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AN EXTENSION OF ROURKE'S PROOF THAT $\Omega_3 = 0$ TO NONORIENTABLE MANIFOLDS

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ABSTRACT. A classical result in manifold theory states that every closed 3-manifold bounds a compact 4-manifold. In 1985 C. Rourke discovered a strikingly short and elementary proof of the orientable case of this theorem ($\Omega_3 = 0$). In this note we show that Rourke's approach can be extended to include nonorientable 3-manifolds.

1. INTRODUCTION

In [4] Rourke gives a brief clever proof of the classical result of Rokhlin [3] that every closed orientable 3-manifold bounds a compact orientable 4-manifold (i.e., $\Omega_3 = 0$). The nonorientable version of Rohklin's theorem, originally proven by Thom [5], guarantees that every closed nonorientable 3-manifold bounds a compact nonorientable 4-manifold ($\mathscr{N}_3 = 0$). In this note, we show that Rourke's approach extends to give a short proof of this latter theorem.

In [4] $\Omega_3 = 0$ is deduced as a corollary of a stronger theorem (proven earlier in [6, 1]) that every closed orientable 3-manifold can be reduced to S^3 by a finite number of elementary Dehn surgeries. Here *elementary* means that a meridian of the attached solid torus is identified with a curve in the boundary of the removed solid torus that is homotopic to the core of the removed solid torus. Then $\Omega_3 = 0$ follows from the observation that any two closed orientable 3-manifolds that differ by an elementary Dehn surgery cobound a compact orientable 4-manifold.

We similarly deduce $\mathscr{N}_3 = 0$ from a stronger theorem (first proven in [2]) about the reducibility by surgery of every nonorientable 3-manifold to a simple model. In the nonorientable situation the simple model that replaces S^3 is the nonorientable 2-sphere bundle over S^1, which we denote $\mathbb{T}$. Our basic theorem is

Theorem 1. *Every closed nonorientable 3-manifold can be reduced to $\mathbb{T}$ by a finite number of elementary Dehn surgeries.*

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Since $\mathbb{T}$ bounds the nonorientable B^3 bundle over S^1, and since any two closed 3-manifolds (orientable or not) that differ by an elementary Dehn surgery cobound a compact 4-manifold, we have

Corollary. ($\mathcal{N}_3 = 0$) *Every closed* 3*-manifold bounds a compact* 4*-manifold.*

In this paper we extend Rourke's techniques to give an elementary proof of Theorem 1.

2. Terminology

As in [4] we will use an induction argument based on a complexity assigned to Heegaard diagrams.

Suppose $M = H_1 \cup H_2$ is a Heegaard splitting of a nonorientable 3-manifold M. Then H_1 and H_2 are nonorientable handlebodies meeting along a nonorientable surface S. If the H_i's are of genus n then S has Euler characteristic $2 - 2n$, and we will call S a nonorientable surface of genus n. A set of n disjoint two-sided (i.e., having an annular regular neighborhood) simple closed curves on S whose complement is a punctured disk is called a *complete system of curves on* S. (Every nonorientable surface of genus n has a complete system of curves.) It is easy to see that if X and Y are complete systems of curves on S with the property that each element of X bounds a disk in H_1 and each element of Y bounds a disk in H_2, then M is completely determined by S, X, and Y. We then call $S(X, Y)$ a Heegaard diagram for M. Moreover, any Heegaard diagram, $S(X, Y)$, uniquely determines a 3-manifold, which we will denote $M(X, Y)$.

A two-sided curve x on a nonorientable surface S is called *exceptional* if $S - x$ is orientable, otherwise it is called *ordinary.* A complete system of curves on S is called *uniform* if it contains only ordinary curves or if genus$(S) = 1$. Every nonorientable surface of genus n has a uniform complete system of curves. (See Observation 1 below.) Note that a genus 1 complete system necessarily contains a single exceptional curve. A Heegaard diagram $S(X, Y)$ will be called uniform if both X and Y are uniform systems.

Remark. The assumption of two-sidedness for all curves used in a Heegaard diagram is of utmost importance. While this property is automatic for a curve on an orientable surface, the situation is much different for nonorientable surfaces. Much of the work in this paper is aimed at preserving two-sidedness when choosing new curves (see, e.g., §4 Lemma 2).

Remark. Although complete systems with one exceptional curve always exist and may seem more natural, the proof given here depends for its simplicity on the consistent use of uniform systems comprised solely of ordinary curves (except of course in the genus 1 case).

3. Observations

Here we list several basic facts about nonorientable handlebodies and their boundaries.

Observation 1. *Every nonorientable handlebody* H *of genus* n *is homeomorphic to a* 3*-ball with* n 1*-handles, all attached in a nonorientable fashion.*

Proof. Clearly H has at least one nonorientable handle h_1. Sliding each of the orientable handles over h_1 gives us the desired realization of H. □

Observation 2. *If $H_1 \cup H_2$ is a Heegaard splitting of a nonorientable manifold M then there is an associated uniform Heegaard diagram for this splitting.*

Proof. If $\text{genus}(H_1) = 1$ there is nothing to prove. Otherwise, use Observation 1 to view both H_1 and H_2 as 3-balls with nonorientable handles attached. Let X be the collection of cocore boundaries of H_1 and Y be the cocore boundaries of H_2. □

Observation 3. *If $S(X, Y)$ is a Heegaard diagram for M, $x \in X$, $y \in Y$ and x intersects y transversally in a single point, then M has a Heegaard splitting of genus $n-1$.*

Proof. Let N be a regular neighborhood of a disk in H_1 bounded by x. Then $[H_1 \backslash \text{int}(N)] \cup [H_2 \cup N]$ is a genus $n-1$ splitting of M. □

Observation 4. *A curve x on a nonorientable genus n surface S is two-sided iff it intersects any given exceptional curve (transverse to it) an even number of times.*

Proof. Let y be an exceptional curve on S transverse to x, and let S_y be the (orientable) manifold obtained by cutting S open along y. Form the orientable double cover $\widetilde{S}$ of S by gluing together two copies of S_y in the proper fashion. By construction, the two lifts of y, called them $\tilde{y}$ and $\tilde{y}'$, will together separate $\widetilde{S}$ into two components each projecting homeomorphically onto $S \backslash y$.

By the nature of orientable double covers, a simple closed curve in S lifts to a loop in $\widetilde{S}$ iff that curve is two-sided. Let $f: ([0, 1], \{0, 1\}) \to (S, *)$ be a parametrization of x with $* \notin y$, and let $\tilde{f}$ be a lift of f. Since each component of $\widetilde{S} \backslash (\tilde{y} \cup \tilde{y}')$ contains one preimage of $*$, it is clear that $\tilde{f}(0) = \tilde{f}(1)$ iff $\tilde{f}(0, 1])$ intersects $\tilde{y} \cup \tilde{y}'$ an even number of times. Furthermore, each of these intersections corresponds to a unique point of $x \cap y$. Therefore x lifts to a loop in $\widetilde{S}$ iff x intersects y an even number of times. □

Observation 5. *If x and y are disjoint exceptional curves in a nonorientable genus n surface S then $x \cup y$ separates S.*

Proof. If $S \backslash (x \cup y)$ were connected then there would be a curve in S that intersects x once and misses y. This would contradict Observation 4. □

Observation 6. *A curve x on a surface S that is a boundary component of a codimension 0 submanifold of S is necessarily two-sided.*

Proof. Note that x is bicollared. □

4. Lemmas

In this section we do most of the work necessary to prove Theorem 1. Lemmas 1 and 2 are nearly the same as those used by Rourke in [4]. A good deal of our effort is spent making Lemma 2 work on a nonorientable surface. The rest of the lemmas are straightforward and deal with special situations we must face due to nonorientability.

Lemma 1. *Suppose $S(X, Y)$ is a Heegaard diagram and Z is a third complete collection of curves on S. Let $\chi(M, Z)$ denote the result of performing surgery on $M(X, Y)$ using the curves of Z (with framings given by parallel curves in S). Then $\chi(M, Z)$ is homeomorphic to the connected sum $M(X, Z) \# M(Y, Z)$.*

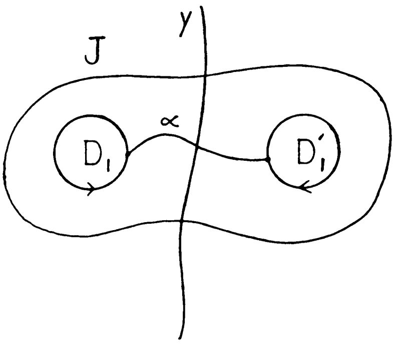

FIGURE 1A

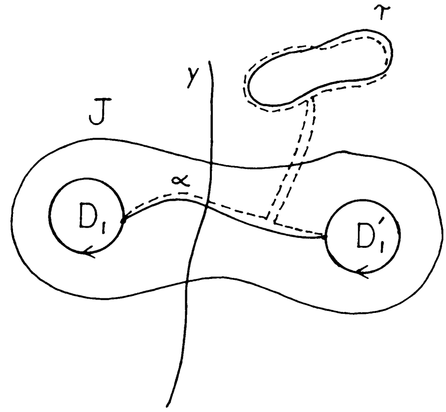

FIGURE 1B

Proof. Rourke's proof of this lemma applies equally well to nonorientable manifolds (see [4, Lemma 1]). □

Lemma 2. *Suppose x and y are two nonseparating two-sided curves on a nonorientable genus n surface S and that x meets y transversally. Let $|x \cap y|$ denote the number of intersection points.*

(a) *If $|x \cap y| = 0$ and both x and y are ordinary then there is a (necessarily ordinary) nonseparating two-sided curve z on S that meets each of x and y transversally in a single point.*

(b) *If $|x \cap y| > 1$ then there is a nonseparating two-sided curve z on S with $|x \cap z| < |x \cap y|$ and $|y \cap z| < |x \cap y|$. Moreover, if x and y are ordinary then z can be chosen to be ordinary.*

Proof. (a) Cut S open along x and glue in disks D_1 and D_1'. Call the resulting surface S_x.

Subcase (a_1). y separates S_x. Then D_1 and D_1' lie on opposite sides of y in S_x; otherwise, y would separate S. Choose an arc α in S_x between corresponding points of ∂D_1 and $\partial D_1'$ meeting y transversally in a single point. Let J be a disk neighborhood of $D_1 \cup \alpha \cup D_1'$ intersecting y in a small arc transverse to α. We then have one of the situations pictured in Figures 1a and 1b. The arrows in the figures indicate identifications that occur when reconstructing S from S_x. In Figure 1A reidentification of ∂D_1 and $\partial D_1'$ turns α into a two-sided curve. Since this curve hits x (and y) once, it can neither separate S nor be exceptional (see Observation 4). This is the required curve z. In the case of Figure 1B, we must modify α in order to achieve two-sidedness. Since x is ordinary, S_x is nonorientable, so one of the components of $S_x - y$ is nonorientable. Choose a one-sided curve τ in $S_x - y$ missing J and "band sum" it to α as shown. After reidentification, this band sum becomes the required curve z.

Subcase (a_2). y does not separate S_x. In this case, cut S_x open along y and glue in disks D_2 and D_2' creating a surface S_{xy}. Let J be a disk in S_{xy} containing each of D_1, D_1', D_2, and D_2'. Choose disjoint arcs α and β in J with α running from ∂D_1 to ∂D_2 and β running between the corresponding points on $\partial D_1'$ and $\partial D_2'$. Figures 2A-D illustrate the possible identifications for recreating S from S_{xy}. In both Figures 2A and 2B the curve on S arising from

$\alpha \cup \beta$ under the described identifications is two-sided and intersects each of x and y once. This is the curve z that our lemma promises. In the case of Figure 2C, $\alpha \cup \beta$ produces a one-sided curve. To find an appropriate modification of $\alpha \cup \beta$, we let K denote $J \backslash (\operatorname{int} D_1 \cup \operatorname{int} D_1')$ with ∂D_1 and $\partial D_1'$ identified. Then K is an orientable submanifold of S_y. S_y is nonorientable because y is ordinary; hence $S_y - K$ is nonorientable. Since $S_y - K = S_{xy} - J$, there is a one-sided curve τ in $S_{xy} - J$. Replace α with the band sum of α and τ. This new arc together with β gives the desired curve z upon reidentification. The situation in Figure 2D is handled similarly.

(b) Let N_x be an annular neighborhood of x chosen sufficiently small that $y \cap N_x$ consists of a finite set of arcs $\{\lambda_A\}$, one for each point A in $x \cap y$. Let ∂_0 and ∂_1 denote the two boundary components of N_x. Assign an orientation to the curve y. A point $A \in x \cap y$ is called a $+1$ intersection if λ_A runs from ∂_0 to ∂_1 and a -1 intersection if λ_A runs from ∂_1 to ∂_0, where orientation on the λ's is induced by that on y. Now choose points $A, B \in x \cap y$ that are consecutive on y, and let α be an arc of y between them containing no other

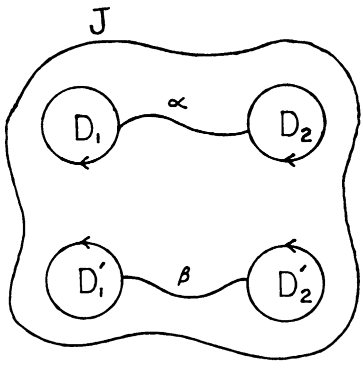

FIGURE 2A

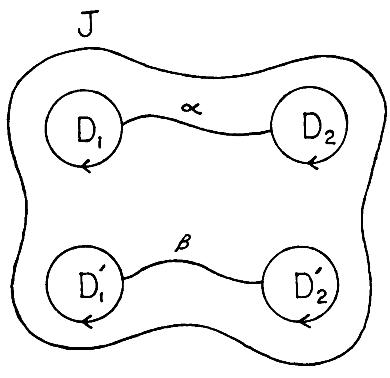

FIGURE 2B

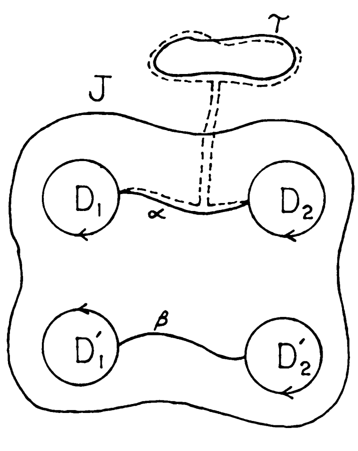

FIGURE 2C

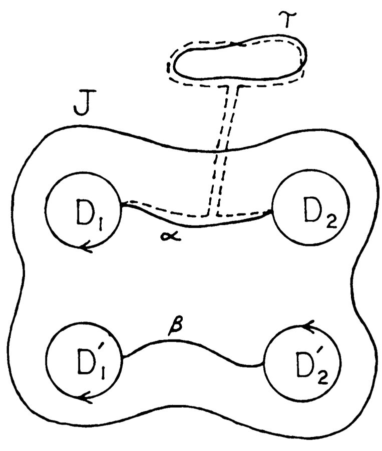

FIGURE 2D

points of $x \cap y$. Let L be a regular neighborhood of $x \cup \alpha$ consisting of N_x and a small "strip" about α connected to N_x at each end.

Subcase (b_1). A and B have opposite sign. Then Figures 3A and 3B illustrate the possible situations. In case L is nonorientable (see Figure 3A), let z be the indicated component of ∂L. Note that $|z \cap x| = 0$ and $|z \cap y|$ is two less than $|x \cap y|$. By Observation 6, z is two-sided. Furthermore, since z and x cobound a submanifold of S and x does not separate S, then z cannot separate S. Moreover, since L itself is nonorientable, z is ordinary.

In the case of Figure 3B (i.e., L orientable), we have two candidates for z. Let z_1 and z_2 denote the boundary components of L parallel to $\alpha \cup \beta_1$ and $\alpha \cup \beta_2$, respectively; where β_1 and β_2 are the subarcs of x with endpoints A and B. Each z_i intersects both x and y in less than $|x \cap y|$ points, and by Observation 6 both z_1 and z_2 are two-sided. Now we focus on $\alpha \cup \beta_1$ and $\alpha \cup \beta_2$, keeping mind that these are just parallel copies of z_1 and z_2. At least one of the $\alpha \cup \beta_i$'s does not separate. This can be seen by considering a curve γ, transverse to both x and y, meeting x at a single point. (Recall x does not separate.) Then, one of the $\alpha \cup \beta_i$'s hits γ an odd number of times, and the other hits γ an even number of times. The $\alpha \cup \beta_i$ that hits γ an odd number of times cannot separate S. Therefore, the corresponding z_i does not separate S. If "ordinary-ness" is not required, simply choose z to be this z_i. If x and y are ordinary and z is required to be ordinary, we must be more selective. Applying Observation 4 to the curve γ shows that the $\alpha \cup \beta_i$'s cannot both be exceptional. Since we are free to use either z_i provided it is nonseparating and ordinary, we need only rule out the possibility that $\alpha \cup \beta_1$ is exceptional and $\alpha \cup \beta_2$ separates (or vice versa). Presuming for the moment that this occurs, let C be the component of $S - (\alpha \cup \beta_2)$ that does not intersect β_1. Since C lies in the complement of $\alpha \cup \beta_1$, C is orientable. Similarly, $S - (\beta_1 \cup \overline{C})$ is orientable. Therefore $S - x$, which is the union of C and $S - (\beta_1 \cup \overline{C})$ along α, must be orientable. This contradicts the assumption that x is ordinary. Thus we are assured that one of the $\alpha \cup \beta_i$'s is both nonseparating

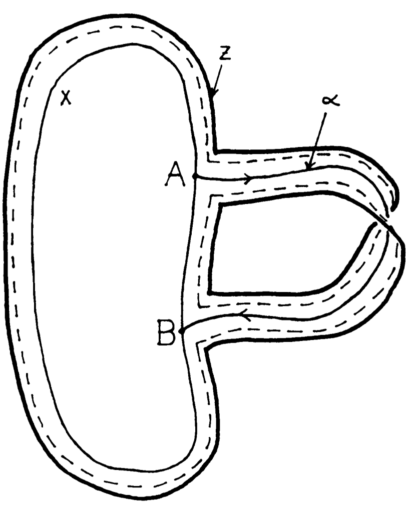

FIGURE 3A

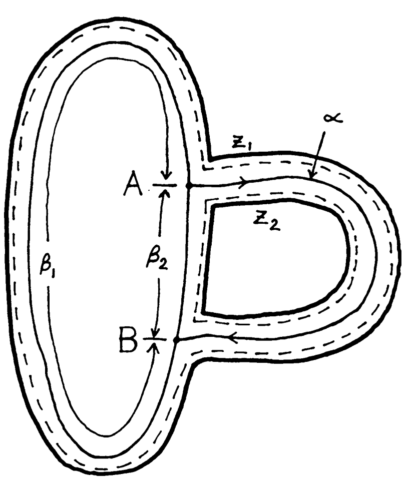

FIGURE 3B

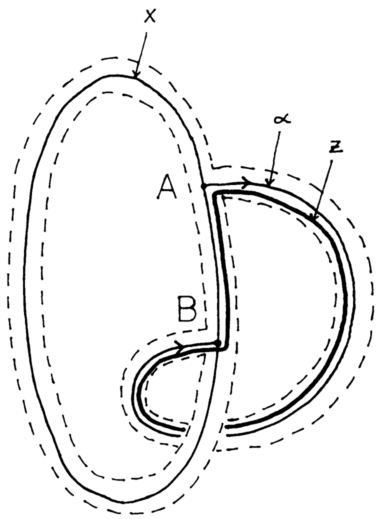

FIGURE 4A

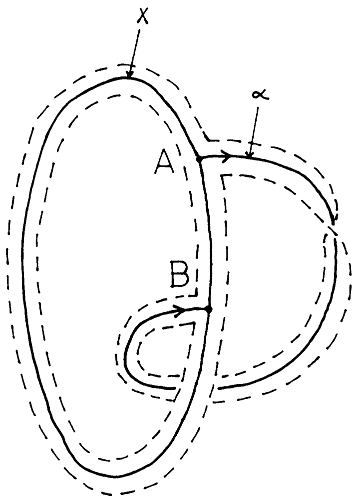

FIGURE 4B

and ordinary. The corresponding z_i is our choice for z.

Subcase (b_2). A and B have the same sign. If L is orientable then Figure 4A illustrates the curve z we will choose. It is easy to see that z is two-sided and $|z \cap y| < |x \cap y|$. Furthermore, $|z \cap x| = 1$, implying that $|z \cap x| < |x \cap y|$ and z is nonseparating and ordinary (use Observation 4 for the latter). Finally, in case L is nonorientable, we reverse the roles of curves x and y, assigning ± 1's to the points of $x \cap y$ according to the way that an oriented x crosses an annular neighborhood of y. Then A and B will have opposite signs. (See Figure 4B). Hence, there are points of $x \cap y$, consecutive on x, with opposite sign. So we find ourselves back in Subcase (b_1) with the roles of x and y reversed. □

Lemma 3. *If* $S(\{x\}, \{y\})$ *is a genus* 1 *Heegaard diagram for a nonorientable manifold* M *and* $x \cap y = \varnothing$, *then* $M \approx \mathbb{T}$.

Proof. By definition of Heegaard diagram, M is the union of two nonorientable genus 1 handlebodies (i.e., nonorientable disk bundles over S^1), identified by some homeomorphism of their boundaries. This homeomorphism is determined, up to isotopy, by x and y. Since $S - x$ and $S - y$ are punctured disks and y cannot bound a disk in S, y must wind once around the puncture in $S - x$. It is then clear that x and y are isotopic in S. Thus, the homeomorphism of S determined by x and y is isotopic to the identity. In particular, M is just the double of a nonorientable disk bundle over S^1. This is $\mathbb{T}$. □

Lemma 4. *If* z *is an ordinary nonseparating two-sided curve on a nonorientable surface* S *of genus* $n > 1$ *then there is a uniform complete system of curves* Z, *on* S, *containing* z.

Proof. Let S_z denote the nonorientable surface of genus $n - 1$ obtained by cutting S open along z and sewing in disks D and D'. Let J be a disk in S_z containing D and D'. If $n = 2$ then S_z is a Klein bottle and two cases must be considered depending on how ∂D and $\partial D'$ get identified when

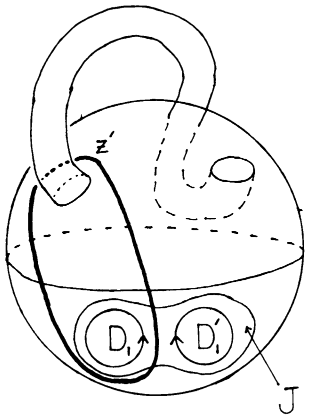

FIGURE 5A

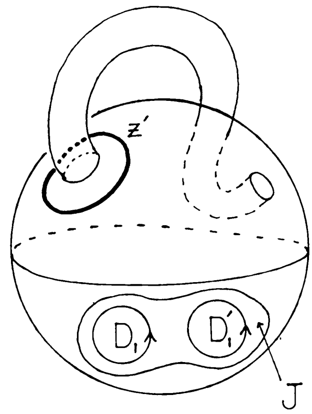

FIGURE 5B

S is recreated from S_z. These two cases are illustrated in Figures 5A and 5B. The correct choice of a second curve z' needed to form a complete uniform system on S is also illustrated in these two figures. If $n > 2$, choose a uniform complete system of curves Z' for S_z with no elements of Z' intersecting J. Since $\text{genus}(S_z) > 1$, each element of Z' is ordinary in S_z. Now regarding Z' as a subset of S, we see that each element of Z' is ordinary in S. So $Z = Z' \cup z$ is a uniform complete system for S. □

Lemma 5. $\mathbb{T} \# \mathbb{T}$ *can be reduced to* $\mathbb{T}$ *with one elementary Dehn surgery.*

Proof. Let H denote a 3-ball with two nonorientable handles attached. It is easy to see that $\text{Double}(H) \approx \mathbb{T} \# \mathbb{T}$. Let H' denote a 3-ball with one orientable and one nonorientable handle attached. It is also easy to see that $\text{Double}(H') \approx \mathbb{T} \# (S^1 \times S^2)$. Since $H \approx H'$ (see Observation 1), $\text{Double}(H) \approx \text{Double}(H')$; therefore $\mathbb{T} \# \mathbb{T} \approx \mathbb{T} \# (S^1 \times S^2)$. An elementary surgery along an S^1 fiber changes $S^1 \times S^2$ to S^3 and thus converts $\mathbb{T} \# (S^1 \times S^2)$ to $\mathbb{T} \# S^3 \approx T$. □

5. PROOF OF THEOREM 1

To a uniform Heegaard diagram $S(X, Y)$, where S is nonorientable, assign a complexity $\mathscr{C}(X, Y) = (n, k)$ where $n = \text{genus}(S)$ and $k = \min\{|x \cap y| : x \in X,\ y \in Y\}$. Note that since S is nonorientable, $n \geq 1$. Our proof will be by induction on the complexity of these uniform Heegaard diagrams under the lexicographic ordering.

If $\mathscr{C}(X, Y) = (1, 0)$, Lemma 3 shows that $M(X, Y)$ is homeomorphic to $\mathbb{T}$. Next, consider the situation where $(n, k) > (1, 0)$ and assume the conclusion of Theorem 1 holds for any 3-manifold described by a uniform Heegaard diagram of lower complexity.

Case 1. $k = 1$. By Observation 3, $M(X, Y)$ has a Heegaard splitting of genus $n - 1$. Observation 2 guarantees an associated uniform Heegaard diagram that necessarily has lower complexity.

Case 2. $k = 0$ (and $n > 1$). Choose a pair of curves $x \in X$ and $y \in Y$ that do not intersect. By Lemma 2(a) there is a nonseparating ordinary two-sided curve z on S that intersects each of x and y transversally in a single point. By Lemma 4 there is a uniform complete system Z on S containing z.

Perform surgery on the elements of Z as prescribed in Lemma 1, thus obtaining a manifold $\chi(M, Z) \approx M(X, Z) \# M(Y, Z)$. Observation 3, followed by Observation 2 and the inductive hypothesis, shows that each of $M(X, Z)$ and $M(Y, Z)$ can be reduced to $\mathbb{T}$ by a finite number of surgeries. Hence, M can be reduced to $\mathbb{T} \# \mathbb{T}$ by finitely many surgeries. An application of Lemma 5 completes the argument.

Case 3. $k > 1$. Again choose curves $x \in X$ and $y \in Y$ such that $|x \cap y| = k$. Then apply Lemmas 2(b) and 4 to obtain a uniform complete system Z on S that contains a curve z with the property that $|x \cap z|$ and $|y \cap z|$ are less than k. Apply Lemmas 1 and 5 as in Case 2. □

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