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Mapping swirls and pseudo-spines of compact 4-manifolds

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Abstract

A compact subset X of the interior of a compact manifold M is a *pseudo-spine* of M if $M - X$ is homeomorphic to $(\partial M) \times [0, \infty)$. It is proved that a 4-manifold obtained by attaching k essential 2-handles to a $B^3 \times S^1$ has a pseudo-spine which is obtained by attaching k B^2's to an S^1 by maps of the form $z \mapsto z^n$. This result recovers the fact that the Mazur 4-manifold has a disk pseudo-spine (which may then be shrunk to an arc). To prove this result, the *mapping swirl* (a "swirled" mapping cylinder) of a map to a circle is introduced, and a fundamental property of mapping swirls is established: homotopic maps to a circle have homeomorphic mapping swirls.

Several conjectures concerning the existence of pseudo-spines in compact 4-manifolds are stated and discussed, including the following two related conjectures: every compact contractible 4-manifold has an arc pseudo-spine, and every compact contractible 4-manifold has a handlebody decomposition with no 3- or 4-handles. It is proved that an important class of compact contractible 4-manifolds described by Poénaru satisfies the latter conjecture.

Keywords: Pseudo-spine; Mazur 4-manifold; Mapping swirl; Poénaru 4-manifolds

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1. Introduction

A compact subset X of the interior of a compact manifold M is a called a (*topological*) *spine* of M if M is homeomorphic to the mapping cylinder of a map from ∂M to X. X is called a *pseudo-spine* of M if $M - X$ is homeomorphic to $(\partial M) \times [0, \infty)$.

It is proved in [1] that for $n \geqslant 5$, every compact contractible n-manifold has a wild arc spine. It is observed, however, that in general compact contractible 4-manifolds don't have arc spines. In fact, a compact contractible 4-manifold with an arc spine must be

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either a 4-ball or the cone over a nontrivial homotopy 3-sphere (if one exists). Thus, a compact contractible 4-manifold with a nonsimply connected boundary can't have an arc spine.

The Mazur 4-manifold [6] is a compact contractible 4-manifold with a nonsimply connected boundary. It is a well-known consequence of [5,3] that the Mazur 4-manifold has an arc pseudo-spine.

The naively optimistic conjecture motivating this paper is: every compact contractible 4-manifold has an arc pseudo-spine. The mathematical content of the paper arises from the introduction of the *mapping swirl* construction which allows us to reinterpret and generalize the method of [5]. In Section 2 of this article the *mapping swirl* of a map to S^1 is defined and two fundamental theorems about it are proved: Theorem 1: Homotopic maps from a compact metric space to S^1 have homeomorphic mapping swirls. Theorem 2: For a compact metric space X and an integer $n \neq 0$, the mapping swirl and the mapping cylinder of the map $(x, z) \mapsto z^n : X \times S^1 \to S^1$ are homeomorphic. Section 3 applies these theorems to produce simple pseudo-spines for the special class of 4-manifolds obtained by adding finitely many essential 2-handles to $B^3 \times S^1$. This approach recovers the previously known result that Mazur's compact contractible 4-manifold has an arc pseudo-spine. Section 4 speculates about the possibility of finding simple pseudo-spines for all compact 4-manifolds. In particular, it includes the conjecture that every compact contractible 4-manifold has an arc pseudo-spine. It also states a closely related conjecture: every compact contractible 4-manifold has a handlebody decomposition with no 3- or 4-handles. It then presents a proof that this conjecture holds for an important class of compact contractible 4-manifolds described by Poénaru.

2. Mapping swirls

Let $f : X \to S^1$ be a map from a compact metric space X to S^1. Intuitively, the *mapping swirl* of f is obtained from the mapping cylinder of f by "swirling" the fibers of the mapping cylinder around infinitely many times in the S^1-direction as they approach the S^1-end of the mapping cylinder. To make this informal definition precise, we use the fact that the mapping cylinder of f embeds naturally in $(CX) \times S^1$, where CX is the cone on X. The mapping swirl of f is defined as a subset of $(CX) \times S^1$. The swirling effect is achieved by using the S^1-factor. We also define the *double mapping swirl* of f, by "swirling" the fibers of the double mapping cylinder of f at both ends. The double mapping swirl of f is defined as a subset of $(\Sigma X) \times S^1$, where ΣX is the suspension of X. The S^1-factor is again used to achieve the swirling effect.

Simple examples show that two maps from a compact metric space to S^1 may differ by only a slight homotopy and yet have nonhomeomorphic mapping cylinders and double mapping cylinders. In contrast, our principal result, Theorem 1, says that homotopic maps from a compact metric space to S^1 have homeomorphic mapping swirls. The swirling process kills the topological difference between the mapping cylinders of homotopic maps.

Definition. Let X be a compact metric space. The *suspension* of X, denoted ΣX, is the quotient space $[-\infty,\infty] \times X/\{\{-\infty\} \times X,\ \{\infty\} \times X\}$. Let $q:[-\infty,\infty] \times X \to \Sigma X$ denote the quotient map. For $(t,x) \in [-\infty,\infty] \times X$, let $tx = q((t,x))$; and let $\pm\infty = q(\{\pm\infty\} \times X)$. The *cone* on X, denoted CX, is $q([0,\infty] \times X)$.

Definition. Let $f:X \to Y$ be a map between compact metric spaces. The *mapping cylinder* of f, denoted $\mathrm{Cyl}(f)$, is the subspace

$$\{(tx, f(x)) \in (CX) \times Y\colon\ (t,x) \in [0,\infty) \times X\} \cup (\{\infty\} \times Y)$$

of $(CX) \times Y$. The *double mapping cylinder* of f, denoted $\mathrm{DblCyl}(f)$, is the subspace

$$\{(tx, f(x)) \in (\Sigma X) \times Y\colon\ (t,x) \in (-\infty,\infty) \times X\} \cup (\{-\infty,\infty\} \times Y)$$

of $(\Sigma X) \times Y$. For $-\infty < t < \infty$, call the set $\{(tx, f(x))\colon\ x \in X\}$ the *t-level* of $\mathrm{DblCyl}(f)$; it is homeomorphic to X. Call $\{\pm\infty\} \times Y$ the $\pm\infty$*-level* of $\mathrm{DblCyl}(f)$. Observe that the union of the t-levels of $\mathrm{DblCyl}(f)$ for $0 \leqslant t \leqslant \infty$ is precisely $\mathrm{Cyl}(f)$.

To reconcile this definition of the mapping cylinder of f with the usual definition, consider the map from the disjoint union $([0,\infty] \times X) \cup Y$ onto the subset of $(CX) \times Y$ which we have called $\mathrm{Cyl}(f)$ which sends $(t,x) \in ([0,\infty] \times X)$ to $(tx, f(x))$ and sends $y \in Y$ to (∞, y). The set of inverse images of this map determines the decomposition of $([0,\infty] \times X) \cup Y$ in which the only nonsingleton elements are sets of the form $(f^{-1}(y) \times \{\infty\}) \cup \{y\}$ for $y \in Y$. This is exactly the decomposition which is determined by the inverse images of the "usual" quotient map from $([0,\infty] \times X) \cup Y$ to the "usual" mapping cylinder of f. Consequently, $\mathrm{Cyl}(f)$ is homeomorphic to the "usual" mapping cylinder of f. Similarly, $\mathrm{DblCyl}(f)$ is homeomorphic to the "usual" double mapping cylinder of f.

Definition. Let X be a compact metric space and let $f:X \to S^1$ be a map. The *mapping swirl* of f, denoted $\mathrm{Swl}(f)$, is the subspace

$$\{(tx, \mathrm{e}^{2\pi \mathrm{i} t} f(x)) \in (CX) \times S^1\colon\ (t,x) \in [0,\infty) \times X\} \cup (\{\infty\} \times S^1)$$

of $(CX) \times S^1$. The *double mapping swirl* of f, denoted $\mathrm{DblSwl}(f)$, is the subspace

$$\{(tx, \mathrm{e}^{2\pi \mathrm{i} t} f(x)) \in (\Sigma X) \times S^1\colon\ (t,x) \in (-\infty,\infty) \times X\} \cup (\{-\infty,\infty\} \times S^1)$$

of $(\Sigma X) \times S^1$. For $-\infty < t < \infty$, call the set $\{(tx, \mathrm{e}^{2\pi \mathrm{i} t} f(x))\colon\ x \in X\}$ the *t-level* of $\mathrm{DblSwl}(f)$; it is homeomorphic to X. Call $\{\pm\infty\} \times S^1$ the $(\pm\infty)$*-level* of $\mathrm{DblSwl}(f)$. Observe that the union of the t-levels of $\mathrm{DblSwl}(f)$ for $0 \leqslant t \leqslant \infty$ is precisely $\mathrm{Swl}(f)$. For $x \in X$, call the set $\{(tx, \mathrm{e}^{2\pi \mathrm{i} t} f(x))\colon\ -\infty < t < \infty\}$ the *x-fiber* of $\mathrm{DblSwl}(f)$, and call the set $\{(tx, \mathrm{e}^{2\pi \mathrm{i} t} f(x))\colon\ 0 \leqslant t < \infty\}$ the *x-fiber* of $\mathrm{Swl}(f)$. If $g:X \to S^1$ is another map and $x \in X$, then the x-fiber of $\mathrm{Swl}(f)$ ($\mathrm{DblSwl}(f)$) and the x-fiber of $\mathrm{Swl}(g)$ ($\mathrm{DblSwl}(g)$) are called *corresponding fibers*.

Theorem 1. *If X is a compact metric space, and $f, g:X \to S^1$ are homotopic maps, then* $\mathrm{Swl}(f)$ *is homeomorphic to* $\mathrm{Swl}(g)$. *Furthermore, the homeomorphism maps the*

0- and ∞-levels of Swl(f) *onto the* 0- *and ∞-levels of* Swl(g), *respectively, and maps each fiber of* Swl(f) *onto the corresponding fiber of* Swl(g).

Proof. The proof has two steps. First we find a homeomorphism of $(\Sigma X)\times S^1$ which carries DblSwl(f) onto DblSwl(g). This homeomorphism moves Swl(f) into DblSwl(g), because Swl$(f)\subset$ DblSwl(f). Second we find a homeomorphism of $(\Sigma X)\times S^1$ which "twists" the image of Swl(f) onto Swl(g) within DblSwl(g).

Step 1. For each $x\in X$, set $\mathcal{F}(x)=\{(tx,\mathrm{e}^{2\pi\mathrm{i}t}f(x))\colon -\infty<t<\infty\}$ and set $\mathcal{G}(x)=\{(tx,\mathrm{e}^{2\pi\mathrm{i}t}g(x))\colon -\infty<t<\infty\}$. $\mathcal{F}(x)$ and $\mathcal{G}(x)$ are the x-fibers of DblSwl(f) and DblSwl(g), respectively. Both lie in $((-\infty,\infty)x)\times S^1\subset(\Sigma(X))\times S^1$.

For each $x\in X$, the x-fibers $\mathcal{F}(x)$ and $\mathcal{G}(x)$ form a "double helix" in the cylinder $((-\infty,\infty)x)\times S^1$. The angle $\theta(x)$ between $\mathcal{F}(x)$ and $\mathcal{G}(x)$ in the S^1-direction is precisely the angle between $f(x)$ and $g(x)$ in S^1, and a twist of the cylinder $((-\infty,\infty)x)\times S^1$ in the S^1-direction through the angle $\theta(x)$ would move $\mathcal{F}(x)$ to $\mathcal{G}(x)$. Unfortunately, one can't form the "union" of these twists over all the cylinders $((-\infty,\infty)x)\times S^1$ to move DblSwl(f) to DblSwl(g) in $(\Sigma X)\times S^1$, because $\theta(x)$ may vary with x, so that there is no single rotation of $\{-\infty,\infty\}\times S^1$ that extends the twists of all the cylinders. Instead of using a twist, one observes that the helix $\mathcal{F}(x)$ can be moved to the helix $\mathcal{G}(x)$ by a slide of the cylinder $((-\infty,\infty)x)\times S^1$ in the $(-\infty,\infty)x$-direction. The length of the slide in the $(-\infty,\infty)x$-direction varies with x and is essentially determined by lifting the homotopy joining f to g in S^1 to a homotopy in $(-\infty,\infty)$. Unlike the previously considered twist, this slide extends to $\{-\infty,\infty\}\times S^1$ via the identity. This is because the slide makes no motion in the S^1-direction and preserves the "ends" of $(-\infty,\infty)x$. The details follow.

Suppose $h\colon X\times[0,1]\to S^1$ is a homotopy such that $h(x,0)=g(x)$ and $h(x,1)=f(x)$. We exploit the fact that S^1 is a group under complex multiplication to define the map $k\colon X\times[0,1]\to S^1$ by $k(x,t)=h(x,t)/h(x,0)$. Thus, $k(x,0)=1$ and $k(x,1)g(x)=f(x)$ for $x\in X$. Let $e\colon(-\infty,\infty)\to S^1$ denote the exponential covering map $e(t)=\mathrm{e}^{2\pi\mathrm{i}t}$. Let $\tilde{k}\colon X\times[0,1]\to(-\infty,\infty)$ be the lift of k (i.e., $e\circ\tilde{k}=k$) such that $\tilde{k}(x,0)=0$ for all $x\in X$. Define $\sigma\colon X\to(-\infty,\infty)$ by $\sigma(x)=\tilde{k}(x,1)$. Observe that for each $x\in X$, $f(x)/\mathrm{e}^{2\pi\mathrm{i}\sigma(x)}=f(x)/e(\tilde{k}(x,1))=f(x)/k(x,1)=g(x)$. Since X is compact, there is a $b\in(0,\infty)$ such that $\sigma(X)\subset(-b,b)$. As we will see, $\sigma(x)$ specifies the length of the slide of the cylinder $((-\infty,\infty)x)\times S^1$ in the $(-\infty,\infty)x$-direction that moves $\mathcal{F}(x)$ to $\mathcal{G}(x)$.

Now define the function $\Phi\colon(\Sigma X)\times S^1\to(\Sigma X)\times S^1$ by setting $\Phi(tx,z)=((t+\sigma(x))x,z)$ for $(t,x)\in(-\infty,\infty)\times X$ and $z\in S^1$, and by requiring that $\Phi|\{-\infty,\infty\}\times S^1=\mathrm{id}$. Clearly Φ is continuous at each point of $(\Sigma X)\times S^1-\{-\infty,\infty\}\times S^1$. For each $z\in S^1$, the continuity of Φ at the points $(\pm\infty,z)$ follows from the inclusions

$$\Phi\big(([t,\infty]x)\times\{z\}\big)\subset\big((t-b,\infty]x\big)\times\{z\},$$
$$\Phi\big(([-\infty,t]x)\times\{z\}\big)\subset\big([-\infty,t+b)x\big)\times\{z\}.$$

Next we verify that $\Phi(\mathrm{DblSwl}(f))\subset\mathrm{DblSwl}(g)$. To this end, let $x\in X$ and consider a typical point $(tx,\mathrm{e}^{2\pi\mathrm{i}t}f(x))$ of the fiber $\mathcal{F}(x)$. Φ moves this point to the point

$$\begin{aligned}\big((t+\sigma(x))x, \mathrm{e}^{2\pi\mathrm{i}t}f(x)\big) &= \big((t+\sigma(x))x, \mathrm{e}^{2\pi\mathrm{i}(t+\sigma(x))}f(x)/\mathrm{e}^{2\pi\mathrm{i}\sigma(x)}\big)\\ &= \big((t+\sigma(x))x, \mathrm{e}^{2\pi\mathrm{i}(t+\sigma(x))}g(x)\big)\end{aligned}$$

which is a point of the fiber $\mathcal{G}(x)$. Consequently, $\Phi(\mathcal{F}(x)) \subset \mathcal{G}(x)$. Thus, Φ maps each fiber of DblSwl(f) into the corresponding fiber of DblSwl(g). Also $\Phi(\{-\infty,\infty\}\times S^1) = \{-\infty,\infty\}\times S^1$. Since DblSwl($f$) and DblSwl($g$) are the unions of their fibers and of $\{-\infty,\infty\}\times S^1$, we conclude that $\Phi(\mathrm{DblSwl}(f)) \subset \mathrm{DblSwl}(g)$.

To complete Step 1, we must verify that Φ is a homeomorphism and that $\Phi(\mathrm{DblSwl}(f)) = \mathrm{DblSwl}(g)$. We accomplish this by defining the function $\overline{\Phi}:(\Sigma X)\times S^1 \to (\Sigma X)\times S^1$ by setting $\overline{\Phi}(tx,z) = ((t-\sigma(x))x, z)$ for $(t,x)\in(-\infty,\infty)\times X$ and $z\in S^1$, and by requiring that $\overline{\Phi}|\{-\infty,\infty\}\times S^1 = \mathrm{id}$. Arguments similar to those just given show that $\overline{\Phi}$ is continuous and that $\overline{\Phi}(\mathrm{DblSwl}(g)) \subset \mathrm{DblSwl}(f)$. Also it is easily checked that the composition of Φ and $\overline{\Phi}$ in either order is the identity. Hence, Φ is a homeomorphism, and $\Phi(\mathrm{DblSwl}(f)) \supset \Phi(\overline{\Phi}(\mathrm{DblSwl}(g))) = \mathrm{DblSwl}(g)$. So $\Phi(\mathrm{DblSwl}(f)) = \mathrm{DblSwl}(g)$.

Step 2. Here we will find a homeomorphism Ψ of $(\Sigma X)\times S^1$ such that

$$\Psi\big(\Phi(\mathrm{Swl}(f))\big) = \mathrm{Swl}(g).$$

For each $x\in X$, set $\mathcal{F}^+(x) = \{(tx, \mathrm{e}^{2\pi\mathrm{i}t}f(x)){:}\ 0\leqslant t<\infty\}$ and set $\mathcal{G}^+(x) = \{(tx, \mathrm{e}^{2\pi\mathrm{i}t}g(x)){:}\ 0\leqslant t<\infty\}$. $\mathcal{F}^+(x)$ and $\mathcal{G}^+(x)$ are the x-fibers of Swl(f) and Swl(g), respectively.

For $x\in X$, since $\mathcal{F}^+(x)\subset\mathcal{F}(x)$, then $\Phi(\mathcal{F}^+(x))\subset\Phi(\mathcal{F}(x))\subset\mathcal{G}(x)$; also $\mathcal{G}^+(x)\subset\mathcal{G}(x)$. We will describe a homeomorphism Ψ which gives the cylinder $((-\infty,\infty)x)\times S^1$ a screw motion that carries the fiber $\mathcal{G}(x)$ onto itself and moves $\Phi(\mathcal{F}^+(x))$ onto $\mathcal{G}^+(x)$. Also Ψ will restrict to the identity on a neighborhood of $\{-\infty,\infty\}\times S^1$.

Recall that $b\in(0,\infty)$ such that $\sigma(X)\subset(-b,b)$. There is a map $\tau:(-\infty,\infty)\times X\to(-\infty,\infty)$ such that for each $x\in X$, $t\mapsto\tau(t,x):(-\infty,\infty)\to(-\infty,\infty)$ is an order preserving piecewise linear homeomorphism which restricts to the identity on $(-\infty,-b]\cup[b,\infty)$ and which moves $\sigma(x)$ to 0. For example, τ can be defined by the formulas:

$$\begin{aligned}&\tau(t,x) = \big(b/\big(b+\sigma(x)\big)\big)\big(t-\sigma(x)\big) && \text{for } t\in\big[-b,\sigma(x)\big],\\ &\tau(t,x) = \big(b/\big(b-\sigma(x)\big)\big)\big(t-\sigma(x)\big) && \text{for } t\in\big[\sigma(x),b\big],\\ &\tau(t,x) = t && \text{for } t\in(-\infty,-b]\cup[b,\infty).\end{aligned}$$

Now define the function $\Psi:(\Sigma X)\times S^1\to(\Sigma X)\times S^1$ by setting

$$\Psi(tx,z) = \big(\tau(t,x)x, \mathrm{e}^{2\pi\mathrm{i}(\tau(t,x)-t)}z\big) \quad \text{for } (t,x)\in(-\infty,\infty)\times X \text{ and } z\in S^1,$$

and by requiring that $\Psi|\{-\infty,\infty\}\times S^1 = \mathrm{id}$. Since τ is continuous, then Ψ is continuous at each point of $(\Sigma X)\times S^1 - \{-\infty,\infty\}\times S^1$. Also since $\tau(t,x) = t$ for $t\in(-\infty,-b]\cup[b,\infty)$, then Ψ restricts to the identity on the neighborhood of $\{-\infty,\infty\}\times S^1$ in $(\Sigma X)\times S^1$ consisting of all points of the form (tx,z) where $t\in[-\infty,-b]\cup[b,\infty]$, $x\in X$ and $z\in S^1$. Hence, Ψ is continuous at each point of $\{-\infty,\infty\}\times S^1$.

Next we verify that $\Psi(\Phi(\mathrm{Swl}(f))) \subset \mathrm{Swl}(g)$. To this end, let $x \in X$ and consider a typical point p of $\mathcal{F}^+(x)$. p has the form $(tx, \mathrm{e}^{2\pi \mathrm{i} t} f(x))$ where $0 \leqslant t < \infty$. Thus $\Phi(p) = ((t + \sigma(x))x, \mathrm{e}^{2\pi\mathrm{i}(t+\sigma(x))} g(x))$. Hence,

$$\begin{aligned}\Psi\big(\Phi(p)\big) &= \big(\tau\big(t+\sigma(x),x\big)x, \mathrm{e}^{2\pi\mathrm{i}(\tau(t+\sigma(x),x)-t-\sigma(x))}\mathrm{e}^{2\pi\mathrm{i}(t+\sigma(x))}g(x)\big)\\ &= \big(\tau\big(t+\sigma(x),x\big)x, \mathrm{e}^{2\pi\mathrm{i}\tau(t+\sigma(x),x)}g(x)\big).\end{aligned}$$

Since $u \mapsto \tau(u,x) : (-\infty,\infty) \to (-\infty,\infty)$ is an order preserving homeomorphism, $\tau(\sigma(x),x) = 0$ and $t \geqslant 0$, then $\tau(t+\sigma(x),x) \geqslant 0$. It follows that $\Psi(\Phi(p))$ belongs to the fiber $\mathcal{G}^+(x)$. This proves $\Psi(\Phi(\mathcal{F}^+(x))) \subset \mathcal{G}^+(x)$. So $\Psi \circ \Phi$ maps each fiber of $\mathrm{Swl}(f)$ into the corresponding fiber of $\mathrm{Swl}(g)$. Also $\Psi \circ \Phi(\{\infty\} \times S^1) = \{\infty\} \times S^1$. Since $\mathrm{Swl}(f)$ and $\mathrm{Swl}(g)$ are the unions of their fibers and of $\{\infty\} \times S^1$, we conclude that $\Psi(\Phi(\mathrm{Swl}(f))) \subset \mathrm{Swl}(g)$.

It remains to establish that $\Psi : (\Sigma X) \times S^1 \to (\Sigma X) \times S^1$ is a homeomorphism and that $\Psi \circ (\mathrm{Swl}(f)) = \mathrm{Swl}(g)$. To this end, first note that there is a map $\bar\tau : (-\infty,\infty) \times X \to (-\infty,\infty)$ such that for each $x \in X$, $t \mapsto \bar\tau(t,x) : (-\infty,\infty) \to (-\infty,\infty)$ is the inverse of the homeomorphism $t \mapsto \tau(t,x) : (-\infty,\infty) \to (-\infty,\infty)$. (Thus, for each $x \in X$, $t \mapsto \bar\tau(t,x) : (-\infty,\infty) \to (-\infty,\infty)$ is an order preserving piecewise linear homeomorphism which restricts to the identity on $(-\infty,-b] \cup [b,\infty)$ such that $\bar\tau(0,x) = \sigma(x)$, and $\bar\tau(\tau(t,x),x) = t$ and $\tau(\bar\tau(t,x),x) = t$ for $-\infty < t < \infty$.) Then define the function $\overline{\Psi} : (\Sigma X) \times S^1 \to (\Sigma X) \times S^1$ by setting

$$\overline{\Psi}(tx,z) = \big(\bar\tau(t,x)x, \mathrm{e}^{2\pi\mathrm{i}(\bar\tau(t,x)-t)}z\big) \quad \text{for } (t,x) \in (-\infty,\infty) \times X \text{ and } z \in S^1,$$

and by requiring that $\overline{\Psi}|\{-\infty,\infty\} \times S^1 = \mathrm{id}$. The proof of the continuity of $\overline{\Psi}$ is similar to the proof of the continuity of Ψ.

Next we verify that $\overline{\Psi}(\mathrm{Swl}(g)) \subset \Phi(\mathrm{Swl}(f))$. To this end, let $x \in X$ and consider a typical point p of $\mathcal{G}^+(x)$. p has the form $(tx, \mathrm{e}^{2\pi\mathrm{i}t} g(x))$ where $0 \leqslant t < \infty$, and

$$\overline{\Psi}(p) = \big(\bar\tau(t,x)x, \mathrm{e}^{2\pi\mathrm{i}(\bar\tau(t,x)-t)}\mathrm{e}^{2\pi\mathrm{i}t}g(x)\big) = \big(\bar\tau(t,x)x, \mathrm{e}^{2\pi\mathrm{i}\bar\tau(t,x)}g(x)\big).$$

Since $u \mapsto \bar\tau(u,x) : (-\infty,\infty) \to (-\infty,\infty)$ is an order preserving homeomorphism, $\bar\tau(0,x) = \sigma(x)$ and $t \geqslant 0$, then $\bar\tau(t,x) = u + \sigma(x)$ for some $u \geqslant 0$. Hence,

$$\overline{\Psi}(p) = \big(\big(u+\sigma(x)\big)x, \mathrm{e}^{2\pi\mathrm{i}(u+\sigma(x))}g(x)\big).$$

Since $u \geqslant 0$, then the point $(ux, \mathrm{e}^{2\pi\mathrm{i}u} f(x))$ belongs to $\mathcal{F}^+(x)$, and $\Phi(ux, \mathrm{e}^{2\pi\mathrm{i}u} f(x)) = ((u+\sigma(x))x, \mathrm{e}^{2\pi\mathrm{i}(u+\sigma(x))}g(x))$. Consequently,

$$\overline{\Psi}(p) = \Phi\big(ux, \mathrm{e}^{2\pi\mathrm{i}u} f(x)\big) \in \Phi\big(\mathcal{F}^+(x)\big).$$

This proves $\overline{\Psi}(\mathcal{G}^+(x)) \subset \Phi(\mathcal{F}^+(x))$. So $\overline{\Psi}$ maps each fiber of $\mathrm{Swl}(g)$ into the Φ-image of the corresponding fiber of $\mathrm{Swl}(f)$. Also $\overline{\Psi}(\{\infty\} \times S^1) = \{\infty\} \times S^1 = \Phi(\{\infty\} \times S^1)$. Since $\mathrm{Swl}(g)$ and $\mathrm{Swl}(f)$ are the unions of their fibers and of $\{\infty\} \times S^1$, we conclude that $\overline{\Psi}(\mathrm{Swl}(g)) \subset \Phi(\mathrm{Swl}(f))$.

It is easy to verify that the composition of Ψ and $\overline{\Psi}$ in either order is the identity. (Remember that $\bar\tau(\tau(t,x)x) = t$ and $\tau(\bar\tau(t,x),x) = t$ for $x \in X$ and $-\infty < t < \infty$.) Hence, Ψ is a homeomorphism.

We have seen that $\Psi(\Phi(\mathrm{Swl}(f))) \subset \mathrm{Swl}(g)$ and $\overline{\Psi}(\mathrm{Swl}(g)) \subset \Phi(\mathrm{Swl}(f))$. So

$$\Psi\big(\Phi\big(\mathrm{Swl}(f)\big)\big) \supset \Psi\big(\overline{\Psi}\big(\mathrm{Swl}(g)\big)\big) = \mathrm{Swl}(g).$$

Thus, the homeomorphism $\Psi \circ \Phi$ maps $\mathrm{Swl}(f)$ onto $\mathrm{Swl}(g)$.

In the course of the proof, we have seen that for each $x \in X$, $\Psi(\Phi(\mathcal{F}^+(x))) \subset \mathcal{G}^+(x)$ and $\overline{\Psi}(\mathcal{G}^+(x)) \subset \Phi(\mathcal{F}^+(x))$. So $\Psi(\Phi(\mathcal{F}^+(x))) \supset \Psi(\overline{\Psi}(\mathcal{G}^+(x))) = \mathcal{G}^+(x)$. Thus, $\Psi(\Phi(\mathcal{F}^+(x))) = \mathcal{G}^+(x)$. In other words, the homeomorphism $\Psi \circ \Phi$ maps each fiber of $\mathrm{Swl}(f)$ onto the corresponding fiber of $\mathrm{Swl}(g)$.

A typical point of the 0-level of $\mathrm{Swl}(f)$ has the form $(0x, f(x))$ and

$$\begin{aligned}\Psi\big(\Phi\big(0x, f(x)\big)\big) &= \Psi\big(\sigma(x)x, \mathrm{e}^{2\pi \mathrm{i}\sigma(x)}g(x)\big)\\ &= \big(\tau\big(\sigma(x), x\big)x, \mathrm{e}^{2\pi\mathrm{i}(\tau(\sigma(x),x)-\sigma(x))}\mathrm{e}^{2\pi\mathrm{i}\sigma(x)}g(x)\big) = \big(0x, g(x)\big)\end{aligned}$$

because $\tau(\sigma(x), x) = 0$. Thus, $\Psi \circ \Phi$ maps the 0-level of $\mathrm{Swl}(f)$ onto the 0-level of $\mathrm{Swl}(g)$.

Since $\Psi(\Phi(\{-\infty\} \times S^1)) = \Psi(\{-\infty\} \times S^1) = \{-\infty\} \times S^1$, then $\Psi \circ \Phi$ maps the ∞-level of $\mathrm{Swl}(f)$ onto the ∞-level of $\mathrm{Swl}(g)$. □

The next theorem and its corollaries make it possible to identify the mapping swirls of a special types of maps.

Theorem 2. *If X is a compact metric space, n is a nonzero integer, and $f : X \times S^1 \to S^1$ is the map $f(x, z) = z^n$, then $\mathrm{Cyl}(f)$ is homeomorphic to $\mathrm{Swl}(f)$. Furthermore, the homeomorphism maps the t-level of $\mathrm{Cyl}(f)$ onto the t-level of $\mathrm{Swl}(f)$ for $0 \leqslant t \leqslant \infty$.*

Proof. We will find a homeomorphism from $C(X \times S^1) \times S^1$ to itself which carries $\mathrm{Cyl}(f)$ onto $\mathrm{Swl}(f)$ by twisting motion in the S^1-direction in the $C(X \times S^1)$ factor of $C(X \times S^1) \times S^1$. This is possible because of the S^1-factor in the domain of f and the special form of f.

Define the function

$$\phi : C\big(X \times S^1\big) \to C\big(X \times S^1\big)$$

by setting $\phi(t(x, z)) = t(x, \mathrm{e}^{-2\pi\mathrm{i}t/n}z)$ for $t \in [0, \infty)$ and $(x, z) \in X \times S^1$ and $\phi(\infty) = \infty$. ϕ is clearly continuous on $C(X \times S^1) - \{\infty\}$; and because ϕ maps the t-level of $C(X \times S^1)$ into itself, then ϕ is continuous at ∞. We show that ϕ is a homeomorphism of $C(X \times S^1)$ by exhibiting its inverse. Indeed, let us define the function $\bar{\phi} : C(X \times S^1) \to C(X \times S^1)$ by setting $\bar{\phi}(t(x, z)) = t(x, \mathrm{e}^{2\pi\mathrm{i}t/n}z)$ for $t \in [0, \infty)$ and $(x, z) \in X \times S^1$ and $\bar{\phi}(\infty) = \infty$. Then $\bar{\phi}$ is continuous by an argument similar to the one just given. Also it is easily checked that the composition of ϕ and $\bar{\phi}$ in either order is the identity. So ϕ and $\bar{\phi}$ are homeomorphisms.

Next define a homeomorphism $\Phi : C(X \times S^1) \times S^1 \to C(X \times S^1) \times S^1$ by $\Phi = \phi \times \mathrm{id}$. Clearly $\overline{\Phi} = \bar{\phi} \times \mathrm{id}$ defines the homeomorphism of $C(X \times S^1) \times S^1$ which is the inverse of Φ.

We now prove that $\Phi(\mathrm{Cyl}(f)) = \mathrm{Swl}(f)$. Let $0 \leqslant t < \infty$, and consider a typical point $p = (t(x,z), f(x,z)) = (t(x,z), z^n)$ of the t-level of $\mathrm{Cyl}(f)$ where $(x,z) \in X \times S^1$. Set $z' = \mathrm{e}^{-2\pi \mathrm{i} t/n} z$. Then

$$\begin{aligned}\Phi(p) &= \big(\phi\big(t(x,z), z^n\big)\big) = \big(t\big(x, \mathrm{e}^{-2\pi \mathrm{i} t/n} z\big), z^n\big)\\ &= \big(t\big(x, \mathrm{e}^{-2\pi \mathrm{i} t/n} z\big), \mathrm{e}^{2\pi \mathrm{i} t}\big(\mathrm{e}^{-2\pi \mathrm{i} t/n} z\big)^n\big)\\ &= \big(t(x,z'), \mathrm{e}^{2\pi \mathrm{i} t}(z')^n\big) = \big(t(x,z'), \mathrm{e}^{2\pi \mathrm{i} t} f(x,z')\big).\end{aligned}$$

So $\Phi(p)$ belongs to the t-level of $\mathrm{Swl}(f)$. Also $\Phi(\{\infty\} \times S^1) = \{\infty\} \times S^1$. It follows that $\Phi(\mathrm{Cyl}(f)) \subset \mathrm{Swl}(f)$, and Φ maps the t-level of $\mathrm{Cyl}(f)$ into the t-level of $\mathrm{Swl}(f)$ for $0 \leqslant t \leqslant \infty$.

A similar argument shows that $\overline{\Phi}$ maps the t-level of $\mathrm{Swl}(f)$ into the t-level of $\mathrm{Cyl}(f)$ for $0 \leqslant t \leqslant \infty$. Indeed, if $0 \leqslant t < \infty$ and $p = (t(x,z)\mathrm{e}^{2\pi \mathrm{i} t} f(x,z)) = (t(x,z), \mathrm{e}^{2\pi \mathrm{i} t} z^n)$ is a typical point of the t-level of $\mathrm{Swl}(f)$, and we set $z' = \mathrm{e}^{2\pi \mathrm{i} t/n} z$, then

$$\overline{\Phi}(p) = \big(\bar{\phi}\big(t(x,z)\mathrm{e}^{2\pi \mathrm{i} t} z^n\big)\big) = \big(t\big(x, \mathrm{e}^{2\pi \mathrm{i} t/n} z\big), \big(\mathrm{e}^{2\pi \mathrm{i} t/n} z\big)^n\big) = \big(t(x,z'), f(x,z)\big)$$

which is a point of the t-level of $\mathrm{Cyl}(f)$. Also $\overline{\Phi}(\{\infty\} \times S^1) = \{\infty\} \times S^1$. Hence, $\overline{\Phi}(\mathrm{Swl}(f)) \subset \mathrm{Cyl}(f)$. Since $\overline{\Phi} = \Phi^{-1}$, it follows that $\Phi(\mathrm{Cyl}(f)) = \mathrm{Swl}(f)$, and Φ maps the t-level of $\mathrm{Cyl}(f)$ onto the t-level of $\mathrm{Swl}(f)$ for $0 \leqslant t \leqslant \infty$. □

We now exploit Theorems 1 and 2 together to state two corollaries which allows us to identify the mapping swirls of certain kinds of maps.

Corollary 1. *If X is a compact metric space, $f : X \times S^1 \to S^1$ and $g : X \times S^1 \to S^1$ are homotopic maps, and $g(x,z) = z^n$ where n is a nonzero integer, then* $\mathrm{Swl}(f)$ *is homeomorphic to* $\mathrm{Cyl}(g)$. *Furthermore, the homeomorphism maps the* 0- *and* ∞-*levels of* $\mathrm{Swl}(f)$ *onto the* 0- *and* ∞-*levels of* $\mathrm{Cyl}(g)$.

Corollary 2. *If $f : S^1 \to S^1$ is a map of degree $n \neq 0$, then* $\mathrm{Swl}(f)$ *is homeomorphic to* $\mathrm{Cyl}(z \mapsto z^n)$. *Furthermore, the homeomorphism maps the* 0- *and* ∞-*levels of* $\mathrm{Swl}(f)$ *onto the* 0- *and* ∞-*levels of* $\mathrm{Cyl}(z \mapsto z^n)$. *In particular,* $\mathrm{Swl}(f)$ *is an annulus if $n = \pm 1$, and* $\mathrm{Swl}(f)$ *is a Möbius strip if $n = \pm 2$.*

The last assertion of this corollary follows from the observation that the mapping cylinder of the map $z \mapsto z^n : S^1 \to S^1$ is an annulus if $n = \pm 1$, and it is a Möbius strip if $n = \pm 2$.

3. Pseudo-spines of 4-manifolds

Recall that a compact subset X of the interior of a compact manifold M is a *pseudo-spine* of M if $M - X$ is homeomorphic to $(\partial M) \times [0, \infty)$.

Let $\|\ \|$ denote the Euclidean norm on $\mathbb{R}^n$: $\|x\| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$. Set $B^n = \{x \in \mathbb{R}^n\colon \|x\| \leqslant 1\}$ and $S^n = \{x \in \mathbb{R}^{n+1}\colon \|x\| = 1\}$.

For each integer n, let $\gamma_n : S^1 \to S^1$ denote the map $\gamma_n(z) = z^n$, and let $X(n)$ denote the adjunction space $B^2 \cup_{\gamma_n} S^1$. Thus, $X(\pm 1)$ is a 2-dimensional disk, and $X(\pm 2)$ is a projective plane. If $|n| > 2$, then $X(n)$ is a 2-dimensional polyhedron which is not a 2-manifold. For nonzero integers $n_1, n_2, \ldots, n_k$, let $X(n_1, n_2, \ldots, n_k)$ denote the adjunction space $(B^2 \times \{1, 2, \ldots, k\}) \cup_\Gamma S^1$ where $\Gamma : S^1 \times \{1, 2, \ldots, k\} \to S^1$ is the map defined by $\Gamma(z, i) = \gamma_{n_i}(z)$ for $z \in S^1$ and $1 \leqslant i \leqslant k$. Thus, $X(n_1, n_2, \ldots, n_k)$ is homeomorphic to a union of $X(n_1), X(n_2), \ldots, X(n_k)$ in which all the "edge circles" of the $X(n_i)$'s are identified with a single copy of S^1.

A simple closed curve C in the boundary of a manifold N is called *essential* if it is not homotopically trivial in ∂N. If C is essential, then any 2-handle attached to N along C is also called *essential.*

Theorem 3. *Suppose $C_1, C_2, \ldots, C_k$ are disjoint essential simple closed curves in $\partial B^3 \times S^1$, and M^4 is the 4-manifold obtained by attaching disjoint 2-handles to $B^3 \times S^1$ along $C_1, C_2, \ldots, C_k$. Let $\pi : \partial B^3 \times S^1 \to S^1$ denote the projection map. For $1 \leqslant i \leqslant k$, let n_i denote the degree of the map $\pi | C_i : C_i \to S^1$. Then M^4 has a pseudo-spine which is homeomorphic to $X(n_1, n_2, \ldots, n_k)$.*

Proof. Note that $n_i \neq 0$ because C_i is essential for $1 \leqslant i \leqslant k$. We write $M^4 = (B^3 \times S^1) \cup (H_1 \cup H_2 \cup \cdots \cup H_k)$ where H_i is the 2-handle attached to $B^3 \times S^1$ along C_i. Thus, for $1 \leqslant i \leqslant k$, there is a homeomorphism $h_i : B^2 \times B^2 \to H_i$ such that $(B^3 \times S^1) \cap H_i = h_i((\partial B^2) \times B^2) \subset \partial B^3 \times S^1$ and $h_i((\partial B^2) \times \{0\}) = C_i$. For $1 \leqslant i \leqslant k$, set $D_i = h_i(B^2 \times \{0\})$; then $\partial D_i = C_i$ and D_i is the "core disk" of H_i.

Clearly $B^3 \times S^1$ is homeomorphic to $\mathrm{Cyl}(\pi)$ by a homeomorphism that takes $\partial B^3 \times S^1$ onto the 0-level of $\mathrm{Cyl}(\pi)$. In addition, Theorem 2 provides a homeomorphism from $\mathrm{Cyl}(\pi)$ to $\mathrm{Swl}(\pi)$ which takes the 0-level of $\mathrm{Cyl}(\pi)$ to the 0-level of $\mathrm{Swl}(\pi)$. The composition of these homeomorphisms allows us to identify $B^3 \times S^1$ with $\mathrm{Swl}(\pi)$ so that $\partial B^3 \times S^1$ is identified with the 0-level of $\mathrm{Swl}(\pi)$. Thus, we can regard $C_1, C_2, \ldots, C_k$ as disjoint simple closed curves lying in the 0-level of $\mathrm{Swl}(\pi)$.

Let $1 \leqslant i \leqslant k$. Observe that $\mathrm{Swl}(\pi | C_i)$ can be naturally identified with a subset of $\mathrm{Swl}(\pi)$ so that the 0-level of $\mathrm{Swl}(\pi | C_i)$ is the subset of the 0-level of $\mathrm{Swl}(\pi)$ identified with C_i, and ∞-levels of $\mathrm{Swl}(\pi | C_i)$ and $\mathrm{Swl}(\pi)$ coincide. Since $\pi | C_i : C_i \to S^1$ is a map of degree n_i, then Corollary 3 provides a homeomorphism from $\mathrm{Swl}(\pi | C_i)$ to the mapping cylinder of the map $z \mapsto z^{n_i} : S^1 \to S^1$ which preserves 0-levels and ∞-levels. Since C_i is the 0-level of $\mathrm{Swl}(\pi | C_i)$ and $C_i = \partial D_i$, then clearly $\mathrm{Swl}(\pi | C_i) \cup D_i$ is homeomorphic to $X(n_i)$.

Set $X = \bigcup_{i=1}^k \mathrm{Swl}(\pi | C_i) \cup D_i$. Then X is a compact subset of $\mathrm{int}(M^4)$, and X is clearly homeomorphic to $X(n_1, n_2, \ldots, n_k)$.

It remains to prove that $M^4 - X$ is homeomorphic to $(\partial M^4) \times [0, \infty)$. Observe that $M^4 - X$ is the union of $\mathrm{Swl}(\pi) - \bigcup_{i=1}^k \mathrm{Swl}(\pi | C_i)$ and the sets $H_i - D_i$ for $1 \leqslant i \leqslant k$. Furthermore, $\mathrm{Swl}(\pi) - \bigcup_{i=1}^k \mathrm{Swl}(\pi | C_i)$ is the union of the fibers of $\mathrm{Swl}(\pi)$ that emanate from the points of $(\partial B^3 \times S^1) - \bigcup_{i=1}^k C_i$, and each of these fibers is homeomorphic to $[0, \infty)$. We will "extend" these fibers to fill the sets $H_i - D_i$, $1 \leqslant i \leqslant k$.

We will define a homeomorphism $G : (\partial M^4) \times [0, \infty) \to M^4 - X$. To begin, there is clearly a homeomorphism $F : (\partial B^3 \times S^1) \times [0, \infty) \to \mathrm{Swl}(\pi) - (\{\infty\} \times S^1)$ which takes $\{(x, z)\} \times [0, \infty)$ onto the (x, z)-fiber of $\mathrm{Swl}(\pi)$, for $(x, z) \in \partial B^3 \times S^1$. Indeed, the formula $F((x, z), t) = (t(x, z), \mathrm{e}^{2\pi i t} z)$ for $((x, z), t) \in (\partial B^3 \times S^1) \times [0, \infty)$ determines such a homeomorphism.

For each i, $1 \leqslant i \leqslant k$, set

$$A_i = h_i\big((\partial B^2) \times B^2\big) \quad \text{and} \quad B_i = h_i\big(B^2 \times (\partial B^2)\big).$$

A_i is called the *attaching tube* of H_i, and B_i is called the *belt tube* of H_i. Then $A_i = \mathrm{Swl}(\pi) \cap H_i$ and

$$\partial M^4 = \left((\partial B^3 \times S^1) - \bigcup_{i=1}^{k} \mathrm{int}(A_i) \right) \cup \left(\bigcup_{i=1}^{k} B_i \right).$$

We set

$$G \Big| \left((\partial B^3 \times S^1) - \bigcup_{i=1}^{k} \mathrm{int}(A_i) \right) \times [0, \infty)$$
$$= F \Big| \left((\partial B^3 \times S^1) - \bigcup_{i=1}^{k} \mathrm{int}(A_i) \right) \times [0, \infty).$$

It remains to define $G | B_i \times [0, \infty)$ for $1 \leqslant i \leqslant k$. Consider a point $p \in B_i$. Then $p = h_i(x, y)$ where $(x, y) \in B^2 \times (\partial B^2)$. If $x = 0$, then $G(\{p\} \times [0, \infty))$ is the "deleted radius" $h_i(\{(0, ty)\colon\ 0 < t \leqslant 1\})$ of the disk $h_i(\{0\} \times B^2)$ joining the center point $h_i(0, 0)$ to p. If $x \neq 0$, then $G(\{p\} \times [0, \infty))$ is the union of an arc in H_i joining the point p to a point $q \in A_i$ together with the ray $F(\{q\} \times [0, \infty))$. Moreover, the arc in H_i joining p to q is the h_i-image of the subarc of the "hyperbola" $\{(sx, ty)\colon\ st = 1\}$ joining the point (x, y) to the point $(x/\|x\|, \|x\| y)$. So $q = h_i(x/\|x\|, \|x\| y)$.

The precise definition of G follows. As we stated earlier,

$$G \Big| \left((\partial B^3 \times S^1) - \bigcup_{i=1}^{k} \mathrm{int}(A_i) \right) \times [0, \infty)$$
$$= F \Big| \left((\partial B^3 \times S^1) - \bigcup_{i=1}^{k} \mathrm{int}(A_i) \right) \times [0, \infty).$$

Now suppose $1 \leqslant i \leqslant k$, $p \in B_i$ and $p = h_i(x, y)$, where $(x, y) \in B^2 \times (\partial B^2)$. If $x = 0$, then

$$G(p, t) = h_i\left(0, \left(\frac{1}{t+1}\right) y\right) \quad \text{for } 0 \leqslant t < \infty.$$

If $x \neq 0$, then

$$G(p, t) = \begin{cases} h_i\left((t+1)x, \left(\dfrac{1}{t+1}\right) y\right), & \text{if } 0 \leqslant t \leqslant \dfrac{1}{\|x\|} - 1, \\[2ex] F\left(h_i\left(\dfrac{x}{\|x\|}, \|x\| y\right), t + 1 - \dfrac{1}{\|x\|}\right), & \text{if } \dfrac{1}{\|x\|} - 1 \leqslant t < \infty. \end{cases}$$

The following remarks are intended to further clarify the properties of G. G maps

$$\left((\partial B^3 \times S^1) - \bigcup_{i=1}^{k} \operatorname{int}(A_i)\right) \times [0,\infty)$$

onto

$$\operatorname{Swl}\left(\pi \Big| \left((\partial B^3 \times S^1) - \bigcup_{i=1}^{k} \operatorname{int}(A_i)\right)\right).$$

For $1 \leqslant i \leqslant k$, G maps

$$\left\{(p,t) \in B_i \times [0,\infty)\colon\ 0 \leqslant t \leqslant \frac{1}{\|x\|} - 1,\ p = h_i(x,y),\ (x,y) \in B^2 \times (\partial B^2)\right\}$$

onto $H_i - D_i$, and G maps

$$\left\{(p,t) \in B_i \times [0,\infty)\colon\ \frac{1}{\|x\|} - 1 \leqslant 1 < \infty,\ p = h_i(x,t),\ (x,y) \in B^2 \times (\partial B^2)\right\}$$

onto $\operatorname{Swl}(\pi|A_i - C_i)$. □

Corollary 3. *Suppose C is a simple closed curve in $(\partial B^3) \times S^1$, and $M^4 = (B^3 \times S^1) \cup H$ where H is a 2-handle attached to $B^3 \times S^1$ along C. Let $\pi : B^3 \times S^1 \to S^1$ denote the projection map, and suppose that the map $\pi|C : C \to S^1$ is degree one. Then M^4 has an arc pseudo-spine.*

Proof. Theorem 3 provides M^4 with a pseudo-spine X that is homeomorphic to the 2-dimensional disk $X(1)$. According to [3], X can be "squeezed" to an arc in $\operatorname{int}(M^4)$. In other words, there is an arc A in $\operatorname{int}(M^4)$ and an onto map $f : M^4 \to M^4$ such that $f(X) = A$ and f maps $M^4 - X$ homeomorphically onto $M^4 - A$. (Interpreted literally, [3] applies only in manifolds of dimension 3. However, the methods of [3] work in manifolds of all dimensions $\geqslant 3$. This is fully explained on p. 95 of [2].) Consequently, $M^4 - A$ is homeomorphic to $\partial M^4 \times [0,\infty)$, making A an arc pseudo-spine of M^4. □

Since Mazur's compact contractible 4-manifold [6] is obtained by attaching a 2-handle to $B^3 \times S^1$ along a degree one curve, we recover the result of [5,3].

Corollary 4. *Mazur's compact contractible 4-manifold has an arc pseudo-spine.*

4. Conjectures

The results proved in this paper exhibit simple pseudo-spines for a very modest collection of 4-manifolds: those obtained by attaching essential 2-handles to $B^3 \times S^1$. The following conjectures are founded on the possibly naive hope that these results can be extended to a more general class of compact 4-manifolds.

Conjecture 1. If a compact 4-manifold with boundary is homotopy equivalent to $X(n_1, n_2, \ldots, n_k)$ (where $n_1, n_2, \ldots, n_k$ are nonzero integers), then it has a pseudo-spine which is homeomorphic to $X(n_1, n_2, \ldots, n_k)$.

In the case of a compact contractible 4-manifold, Conjecture 1 combined with the result of [3] would yield:

Conjecture 2. Every compact contractible 4-manifold has an arc pseudo-spine.

Corollary 3 provides an arc pseudo-spine for every compact contractible 4-manifold that is obtained by attaching a 2-handle to $B^3 \times S^1$. Such a 4-manifold has a handlebody decomposition consisting of a single 0-handle, a single 1-handle and a single 2-handle. No 3- or 4-handles are needed. This suggests breaking Conjecture 2 into the following two parts.

Conjecture 2A. Every piecewise linear compact contractible 4-manifold has a handlebody decomposition with no 3- or 4-handles.

Conjecture 2B. Every compact contractible 4-manifold that has a handlebody decomposition with no 3- or 4-handles has an arc pseudo-spine.

Here is a less general and apparently more elementary question than those raised by the previous conjectures. If M^4 and N^4 are 4-manifolds with boundary, define their *boundary connected sum* $M^4 \cup_\partial N^4$ to be the adjunction space $M^4 \cup_h N^4$ where h is a homeomorphism from a collared 3-ball in ∂M^4 to a collared 3-ball in ∂N^4.

Conjecture 3. If two compact 4-manifolds have arc pseudo-spines, then so does their boundary-connected sum.

If two compact contractible 4-manifolds are each obtained by attaching a single 2-handle to $B^3 \times S^1$, then their boundary connected sum has a tree pseudo-spine which is homeomorphic to the letter "H". This is proved by using the methods of the proof of Theorem 3 and [3]. (Recall that a *tree* is a compact contractible 1-dimensional polyhedron.) This raises the question of whether a tree pseudo-spine can be simplified to an arc pseudo-spine. We can ask, more generally, whether a compact 1-dimensional polyhedral pseudo-spine be simplified to a homotopy equivalent canonical model.

Conjecture 4. If a compact 4-manifold has a tree pseudo-spine, then it has an arc pseudo-spine.

Conjecture 5. If a compact noncontractible 4-manifold has a pseudo-spine which is a compact 1-dimensional polyhedron, then it has a pseudo-spine which is a wedge of circles.

There are clear limitations on the amount to which a pseudo-spine can be simplified within its homotopy class. If a compact 4-manifold has a point pseudo-spine, then it is a cone over its boundary, which implies that its boundary is simply connected. On the other hand, there are compact contractible 4-manifolds with nonsimply connected boundaries which have arc pseudo-spines (e.g., the Mazur manifold). Clearly, the arc pseudo-spines of such manifolds can't be simplified to points.

The study of spines and pseudo-spines pursued in this paper and in [1] was partially motivated by the question of whether a compact contractible n-manifold other than the n-ball can have disjoint spines. (The existence of disjoint spines is equivalent to the existence of disjoint pseudo-spines.) In [4] it is shown that for $n \geqslant 9$, there is a large family of distinct compact contractible n-manifolds with disjoint spines. We conjecture a different situation in dimension 4.

Conjecture 6. The only compact contractible 4-manifold that has disjoint spines is the 4-ball.

We conclude with some remarks concerning Conjectures 2, 2A and 2B. The "classical" examples of compact contractible 4-manifolds include, in addition to the Mazur 4-manifold, those described by Poénaru in [7]. We will sketch the construction of Poénaru's examples, and we will explain why many Poénaru 4-manifolds have handlebody decompositions with no 3- or 4-handles. Hence, they provide some evidence for Conjecture 2A. The authors, however, do not know whether Poénaru's examples have arc pseudo-spines. These manifolds are, thus, a likely place to take up the study of Conjectures 2 and 2B.

The following discussion fits most naturally into the piecewise linear category. For this reason we identify the n-ball B^n with $[0,1]^n$ for the remainder of the paper. A locally unknotted piecewise linearly embedded 2-dimensional disk D in B^4 such that $D \cap (\partial B^4) = \partial D$ is called a *slice disk* in B^4 and ∂D is called a *slice knot* in ∂B^4. A piecewise linear simple closed curve J is ∂B^4 is called a *ribbon knot* if there is a piecewise linear map $f: B^2 \to \partial B^4$ which maps ∂B^2 onto J such that the singular set of f—

$$\{p \in \partial B^4\colon\ f^{-1}(p) \text{ contains more than one point}\}$$

—is the union of a pairwise disjoint collection of piecewise linear arcs $A_1, A_2, \ldots, A_k$ in ∂B^4 and for $1 \leqslant i \leqslant k$, $f^{-1}(A_i)$ is the union of two disjoint piecewise linear arcs A_i' and A_i'' in B^2 where $A_i' \subset \mathrm{int}(B^2)$, $A_i'' \cap (\partial B^2) = \partial A_i''$, and f maps each of A_i' and A_i'' homeomorphically onto A_i. Clearly f can be homotoped rel ∂B^2 to a piecewise linear embedding whose image is a slice disk by pushing $f|\mathrm{int}(B^2)$ radially into $\mathrm{int}(B^4)$ and pushing $f|A_i'$ "deeper" than the rest of $f|\mathrm{int}(B^2)$. The slice disk formed in this manner is called a *ribbon disk*. Thus, every ribbon knot is a slice knot. The converse assertion: every slice knot is a ribbon knot, is one of the fundamental unresolved problems of knot theory.

Poénaru's construction of a compact contractible 4-manifold begins with a slice disk D in B^4 such that ∂D is knotted in ∂B^4 and with a knotted piecewise linear simple closed curve K in the boundary of a second 4-ball $\widetilde{B}^4$. Let N be a regular neighborhood of D in B^4 such that $N \cap (\partial B^4)$ is a regular neighborhood of ∂D in ∂B^4. Set $A = \mathrm{cl}(B^4 - N) \cap N$. Then A is a solid torus (i.e., A is piecewise linearly homeomorphic to $S^1 \times B^2$), and we can think of N as a 2-handle attached to $\mathrm{cl}(B^4 - N)$ along A to yield B^4. Let T be a regular neighborhood of K in $\partial \widetilde{B}^4$. Then T is a solid torus. Let $g : T \to A$ be a piecewise linearly homeomorphism. Now define the Poénaru 4-manifold $P^4(D, K)$ to be the adjunction space $\widetilde{B}^4 \cup_g \mathrm{cl}(B^4 - N)$. We can think of $\widetilde{B}^4$ as a "knotted 2-handle" with knotted attaching tube T which is attached to $\mathrm{cl}(B^4 - N)$ by the homeomorphism $g : T \to A$ to yield $P^4(D, K)$. To see that $P^4(D, K)$ is contractible, notice that $\mathrm{cl}(B^4 - N)$ becomes contractible if the core curve of A is "killed", and attaching $\widetilde{B}^4$ to $\mathrm{cl}(B^4 - N)$ by g "kills" this curve. However, $\partial P^4(D, K)$ is not simply connected because it is the union of the two nontrivial knot complements $\mathrm{cl}(\partial B^4 - (N \cap (\partial B^4)))$ and $\mathrm{cl}(\partial B^4 - T)$. See [7] for further details.

Finally we verify that some Poénaru 4-manifolds have handlebody decompositions with no 3- or 4-handles.

Proposition. *If D is a ribbon disk in B^4 and K is a piecewise linear knot in $\partial \widetilde{B}^4$, then the Poénaru 4-manifold $P^4(D, K)$ has a handlebody decomposition with no 3- or 4-handles.*

Proof. Let N, A, T and g be as in the paragraph describing the construction of $P^4(D, K)$. To prove the Proposition, we will established two assertions.

(a) $\mathrm{cl}(B^4 - N)$ has a handlebody decomposition with no 3- or 4-handles.

(b) There is a piecewise linear homeomorphism from $\widetilde{B}^4$ to $B^3 \times [0, 1]$ which identifies T with a subset $T_0 \times \{0\}$ of $B^3 \times \{0\}$ so that $B^3 \times [0, 1]$ is obtained from $T_0 \times [0, 1]$ by attaching 1- and 2-handles to $(\partial T_0) \times [0, 1]$.

The proof of the Proposition is then completed by noting that since $\mathrm{cl}(B^4 - N)$ is piecewise linearly homeomorphic to $(T_0 \times [0, 1]) \cup_g \mathrm{cl}(B^4 - N)$, then by assertion (a), $(T_0 \times [0, 1]) \cup_g \mathrm{cl}(B^4 - N)$ has a handlebody decomposition with no 3- or 4-handles. Furthermore, by assertion (b), $(B^3 \times [0, 1]) \cup_g \mathrm{cl}(B^4 - N)$ is obtained from $(T_0 \times [0, 1]) \cup_g \mathrm{cl}(B^4 - N)$ by attaching 1- and 2-handles. We conclude that $(B^3 \times [0, 1]) \cup_g \mathrm{cl}(B^4 - N) = P^4(D, K)$ has a handlebody decomposition with no 3- or 4-handles.

We now demonstrate assertion (a): $\mathrm{cl}(B^4 - N)$ has a handlebody decomposition with no 3- or 4-handles. (Evidently, a related fact is proved in [8], though the language there is quite different.) We can identify B^4 with $B^3 \times [0, 1]$ so that $\partial D \subset \mathrm{int}(B^3) \times \{1\}$. Furthermore, we can assume that the ribbon disk D is positioned in a special way that we now describe. D arises from a map $f : B^2 \to \mathrm{int}(B^3) \times \{1\}$ with singular set equal to the union of a pairwise disjoint collection of arcs $A_1, A_2, \ldots, A_k$ such that for $1 \leqslant i \leqslant k$, $f^{-1}(A_i)$ is the union of two disjoint arcs A_i' and A_i'' in B^2 where $A_i' \subset \mathrm{int}(B^2)$, $A_i'' \cap (\partial B^2) = \partial A_i''$, and f maps each of A_i' and A_i'' homeomorphically onto A_i.

We impose a "collared" handlebody decomposition on B^2 as follows. The 0-handles are disjoint disks $E_1, E_2, \ldots, E_k$ in $\mathrm{int}(B^2)$ such that $A_i' \subset \mathrm{int}(E_i)$ and $E_i \cap A_j'' = \emptyset$ for $1 \leqslant i, j \leqslant k$. For $1 \leqslant i \leqslant k$, we add an exterior collar to E_i to obtain a slightly larger disk E_i^+ in $\mathrm{int}(B^2)$ so that $E_1^+, E_2^+, \ldots, E_k^+$ are pairwise disjoint and are disjoint from $A_1'', A_2'', \ldots, A_k''$. Next we connect the k disks $E_1^+, E_2^+, \ldots, E_k^+$ with $k-1$ disjoint 1-handles or "bands" $F_1, F_2, \ldots, F_{k-1}$ in $\mathrm{int}(B^2) - \bigcup_{i=1}^k \mathrm{int}(E_i^+)$. Set

$$G = \left(\bigcup_{i=1}^{k} E_i^+\right) \cup \left(\bigcup_{j=1}^{k-1} F_j\right).$$

Then G is a disk in $\mathrm{int}(B^2)$. For $1 \leqslant j \leqslant k-1$, each of the sets $(\partial F_j) \cap (\bigcup_{j=1}^k E_i^+)$ and $(\partial F_j) \cap (\partial G)$ is the union of two disjoint arcs in ∂F_j, and these four arcs subdivide ∂F_j and have disjoint interiors. We add an exterior collar to G to obtain a slightly larger disk G^+ in $\mathrm{int}(B^2)$. Of course, $B^2 - \mathrm{int}(G^+)$ is an annulus.

To form the ribbon disk D from the map f, we push f "vertically" down the $[0,1]$-fibers of $B^3 \times [0,1]$ and make some minor "horizontal" adjustments to achieve an embedding with the following properties. (We now identify B^2 with its image D.) The 0-handles $E_1, E_2, \ldots, E_k$ lie in the level $B^3 \times \{1/4\}$. For $1 \leqslant i \leqslant k$, the collar $E_i^+ - \mathrm{int}(E_i)$ lies vertically over ∂E_i in the product $B^3 \times [1/4, 1/2]$ so that ∂E_i^+ lies in the level $B^3 \times \{1/2\}$. The 1-handles $F_1, F_2, \ldots, F_{k-1}$ lie in the level $B^3 \times \{1/2\}$. The collar $G^+ - \mathrm{int}(G)$ lies vertically over ∂G in the product $B^3 \times [1/2, 3/4]$ so that ∂G^+ lies in the level $B^3 \times \{3/4\}$. The annulus $D - \mathrm{int}(G^+)$ lies in the product $B^3 \times [3/4, 1]$ so that each level circle of the annulus lies in a $B^3 \times \{t\}$-level and, of course, ∂D lies in $B^3 \times \{1\}$.

Let $\pi : B^3 \times [0,1] \to B^3$ denote projection. The regular neighborhood N of D can be assumed to have the following form:

$$\begin{aligned} N = {} & \big(N_1 \times [1/4 - \delta, 1/4 + \delta]\big) \cup \big(N_2 \times [1/4 + \delta, 1/2 - \delta]\big) \\ & \cup \big(N_3 \times [1/2 - \delta, 1/2 + \delta]\big) \cup \big(N_4 \times [1/2 + \delta, 3/4]\big) \cup N_5 \end{aligned}$$

where N_1, N_2, N_3 and N_4 are regular neighborhoods of $\pi(D \cap (B^3 \times \{t\}))$ in $\mathrm{int}(B^3)$ for $t = 1/4, 3/8, 1/2$ and $5/8$, respectively. N_5 is a regular neighborhood of the annulus $D - \mathrm{int}(G^+)$ in $B^3 \times [3/4, 1]$, and $0 < \delta < 1/8$.

N_1 is a regular neighborhood of the union of the k disks $\pi(E_i)$, $1 \leqslant i \leqslant k$; and N_2 is a regular neighborhood of the union of the k simple closed curves $\pi(\partial E_i)$, $1 \leqslant i \leqslant k$. Thus, N_1 has k components each of which is a 3-ball containing one of the disks $\pi(E_i)$, and N_2 has k components each of which is a solid torus containing one of the simple closed curves $\pi(\partial E_i)$. Moreover, we can assume that $N_2 \subset N_1$, and that $\mathrm{cl}(N_1 - N_2)$ has k components each of which is a 3-ball that intersects $\mathrm{cl}(B^3 - N_1)$ in a pair of disjoint boundary disks. This allows us to view each component of $\mathrm{cl}(N_1 - N_2)$ as a 3-dimensional 1-handle attached to $\mathrm{cl}(B^3 - N_1)$. Hence, $\mathrm{cl}(B^3 - N_2)$ is obtained by attaching k 3-dimensional 1-handles (the components of $\mathrm{cl}(N_1 - N_2)$) to $\mathrm{cl}(B^3 - N_1)$.

Let X denote the union of the simple closed curves ∂E_i^+, $1 \leqslant i \leqslant k$, and the "bands" F_j, $1 \leqslant j \leqslant k$. N_3 is a regular neighborhood of $\pi(X)$. Hence, we can assume that $N_2 \subset N_3$ and that N_3 is obtained from N_2 by attaching $k-1$ 3-dimensional 1-handles, each 1-handle containing one of the disks $\pi(F_j)$. N_4 is a regular neighborhood of $\pi(\partial G)$, and ∂G is obtained from X by removing from X all of F_j except for the two arcs comprising $F_j \cap (\partial G)$ for $1 \leqslant j \leqslant k-1$. It follows that we can assume that $N_4 \subset N_3$, and that $\mathrm{cl}(N_3 - N_4)$ has $k-1$ components each of which is a 3-ball that intersects $\mathrm{cl}(B^3 - N_3)$ in a boundary annulus. This allows us to view each component of $\mathrm{cl}(N_3 - N_4)$ as a 3-dimensional 2-handle attached to $\mathrm{cl}(B^3 - N_3)$. Hence, $\mathrm{cl}(B^3 - N_4)$ is obtained by attaching $k-1$ 3-dimensional 2-handles (the components of $\mathrm{cl}(N_3 - N_4)$) to $\mathrm{cl}(B^3 - N_3)$.

The following seven assertions clearly imply that $\mathrm{cl}(B^4 - N)$ has a handlebody decomposition involving no 3- or 4-handles.

(i) $Y_0 = B^3 \times [0, 1/4 - \delta]$ is a 4-ball and can, thus, be regarded as a 0-handle.
(ii) $Y_{0+} = Y_0 \cup (\mathrm{cl}(B^3 - N_1) \times [1/4 - \delta, 1/2 - \delta])$ is homeomorphic to Y_0.
(iii) $Y_1 = \mathrm{cl}(B^3 \times [0, 1/2 - \delta] - N)$ is obtained from Y_{0+} by attaching 1-handles.
(iv) $Y_{1+} = Y_1 \cup (\mathrm{cl}(B^3 - N_3) \times [1/2 - \delta, 3/4])$ is homeomorphic to Y_1.
(v) $Y_2 = \mathrm{cl}(B^3 \times [0, 3/4] - N)$ is obtained from Y_{1+} by attaching 2-handles.
(vi) $Y_{2+} = Y_2 \cup (\mathrm{cl}(B^3 - N_4) \times [3/4, 1])$ is homeomorphic to Y_2.
(vii) $\mathrm{cl}(B^4 - N)$ is homeomorphic to Y_{2+}.

Assertions (i), (ii), (iv) and (vi) are immediate.

To prove assertion (iii), observe that $Y_1 = Y_{0+} \cup (\mathrm{cl}(N_1 - N_2) \times [1/4 + \delta, 1/2 - \delta])$. Since $\mathrm{cl}(N_1 - N_2)$ can be viewed as the union of k 3-dimensional 1-handles attached to $\mathrm{cl}(B^3 - N_1)$, then $\mathrm{cl}(N_1 - N_2) \times [1/4 + \delta, 1/2 - \delta]$ can be viewed as the union of k 4-dimensional 1-handles attached to Y_{0+} along $(\partial \mathrm{cl}(B^3 - N_1)) \times [1/4 + \delta, 1/2 - \delta]$. Hence, Y_1 is obtained from Y_{0+} by attaching k 4-dimensional 1-handles.

To prove assertion (v), observe that $Y_2 = Y_{1+} \cup (\mathrm{cl}(N_3 - N_4) \times [1/2 + \delta, 3/4])$. Since $\mathrm{cl}(N_3 - N_4)$ can be viewed as the union of $k-1$ 3-dimensional 2-handles attached to $\mathrm{cl}(B^3 - N_3)$, then $\mathrm{cl}(N_3 - N_4) \times [1/2 + \delta, 3/4]$ can be viewed as the union of $k-1$ 4-dimensional 2-handles attached to Y_{1+} along $(\partial \mathrm{cl}(B^3 - N_3)) \times [1/2 + \delta, 3/4]$. Hence, Y_2 is obtained from Y_{1+} by attaching $k-1$ 4-dimensional 2-handles.

Finally, to prove assertion (vii), we observe that the original map $f : B^2 \to B^3 \times \{1\}$ embeds the annulus $B^2 - \mathrm{int}(G^+)$. Hence, there is a piecewise linear ambient isotopy of $B^3 \times \{1\}$ which "drags" $f(\partial G^+)$ through the level circles of the annulus $f(B^2 - \mathrm{int}(G^+))$. This ambient isotopy can be "spread out" as a level preserving piecewise linear homeomorphism h of $B^3 \times [3/4, 1]$ which restricts to the identity on $B^3 \times \{3/4\}$, which carries the "cylinder" $\pi(\partial G^+) \times [3/4, 1]$ onto the annulus $D - \mathrm{int}(G^+)$, and which carries $N_4 \times [3/4, 1]$ onto N_5. (If $h(N_4 \times [3/4, 1]) \neq N_5$ initially, we correct this by redefining N_5.) We extend h over $B^3 \times [0, 3/4]$ via the identity. Then h carries Y_{2+} onto

$$Y_2 \cup \mathrm{cl}\bigl(B^3 \times [3/4, 1] - N_5\bigr) = \mathrm{cl}\bigl(B^3 \times [0, 1] - N\bigr) = \mathrm{cl}(B^4 - N).$$

This completes the proof of assertion (a): $\mathrm{cl}(B^4 - N)$ has a handlebody decomposition with no 3- or 4-handles.

It remains to demonstrate assertion (b): there is a piecewise linear homeomorphism from $\widetilde{B}^4$ to $B^3 \times [0, 1]$ which identifies T with a subset $T_0 \times \{0\}$ of $B^3 \times \{0\}$ so that $B^3 \times [0, 1]$ is obtained from $T_0 \times [0, 1]$ by attaching 1- and 2-handles to $(\partial T_0) \times [0, 1]$. Let C^3 be a 3-ball in $\partial \widetilde{B}^4$ such that $T \subset \mathrm{int}(C^3)$. $C^3 - \mathrm{int}(T)$ has a handlebody decomposition based on T; in other words, C^3 can be obtained by attaching 0-, 1-, 2-

and 3-handles to T. The 0-handles of this decomposition can be eliminated by cancelling them with some 1-handles, and the 3-handles can be eliminated by cancelling them with some 2-handles. These cancellations can be performed without moving T, but then C^3 may be forced to move. At the end of the process, T is still a subset of the (possibly repositioned) 3-ball C^3. (T may no longer be interior to C^3.) Now C^3 is obtained by attaching 1- and 2-handles to T. Since C^3 is a piecewise linear 3-ball in $\partial\widetilde{B}^4$, there is a piecewise linear homeomorphism $k: B^3 \times [0,1] \to \widetilde{B}^4$ such that $k(B^3 \times \{0\}) = C^3$. There is a solid torus T_0 in B^3 such that $k(T_0 \times \{0\}) = T$. It follows that B^3 can be obtained from T_0 by adding 3-dimensional 1- and 2-handles. By "crossing" each of these handles with $[0,1]$, we see that $B^3 \times [0,1]$ can be obtained from $T_0 \times [0,1]$ by attaching 4-dimensional 1- and 2-handles to $(\partial T_0) \times [0,1]$. This proves assertion (b). $\square$

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