Symmetry

and the

ALHAMBRA MOSAICS

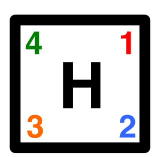
Homework Problems with Hints

1. Fill in the blanks in the multiplication table for the symmetry group { **id**, R_v, R_h, $T_{1/2}$ }.

	Id	R_v	R_h	$T_{1/2}$
Id	Id	R_v	R_h	$T_{1/2}$
R_v	R_v	Id	$T_{1/2}$	
R_h	R_h	$T_{1/2}$	Id	
$T_{1/2}$	$T_{1/2}$			Id

Hint. There are two different ways to compute $T_{1/2}R_h$: the ***geometric approach*** and the ***algebraic approach***.

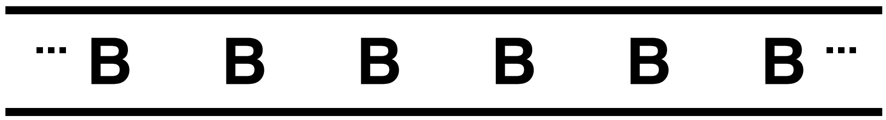
The ***geometric approach****:* Start with the box.



First apply R_h to this box. Then apply $T_{1/2}$ to the resulting box. Which element of the symmetry group gives the same final result when applied to the initial box?

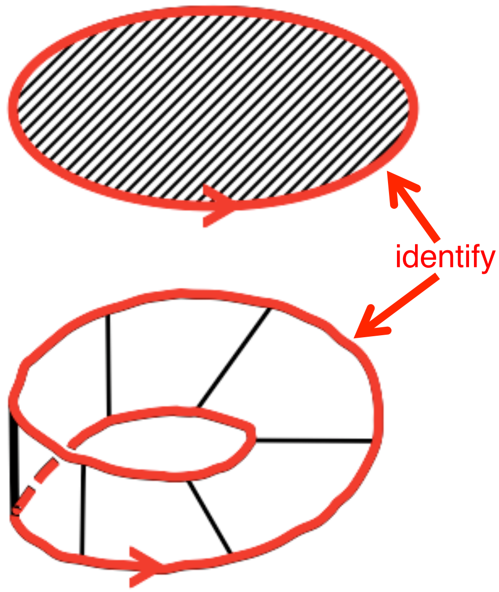
The ***algebraic approach****:* Observe that the multiplication table tells us that $R_vR_h = T_{1/2}$. Hence, $T_{1/2}R_h = R_vR_hR_h = R_v(\text{Id}) = R_v$. Use the same technique to fill in the other blanks.

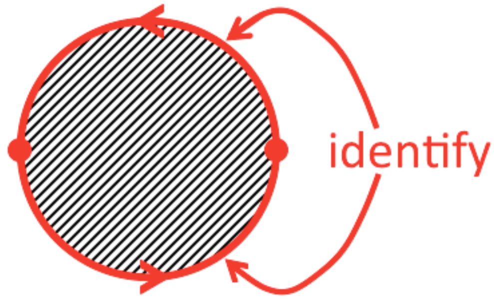
2. Find the orbifolds of the following two friezes.



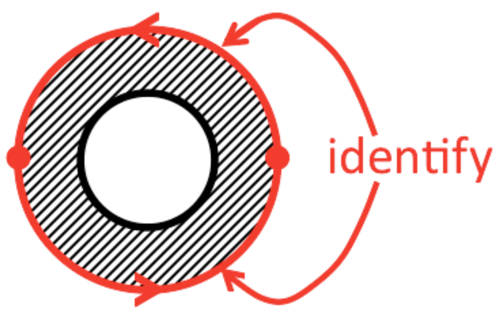
Hint: First find the fundamental chamber of each frieze. Then let the symmetries of each frieze identify parts of the fundamental chamber to create the orbifold. Finally label the orbifold's cone points, mirrors and corner points (if any).

3. Prove that the projective plane (pictured on the left as the surface obtained from a disk and a Mobius strip by identifying their boundary loops) is homeomorphic to the surface (pictured on the right) obtained from a disk by identifying diametrically opposed points in its boundary.

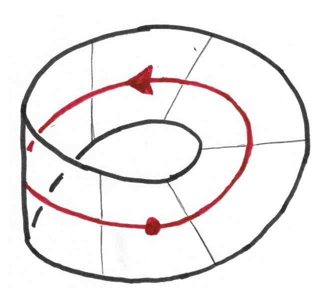




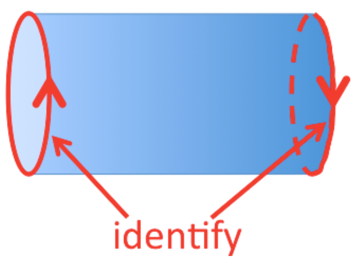
Hint: Remove a disk from the center of the surface on the right, leaving an annulus with identifications on its outer boundary loop (shown below). Prove that this annulus with identifications on its outer boundary loop is a Mobius strip.

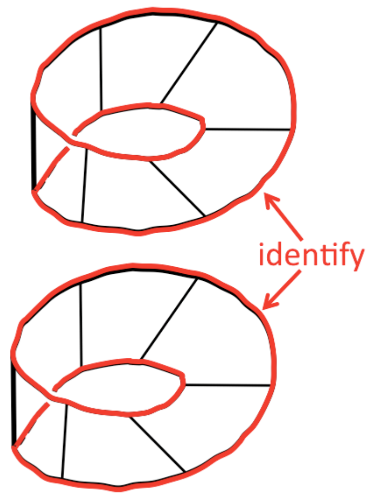


To help with this proof, draw a red loop around the center of a Mobius strip, indicate a point on the loop with a dot, and orient the loop with an arrow. Then cut the Mobius strip open along the red loop. What do you get?

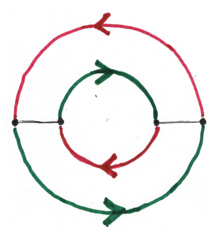


4. Prove that the *Klein bottle* (pictured on the left as the surface obtained from the annulus by identifying its two boundary loops in opposite directions) is homeomorphic to the surface (pictured on the right) which is obtained from two Mobius strips by identifying their boundary loops.

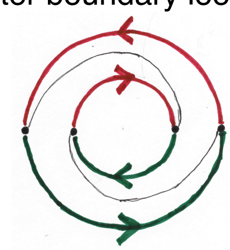




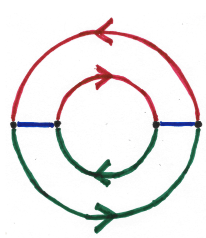
Hint: Flatten the annulus-with-identifications that yields the Klein bottle, and bisect its boundary loops into two arcs (one red, one green).



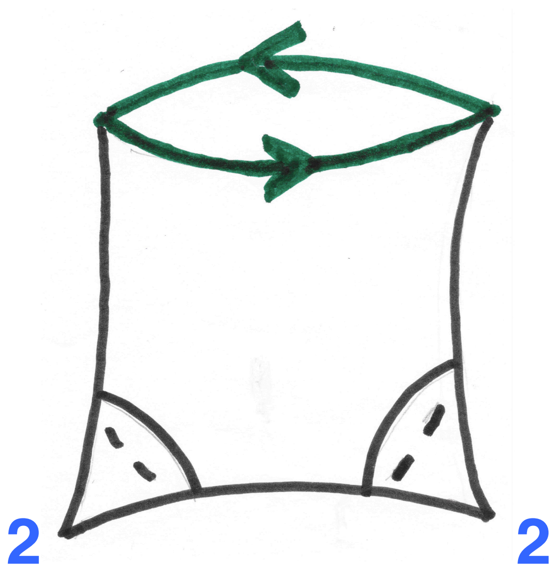
There is a homeomorphism of this annulus-with-identifications that rotates the inner boundary loop through a ½-twist while keeping the outer boundary loop fixed.



Cut the resulting annulus-with-identification into two pieces along the horizontal blue arcs. What are the two pieces?



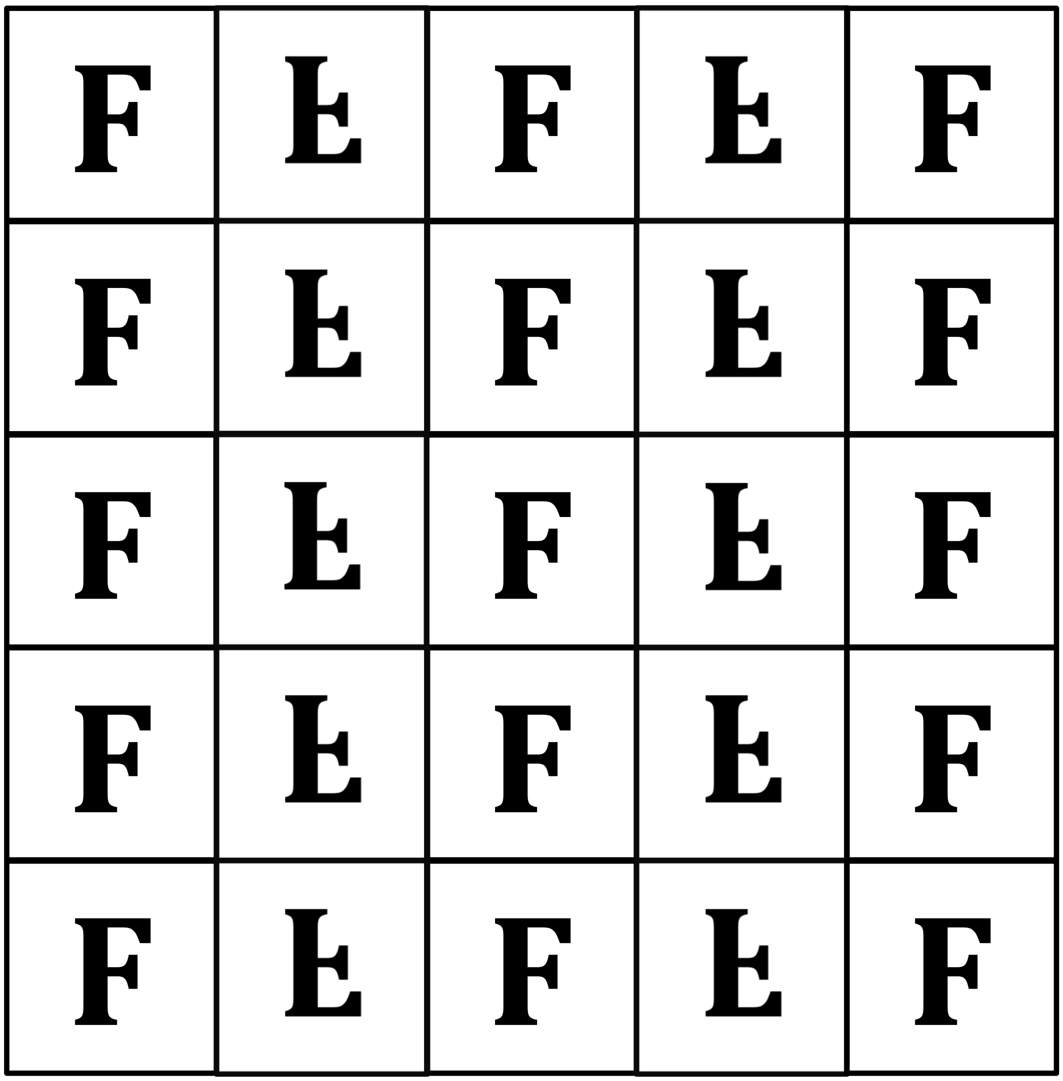
5. Find the fundamental chamber of the following wallpaper pattern, and verify that the orbifold is the "purse with the twisted zipper" shown below – orbifold symbol **P**22.



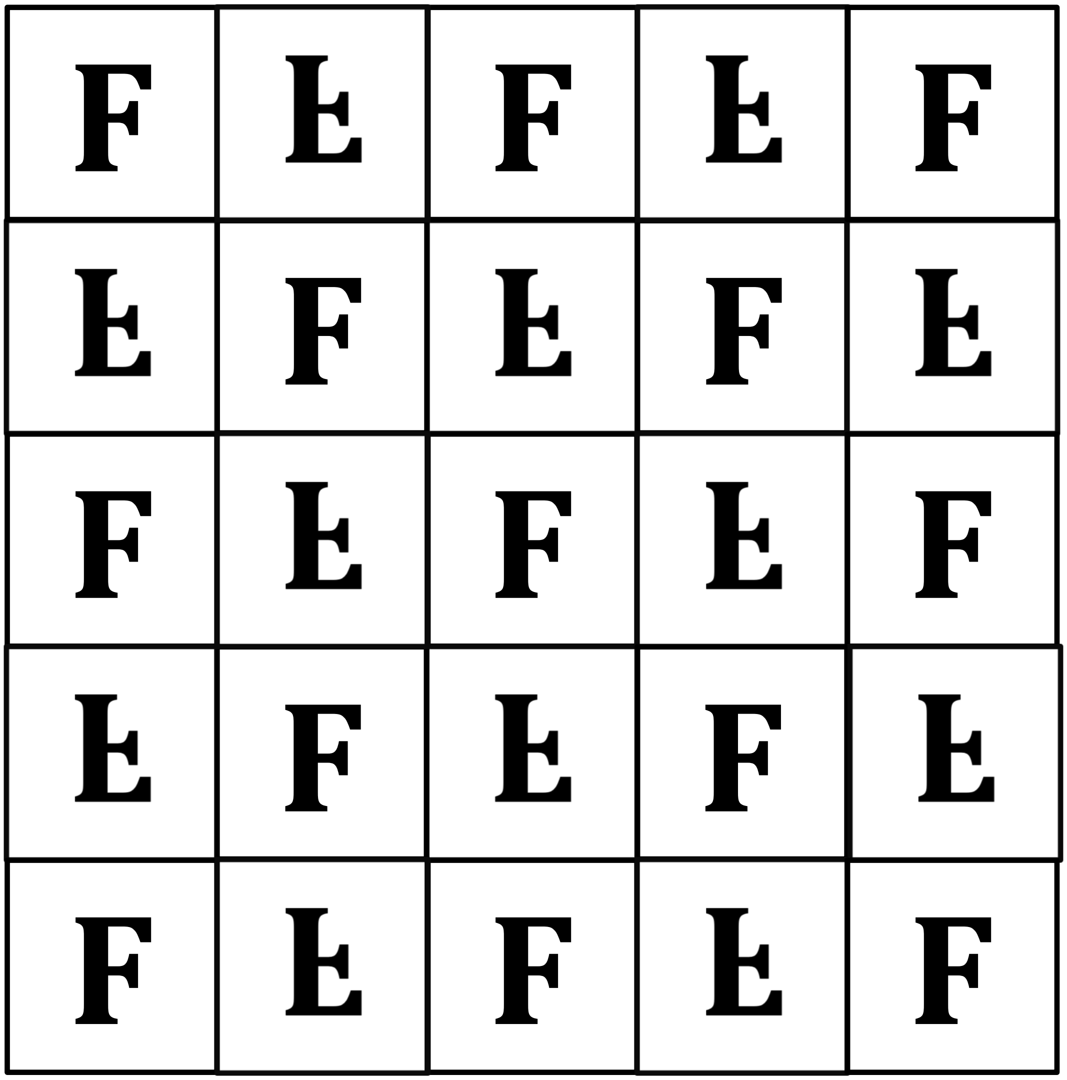
The purse with the twisted zipper.

6. Find the orbifolds of the three wallpaper patterns below.

a)



b)



c)