Compact contractible n-manifolds have arc spines (n≥5)

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COMPACT CONTRACTIBLE n-MANIFOLDS HAVE ARC SPINES ($n \geq 5$)

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The following two theorems were motivated by questions about the existence of disjoint spines in compact contractible manifolds.

THEOREM 1. *Every compact contractible n-manifold* ($n \geq 5$) *is the union of two n-balls along a contractible* $(n-1)$*-dimensional submanifold of their boundaries.*

A compactum X is a *spine* of a compact manifold M if M is homeomorphic to the mapping cylinder of a map from ∂M to X.

THEOREM 2. *Every compact contractible n-manifold* ($n \geq 5$) *has a wild arc spine.*

Also a new proof is given that for $n \geq 6$, every homology $(n-1)$-sphere bounds a compact contractible n-manifold. The implications of arc spines for compact contractible manifolds of dimensions 3 and 4 are discussed in §5. The questions about the existence of disjoint spines in compact contractible manifolds which motivated the preceding theorems are stated in §6.

1. Introduction. Let M be a compact manifold with boundary. A compactum X is a *spine* of M if there is a map $f : \partial M \to X$ and a homeomorphism $h : M \to \mathrm{Cyl}(f)$ such that $h(x) = q((x, 0))$ for $x \in \partial M$. Here $\mathrm{Cyl}(f)$ denotes the mapping cylinder of f and $q : (\partial M \times [0,1]) \cup X \to \mathrm{Cyl}(f)$ is the natural quotient map. Thus $q_{|\partial M \times [0,1)}$ and $q|X$ are embeddings and $q(x, 1) = q(f(x))$ for $x \in \partial M$. So h carries ∂M homeomorphically onto $q(M \times \{0\})$, $h^{-1} \circ q_{|X}$ embeds in X int M, and $M - h^{-1}(q(X)) \cong \partial M \times [0, 1)$.

An arc A in the interior of an n-manifold M is *tame* if A has a neighborhood U in M such that (U, A) is homeomorphic to $(\mathbb{R}^n, [-1,1] \times (0, 0 \ldots, 0))$. An arc in the interior of a manifold is *wild* if it is not tame.

Recall that a space is *contractible* if it is homotopy equivalent to a point, and it is *acyclic* if its homology groups are isomorphic to the homology groups of a point. A *homotopy n-sphere* is a closed n-manifold which is homotopy equivalent to an n-sphere. A *homology n-sphere* is a closed n-manifold whose homology groups are isomorphic to the homology groups of an n-sphere. (Throughout this paper all homology groups have integer coefficients.) We now list some elementary facts about these terms that will be used without comment in the proofs below.

(1) Every contractible space is acyclic, and every homotopy n-sphere is a homology n-sphere.
(2) The boundary of every compact acyclic n-manifold is a homology $(n-1)$-sphere.
(3) Every bicollared homology $(n-1)$-sphere in a homology n-sphere separates the homology n-sphere into two compact acyclic n-manifolds.
(4) Conversely, a closed n-manifold is a homology n-sphere if it is the union of two compact acyclic n-manifolds which intersect in their common boundary.
(5) A compact n-manifold is acyclic if it is the union of two compact acyclic n-manifolds which intersect in a compact acyclic $(n-1)$- dimensional submanifold of the boundary of each.
(6) A simply connected acyclic manifold is contractible.
(7) A simply connected homology n-sphere is a homotopy n-sphere.
(8) For $n \geq 4$, a compact contractible n-manifold is an n-ball if its boundary is an $(n-1)$-sphere.
(9) For $n \geq 4$, a homotopy n-sphere is an n-sphere.

Facts (1) - (5) follow from well known results of homology theory including homotopy invariance, excision, the Mayer-Vietoris sequence, and universal coefficient and duality theorems. Facts (6) and (7) follow from the Hurewicz isomorphism theorem and a theorem of Whitehead. Facts (8) and (9) follow from the Poincaré conjecture for topological manifolds, [**2**] and [**7**].

In the following proofs, all homomorphisms between homology groups or homotopy groups are inclusion induced unless otherwise specified.

We state two lemmas which play essential roles in the subsequent proofs of the theorems.

LEMMA 1. *For each $n \geq 4$, if Σ^n is a homology n-sphere, then there is a bicollared embedding of a homology $(n-1)$-sphere Σ^{n-1} in Σ^n such that $\pi_1(\Sigma^{n-1}) \to \pi_1(\Sigma^n)$ is onto.*

Proof. The $n = 4$ case is just Proposition 2 of [**4**]. For $n \geq 5$, this result is known. (J. C. Hausmann has called it the "Newman construction" because it generalizes the method of [**15**].) We sketch the argument because we will refer to it in the proofs of the theorems. Since $H^4(\Sigma^n; \mathbb{Z}_2) = 0$, then by [**12**] Σ^n has a PL structure. Since $H_1(\Sigma^n) = 0$, then $\pi_1(\Sigma^n)$ is a finitely presented perfect group. According to §2.1 of [**8**] (or see Proposition 4.4 of [**3**]), $\pi_1(\Sigma^n)$ is the homomorphic image of a finitely presented perfect group G of deficiency 0. (A group presentation has *deficiency* 0 if the number of generators equals the number of relators.) Use the presentation of G to construct a finite 2-complex K with $\pi_1(K) \approx G$. An analysis of the cellular homology sequence of K reveals that K is acyclic. (The fact that the presentation of G has defiency 0 is used here.) The epimorphism $\pi_1(K) \approx G \to \pi_1(\Sigma^n)$ determines an embedding of K in Σ^n so that the inclusion induced homomorphism $\pi_1(K) \to \pi_1(\Sigma^n)$ is onto. Let N be a regular neighborhood of K in Σ^n. Then N is acyclic and, hence, ∂N is a homology $(n-1)$-sphere. Since $n \geq 5$, general position arguments show that $\pi_1(\partial N) \to \pi_1(N)$ is an isomorphism. Since $\pi_1(K) \to \pi_1(N)$ is an isomorphism, and $\pi_1(K) \to \pi_1(\Sigma^n)$ is onto, it follows that $\pi_1(\partial N) \to \pi_1(\Sigma^n)$ is onto. □

LEMMA 2. *Every compact contractible n-manifold $(n \geq 4)$ is determined by its boundary.*

Proof. Suppose C and D are compact contractible n-manifolds with $\partial C = \partial D = \Sigma$. We must prove C is homeomorphic to D. Observe that $S = (C \times \{0\}) \cup (\Sigma \times [1,0]) \cup (D \times \{1\})$ is homology n-sphere which, by Van Kampen's Theorem, is simply connected. Thus, S is a homotopy n-sphere and, hence an n-sphere by the Poincaré conjecture. So S is the boundary of an $(n+1)$-ball B. Therefore, $(B, C \times \{0\}, D \times \{1\})$ is a simply connected h-cobordism which already has a product structure joining $\partial C \times \{0\}$ to $\partial D \times \{1\}$

in ∂B. Hence, the known h-cobordism theorems imply there is a product structure on B joining $C \times \{0\}$ to $D \times \{1\}$. This product structure induces a homeomorphism from C to D. □

Another crucial ingredient in the proofs of Theorems 1 and 2 is the following result.

THEOREM 0. *Every homology n-sphere bounds a compact contractible $(n+1)$-manifold.*

This is proved for $n \geq 4$ in [**11**] and for $n = 3$ in [**7**]. Later we will give a new proof for $n \geq 5$, which relies on the n-dimensional Poincaré conjecture.

2. Proof of Theorem 1. Suppose C is a compact contractible n-manifold ($n \geq 5$). Then Lemma 1 provides a bicollared embedding of a homology $(n-2)$- sphere Σ in ∂C such that $\pi_1(\Sigma) \to \pi_1(\partial C)$ is onto. Σ separates ∂C into two compact acyclic $(n-1)$-manifolds Q_1 and Q_2. Each loop in Q_1 is homotopic in ∂C to a loop in Σ. If such a homotopy is cut off on $\partial Q_1 = \Sigma$, then we see that each loop in Q_1 is a boundary component of a (singular) punctured disk in Q_1 which has its other boundary components in Σ. The same is true of the loops in Q_2. In algebraic language: every element of $\pi_1(Q_i)$ lies in the normal closure of the image of $\pi_1(\Sigma) \to \pi_1(Q_i)$.

Theorem 0 implies that Σ bounds a compact contractible $(n-1)$-manifold D. Hence, for $i = 1$ or 2, $Q_i \cup_\Sigma D$ is a homology $(n-1)$-sphere. Since every element of $\pi_1(Q_i)$ is in the normal closure of the image of $\pi_1(\Sigma) \to \pi_1(Q_i)$, and since $\pi_1(D) = 1$, then every element of $\pi_1(Q_i)$ includes trivially into $\pi_1(Q_i \cup_\Sigma D)$. Consequently, $\pi_1(Q_i \cup_\Sigma D) = 1$. Thus each $Q_i \cup_\Sigma D$ is a homotopy $(n-1)$-sphere and, hence, an $(n-1)$-sphere by the Poincaré conjecture. So each $(Q_i \cup_\Sigma D)$ bounds an n-ball B_i. Observe that $B_1 \cup_D B_2$ is a compact acyclic n-manifold which is simply connected and, hence contractible. Furthermore, $\partial(B_1 \cup_D B_2) = Q_1 \cup_\Sigma Q_2 = \partial C$. Lemma 2 now implies that C is homeomorphic to $B_1 \cup_D B_2$. □

3. Proof of Theorem 2. Let C be a compact contractible n-manifold ($n \geq 5$). Then Lemma 1 provides an embedding of $\Sigma \times [0,1]$ in ∂C where Σ is homology $(n-2)$-sphere such that for each $t \in [0,1]$, the inclusion of $\Sigma \times \{t\}$ into ∂C induces an epimorphism of fundamental groups. Therefore, $\partial C - \Sigma \times (0,1) = Q_0 \cup Q_1$ where each Q_i is a compact acyclic $(n-1)$-manifold with $\partial Q_i = \Sigma \times \{i\}$. Define the map f: $\partial C \to [0,1]$ by $f(Q_0) = 0$, $f(\Sigma \times \{t\}) = t$ for $0 < t < 1$, and $f(Q_1) = 1$. We will argue below that $\mathrm{Cyl}(f)$ is the cell-like image of C and that $\mathrm{Cyl}(f)$ satisfies the disjoint disks property. Since $\mathrm{Cyl}(f)$ is clearly finite dimensional, it will then follow by Edwards' theorem [**6**] that C is homeomorphic to $\mathrm{Cyl}(f)$, proving that C has an arc spine.

We make some preliminary remarks about the topology of $\mathrm{Cyl}(f)$. $\mathrm{Cyl}(f)$ is the image of a metrizable space via a quotient map which is a closed map with compact point inverses. According to Theorem XI.5.2 on page 235 of [**5**], this makes $\mathrm{Cyl}(f)$ metrizable. $\mathrm{Cyl}(f)$ is an ANR because, in the terminology of Theorem VI.1.2 on page 178 of [**10**], $\mathrm{Cyl}(f)$ is an "adjunction space" which is formed from spaces that are all ANR's.

The construction of a cell-like map from C to $\mathrm{Cyl}(f)$ is similar to the proof of Theorem 1. By Theorem 0, Σ bounds a compact contractible $(n-1)$-manifold D. As in the proof of Theorem 1, $Q_i \cup_{\Sigma\times\{i\}} (D \times \{i\})$ is an $(n-1)$-sphere, for $i = 0$ or 1. So each $Q_i \cup_{\Sigma\times\{i\}} (D \times \{i\})$ bounds an n-ball B_i. Consequently, $B_0 \cup_{D\times\{0\}} D \times [0,1] \cup_{D\times\{1\}} B_1$ is a compact contractible n-manifold. Moreover, $\partial(B_0 \cup_{D\times\{0\}} D \times [0,1] \cup_{D\times\{1\}} B_1) = Q_0 \cup_{\Sigma\times\{0\}} \Sigma \times [0,1] \cup_{\Sigma\times\{1\}} Q_1 = \partial C$. So Lemma 2 implies $B_0 \cup_{D\times\{0\}} D \times [0,1] \cup_{D\times\{1\}} B_1$ is homeomorphic to C. Let $\partial C \times [0,1]$ be an exterior collar on $B_0 \cup_{D\times\{0\}} D \times [0,1] \cup_{D\times\{1\}} B_1$ so that $(x,1)$ identified with x for each $x \in \partial C$. Then the union of $B_0 \cup_{D\times\{0\}} D \times [0,1] \cup_{D\times\{1\}} B_1$ and the exterior collar $\partial C \times [0,1]$ is homeomorphic to C. Let us identify C with this union. Then ∂C is identified with $\partial C \times \{0\}$ in the collar $\partial C \times [0,1]$. We define a cell-like map $g : C \to \mathrm{Cyl}(f)$ as follows. Let $q : (\partial C \times [0,1]) \cup [0,1] \to \mathrm{Cyl}(f)$ be the natural quotient map. For a point (x,t) in the collar $\partial C \times [0,1]$, set $g((x,t)) = q((x,t))$. Set $g(B_0) = q(0)$, $g(D \times \{t\}) = q(t)$ for $0 \le t \le 1$, and $g(B_1) = q(1)$. Hence, for $x \in \partial C, g(x) = q((x,0))$. g is cell-like because the only non-singleton point inverses of g are the contractible sets $g^{-1}(q(0)) = B_0$, $g^{-1}(q(t)) = D \times \{t\}$ for $0 < t < 1$, and $g^{-1}(q(1)) = B_1$.

We now verify that $\mathrm{Cyl}(f)$ has the disjoint disks property. Assign $\mathrm{Cyl}(f)$ a metric. Let $A = q([0,1])$. First we will prove the following assertion: if Y is a dense subset of $A, \phi : B^2 \to \mathrm{Cyl}(f)$ is a map, and $\varepsilon > 0$, then ϕ is within ε of a map $\phi' : B^2 \to \mathrm{Cyl}(f)$ such that $\phi'(B^2) \cap A \subset Y$. Then we will show that this assertion easily implies the disjoint disks property.

There is a finite sequence $0 = t_0 < t_1 < t_2 < \ldots < t_{k-1} < t_k = 1$ and a $\delta > 0$ such that the sets $U_0 = q((Q_0 \cup (\Sigma \times [0, t_1))) \times (\delta, 1])$, $U_i = q((\Sigma \times (t_{i-1}, t_{i+1})) \times (\delta, 1])$ for $0 < i < k$, and $U_k = q(((\Sigma \times (t_{k-1}, 1]) \cup Q_1) \times (\delta, 1])$ are of diameter $< \varepsilon/3$. Clearly $\{U_i : 0 \leq i \leq k\}$ is a collection of contractible open subsets of $\mathrm{Cyl}(f)$ which covers A. Observe that $U_0 - A = q((Q_0 \cup (\Sigma \times [0, t_1))) \times (\delta, 1))$, $U_i - A = q((\Sigma \times (t_{i-1}, t_{i+1})) \times (\delta, 1))$ for $0 < i < k$, and $U_k - A = q(((\Sigma \times (t_{k-1}, 1]) \cup Q_1) \times (\delta, 1))$. Hence, each $U_i - A$ is a non-empty, connected, dense subset of U_i. Let T be a triangulation of B^2 which is so fine that if $\sigma \in T$ and $\phi(\sigma)$ intersects A, then $\phi(\sigma)$ is contained in some U_i. Then ϕ can be perturbed by less than $\varepsilon/3$ so that it maps the 1-skeleton of T into $\mathrm{Cyl}(f) - A$. Since $\pi_1(\Sigma \times \{t\}) \to \pi_1(\partial C)$ is onto for $0 \leq t \leq 1$, then the argument given in the proof of Theorem 1 can be used here to show that if $0 \leq u < t_1$, then every element of $\pi_1(Q_0 \cup (\Sigma \times [0, t_1)))$ is in the normal closure of the image of $\pi_1(\Sigma \times \{u\}) \to \pi_1(Q_0 \cup (\Sigma \times [0, t_1)))$. Consequently, if $0 \leq u < t_1$, then every element of $\pi_1(U_0 - A)$ is in the normal closure of the image of $\pi_1(q((\Sigma \times \{u\}) \times (\delta, 1))) \to \pi_1(U_0 - A)$. Similarly, if $t_{k-1} < u \leq 1$, then every element of $\pi_1(U_k - A)$ is in the normal closure of the image of $\pi_1(q((\Sigma \times \{u\}) \times (\delta, 1))) \to \pi_1(U_k - A)$. Also for $0 < i < k$, if $t_{i-1} < u < t_{i+1}$, then $\pi_1(q((\Sigma \times \{u\}) \times (\delta, 1))) \to \pi_1(U_i - A)$ is onto. Since Y is a dense subset of A, we can choose $u_0 \in [0, t_1)$, $u_i \in (t_{i-1}, t_{i+1})$ for $0 < i < k$, and $u_k \in (t_{k-1}, 1]$ such that $q(u_i) \in Y$ for $0 \leq i \leq k$. Set $y_i = q(u_i)$ for $0 \leq i \leq k$. Now suppose $\sigma \in T$ is a 2-simplex such that $\phi(\sigma) \cap A \neq \varnothing$. Then $\phi(\sigma) \subset U_i$ for some i between 0 and k, and $\phi(\partial\sigma) \cap A = \varnothing$. It follows from the facts cited above that there is a punctured disk $\tau \subset \sigma$ such that $\partial\sigma$ is a component of $\partial\tau$ and there is a map $\psi_\sigma : \tau \to U_i - A$ such that $\psi_{\sigma|\partial\sigma} = \psi_{|\partial\sigma}$ and $\psi_\sigma(\partial\tau - \partial\sigma) \subset q((\Sigma \times \{u_i\}) \times (\delta, 1))$. (If $0 < i < k$, then τ can be taken to be an annulus.) Since the components of the closure of $\sigma - \tau$ are disks in the $\mathrm{int}(\sigma)$ and since $q((\Sigma \times \{u_i\}) \times (\delta, 1])$ is the interior of a cone with vertex $q(u_i) = y_i$ and base $q((\Sigma \times \{u_i\}) \times \{\delta\})$, then ψ_σ extends to a map $\chi_\sigma : \sigma \to (U_i - A) \cup \{y_i\}$. Now define $\phi' : B^2 \to \mathrm{Cyl}(f)$ as follows. Let $\sigma \in T$ be a 2-simplex. If $\phi(\sigma) \cap A = \varnothing$, set $\phi'_{|\sigma} = \phi_{|\sigma}$. If $\phi(\sigma) \cap A \neq \varnothing$, set

$\phi'_{|\sigma} = \chi_\sigma$. Then ϕ' is within ε of ϕ, and $\phi'(B^2) \subset (\mathrm{Cyl}(f) - A) \cup Y$. This establishes our first assertion.

Now to verify that $\mathrm{Cyl}(f)$ has the disjoint disks property, suppose $\phi : B^2 \to \mathrm{Cyl}(f)$ and $\psi : B^2 \to \mathrm{Cyl}(f)$ are maps and $\varepsilon > 0$. Let Y and Z be disjoint dense subsets of the arc A. Our previous assertion provides maps $\phi' : B^2 \to (\mathrm{Cyl}(f) - A) \cup Y$ and $\psi' : B^2 \to (\mathrm{Cyl}(f) - A) \cup Z$ such that ϕ' is within $\varepsilon/2$ of ϕ and ψ' is within $\varepsilon/2$ of ψ. Thus $\phi'(B^2) \cap \psi'(B^2) \cap A = \varnothing$. Since q carries $\partial C \times [0,1)$ homeomorphically onto $\mathrm{Cyl}(f) - A$, then $\mathrm{Cyl}(f) - A$ is an n-manifold. Since $n \geq 5$, then $\phi'_{|\phi'^{-1}(\mathrm{Cyl}(f)-A)}$ and $\psi'_{|\psi'^{-1}(\mathrm{Cyl}(f)-A)}$ can be perturbed into "general position", thereby producing maps $\phi'' : B^2 \to \mathrm{Cyl}(f)$ and $\psi'' : B^2 \to \mathrm{Cyl}(f)$ with disjoint images such that ϕ'' is within ε of ϕ and ψ'' is within ε of ψ.

Now Edwards' theorem [6] implies that the cell-like map $g : C \to \mathrm{Cyl}(f)$ can be approximated by homeomorphisms. Moreover, the approximating homeomorphisms can be chosen to agree with g over any closed subset of $\mathrm{Cyl}(f)$ which is interior to the subset of $\mathrm{Cyl}(f)$ over which g is already a homeomorphism. In particular, there is a homeomorphism $h : C \to \mathrm{Cyl}(f)$ which agrees with g on ∂C. So $h(x) = g(x) = q((x, 0))$ for $x \in \partial C$. We conclude that C has an arc spine.

We now prove that the arc spine of C is wild, or equivalently that $A = q([0,1])$ is a wild arc in $\mathrm{Cyl}(f)$. In fact, we will argue that A is wild as long as Σ is not simply connected. It is automatically the case that Σ is not simply connected if ∂C is not simply connected, because $\pi_1(\Sigma \times \{t\}) \to \pi_1(\partial C)$ is onto for $0 \leq t \leq 1$. However if ∂C is simply connected (i.e., if C is an n-ball), then we must explicitly choose Σ to be a non-simply connected homology $(n-2)$-sphere such that $\Sigma \times [0,1]$ embeds in ∂C. This is easily accomplished because, in fact, every homology $(n-2)$-sphere Σ has a collared embedding in S^{n-1}. (Proof: Σ bounds a compact contractible $(n-1)$-manifold D whose double $D \cup_\Sigma D$ is an $(n-1)$-sphere by the Poincaré conjecture.) Thus, we may assume (after taking special care to choose Σ appropriately in the case that C is an n-ball) that $\pi_1(\Sigma) \neq 1$. If A were tame, then the point $q(1/2) \in A$ would have a neighborhood U in $q((\Sigma \times (0,1)) \times (0,1])$ such that $U - A$ is simply connected. Let U be any neighborhood of $q(1/2)$ in $q((\Sigma \times (0,1)) \times (0,1])$. There is a $\delta \in (0,1)$ such that $q((\Sigma \times \{1/2\}) \times$

$\{\delta\}) \subset U$. Since $q_{|(\Sigma\times(0,1))\times(0,1)}$ is a homeomorphism, then $\pi_1(q((\Sigma\times\{1/2\})\times\{\delta\})) \to \pi_1(q((\Sigma\times(0,1))\times(0,1)))$ is an isomorphism. Since this isomorphism factors through $\pi_1(U-A)$,and since $\pi_1(q((\Sigma\times\{1/2\})\times\{\delta\})) \approx \pi_1(\Sigma) \neq 1$, then $U-A$ can't be simply connected. Consequently, A must be wild in $\mathrm{Cyl}(f)$. □

4. Proof of Theorem 0 for n $\geq$ 5. Let Σ be a homology n-sphere ($n \geq 5$). As is the Proof of Lemma 1, there is an acyclic finite 2-complex K embedded in Σ such that $\pi_1(K) \to \pi_1(\Sigma)$ is onto. Furthermore, if N is a regular neighborhood of K in Σ, then ∂N is a homology $(n-1)$-sphere such that $\pi_1(\partial N) \to \pi_1(\Sigma)$ is onto. Set $Q = \Sigma - \operatorname{int} N$. Then Q is a compact acyclic n-manifold.

Let K' be a copy of K embedded in S^n, and let N' be a regular neighborhood of K' in S^n. Then N' is acyclic and, hence, $\partial N'$ is a homology $(n-1)$-sphere. Set $D' = S^n - \operatorname{int} N'$. Then D' is a compact acyclic n-manifold. $\pi_1(D') \approx \pi_1(S^n - K)$, and $\pi_1(S^n - K) = 1$ by general position since K is 2-dimensional and $n \geq 5$. So D' is simply connected. Therefore D' is a compact contractible n-manifold with $\partial D' = \partial N'$.

We will now prove that N is homeomorphic to N'. Since $n \geq 5$, then according to [**14**], regular neighborhoods of the 2-complex K in manifolds of dimension n are classified up to homeomorphism by homotopy classes of maps from K to BPL. We will argue that any two maps from K to BPL are homotopic. Then $N \cong N'$ will follow. Let $\phi, \psi : K \to \mathrm{BPL}$ be any two maps. Since K is acyclic, then $\pi_1(K)$ is perfect. Also $\pi_1(\mathrm{BPL}) \approx \mathbb{Z}_2$. ($\pi_1(\mathrm{BPL}) \approx \pi_1(\mathrm{BO})$ by the Hirsch-Mazur Theorem stated at the bottom of page 384 of [**13**], and $\pi_1(\mathrm{BO}) \approx \pi_0(\mathrm{O}) \approx \mathbb{Z}_2$ by the homotopy exact sequence of a bundle and the fact that the orthogonal group has two components.) Hence, $\phi_\# : \pi_1(K) \to \pi_1(\mathrm{BPL})$ and $\psi_\# : \pi_1(K) \to \pi_1(\mathrm{BPL})$ are zero maps. Therefore, ϕ and ψ lift to maps $\tilde{\phi}, \tilde{\psi} : K \to \widetilde{\mathrm{BPL}}$ where $\widetilde{\mathrm{BPL}}$ is the universal cover of BPL. Since $\widetilde{\mathrm{BPL}}$ is simply connected it is n-simple for all $n \geq 1$ (i.e., $\pi_1(\widetilde{\mathrm{BPL}})$ acts trivially on $\pi_n(\widetilde{\mathrm{BPL}})$ for $n \geq 1$). Consequently, obstruction theory applies routinely to the problem of finding homotopies between maps into $\widetilde{\mathrm{BPL}}$. Since $\widetilde{\mathrm{BPL}}$ is simply connected, there is a homotopy $h : K^1 \times [0,1] \to \widetilde{\mathrm{BPL}}$ joining $\tilde{\phi}_{|K^1}$ to $\tilde{\psi}_{|K^1}$. (Here K^i denotes the i-skeleton of K for $i = 0, 1, 2$.) According to Eilenberg's homotopy theorem (Theorem 8.3 on page

184 of [9]), $\tilde{\phi}$ is homotopic to $\tilde{\psi}$ via a homotopy which extends $h_{|K^0\times[0,1]}$ if and only if an obstruction $\delta^2(\tilde{\phi},\tilde{\psi},h)\in H^2(K;\pi_2(\widetilde{\mathrm{BPL}}))$ vanishes. Since K is acyclic, then a universal coefficient theorem implies $H^2(K;\pi_2(\widetilde{\mathrm{BPL}}))=0$. We conclude that $\tilde{\phi}$ is homotopic to $\tilde{\psi}$. Therefore, ϕ is homotopic to ψ. So N is homeomorphic to N'.

Since ∂N is homeomorphic to $\partial N'$, then ∂N bounds a compact contractible n-manifold D. As in the proof of Theorem 1, $N\cup_{\partial N} D$ and $Q\cup_{\partial N} D$ are homotopy n-spheres and, hence, n-spheres by the Poincaré conjecture. So $N\cup_{\partial N} D$ and $Q\cup_{\partial N} D$ bound $(n+1)$-balls B_1 and B_2, respectively. Then $B_1\cup_D B_2$ is a compact contractible n-manifold with $\partial(B_1\cup_D B_2)=N\cup_{\partial N} Q=\Sigma$. □

5. Arc spines in dimensions 3 and 4. Suppose C is a compact contractible manifold of dimension 3 or 4 and A is an arc spine of C. Then [1] and [16] imply that A is tame. Hence, A can be shrunk to a point, revealing that C is just a cone on its boundary and that ∂C must be simply connected. We conclude that the only compact contractible 3-manifold that admits an arc spine is the 3-ball and its spine must be tame. We also conclude that the only compact contractible 4-manifolds that admit arc spines are either the 4-ball or cones on exotic homotopy 3-spheres (if they exist) and again the spines must be tame.

6. Questions.

1. Is the n-ball the only compact contractible n-manifold $(n\geq 4)$ that has two disjoint spines?
2. Does every compact contractible n-manifold $(n\geq 4)$ have two disjoint spines?

We remark that although Theorem 2 provides each high dimensional compact contractible manifold with a wild arc spine, the existence of disjoint spines does not follow directly, because a wild arc can't necessarily be pushed off itself by ambient homeomorphism. Indeed, according to [17], for each $n\geq 4$, there is an arc A in $\mathbb{R}^n$ which is *sticky* in the sense that there is an $\varepsilon>0$ such that no homeomorphism of $\mathbb{R}^n$ which is within ε of the identity moves A off itself.

References

[1] J. L. Bryant and R. C. Lacher, *Mapping cylinder neighborhoods on one-complexes in four-space*, Trans. Amer. Math. Soc. **164** (1972), 333-339.

[2] E. H. Connell, *A topological h-cobordism theorem*, Illinois J. Math., **11** (1967), 300-309.

[3] R. J. Daverman and F. C. Tinsley, *Laminations, finitely generated perfect groups, and acyclic maps*, Michigan Math. J., **33** (1986), 343-351.

[4] ———, *Acyclic maps whose mapping cylinders embed in 5-manifolds*, Houston J. of Math, **16** (1990), 255-270.

[5] J. Dugundji, *Topology*, Allyn and Bacon, 1966.

[6] R. D. Edwards, *The topology of manifolds and cell-like maps*, Proc. Internat. Congress of Mathematicians, Helsinki, 1978 (Lethi, O., ed.) Acad. Sci. Fenn. 1980, Helsinki, 111-127.

[7] M. H. Freedman, *The topology of four-dimensional manifolds*, J. Differential Geom., **17** (1982), 357-453.

[8] J. C. Hausmann, *Homological surgery*, Annals of Math., **104** (1976), 573-584.

[9] S. T. Hu, *Homotopy Theory*, Academic Press, 1959.

[10] ———, *Theory of Retracts*, Wayne State University Press, 1965.

[11] M. A. Kervaire, *Smooth homology spheres and their fundamental groups*, Trans. Amer. Math. Soc., **144** (1969), 67-72.

[12] R. C. Kirby and L. C. Siebenmann, *On the triangulation of manifolds and the Hauptvermutung*, Bull. Amer. Math. Soc., **75** (1969), 742-749.

[13] R. Lashof and M. Rothenberg, *Microbundles and smoothing*, Topology, **3** (1964), 357-388.

[14] W. B. R. Lickorish and L. C. Siebenmann, *Regular neighborhoods and the stable range*, Trans. Amer. Math. Soc., **139** (1969), 207-230.

[15] M. H. A. Newman, *Boundaries of ULC sets in Euclidean n-space*, Proc. Natl. Acad. Sci., USA **34** (1948), 193-196.

[16] V. Nicholson, *Mapping cylinder neighborhoods*, Trans. Amer. Math. Soc., **143** (1969), 259-268.

[17] D. G. Wright, *Sticky arcs in* $E^n (n \geq 4)$, Proc. Amer. Math. Soc., **66** (1977), 181-182.

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