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CELL-LIKE MAPS AND THE KOZLOWSKI-WALSH THEOREM
- SOME ALTERNATIVE PROOFS -

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We present a proof of the following result of George Kozlowski and John Walsh.

Theorem: If M^3 is a 3-manifold and $f: M^3 \to Y$ is a cell-like map, then $\dim Y \leq 3$.

A sketch of a proof: Our strategy is revealed by Lemma A, page 6. We shall show that the map $f: M^3 \to Y$ has "approximate inverses"; i.e., given an open cover $\mathcal{U}$ of Y, we shall find a map $g: Y \to M^3$ such that $f \circ g$ is within $\mathcal{U}$ of 1_Y.

Step 1: Invoking some 3-manifold topology (Corollaries F2 and F3, page 37) we obtain an open cover $\mathcal{W}$ of M^3 such that (1) $\mathcal{W}$ is a fine refinement of $f^{-1}\mathcal{U}$ (details on page 19), (2) each point inverse of f is contained in an element of $\mathcal{W}$, and (3) the intersection of any finite number of elements of $\mathcal{W}$ is aspherical (no higher homotopy). ((3) depends on the cell-likeness of the point inverses of f.)

<u>Step 2</u>: Lemma B (page 13) provides a star-finite open cover $\mathcal{V}$ of Y and a map $\psi: |K^{(2)}| \to M$ where K is the nerve of $\mathcal{V}$, $|K^{(2)}|$ is the polyhedron underlying the 2-skeleton of K, and the map ψ has the following properties. (1) ψ maps the 2-skeleton of each simplex of K into an element of $\mathcal{W}$. (2) If $\varphi: Y \to |K|$ is one of the natural "barycentric maps" available, then $\psi \circ \varphi | \varphi^{-1}|K^{(2)}| : \varphi^{-1}|K^{(2)}| \to M$ is the "germ" of an approximate inverse of f (i.e., $f \circ \psi \circ \varphi | \varphi^{-1}|K^{(2)}|$ is close to $1|\varphi^{-1}|K^{(2)}|$).

In the proof of Lemma B, ~~the~~ we arrive at the open cover $\mathcal{V}$ of Y as follows. We successively find three open covers $\mathcal{V}_2, \mathcal{V}_1, \mathcal{V}_0$ of Y such that for each $y \in Y$, (1) $f^{-1}(\mathrm{Star}(y, \mathcal{V}_2))$ is contained in some element of $\mathcal{W}$, and (2) for $i = 0, 1$, $f^{-1}(\mathrm{Star}(y, \mathcal{V}_i))$ contracts to a point in some element of $f^{-1}\mathcal{V}_{i+1}$. ((2) depends on the cell-likeness of the point inverses of f.) $\mathcal{V}$ is a star-finite open cover of Y which refines $\mathcal{V}_0$. **This 3-deep nesting — $\mathcal{V}_2, \mathcal{V}_1, \mathcal{V}_0$ — allows us to construct the map $\psi: |K^{(2)}| \to M$ as desired. (If we had similarly built up $(n+1)$-deep nesting — $\mathcal{V}_n, \mathcal{V}_{n-1}, \dots, \mathcal{V}_0$ — we could have mapped the n-skeleton of K into M.)**

<u>Step 3</u>: ψ maps the 2-skeleton of each simplex of K into an element of $\mathcal{W}$, and finite

intersections of elements of $\mathcal{W}$ are aspherical.
It follows (remarkably!) from Lemma C (page 16) that $\psi : |K^{(2)}| \to M$ extends to a map $\bar{\psi} : |K| \to M$ defined on <u>all</u> of $|K|$ (the polyhedron underlying K) such that $\bar{\psi}$ maps each simplex of K into an element of $\mathcal{W}$.

<u>Step 4</u>: We define the map $g : Y \to M$ by $g = \psi \circ \varphi$, and we verify that $f \circ g$ is within $\mathcal{U}$ of $1|Y$. (This verification depends on the fact that $\mathcal{W}$ was originally chosen to be a sufficiently fine refinement of $f^{-1}\mathcal{U}$.) ∎

1. <u>General</u> <u>results</u> <u>on</u> <u>cell-like</u> <u>maps</u>

All spaces are metrizable.

<u>Definition</u> Y is an <u>ANR</u> (<u>absolute</u> <u>neighborhood</u> <u>retract</u>) if it satisfies either of the following equivalent conditions:
(1) If A closed $\subset X$ and $f : A \to Y$ is a map, then f extends to a map $g : U \to Y$ where $A \subset U$ open $\subset X$.
(2) If Y is embedded as a closed subset of a space X then a neighborhood of Y in X retracts onto Y.

<u>Definition</u> X is <u>cell-like</u> if X is compact and satisfies any one of the following equivalent conditions:

(1) Any map of X into any ANR is homotopically trivial.
(2) Under some embedding of X in some space Y, X is a UV^∞ subset of Y.
(3) Under any embedding of X in any ANR Y, X is a UV^∞ subset of Y.
(X is a <u>UV^∞</u> subset of Y if every neighborhood U of X in Y contains a neighborhood V of X in Y such that V contracts to a point in U.)
(4) Under some embedding of X in some space Y, X contracts to a point in each of its neighborhoods in Y.
(5) Under any embedding of X in any ANR Y, X contracts to a point in each of its neighborhoods in Y.
(6) X embeds as a shrinkable subset of $\begin{cases} Q, \text{ the Hilbert cube, always} \\ \text{some } \mathbb{R}^n, \text{ if } \dim X < \infty \end{cases}$
(X is a <u>shrinkable</u> subset of Y if there is a map $f: Y \to Y$ and a point $x \in Y$ such that $f^{-1}x = X$ and $f|Y-X$ takes $Y-X$ homeomorphically onto $Y - \{x\}$. This is equivalent to saying the decomposition space Y/X is homeomorphic to Y. A compact subset of $\mathbb{R}^n$ is shrinkable if and only if it is cellular.)

<u>Definition</u>: A map $f: X \to Y$ is <u>proper</u> if it satisfies either of the following equivalent conditions:
(1) If K is compact $\subset Y$, then $f^{-1}K$ is also compact
(2) $f: X \to Y$ is a closed map, and $f^{-1}y$ is compact for each $y \in Y$.

<u>Definition</u>: Let X be an ANR. A map $f: X \to Y$ is <u>cell-like</u> if f is proper and onto and $f^{-1}y$ is a cell-like set for each $y \in Y$.

<u>Definition</u>: Suppose $f, g : X \to Y$ are maps, $h : X \times [0,1] \to Y$ is a homotopy, and $\mathcal{U}$ is a collection of subsets of Y. If for each $x \in X$, there is a $U \in \mathcal{U}$ for which $f(x), g(x) \in U$, then we say f is <u>within $\mathcal{U}$</u> of g and that f and g are <u>$\mathcal{U}$-close</u>. If for each $x \in X$, there is a $U \in \mathcal{U}$ for which $h(x \times [0,1]) \subset U$, then we say h is <u>limited by $\mathcal{U}$</u> and we call h a <u>$\mathcal{U}$-homotopy</u>. If $h_0 = f$ and $h_1 = g$, and h is limited by $\mathcal{U}$, then we say that f is <u>$\mathcal{U}$-homotopic</u> to g. We let <u>$f^{-1}\mathcal{U}$</u> $= \{f^{-1}U : U \in \mathcal{U}\}$.

<u>Definition</u> A map $f : X \to Y$ is a <u>fine homotopy equivalence</u> if f is proper and onto, and if for every open cover $\mathcal{U}$ of Y, there is a map $g : Y \to X$ such that $f \circ g$ is $\mathcal{U}$-homotopic to $1|Y$ and $g \circ f$ is $f^{-1}\mathcal{U}$ homotopic to $1|X$.

<u>Definition</u>: <u>$\dim X \leq n$</u> if every open cover $\mathcal{U}$ of X is refined by a (locally finite) open cover $\mathcal{V}$ of X which has <u>order</u> $\leq n+1$: i.e., no point of X belongs to more than $n+1$ distinct elements of $\mathcal{V}$. <u>$\dim X = n$</u> if $\dim X \leq n$ but $\dim X \not\leq n-1$.

<u>Definition</u>: X is <u>countable dimensional</u> if X is the union of countably many finite dimensional spaces.

<u>Theorem</u> (Smale, Lacher, Armentrout-Price, Haver, Kozlowski, Ancel): Suppose X is an ANR and $f : X \to Y$ is a cell-like map. If Y is countable dimensional, then Y is an ANR and f is a fine homotopy equivalence.

<u>Theorem</u> (same group): Suppose X is an ANR and $f: X \to Y$ is a cell-like map. Then Y is an ANR if and only if f is a fine homotopy equivalence.

<u>Example</u> (Taylor): There is a cell-like map $f: \mathbb{Q} \to T$ where $\mathbb{Q}$ is the Hilbert cube and T is not an ANR.

Is there a cell-like map from a finite dimensional ANR to a non-ANR? Equivalently, is there a cell-like map whose domain is a finite dimensional ANR but which is not a fine homotopy equiv.

<u>The Big Question</u>: Can a cell-like map raise dimension? Is there a cell-like map $f: X \to Y$ where X is an ANR, $\dim X < \infty$ and $\dim Y > \dim X$?

The <u>Kozlowski – Walsh Theorem</u> says that the answer to this question is "no" whenever X is a subset of a 3-manifold.

Our strategy for proving the Kozlowski-Walsh Theorem is revealed by the following lemma. Given a cell-like map $f: X \to Y$, in order to show that $\dim Y \leq \dim X$, we seek an "approximate inverse" to f — a map $g: Y \to X$ such that $f \circ g$ is close to $1\!\!1_Y$.

<u>Lemma A</u>: Suppose $f: X \to Y$ is a proper onto map. If for every open cover $\mathcal{W}$ of Y, there is a map $g: Y \to X$ such that $f \circ g$ is within $\mathcal{W}$ of $1\!\!1_Y$, then $\dim Y \leq \dim X$.

<u>Proof</u>: Suppose $\dim X = n$. Let $\mathcal{U}$ be an open cover of Y. Then $f^{-1}\mathcal{U}$ is an open cover of X. So there is a locally finite open cover $\mathcal{V}$ of X of order $\leq n+1$ such that $cl\,\mathcal{V} = \{cl\,V : V \in \mathcal{V}\}$ refines $f^{-1}\mathcal{U}$. Since f is proper and onto, then $f\,cl\,\mathcal{V} = \{f(cl\,V) : V \in \mathcal{V}\}$ is a locally finite closed cover of Y which refines $\mathcal{U}$. (However, $f\,cl\,\mathcal{V}$ does <u>not</u> necessarily have order $\leq n+1$.) For each $V \in \mathcal{V}$, choose a $U(V) \in \mathcal{U}$ for which $f(cl\,V) \subset U(V)$. Since $f\,cl\,\mathcal{V}$ is locally finite, there is an open cover $\mathcal{W}$ of Y such that for each $V \in \mathcal{V}$, $\cup\{W \in \mathcal{W} : f(cl\,V) \cap W \neq \emptyset\} \subset U(V)$. (For each $y \in Y$, let $W_y =$
$\cap\{U(V) : V \in \mathcal{V} \text{ and } y \in f(cl\,V)\} - \cup\{f(cl\,V) : V \in \mathcal{V} \text{ and } y \notin f(cl\,V)\}$. Let $\mathcal{W} = \{W_y : y \in Y\}$. $\mathcal{W}$ is the desired open cover of Y.)

By hypothesis, there is a map $g : Y \to X$ such that $f \circ g$ is within $\mathcal{W}$ of $1|Y$. Now $g^{-1}\mathcal{V}$ is a locally finite open cover of Y of order $\leq n+1$. It remains to show that $g^{-1}\mathcal{V}$ refines $\mathcal{U}$. In fact, we shall show that $g^{-1}(V) \subset U(V)$ for each $V \in \mathcal{V}$. Let $V \in \mathcal{V}$. Let $y \in g^{-1}(V)$. There is a $W \in \mathcal{W}$ such that $y, f\circ g(y) \in W$. Since $g(y) \in V$, then $f\circ g(y) \in f(cl\,V)$. So $f(cl\,V) \cap W \neq \emptyset$. Now our choice of $\mathcal{W}$ guarantees that $W \subset U(V)$. So $y \in U(V)$. ∎

Since fine homotopy equivalences have approximate inverses, we get:

<u>Corollary</u>: A fine homotopy equivalence can't raise dimension; i.e., if $f: X \to Y$ is a fine homotopy equivalence, then $\dim Y \leq \dim X$.

<u>Corollary</u>: Suppose X is an ANR, $\dim X < \infty$, and $f: X \to Y$ is a cell-like map. Then the following are equivalent:

(1) $\dim Y \leq \dim X$.

(2) $\dim Y < \infty$.

(3) Y is countable dimensional.

(4) Y is an ANR.

(5) f is a fine homotopy equivalence.

<u>Proof</u>: (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are obvious. Two theorems attributed to Smale et. al. were stated above. The first entails (3) $\Rightarrow$ (4), and the second entails (4) $\Rightarrow$ (5). The preceding Corollary entails (5) $\Rightarrow$ (1). ∎

2. A proof of the Kozlowski - Walsh Theorem

As a warm-up for the proof of the Kozlowski-Walsh Theorem, we will give a self-contained proof of the following result.

Theorem 0: Suppose X is an ANR which is either separable or locally compact, and $f: X \to Y$ is a cell-like map. If $\dim Y < \infty$, then $\dim Y \leq \dim X$.

Before giving the proof of Theorem 0, we recall a number of definitions which will be used throughout this section.

Let $\mathcal{V}$ be an open cover of a space Y. For $A \subset Y$, we let $\underline{\text{Star}(A, \mathcal{V})} = \cup\{V \in \mathcal{V} : A \cap V \neq \emptyset\}$. We say that $\mathcal{V}$ star-refines an open cover $\mathcal{U}$ of Y if for every $y \in Y$, $\text{Star}(y, \mathcal{V})$ is contained in an element of $\mathcal{U}$. Each open cover of a metric space Y is star-refined by some other open cover of Y.

Let $\mathcal{V}$ be an open cover of a space Y. The nerve of $\mathcal{V}$ is the collection

$$K = \{\sigma \subset \mathcal{V} : \sigma \text{ is finite and } \cap \sigma \neq \emptyset\}.$$

If $\sigma \in K$, then $\underline{\partial \sigma} = \{\tau \subset \sigma : \tau \neq \sigma\}$ and $\underline{\dim \sigma}$ is one less than the number of elements in σ. For $n \geq 0$, $\underline{K^{(n)}} = \{\sigma \in K : \dim \sigma \leq n\}$.

Let $\mathcal{V}$ be an open cover of a space Y, and let K be its nerve. Let $\underline{l_2(\mathcal{V})}$ be the Banach space of all square-summable functions $\alpha : \mathcal{V} \to \mathbb{R}$ with norm $\|\alpha\| = (\Sigma\{(\alpha(V))^2 : V \in \mathcal{V}\})^{1/2}$. The polyhedron underlying K, denoted $\underline{|K|}$, is the subspace of $l_2(\mathcal{V})$ consisting of all $\alpha : \mathcal{V} \to \mathbb{R}$ which satisfy the following three conditions:
(1) $\alpha(\mathcal{V}) \subset [0,1]$, (2) $\alpha^{-1}(0,1] \in K$, and
(3) $\Sigma\{\alpha(V) : V \in \mathcal{V}\} = 1$. For $\sigma \in K$, we let $|\sigma| = \{\alpha \in |K| : \alpha^{-1}(0,1] \subset \sigma\}$, $\operatorname{int}|\sigma| = \{\alpha \in |K| : \alpha^{-1}(0,1] = \sigma\}$, and $\partial|\sigma| = |\sigma| - \operatorname{int}|\sigma|$. Then $\{\operatorname{int}|\sigma| : \sigma \in K\}$ is a disjoint collection whose union is $|K|$. For $L \subset K$, we let $|L| = \cup\{|\sigma| : \sigma \in L\}$. If $\sigma \in K$ and $\dim \sigma = n$, then $|\sigma|$ is an n-cell and $\partial|\sigma| = |\partial\sigma|$ is its $(n-1)$-sphere boundary.

Again let $\mathcal{V}$ be an open cover of a space Y, and let K be its nerve. K is <u>locally finite</u> if for each $\sigma \in K$, the set $\{\tau \in K : \sigma \subset \tau\}$ is finite. $\mathcal{V}$ is <u>star-finite</u> if for each $V \in \mathcal{V}$, the set $\{W \in \mathcal{V} : V \cap W \neq \emptyset\}$ is finite. The following are equivalent: (1) $\mathcal{V}$ is star-finite, (2) K is locally finite, (3) $|K|$ is locally compact. If a metric space Y is either locally compact or separable, then each open cover of Y is refined by a star-finite open cover of Y.

Again let $\mathcal{V}$ be an open cover of a space Y, and let K be its nerve. A map $\varphi: Y \to |K|$ is a <u>barycentric map</u> if it satisfies either of the following equivalent conditions:
(1) if $y \in Y$, $\sigma \in K$ and $\varphi(y) \in \operatorname{int} |\sigma|$, then $y \in \bigcap \sigma$;
(2) if $y \in Y$, $V \in \mathcal{V}$ and $\varphi(y)(V) > 0$, then $y \in V$.
Barycentric maps always exist. For, take $\{\gamma_V : V \in \mathcal{V}\}$ to be a partition of unity subordinate to $\mathcal{V}$; i.e.,
(1) for each $V \in \mathcal{V}$, $\gamma_V : Y \to [0,1]$ is a map with $\gamma_V^{-1}(0,1] \subset V$;
(2) $\{\gamma_V^{-1}(0,1] : V \in \mathcal{V}\}$ is a locally finite open cover of Y; and
(3) $\Sigma\{\gamma_V(y) : V \in \mathcal{V}\} = 1$ for each $y \in Y$.
Then a barycentric map $\varphi: Y \to |K|$ is defined by the formula $\varphi(y)(V) = \gamma_V(y)$ for $y \in Y$, $V \in \mathcal{V}$.

<u>Proof of Theorem Q</u>: We have isolated the central idea of this proof in Lemma B below, because we shall use this idea again in the proof of the Kozlowski–Walsh Theorem.

Let $\mathcal{U}$ be an open cover of Y. By Lemma A, it suffices to find a map $g: Y \to X$ such that $f \circ g$ is within $\mathcal{U}$ of 1_Y.

Let $\mathcal{V}_0$ be the open cover of Y provided by Lemma B corresponding to the open cover $\mathcal{U}$ of Y and the choice $n = \dim Y$. Since X is either separable or locally compact, and f maps X properly onto Y, then Y is either separable or locally compact. Also $\dim Y = n$. Consequently, $\mathcal{V}_0$ is refined by a star-finite open cover $\mathcal{V}$ of Y of order $\leq n+1$. (Proof: $\mathcal{V}_0$ is refined by a star-finite open cover $\mathcal{W}$ of Y, and $\mathcal{W}$ is refined by an open cover $\mathcal{T}$ of Y of order $\leq n+1$. Let $\alpha : \mathcal{T} \to \mathcal{W}$ be a function such that $T \subset \alpha(T)$ for each $T \in \mathcal{T}$. For each $W \in \mathcal{W}$, let $V(W) = \bigcup\{T \in \mathcal{T} : \alpha(T) = W\}$. Then $\mathcal{V} = \{V(W) : W \in \mathcal{W}\}$ is a star-finite open cover of Y which refines $\mathcal{V}_0$ and has order $\leq n+1$.)

Let K be the nerve of $\mathcal{V}$. Then there is a map $\psi : |K^{(n)}| \to X$ such that if $\sigma \in K^{(n)}$, then there is a $U \in \mathcal{U}$ for which $U\sigma \subset U$ and $\psi|\sigma| \subset f^{-1}U$. Since the order of $\mathcal{V}$ is $\leq n+1$, then $K^{(n)} = K$, and ψ maps <u>all</u> of $|K|$ into X.

Let $\varphi : Y \to |K|$ be a barycentric map. Define $g : Y \to X$ by $g = \psi \circ \varphi$. We now argue that $f \circ g$ is within $\mathcal{U}$ of $1|Y$. Let $y \in Y$. Choose $\sigma \in K$ so that $\varphi(y) \in \operatorname{int}|\sigma|$. Then $y \in \bigcap\sigma$ because φ is a barycentric map. There is a $U \in \mathcal{U}$ such that $U\sigma \subset U$ and $\psi|\sigma| \subset f^{-1}U$. It follows that $y \in U$ and $g(y) = \psi\circ\varphi(y) \in f^{-1}U$. So $y, f\circ g(y) \in U$. ∎

<u>Lemma B</u>: Suppose X is an ANR and $f: X \to Y$ is a cell-like map. For every open cover $\mathcal{U}$ of Y, for every $n \geq 0$, there is an open cover $\mathcal{V}_0$ of Y with the following property. If $\mathcal{V}$ is any star-finite open cover of Y which refines $\mathcal{V}_0$ and K is the nerve of $\mathcal{V}$, then there is a map $\psi: |K^{(n)}| \to X$ such that for each $\sigma \in K^{(n)}$, there is a $U \in \mathcal{U}$ such that $U\sigma \subset U$ and $\psi|\sigma| \subset f^{-1}U$.

<u>Proof</u>: Let $\mathcal{V}_n = \mathcal{U}$. We can choose n open covers $\mathcal{V}_{n-1}, \mathcal{V}_{n-2}, \dots, \mathcal{V}_0$ of Y so that for $0 \leq i \leq n-1$, if $y \in Y$, then there is a $V \in \mathcal{V}_{i+1}$ such that $f^{-1}(\mathrm{Star}(y, \mathcal{V}_i))$ contracts to a point in $f^{-1}V$. If $0 \leq i \leq n-1$ and we have $\mathcal{V}_{i+1}$, then we obtain $\mathcal{V}_i$ in two steps. First, since $f: X \to Y$ cell-like and proper, there is an open cover $\mathcal{W}$ of Y such that for each $W \in \mathcal{W}$, there is a $V \in \mathcal{V}_{i+1}$ so that $f^{-1}W$ contracts to a point in $f^{-1}V$. Second take $\mathcal{V}_i$ to be a star-refinement of $\mathcal{W}$. Observe that $\mathcal{V}_i$ must star-refine $\mathcal{V}_{i+1}$ for $0 \leq i \leq n-1$.

Suppose $\mathcal{V}$ is a star-finite open cover of Y which refines $\mathcal{V}_0$ and K is the nerve of $\mathcal{V}$. For $0 \leq i \leq n$ we shall inductively obtain maps $\psi^i: |K^{(i)}| \to X$ with the following property:

$\mathcal{P}_i$: For each $\sigma \in K^{(i)}$, there is a $V_i(\sigma) \in \mathcal{V}_i$ such that $U\sigma \subset V_i(\sigma)$ and $\psi^i|\sigma| \subset f^{-1}V_i(\sigma)$.

Since $\mathcal{V}_n = \mathcal{U}$, we will be done when we have $\psi^n : |K^{(n)}| \to X$ satisfying P_n.

To begin, for each $V \in \mathcal{V}$, choose $\psi^0 | \{V\}|$ to be a point in $f^{-1}V$. This clearly determines $\psi^0 : |K^{(0)}| \to X$ so that it satisfies P_0.

Let $0 \le i \le n-1$ and inductively assume we have a map $\psi^i : |K^{(i)}| \to X$ which satisfies P_i. Now consider a $\sigma \in K^{(i+1)}$ with $\dim \sigma = i+1$. Let $y \in \cap \sigma$ and choose $V_{i+1}(\sigma) \in \mathcal{V}_{i+1}$ so that $f^{-1}(\mathrm{Star}(y, \mathcal{V}_i))$ contracts to a point in $f^{-1}V_{i+1}(\sigma)$. Since $\sigma \subset \mathcal{V}$ and $\mathcal{V}$ refines $\mathcal{V}_i$, then $\cup \sigma \subset \mathrm{Star}(y, \mathcal{V}_i) \subset V_{i+1}(\sigma)$. (This verifies part of P_{i+1}.) For each $\tau \in \partial \sigma$, we prove that $\psi^i |\tau| \subset f^{-1}\mathrm{Star}(y, \mathcal{V}_i)$ as follows: There is a $V_i(\tau) \in \mathcal{V}_i$ such that $\cup \tau \subset V_i(\tau)$ and $\psi^i |\tau| \subset f^{-1} V_i(\tau)$; and $V_i(\tau) \subset \mathrm{Star}(y, \mathcal{V}_i)$ because $y \in \cap \sigma \subset \cap \tau \subset V_i(\tau)$. We conclude that $\psi |\partial \sigma| \subset f^{-1}\mathrm{Star}(y, \mathcal{V}_i)$. Since $f^{-1}\mathrm{Star}(y, \mathcal{V}_i)$ contracts to a point in $f^{-1}V_{i+1}(\sigma)$, then $\psi^i \,|\, |\partial\sigma|$ extends to a map of $|\sigma|$ into $f^{-1}V_{i+1}(\sigma)$. In this way $\psi^i : |K^{(i)}| \to X$ extends to a map $\psi^{i+1} : |K^{(i+1)}| \to X$. The continuity of ψ^{i+1} follows from the local finiteness of K which is guaranteed by the star-finiteness of $\mathcal{V}$. Moreover, ψ^{i+1} satisfies P_{i+1}. Indeed, for $\sigma \in K^{(i+1)}$:

if $\dim \sigma = i+1$, then P_{i+1} holds by construction; and if $\dim \sigma \le i$, P_{i+1} holds because ψ^i satisfies P_i, ψ^{i+1} extends ψ^0, and $\mathcal{V}_i$ refines $\mathcal{V}_{i+1}$. ■

We can use Lemma B to prove Theorem 0 because we know that $\dim Y < \infty$. Suppose we don't initially know that $\dim Y < \infty$, but we make a choice of n and apply Lemma B anyway. Then we obtain an open cover $\mathcal{V}$ of Y and a map $\psi : |K^{(n)}| \to X$ where K is the nerve of $\mathcal{V}$. Since we don't know that $\dim Y \le n$, we can't require that $\mathcal{V}$ have order $\le n+1$, and we can't conclude that $K^{(n)} = K$. Thus, the domain of ψ is not all of $|K|$. Lemma C will show that we can extend ψ to a map of <u>all</u> of $|K|$ into X if:

(1) there is a cover $\mathcal{W}$ of X by open sets whose finite intersections have trivial higher homotopy, and

(2) ψ maps the n-skeleton of each simplex of K into an element of $\mathcal{W}$.

It so happens that if X is a 3-manifold, we can find an open cover $\mathcal{W}$ of X and control ψ so that these two conditions hold. This is how we prove the Kozlowski-Walsh Theorem.

<u>Lemma C</u>: Suppose K is the nerve of an open cover of some space, K is locally finite, $n \geq 1$, $\psi : |K^{(n)}| \to X$ is a map, $\mathcal{W}$ is an open cover of X, and the following two conditions hold:

(1) If $W_1, W_2, \ldots, W_k \in \mathcal{W}$, then for each $p \geq n$, every map of the p-sphere S^p into $\bigcap_{i=1}^{k} W_i$ is homotopically trivial.

(2) For each $\sigma \in K$, there is a $W(\sigma) \in \mathcal{W}$ such that $\psi|\sigma^{(n)}| \subset W(\sigma)$, where $\sigma^{(n)} = \{\tau \subset \sigma : \dim \tau \leq n\}$. ($\sigma^{(n)}$ is the "n-skeleton" of σ.) Then ψ extends to a map $\overline{\psi} : |K| \to X$ such that $\overline{\psi}|\sigma| \subset W(\sigma)$ for each $\sigma \in K$.

<u>Proof</u>: For each $\sigma \in K$, let $W^*(\sigma) = \bigcap\{W(\tau) : \sigma \subset \tau \in K\}$. We verify that $\psi|\sigma^{(n)}| \subset W^*(\sigma)$ for each $\sigma \in K$. We must prove that if $\sigma, \tau \in K$ and $\sigma \subset \tau$, then $\psi|\sigma^{(n)}| \subset W(\tau)$. Simply observe that $\sigma^{(n)} \subset \tau^{(n)}$ and $\psi|\tau^{(n)}| \subset W(\tau)$.

Let $\psi^n = \psi$. For each $p > n$, we shall inductively construct a map $\psi^p : |K^{(p)}| \to X$ such that $\psi^p|\,|K^{(p-1)}| = \psi^{p-1}$ and $\psi^p|\sigma^{(p)}| \subset W^*(\sigma)$ for each $\sigma \in K$.

Assume $p > n$ and we have a map $\psi^{p-1} : |K^{(p-1)}| \to X$ such that $\psi^{p-1}|\sigma^{(p-1)}| \subset W^*(\sigma)$ for each $\sigma \in K$. Now consider a $\sigma \in K^{(p)}$ with $\dim \sigma = p$. Since $\sigma^{(p-1)} = \partial\sigma$, then $\psi^{p-1}|\partial\sigma| \subset W^*(\sigma)$.

$W^*(\sigma)$ is the intersection of finitely many elements of $\mathcal{W}$ because K is locally finite. So each map of a $(p-1)$-sphere into $W^*(\sigma)$ is homotopically trivial. Thus $\psi^{p-1}| \,|\partial\sigma|$ extends to a map of $|\sigma|$ into $W^*(\sigma)$. In this way, $\psi^{p-1}: |K^{(p-1)}| \to X$ extends to a map $\psi^p : |K^{(p)}| \to X$. The continuity of ψ^p follows from the local finiteness of K. To verify that $\psi^p |\sigma^{(p)}| \subset W^*(\sigma)$ for each $\sigma \in K$, we must show that if $\rho, \sigma, \tau \in K$, $\rho \in \sigma^{(p)}$ and $\sigma \subset \tau$, then $\psi^p |\rho| \subset W(\tau)$. Since $\rho \subset \sigma \subset \tau$, then $W^*(\rho) \subset W(\tau)$; so it suffices to show that $\psi^p|\rho| \subset W^*(\rho)$. If $\dim\rho = p$, then $\psi^p|\rho| \subset W^*(\rho)$ by construction. If $\dim\rho \leq p-1$, then $\psi^p|\rho| = \psi^{p-1}|\rho^{(p-1)}| \subset W^*(\rho)$.

Now the desired map $\overline{\psi} : |K| \to X$ is defined by setting $\overline{\psi} \,|\, |K^{(p)}| = \psi^p$ for $p \geq n$. The continuity of ψ follows from the local finiteness of K. ■

<u>The Kozlowski-Walsh Theorem</u>: If M^3 is a 3-manifold and $f: M^3 \to Y$ is a cell-like map, then $\dim Y \leq 3$.

<u>Remark</u>: Kozlowski and Walsh actually obtain a stronger result: If M^3 is a 3-manifold, X is any subset of M^3, and $f: X \to Y$ is a proper onto map such that $f^{-1}y$ is a cell-like set for each $y \in Y$, then $\dim Y \leq \dim X$. They do not assume either that X is an ANR or that X is a closed subset of M^3. In the case that X is an ANR and a closed subset of M^3, then we can easily deduce the stronger result from the Theorem stated above in the following way. Let $\bar{Y}$ be the quotient space obtained from M^3 by identifying the point-inverses of f to points. Then we can regard Y as a subset of $\bar{Y}$, and we can regard the quotient map $\bar{f}: M^3 \to \bar{Y}$ as an extension of $f: X \to Y$. Since $\bar{f}$ is cell-like, the Theorem implies $\dim \bar{Y} \leq 3$. Hence $\dim Y \leq 3$. So Theorem 0 implies that $\dim Y \leq \dim X$. If X is not an ANR, then we can't invoke Theorem 0; so $\dim Y \leq 3$ is the best we can get. If X is not a closed subset of M^3, then the quotient map $\bar{f}: M^3 \to \bar{Y}$ is not necessarily proper, and this simple argument yields nothing. However the proof which we are about to give of this Theorem can be modified to a proof of the stronger result. We will comment on these modifications after we present the proof.

<u>Definition</u>: A space X is <u>aspherical</u> if for every $p \geq 2$, every map of the p-sphere S^p into X is homotopically trivial.

<u>Proof of the Kozlowski-Walsh Theorem</u>: Let $\mathcal{U}$ be an open cover of Y. By Lemma A, it suffices to find a map $g: Y \to M^3$ such that $f \circ g$ is within $\mathcal{U}$ of $1|Y$.

Let $\mathcal{U}^*$ be an open cover of Y which star-refines $\mathcal{U}$; and for each $y \in Y$, choose $U_y \in \mathcal{U}^*$ so that $y \in U_y$. Our ability to prove this Theorem depends crucially on two facts about 3-manifolds which are established in section 3: Corollary F2 and Corollary F3. We now invoke Corollary F2 to obtain for each $y \in Y$ an aspherical open subset W_y of M^3 such that $f^{-1} y \subset W_y \subset f^{-1} U_y$. We also require that if $y, z \in Y$, then $W_y \cup W_z$ is a proper subset of M^3. Let $\mathcal{W} = \{W_y : y \in Y\}$. Since $f: M^3 \to Y$ is a proper map, there is an open cover $\mathcal{U}^{**}$ of Y such that for each $y \in Y$, there is a $W \in \mathcal{W}$ such that $f^{-1}(\mathrm{Star}(y, \mathcal{U}^{**})) \subset W$. Observe that $\mathcal{U}^{**}$ must refine $\mathcal{U}^*$.

Let $\mathcal{V}_0$ be the open cover of Y provided by Lemma B corresponding to the open cover $\mathcal{U}^{**}$

and the choice $n=2$. Since M is separable and f maps M onto Y, then Y is also separable. Hence there is a star-finite open cover $\mathcal{V}$ of Y which refines $\mathcal{V}_0$. Let K be the nerve of $\mathcal{V}$. Then there is a map $\psi : |K^{(2)}| \to M$ such that for each $\sigma \in K^{(2)}$, there is a $U \in \mathcal{U}^{**}$ for which $U\sigma \subset U$ and $\psi|\sigma| \subset f^{-1}U$.

We shall now show that the hypotheses of Lemma C hold for our choice of $n=2$, so that ψ will extend to all of $|K|$. Corollary F3 and induction imply that any finite number of elements of $\mathcal{W}$ have aspherical intersection. We must also argue that for each $\sigma \in K$, there is a $W(\sigma) \in \mathcal{W}$ such that $\psi|\sigma^{(2)}| \subset W(\sigma)$ where $\sigma^{(2)} = \{\tau \subset \sigma : \dim \tau \leq 2\}$. Let $\sigma \in K$. Let $y \in \bigcap \sigma$. Choose $W(\sigma) \in \mathcal{W}$ so that $f^{-1}(\mathrm{Star}(y, \mathcal{U}^{**})) \subset W(\sigma)$. Let $\tau \in \sigma^{(2)}$. We must show $\psi|\tau| \subset W(\sigma)$. There is a $U \in \mathcal{U}^{**}$ such that $U\tau \subset U$ and $\psi|\tau| \subset f^{-1}U$. Since $\bigcap\sigma \subset \bigcap\tau$, then $y \in U$. So $f^{-1}U \subset f^{-1}\mathrm{Star}(y, \mathcal{U}^{**})$. We conclude that $\psi|\tau| \subset W(\sigma)$. Now Lemma C provides a map $\bar{\psi} : |K| \to M$ which extends $\psi : |K^{(2)}| \to M$ such that $\bar{\psi}|\sigma| \subset W(\sigma)$ for each $\sigma \in K$.

Let $\varphi : Y \to |K|$ be a barycentric map. Define $g : Y \to M$ by $g = \bar{\psi} \circ \varphi$. We now argue that $f\circ g$ is within $\mathcal{U}$ of 1_Y. Let $y \in Y$. Choose $\sigma \in K$ so that $\varphi(y) \in \operatorname{int}|\sigma|$. Then $y \in \bigcap\sigma$ because φ is a barycentric map. Let $V \in \sigma$. Then $y \in V$. Since $\{V\} \in K^{(2)}$ and $\mathcal{U}^{**}$ refines $\mathcal{U}^*$, then there is a $U_1 \in \mathcal{U}^*$ such that $V \subset U_1$ and $\psi|\{V\}| \in f^{-1}U_1$. Therefore $y \in U_1$ and $\bar{\psi}|\{V\}| \in f^{-1}U_1$. Since $\mathcal{W}$ refines $f^{-1}\mathcal{U}^*$, there is a $U_2 \in \mathcal{U}^*$ such that $\bar{\psi}|\sigma| \subset f^{-1}U_2$. So $g(y) = \bar{\psi}\circ\varphi(y) \in f^{-1}U_2$, and consequently $f\circ g(y) \in U_2$. Since $\{V\} \subset \sigma$, then $\bar{\psi}|\{V\}| \in \bar{\psi}|\sigma|$. Thus $\bar{\psi}|\{V\}| \in f^{-1}U_1 \cap f^{-1}U_2$. We conclude that $y \in U_1$, $f\circ g(y) \in U_2$ and $U_1 \cap U_2 \neq \emptyset$. Since $\mathcal{U}^*$ star-refines $\mathcal{U}$, there is a $U \in \mathcal{U}$ such that $U_1 \cup U_2 \subset U$. Therefore y, $f\circ g(y) \in U$. ∎

As we noted earlier, Kozlowski and Walsh proved a better theorem than we attributed to them. They actually obtained:

<u>The Strong Kozlowski-Walsh Theorem</u>: If M^3 is a 3-manifold, X is any subset of M^3, and $f: X \to Y$ is a proper onto map such that $f^{-1}y$ is a cell-like set for each $y \in Y$, then $\dim Y \leq \dim X$.

We now indicate how the proof of the Kozlowski-Walsh Theorem given above can be adapted to prove the Strong Kozlowski-Walsh Theorem.

Suppose $X \subset M$ and $f: X \to Y$ is a proper onto map. In this situation we say that a collection $\mathcal{U}$ of open subsets of M is a <u>saturated open cover of X</u> if $\mathcal{U}$ covers X and $f^{-1}f(U \cap X) = U \cap X$ for each $U \in \mathcal{U}$. This condition makes $f\mathcal{U} = \{f(U \cap X) : U \in \mathcal{U}\}$ an open cover of Y.

The principal difference between the proof given above and the proof of the stronger theorem is the following: The role played in the above proof by an open cover of Y must be played in the proof of the stronger result by a saturated open cover of X. Lemmas A and B are transformed accordingly into the following propositions. Lemma C needs no modification.

<u>Lemma A'</u>: Suppose $X \subset M$ and $f: X \to Y$ is a proper onto map. If for every saturated open cover $\mathcal{U}$ of X, there is a map $g: Y \to X$ such that gof is within $\mathcal{U}$ of the inclusion $X \hookrightarrow M$, then $\dim Y \leq \dim X$.

<u>Lemma B'</u>: Suppose M is an ANR, $X \subset M$ and $f: X \to Y$ is a proper map such that $f^{-1}y$ is a cell-like set for each $y \in Y$. For every saturated open cover $\mathcal{U}$ of X, for every $n \geq 0$, there is a saturated open cover $\mathcal{U}_0$ of X with the following

property. If $\mathcal{V}$ is any star-finite open cover of Y which refines $f\mathcal{U}_0$, and K is the nerve of $\mathcal{V}$, then there is a map $\psi : |K^{(n)}| \to M$ such that for each $\sigma \in K^{(n)}$, there is a $U \in \mathcal{U}$ such that $\bigcup \sigma \subset f(U \cap X)$ and $\psi|\sigma| \subset U$.

Modulo these alterations, the proof proceeds in outline as it did above. As might be expected, the details become more intricate, but not in any significant way.

Finally we make some remarks on the possibility of extending the methods of proof used here to higher dimensions. Suppose $n > 3$, M^n is an n-manifold, and $f: M^n \to Y$ is a cell-like map. Unfortunately, the two crucial 3-manifold results, Corollary F2 and Corollary F3, are false in higher dimensions.

The failure of Corollary F2 in higher dimensions is not really significant; for it would suffice to prove the theorem for those maps $f: M^n \to Y$ whose point-inverses are cellular. The point-inverses of such maps come equipped with arbitrarily tight <u>contractible</u> open neighborhoods. So the high-dimensional analogue of Corollary F2 is unneeded. The reason it suffices to prove that maps with cellular point inverses can't raise dimension is the following: If the map $f: M^n \to Y$ is cell-like, then the map $f \times 1|\mathbb{R}: M^n \times \mathbb{R} \to Y \times \mathbb{R}$ has cellular point inverses *, and obviously if $\dim Y \times \mathbb{R} < \infty$, then $\dim Y < \infty$.

The failure of Corollary F3 in high dimensions is a significant obstacle to generalizing the proof. Even if f has cellular point inverses, we can't avoid the failure of Corollary F3. Indeed, it is not hard to visualize two open 4-cells in $\mathbb{R}^4$ which intersect is an open set which has the homotopy type of S^2; and $\pi_i(S^2) \neq 0$ for infinitely many values of i. (Picture the two open 4-cells as open neighborhoods of the northern and southern hemispheres of S^3 in $\mathbb{R}^4$.)

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* If X is a cell-like subset of M^n, then $X \times \{t\}$ is a cellular subset of $M^n \times \mathbb{R}$ for each $t \in \mathbb{R}$.

3. Aspherical Subsets of 3-manifolds

Unless specified otherwise, "n-manifold" means "connected boundaryless PL n-manifold".

There is a very elementary appealingly geometric argument which has many applications in geometric topology, and which we shall use several times, called the one-ended arc argument. Typically, it is employed to establish a particular case of the following general fact.

Lemma D (The one-ended arc argument): The following situation is impossible: M^n is an n-manifold. K^{k+1} is a compact $(k+1)$-manifold with boundary, $f: K \to M$ is a map, and D^k is a PL k-cell in ∂K such that $f|D: D \to M$ is a PL embedding. L^{n-k} is an $(n-k)$-manifold PL embedded as a closed subset of M^n. $f^{-1}(L) \cap \partial K$ consists of a single point, p, in int D; and $f(D)$ intersects L transversely at $f(p)$ (i.e., $f(p)$ has a PL n-cell neighborhood C^n in M such that there is a PL homeomorphism of $(C, f(D) \cap C, L \cap C, f(p))$ onto $(B^k \times B^{n-k}, B^k \times 0, 0 \times B^{n-k}, 0 \times 0)$).

Proof: If this situation were realized, then we could adjust the map $f: K \to M$ slightly, retaining the above properties, so that f becomes a PL map in general position with respect to L. Then $f^{-1}L$ is a disjoint union of finitely many simple closed curves in $\operatorname{int} K$, and finitely many arcs in K whose interiors lie in $\operatorname{int} K$ and whose boundaries lie in ∂K. If A is the arc component of $f^{-1}L$ which has one endpoint at p, then the other endpoint of A must also lie in ∂K. But $f^{-1}L \cap \partial K = \{p\}$. So A must be a "one-end arc". ■

One significant application of Lemma D is the <u>No Retraction Theorem</u>: There is no retraction of B^{n+1} onto S^n. <u>Proof</u>: Apply Lemma D with $M^n = S^n$, $K = B^{n+1}$, $L =$ a point in S^n, and $f: B^{n+1} \to S^n$ a retraction map $(f|S^n = 1|S^n)$.

<u>Corollary D1</u>: Suppose M^n is an n-manifold and F^{n-1} is a compact $(n-1)$-manifold PL embedded in M^n. If F contracts to a point in M, then F separates M and one of the components of $M-F$ has compact closure.

<u>Proof</u>: Let $f: F \times [0,1] \to M$ be a homotopy such that $f_0 = 1|F$ and $f(F \times 1)$ is a point. If F doesn't separate M, then there is a simple closed curve L PL embedded in M

so that $L \cap F$ is a single point at which L "pierces through" F, and $L \cap f(F \times 1) = \emptyset$. This contradicts Lemma D.

Next, if both components of $M - F$ have non-compact closure, then there is a PL embedding of the real line in M as a closed subset, L, of M such that $L \cap F$ consists of a single point at which L "pierces through" F, and $L \cap f(F \times I) = \emptyset$. This contradicts Lemma D. ■

<u>Corollary **D2**</u>: Suppose X is a compact subset of an n-manifold M^n. If X contracts to a point in an open subset U of M^n, then $M - U$ lies in a single component of $M - X$.

<u>Proof</u>: Using the fact that U is an ANR, we deduce that some neighborhood of X in U contracts to a point in U. Thus, there is a compact n-manifold with boundary, N, which is PL embedded in M so that $X \subset \operatorname{int} N$ and N contracts to a point in U.

Let $p, q \in M - U$. Let L be a PL arc in M joining p to q. We can adjust L so that for each component F of ∂N, $L \cap F$

is either empty or a single point at which L "pierces through" F. (If $L \cap F$ contains more than one point, reroute L "parallel" to F between its first and last points of intersection with F.) Now Lemma D rules out the possibility that L intersects any component of ∂N. So $L \cap \partial N = \emptyset$. Since $\partial L \subset M - N$, then $L \cap N = \emptyset$. Thus $L \cap X = \emptyset$. So p and q are joined by an arc in $M - X$. ■

<u>Corollary D3</u>: Suppose X is a cell-like subset of an n-manifold M^n. If U is a connected open neighborhood of X in M^n, then $U - X$ is connected.

<u>Proof</u>: Let $p, q \in U - X$. Since X contracts to a point in $U - \{p, q\}$, then Corollary D2 implies that p and q lie in the same component of $U - X$. ■

<u>Corollary D4</u>: Suppose X is a cell-like subset of an n-manifold M^n. If U is an open neighborhood of X in M^n, then there is a compact n-manifold, N, with <u>connected boundary</u> which is PL embedded in M^n so that $X \subset \operatorname{int} N$ and $N \subset U$.

Proof: There is a compact n-manifold with boundary, P, PL embedded in M^n so that $X \subset \operatorname{int} P$ and $P \subset U$. Let $F_0, F_1, \ldots, F_k$ be the components of ∂P. Corollary D3 implies that $\operatorname{int} P - X$ is connected. Consequently $P - X$ is connected, and we can find disjoint PL arcs $A_1, A_2, \ldots, A_k$ in $P - X$ such that A_i joins F_0 to F_i for $1 \le i \le k$. By "drilling a hole" in P along each A_i, for $1 \le i \le k$, we obtain a compact n-manifold, N, with <u>connected</u> boundary as desired. ∎

We now recall some elementary facts about orientation in a manifold. Let M^n be an n-manifold possibly with boundary. A <u>chart</u> in M is a pair (U, h) where U is an open subset of M and h is a homeomorphism from U onto an open subset of $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n \geq 0\}$. If (U, h) and (V, k) are charts in M, then the homeomorphism

$$k \circ h^{-1} | h(U \cap V) : h(U \cap V) \to k(U \cap V)$$

is called a <u>transition map</u> associated with the two charts (U, h) and (V, k). M is <u>orientable</u> if it can be covered by charts all of whose associated transition maps are orientation-preserving homeomorphisms between open subsets of $\mathbb{R}^n_+$. Let $\alpha : [0,1] \to M$ be a loop in M: $\alpha(0) = \alpha(1)$: α is <u>orientation-preserving</u> if there is a partition $0 = t_0 < t_1 < \cdots < t_k = 1$ of $[0,1]$ and there are charts (U_i, h_i) in M for $0 \le i \le k$ so that $\alpha[t_{i-1}, t_i] \subset U_i$ for $1 \le i \le k$, $(U_0, h_0) = (U_k, h_k)$, and the transition map associated with the two charts (U_{i-1}, h_{i-1}) and (U_i, h_i) is an orientation-preserving homeomorphism between open subsets of $\mathbb{R}^n_+$ for $1 \le i \le k$. If α is not orientation-preserving, it is <u>orientation-reversing</u>.

<u>Fact</u>: M^n is orientable if and only if each loop in M^n is orientation-preserving. <u>Fact</u>: If two loops in M^n are homotopic, and one is orientation-preserving, then so is the other. <u>Fact</u>: A constant loop in M^n is orientation-preserving. <u>Corollary</u>: A simply-connected manifold is orientable. <u>Corollary</u>: If N^n is an n-manifold, possibly with boundary, which is embedded in M^n, and if every loop in N contracts to a point in M, then N is orientable.

A <u>homotopy 3-cell</u> is a compact contractible 3-manifold whose boundary is a 2-sphere. The Poincaré Conjecture is equivalent to the statement: Every homotopy 3-cell is a 3-cell.

Lemma E: A compact simply connected 3-manifold with connected boundary is a homotopy 3-cell.

Proof: Let C be a compact simply connected 3-manifold with connected boundary.

To show that C is contractible, it suffices to prove that $\pi_i(C)=0$ for all $i\geq 1$. For then the map of $C\times\{0,1\}$ into C which is the identity on $C\times 0$ and constant on $C\times 1$ can be extended to a map of $C\times[0,1]$ into C by working up in dimension through the skeleta of a triangulation of $C\times[0,1]$. $\pi_1(C)=0$ by hypothesis. We will show that $\pi_i(C)=0$ for $i\geq 2$ by arguing that $H_i(C)=0$ for $i\geq 2$ and invoking the Hurewicz Isomorphism Theorem.

Since C has boundary, it collapses to a 2-dimensional polyhedron. Thus $H_i(C)=0$ for $i\geq 3$. It remains to show that $H_2(C)=0$, and that ∂C is a 2-sphere.

Algebraic proof that $H_2(C)=0$ and ∂C is a 2-sphere: Since C is simply connected, it is orientable. Therefore the Lefschetz Duality and Universal Co-efficient Theorems tell us that

$$H_2(C,\partial C) \approx H^1(C) \approx FH_1(C) + T.H_0(C)$$

where $FH_1(C)$ is the free submodule of $H_1(C)$ and $TH_0(C)$ is the torsion submodule of $H_0(C)$. Now $H_1(C)=0$ because $\pi_1(C)=0$, and $H_0(C)$ is a free module. So $H_2(C,\partial C)=0$.

Consider the exact sequence

$$H_2(\partial C) \to H_2(C) \to H_2(C,\partial C).$$

Since C is orientable, so is ∂C. Hence $H_2(\partial C)$ is a free module of rank one which is "generated" by ∂C, and ∂C bounds in C. Hence the map $H_2(\partial C) \to H_2(C)$ is the 0-map. Now the exactness of the above sequence tells us that $H_2(C)=0$.

Finally, to see that ∂C is a 2-sphere, consider the exact sequence

$$H_2(C,\partial C) \to H_1(\partial C) \to H_1(C).$$

We argued above that $H_2(C,\partial C)=0$ and $H_1(C)=0$. Consequently, $H_1(\partial C)=0$. We conclude that ∂C must be a 2-sphere.

<u>Sketch of a geometric proof that $H_2(C)=0$ and ∂C is a 2-sphere</u>: Consider an element σ of $H_2(C)$. A "cut-and-paste" construction allows us ~~to~~ represent σ by a 2-manifold L^2 which

is PL embedded in int C. To prove $\sigma = 0$, it suffices to show that L_1 separates C. For then ∂C lies in one of the two components of $C-L$, because ∂C is connected. So the closure of the other component of $C-L$ is a compact 3-manifold whose boundary is precisely L. In other words, L bounds in C.

Assume L_1 doesn't separate C. Then there is a PL simple closed curve K embedded in int C so that $K \cap L$ is a single point at which K "pierces through" L. Since $\pi_1(C) = 0$, there is a homotopy in int C which contracts K to a point which is not in L. This contradicts Lemma D.

Now suppose ∂C is not a 2-sphere. Since $\pi_1 C = 0$, then C and, consequently, ∂C are orientable. So ∂C is a "sphere with handles". In particular, we can find on ∂C two simple closed curves, J and K, such that $J \cap K$ is a single point at which J "pierces through" K. Since $\pi_1(C) = 0$, then $H_1(C) = 0$. Thus J and K bound homologically in C. A "cut-and-paste" construction produces compact 2-manifolds with boundary, F and G, PL embedded in C so that $\partial F = J$, $\partial G = K$, and int $F \cup$ int $G \subset$ int C. If we pull F slightly into ~~int~~ C, we have that int G is a closed subset of int C and $\partial F \cap \text{int}\, G$ consists of a single point at which ∂F "pierces through" int G. This contradicts Lemma D. ■

Recall that a space X is <u>aspherical</u> if for each $n \geq 2$, every map of the n-sphere, S^n, into X is homotopically trivial.

We recall that the various lifting properties of covering spaces entail the following facts. If $p: \tilde{X} \to X$ is a covering projection, then for each $n \geq 2$, p induces an isomorphism of $\pi_n(\tilde{X})$ onto $\pi_n(X)$; so $\tilde{X}$ is aspherical if and only if X is aspherical.

<u>Lemma F</u>: Suppose M^3 is a 3-manifold which is either non-compact or has non-empty boundary. If $\pi_2(M^3) = 0$, then M^3 is aspherical.

<u>Proof</u>: Let $\tilde{M}$ be the universal covering space of M. Then $\pi_1(\tilde{M}) = 0$ and $\pi_n(\tilde{M}) \simeq \pi_n(M)$ for $n \geq 2$. Hence $\pi_2(\tilde{M}) = 0$. Obviously, to finish the proof, it suffices to show that $\pi_n(\tilde{M}) = 0$ for $n \geq 3$. We shall establish this by arguing that $H_n(\tilde{M}) = 0$ for $n \geq 3$ and invoking the Hurewicz Isomorphism Theorem.

Let $n \geq 3$ and let $\sigma \in H_n(\tilde{M})$. σ is represented by a cycle which is supported on some compact set. Since $\tilde{M}$ is either non-compact or has non-empty boundary, then there is a compact 3-manifold with non-empty boundary, P, which is PL embedded in $\tilde{M}$ and which contains the support of the cycle representing σ. Consequently, σ lies in the image of the inclusion induced homomorphism $H_n(P) \to H_n(\tilde{M})$. Since P is a compact 3-manifold with non-empty boundary, P collapses to a 2-dimensional polyhedron. Since $n > 2$, it follows that $H_n(P) = 0$. Therefore $\sigma = 0$. This proves $H_n(\tilde{M}) = 0$. ∎

We are now prepared to present proofs of the facts about aspherical subsets of 3-manifolds needed in the proof of the Kozlowski-Walsh Theorem. These proofs depend on one of the fundamental theorems of 3-manifold topology:

<u>The Sphere Theorem</u> (Papakyriakopolous): Suppose M^3 is an orientable 3-manifold. If $\pi_2(M^3) \neq 0$, then there is a 2-sphere PL embedded in M^3 which does not contract to a point in M^3.

<u>Corollary F1</u>: Suppose M^3 is a 3-manifold and N^3 is a compact 3-manifold with boundary which is PL embedded in M^3. If N^3 contracts to a point in M^3 and ∂N^3 is connected, then N^3 is aspherical.

<u>Proof</u>: According to Lemma F, it is sufficient to show that $\pi_2(N) = 0$. Since each loop in N contracts to a point in M, then each loop in N is orientation-preserving. So N is orientable. Let S be a 2-sphere PL embedded in int N^3. According to the Sphere Theorem, it suffices to show that S contracts to a point in N^3. We will prove this by showing that S bounds a homotopy 3-cell in N.

Since N contracts to a point in M, so does S. It follows from Corollary D1 that S separates M. Since ∂N is connected, it lies in one of the components of $M - S$. Let C be the closure of the component of $M - S$ which does not contain ∂N. Then $C \subset N$. So C is a compact 3-manifold whose boundary is S.

C is simply connected. For any loop in C contracts to a point in M; but this contraction can be "cut off" on the 2-sphere S and redefined to lie totally in C. It now follows from Lemma E that C is a homotopy 3-cell. ∎

<u>Corollary F2</u>: Suppose M^3 is a 3-manifold and X is a cell-like subset of M^3. Then every open neighborhood of X in M^3 contains an aspherical open neighborhood of X in M^3.

<u>Proof</u>: Let U be an open neighborhood of X in M^3. Since X is cell-like, U contains an open neighborhood V of X in M^3 such that V contracts to a point in U. Corollary D4 provides a compact 3-manifold, N, with connected boundary which is PL embedded in M^3 so that $X \subset \operatorname{int} N$ and $N \subset V$. Then N contracts to a point in M. It follows from Corollary F1 that N is aspherical. So $\operatorname{int} N$ is an aspherical open neighborhood of X which lies in U. ∎

We remark that Corollary F2 is Lemma 2 in the paper of Kozlowski and Walsh, without the unnecessary hypothesis that M^3 be non-compact.

<u>Corollary F3</u>: If U and V are aspherical open subsets of a 3-manifold M^3 and $U \cup V$ is a proper subset of M^3, then $U \cap V$ is aspherical.

Proof: According to Lemma F, it suffices to prove that each map of a 2-sphere into $U \cap V$ is homotopically trivial. First we prove this under the restrictive hypothesis that M is simply connected. Under this hypothesis, M and, hence, $U \cap V$ are orientable. So we can apply the Sphere Theorem. To this end, let S be a 2-sphere PL embedded in $U \cap V$. We shall prove that S contracts to a point in $U \cap V$ by showing that S bounds a homotopy 3-cell in $U \cap V$.

Choose $p \in M - (U \cup V)$. Since U is aspherical and $U \subset M$, then S contracts to a point in M. Thus Corollary D1 tells us that S separates M. Let G and H be the components of $M - S$ so that $p \in H$. Let $C = \mathrm{cl}\, G = G \cup S = M - H$. Then C is a 3-manifold with $\partial C = S$.

We now argue that C is compact. Clearly, G and $H-p$ are the components of $(M-p) - S$. Since $U \subset M-p$, ~~then~~ S contracts to a point in $M-p$. Now Corollary D1 tells us that the closure, in $M-p$, of either G or $H-p$ is compact. But the closure, in $M-p$, of $H-p$ can't be compact, because $H-p$

contains a sequence of points which converges to p. So the closure, in $M-p$, of G is compact. Evidently, the closure of G in $M-p$ coincides with C, the closure of G in M. So C is compact.

C is simply connected. For any loop in C contracts to a point in M; but this contraction can be "cut off" on the 2-sphere, S, and redefined to lie totally in C.

It now follows from Lemma E that C is a homotopy 3-cell.

Since S contracts to a point in U, then Corollary D2 tells us that $M-U$ lies in one of the components of $M-S$. Since $p \in M-U$ and $p \in H$, then $M-U \subset H$. The same argument shows that $M-V \subset H$. So $M-(U \cap V) = (M-U) \cup (M-V) \subset H$. Therefore $C = M-H \subset U \cap V$.

Consequently, S bounds a homotopy 3-cell, C, in $U \cap V$. So S contracts to a point in $U \cap V$. It follows that each map of a 2-sphere into $U \cap V$ is homotopically trivial.

Now consider the general case in which

we do not have the hypothesis that M is simply connected. Let $p: \tilde{M} \to M$ be the universal covering space of M. Since the restriction of p to a component of $p^{-1}U$ or $p^{-1}V$ is a covering projection, it follows that $p^{-1}U$ and $p^{-1}V$ are aspherical open subsets of $\tilde{M}$ such that $(p^{-1}U) \cup (p^{-1}V)$ is a proper subset of $\tilde{M}$. Since $\tilde{M}$ is simply connected, then the argument given above shows that $(p^{-1}U) \cap (p^{-1}V)$ is aspherical. Since $(p^{-1}U) \cap (p^{-1}V) = p^{-1}(U \cap V)$, and the restriction of p to each component of $p^{-1}(U \cap V)$ is a covering projection, we conclude that $U \cap V$ is aspherical. ∎

We notice that Corollary F3 is Lemma 3 in the paper of Kozlowski and Walsh.

In their paper, Kozlowski and Walsh deduce their Lemmas 2 and 3 from their Lemma 1. We haven't yet proved anything equivalent to their Lemma 1. The proofs presented here of results corresponding to their Lemmas 2 and 3 (our Corollaries F2 and F3) went by a different route. However, we remark that Lemma 1 of Kozlowski and Walsh contains an unnecessary hypothesis: namely, that the ambient manifold M^3 be <u>non-compact</u>.

So here is a proof of a result equivalent to Lemma 1 of Kozlowski and Walsh except that the hypothesis that M^3 be non-compact has been deleted.

<u>Corollary F4</u>: Suppose M^3 is a 3-manifold and S is a 2-sphere which is PL embedded in M. If S contracts to a point in M, then S bounds a homotopy 3-cell in M.

<u>Proof</u>: First let us assume that M is simply connected. We know by Corollary D1 that S separates M and that one of the components of $M-S$ has compact closure. Let C be the closure of that component of $M-S$. C is simply connected. For any loop in C contracts to a point in M; but this contraction can be "cut off" on the 2-sphere S and redefined to lie totally in C. So C is a homotopy 3-cell by Lemma E.

In general, if M is not simply connected, let $p: \tilde{M} \to M$ be the universal covering space of M. Since the only covering space of a 2-sphere is a homeomorphism of a 2-sphere onto a 2-sphere, then $p^{-1}(S)$

is the union of a discrete collection of 2-spheres PL embedded in $\tilde{M}$, and p carries each of these 2-spheres one-to-one onto S.

Let $\tilde{S}$ be a component of $p^{-1}(S)$. The homotopy which contracts S to a point in M lifts to a homotopy which contracts $\tilde{S}$ to a point in $\tilde{M}$. Since $\tilde{M}$ is simply connected, then the argument given in the first paragraph shows that $\tilde{S}$ bounds a homotopy 3-cell $\tilde{C}$ in $\tilde{M}$.

Since the components of $p^{-1}(S)$ are discrete, at most finitely many of them lie in the compactum $\tilde{C}$. From these we choose one that is "innermost" and call it $\tilde{T}$; i.e., $\tilde{T}$ is a component of $p^{-1}(S)$ which lies in $\tilde{C}$ and which does not separate any other component of $p^{-1}(S)$ from $\tilde{M}-\tilde{C}$. Since $\tilde{C}$ is contractible, $\tilde{T}$ contracts to a point in $\tilde{C}$. So the argument given in the first paragraph implies that $\tilde{T}$ bounds a homotopy 3-cell $\tilde{D}$ in $\tilde{C}$; and the choice of $\tilde{T}$ as "innermost" in $\tilde{C}$ guarantees that $\tilde{D} \cap p^{-1}(S) = \tilde{T}$.

We now argue that $p|\tilde{D}$ is one-to-one. Since $p^{-1}(S) \cap \tilde{D} = \tilde{T}$ and $p|\tilde{T}$ is one-to-one, it suffices to show that $p \mid \operatorname{int} \tilde{D}$ is one-to-one. Suppose $x, y \in \operatorname{int} \tilde{D}$ and $p(x) = p(y)$. Let $\alpha : [0,1] \to \tilde{D}$ be a path with $\alpha(0) \in \tilde{T}$, $\alpha(0,1] \subset \operatorname{int} \tilde{D}$ and $\alpha(1) = x$. Let $\beta : [0,1] \to \tilde{M}$ be a lift of the path $p \circ \alpha : [0,1] \to M$ such that $\beta(1) = y$. Since $p \circ \alpha\,(0,1] \cap S = \emptyset$ and $p\alpha(0) \in S$, then $\beta(0,1] \cap p^{-1}(S) = \emptyset$ and $\beta(1) \in p^{-1}(S)$. So since $\beta(1) = y \in \operatorname{int} \tilde{D}$. and $\partial \tilde{D} = \tilde{T} = p^{-1}(S) \cap \tilde{D}$, then $\beta(0,1] \subset \operatorname{int} \tilde{D}$ and $\beta(0) \in \tilde{T}$. Thus $\alpha(0), \beta(0) \in \tilde{T}$. Since $p \mid \tilde{T}$ is one-to-one and $p \circ \alpha(0) = p \circ \beta(0)$, then $\alpha(0) = \beta(0)$. Since path lifting to a given endpoint is unique, we have $\alpha(1) = \beta(1)$. Thus $x = y$.

Now $p \mid \tilde{D}$ embeds the homotopy 3-cell $\tilde{D}$ in M. So $p(\tilde{D})$ is a homotopy 3-cell in M with $\partial p(\tilde{D}) = p(\tilde{T}) = S$. $\blacksquare$