l_{H,c}$ is within Λ_1 of $p_n \circ \beta \circ r_{H,c}$.) Hence the endpoints $\chi_H(a) = \chi_H(b)$ and $\chi_H(b')$ of $p_n \circ \beta(\tau)$ are at least Λ_2 apart. However, $d(b, b') < \delta_3$, so that $d_n(\chi_H(b), \chi_H(b')) = d_n(p_n \circ \beta(b), p_n \circ \beta(b')) < \Lambda_2$. This contradiction establishes the claim.

There are two cases. Case (i): Suppose there are $d_1, d_2 \in \aleph_H$ with $\partial d_1 = \{a_1, b_1\}$, $\partial d_2 = \{a_2, b_2\}$, $a_1 = a_2 \in c$ and such that x is between d_1 and d_2. (We allow for the possibilities that x is on either d_1 or d_2, or that $d_1 = d_2$.) Let σ_1, σ_2, and $a'_1 = a'_2$ be as in the claim. Let ν be the arc on $r_{H,c}$ between b_1 and b_2. Since each of σ_1 and σ_2 has arclength less than $\delta/4$, then ν has arclength less than $\delta/2$. Also, $d(a'_i, a_i) <$ size $H < \delta/4$. Therefore, each of d_1 and d_2 have length less than $\delta/2$. Choose $x' \in \nu$ such that $\chi_H(x') = \chi_H(x)$. Then $d(x, x') < \delta$ and the proof of the corollary is complete in this case. Case (ii): The alternative is that x is not between those two $d_1, d_2 \in \aleph_H$ with $\partial d_1 = \{a_1, b_1\}$, $\partial d_2 = \{a_2, b_2\}$ for which $\{a_1, a_2\} = \partial c$. This case also follows from above claim, since $a_1 = a'_1$ and $a_2 = a'_2$. The argument now is similar to, but simpler than, that of Case (i). □

We now complete our definition of $\beta : D^2 \to Z$ by addressing the case when dim $Z = 2$. In dimension two, we have assumed a well-balanced defining sequence. Therefore, N^*_n is either one-dimensional or it is an ANR. Accordingly, our final definition splits into two cases:

Definition 6.27. Suppose dim $Z = 2$. Let $H \in \mathcal{H}$ and $n = \operatorname{loc} H$.

(i) If $n = \infty$, we define, as before, $\beta(\operatorname{cl} H)$ to be the singleton $\beta(\operatorname{bdy} H)$.

(ii) If n is finite, then $p_n \circ \beta(\operatorname{bdy} H) \subseteq N^*_n$ by Lemma 6.20; moreover,

- if N^*_n is one-dimensional, we set $p_n \circ \beta|_{\operatorname{cl} H} = \chi_H$;
- if N^*_n is an ANR, we define $p_n \circ \beta|_{\operatorname{cl} H}$ as in Definition 6.24(ii,iii), upon replacing $\delta_{n,k}$ by some $\zeta'_{n,k}$ which are chosen to have the following property: if $x, y \in F$ with $d(x, y) < \zeta'_{n,k}$, then $d_n\big(p_n \circ \beta(x), p_n \circ \beta(y)\big) < \xi_{N^*_n,k}$. Then use Lemma 6.21 instead of Corollary 6.23. Also make the sequence $\big(\zeta'_{n,k}\big)_k$ decrease to zero.

By construction we have:

Lemma 6.28. *Let $H \in \mathcal{H}$ and $n = \operatorname{loc} H$. If $n = \infty$, then $\beta(\operatorname{cl} H)$ is a singleton. If n is finite then $p_n \circ \beta(\operatorname{cl} H) \subseteq N^*_n$.* □

Verifying continuity. The next result will conclude the proof of Theorem 3.1:

Lemma 6.29. *The map $\beta : D^2 \to Z$ is continuous.*

Proof. (A) First we assume that $\dim Z \geq 3$. Let $x_0 \in D^2$ and $\epsilon > 0$ be given. Fix $n \in \mathbb{N}$. We will find a $\delta > 0$ such that if $x \in D^2$ with $d(x, x_0) < \delta$, then $d_n\big(p_n \circ \beta(x), p_n \circ \beta(x_0)\big) < \epsilon$. Since $p_n \circ \beta|_H : H \to M_n$ is continuous for every $H \in \mathcal{H}$, and since H is open in D^2, we may assume that $x_0 \in F$. Since $p_n \circ \beta|_F : F \to M_n$ is continuous, there is a $\delta_1 > 0$ such that if $x \in F$ with $d(x, x_0) < \delta_1$, then $d_n\big(p_n \circ \beta(x), p_n \circ \beta(x_0)\big) < \epsilon/2$. Choose $N > n$ such that $\operatorname{diam} f_{m,n}(D_m) < \epsilon/2$ for all $m \geq N$. Choose $j \in \mathbb{N}$ such that if $m \in \{1, 2, \ldots, N\}$ and $u, v \in N_m^*$ with $d_m(u, v) < 1/j$, then $d_n\big(p_n \circ \iota_m(u), p_n \circ \iota_m(v)\big) < \epsilon/2$. By Lemma 6.18, we may choose a finite subset $\mathcal{H}' \subseteq \mathcal{H}$ such that the size of every $H \in \mathcal{H} \setminus \mathcal{H}'$ is less than $\min\{\delta_{1,j}, \delta_{2,j}, \ldots, \delta_{N,j}, \delta_1/2\}$. Since the map β is continuous when restricted to $F \cup \big(\bigcup\{\operatorname{cl} H \mid H \in \mathcal{H}'\}\big)$, there is $\delta_2 > 0$ such that if $x \in \bigcup\{\operatorname{cl} H \mid H \in \mathcal{H}'\}$ and $d(x, x_0) < \delta_2$, then $d_n\big(p_n \circ \beta(x), p_n \circ \beta(x_0)\big) < \epsilon$. Put $\delta = \min\{\delta_1/2, \delta_2\}$. Now, take $x \in D^2$ with $d(x, x_0) < \delta$. If $x \in F$, then $d_n\big(p_n \circ \beta(x), p_n \circ \beta(x_0)\big) < \epsilon/2 < \epsilon$, because $d(x, x_0) < \delta_1$. We therefore may assume that $x \in H$ for some $H \in \mathcal{H}$. If $H \in \mathcal{H}'$, then $d_n\big(p_n \circ \beta(x), p_n \circ \beta(x_0)\big) < \epsilon$, because $d(x, x_0) < \delta_2$. So, we will assume that $x \in H \in \mathcal{H} \setminus \mathcal{H}'$. Let $m = \operatorname{loc} H$ and let $1/s$ be the size of H. Choose x' in the base of H such that $\varphi_{H,c}(x', t) = x$ for some t. Then $x' \in \operatorname{bdy} H \subseteq F$ and $d(x', x) < 1/s$. Hence, $d(x', x_0) \leq d(x', x) + d(x, x_0) < 1/s + \delta < \delta_1/2 + \delta_1/2 = \delta_1$, so that $d_n\big(p_n \circ \beta(x'), p_n \circ \beta(x_0)\big) < \epsilon/2$. If $m = \infty$, then $\beta(x) = \beta(x')$ by Lemma 6.28, so that $d_n\big(p_n \circ \beta(x), p_n \circ \beta(x_0)\big) = d_n\big(p_n \circ \beta(x'), p_n \circ \beta(x_0)\big) < \epsilon/2 < \epsilon$. If $N < m < \infty$, then $\operatorname{diam} p_n \circ \beta(\operatorname{cl} H) < \epsilon/2$, because $p_m \circ \beta(\operatorname{cl} H) \subseteq N_m^*$ by Lemma 6.28 and $f_{m,m-1}(N_m^*) \subseteq D_{m-1}$. Hence, $d_n\big(p_n \circ \beta(x), p_n \circ \beta(x_0)\big) \leq d_n\big(p_n \circ \beta(x), p_n \circ \beta(x')\big) + d_n\big(p_n \circ \beta(x'), p_n \circ \beta(x_0)\big) \leq \epsilon/2 + \epsilon/2 = \epsilon$. Finally assume $m \leq N$. Since $1/s < \delta_{m,j}$, then the homotopy T in the definition of $\beta|_{\operatorname{cl} H}$ has tracks of diameter less than $1/j$, so that, by choice of j, we have $d_n\big(p_n \circ \beta(x), p_n \circ \beta(x')\big) < \epsilon/2$. Therefore, as above, $d_n\big(p_n \circ \beta(x), p_n \circ \beta(x_0)\big) < \epsilon$.

(B) Now suppose $\dim Z = 2$. We follow the proof of (A) until we have to choose $\mathcal{H}'$. At this point we use Corollary 6.26 to find a finite subset $\mathcal{H}' \subseteq \mathcal{H}$ such that for every $H \in \mathcal{H} \setminus \mathcal{H}'$ with $m = \operatorname{loc} H \in \{1, 2, \ldots, N\}$ the following is true: (i) $\operatorname{size} H < \delta_1/2$; (ii) if N_m^* is an ANR, then $\operatorname{size} H < \zeta'_{m,j}$; (iii) if N_m^* is one-dimensional, then for every $x \in \operatorname{cl} H$ there is an $x'' \in \operatorname{bdy} H$ with $d(x, x'') < \delta_1/2$ and $d_n\big(p_n \circ \beta(x), p_n \circ \beta(x'')\big) = d_n\big(\chi_H(x), \chi_H(x'')\big) < \epsilon/2$. The proof is now the same, including the selection of the point x'. However, in the case where $m \leq N$ we use the point x'' instead of the point x' in the event that N_m^* is one-dimensional. □

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