AMS/IP Studies in Advanced Mathematics
Volume 2, 1997 (Part 1)

CAT(0) REFLECTION MANIFOLDS

F. D. ANCEL, M. W. DAVIS AND C. R. GUILBAULT

Let M^n be a compact, contractible manifold. Its boundary Σ^{n-1} is a homology sphere. We assume that Σ^{n-1} admits a PL triangulation. (This is automatic for $n \neq 5$.)

In [7], Davis showed how to construct an action of a reflection group on an open contractible manifold with fundamental chamber M. In [10], Gromov showed that a modified version of this construction could be given a piecewise Euclidean, CAT(0) metric. (Roughly, a "CAT(0) metric" is the generalization to singular metric spaces of the notion of a complete Riemannian metric of nonpositive sectional curvature on a simply connected manifold. The precise definition is given in [10, §2.4.C].) In Gromov's version, M is replaced by $C\Sigma$, the cone on Σ, and the CAT(0) space on which the group acts is generally, no longer a manifold, but only a polyhedral homology manifold. In [1], Ancel and Guilbault showed that any contractible M^n, with $n \geq 5$, can be written as the union of two cones along the cone on a homology $(n-2)$-sphere. This allows us to use Gromov's idea to put CAT(0) metrics on many of the original examples of [7]. In particular, we get the following result.

Theorem. *Let M^n be a compact, contractible manifold, with $n \geq 5$ and with boundary a PL homology sphere. Then there is an open contractible n-manifold $\mathcal{X}$ with a piecewise Euclidean $CAT(0)$ metric and an isometric action of a reflection group W on $\mathcal{X}$ with fundamental chamber M^n.*

Basically, the proof consists of recalling the constructions of [7], [10] and [1]

The construction of [7] Triangulate Σ and denote the resulting simplicial complex again by Σ. Let S be the set of vertices in Σ and let

Partially supported by NSF Grant DMS 9208071

W be the right-angled Coxeter group generated by S with relations:

$$\begin{aligned} s^2 &= 1\,, && \text{for all } s \in S \\ (ss')^2 &= 1\,, && \text{if } s \text{ and } s' \text{ span an edge of } \Sigma\,. \end{aligned}$$

We identify S with its image in W. "Right-angled" refers to the fact that if s and s' are distinct elements of S, then the order of ss' in W is either 2 or ∞. For each s in S, let Σ_s denote the closed star of s in the barycentric subdivision of Σ. For each $x \in M$, let W_x denote the subgroup of W generated by $\{s \in S \mid x \in \Sigma_s\}$. (If this set is empty, then $W_x = \{1\}$.) Set

$$\mathcal{X}(W, M) = (W \times M)/\sim$$

where the equivalence relation $\sim$ is defined by $(w, x) \sim (w', x')$ if and only if $x = x'$ *and* $w^{-1}w' \in W_x$. Then W acts naturally on $\mathcal{X}(W, M)$. It is proved in [7] that $\mathcal{X}(W, M)$ is a manifold (since M is a manifold with boundary).

More generally, if X is any space and $\{X_s\}_{s \in S}$ is a family of closed subspaces, then we can define a W-space $\mathcal{X}(W, X)$ in exactly the same way. By a *reflection group* we mean an action of a Coxeter group on a space which is equivariantly homeomorphic to some $\mathcal{X}(W, X)$. The space X is called the *fundamental chamber.*

In particular, the construction $\mathcal{X}(W, M)$ could be modified by replacing M by $C\Sigma$. The resulting space $\mathcal{X}(W, C\Sigma)$ is a polyhedral homology manifold, which is W-equivariantly homotopy equivalent to $\mathcal{X}(W, M)$. However, $\mathcal{X}(W, C\Sigma)$ is generally not a topological manifold since there will be singularities at the cone point and its W-translates whenever Σ is not simply connected (and of dimension greater than 1).

A simplicial complex is a *flag complex* if any finite set of vertices, which are pairwise joined by edges, span a simplex.

It is proved in [7] that $\mathcal{X}(W, M)$ is contractible if and only if Σ is a flag complex. (This condition is easy to achieve, for example, the barycentric subdivision of any cell structure on Σ is a flag complex.)

The construction of [10] The cone on the barycentric subdivision of a k-simplex can be identified with a standard simplicial subdivision of a $(k+1)$-cube in a natural way. This gives an identification of the cone on a k-simplex with a $(k+1)$-cube, well-defined up to symmetries. The picture for $k = 2$ is given below.

Gromov puts a piecewise Euclidean structure on $C\Sigma$ by identifying the cone on each simplex of Σ with a regular Euclidean cube of edge length 1. Each translate of $C\Sigma$ in $\mathcal{X}(W, C\Sigma)$ is given a cubical structure

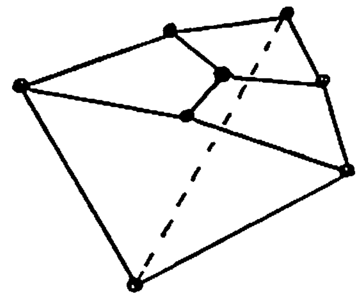

FIGURE 1

in a similar fashion. Further details on this cubification can be found in [5, §6].

The distance between two points in $\mathcal{X}(W, C\Sigma)$ is then defined to be the length of the shortest path between them. This metric gives $\mathcal{X}(W, C\Sigma)$ the structure of a "geodesic space" (also called a "length space"). Since $\mathcal{X}(W, C\Sigma)$ is simply connected, to prove that this metric is CAT(0), it sufficies to show that it satisfies CAT(0) locally (i.e., that it is nonpositively curved), cf., [10, p. 119] and [3, p. 195]. To prove this it is necessary and sufficient to show that the link of each cubical face of $C\Sigma$ in $\mathcal{X}(W, C\Sigma)$ is a flag complex, cf., [10, pp. 120-122] and [4, Lemma 1.3]. But this is true if and only if Σ is a flag complex. (Proof: The link of the cone point is Σ. The link of any other cubical face in the interior of $C\Sigma$ can be identified with the link of a simplex in Σ and this is a flag complex if Σ is. Finally, the link of a cubical face on the boundary of $C\Sigma$ can be identified with an iterated suspension of a link of a simplex in Σ and the suspension of a flag complex is a flag complex.) In the case when the Coxeter group W is not right-angled there is a more refined version of this construction, due to Moussong, [11].

The construction of [1]. Let $\Sigma_0^{n-2} \subset \Sigma^{n-1}$ be a PL-embedded homology sphere of codimension one. Then Σ_0 divides Σ into two homology $(n-1)$-cells, call them N_1 and N_2. It is proved in [1, Lemma 1] and [6, Proposition 2] that, for $n \geq 5$, one can always find a Σ_0 so that the induced homomorphism $\pi_1(\Sigma_0) \to \pi_1(\Sigma)$ is surjective. It follows from van Kampen's Theorem that, for $i = 1, 2$, $\pi_1(N_i)$ is normally generated by the image of $\pi_1(\Sigma_0)$. This result is used in [1] to prove that M can be written as a mapping cylinder of some map from Σ to an arc and in [2] to prove that the interior of M can be given a complete

CAT(ϵ) metric, for any $\epsilon \leq 0$.

Proof of the theorem. As above, let Σ_0 be a PL homology sphere of codimension one in Σ so that, for $i = 1, 2$, $\pi_1(N_i)$ is normally generated by the image of $\pi_1(\Sigma_0)$. Choose a PL triangulation of Σ as a flag complex so that Σ_0 is a full subcomplex. For $i = 1, 2$, set

$$\Sigma_i = N_i \underset{\Sigma_0}{\cup} C\Sigma_0$$

where, as before, CY denotes the cone on Y. By van Kampen's Theorem, Σ_i is simply connected. Let F denote the closed star of the cone point in the barycentric subdivision of $C\Sigma_0$. Glue $C\Sigma_1$ to $C\Sigma_2$ along F and call the result X. Then X is clearly a contractible, polyhedral homology n-manifold with boundary. Since the complement of the interior of F in $C\Sigma_0$ is a collared neighborhood of Σ_0, we have $\partial X = N_1 \cup (\Sigma_0 \times I) \cup N_2$, which is PL-homeomorphic to Σ. For $i = 0, 1, 2$, let v_i denote the cone point of $C\Sigma_i$ and let e denote the union of the edge from v_1 to v_0 and the edge from v_0 to v_2. The triangulation of X (as a union of two cones) has PL singularities only along e. According to a well-known theorem of Edwards [9], a polyhedral homology manifold of dimension $n \geq 5$ (with boundary a manifold) is a topological manifold if and only if the link of each vertex in its interior is simply connected. This holds in our case. (The link of v_0 is the suspension of Σ_0; for $i = 1, 2$, the link of v_i is Σ_i.) Hence, X is a contractible n-manifold. By the h-cobordism Theorem, it is homeomorphic rel boundary to M.

Now apply Gromov's cubification to $C\Sigma_1$ and $C\Sigma_2$, separately. This defines a cubical structure on their union, X. For $i = 0, 1, 2$, let S_i be the set of vertices in Σ_i so that $S_0 = S_1 \cap S_2$ and $S = S_1 \cup S_2$. For $s \in S_i - S_0$, set

$$X_s = (\Sigma_i)_s,$$

the closed star of s in the barycentric subdivision of Σ_i. For $s \in S_0$, set

$$X_s = (\Sigma_1)_s \cup (\Sigma_2)_s,$$

In both cases, X_s is an $(n-1)$-cell in ∂X and a union of cubical faces. Let W be the right-angled Coxeter group generated by S defined as before. Put

$$\mathcal{X} = \mathcal{X}(W, X)$$

It has a cubical structure induced from that of X. One checks, just as in Gromov's construction, that the link of each cubical face is a flag complex. Hence, we see that $\mathcal{X}$ is CAT(0) and the proof is complete. □

Remarks. 1) If Σ^{n-1} is not simply connected, $n > 2$, then the CAT(0) manifold $\mathcal{X}$ is not simply connected at infinity and hence, not homeomorphic to $\mathbb{R}^n$.

2) A very similar construction using reflection groups was mentioned in Remark 5b.2 of [8, p. 384]. The idea there was to use only one of the cones, say $C\Sigma_1$, as the fundamental chamber. Let W' be the right-angled Coxeter group generated by the vertices of Σ_1. It follows, as above, that $\mathcal{X}(W', C\Sigma_1)$ is a CAT(0) manifold. Moreover, if the double of N_1 (along Σ_0) is not simply connected, then $\mathcal{X}(W', C\Sigma_1)$ is not simply connected at infinity. The point of our theorem is that *any* compact, contractible manifold (with PL boundary) can occur as a fundamental chamber.

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