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SYMMETRY IN MOORISH AND OTHER ORNAMENTS†

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Abstract—An investigation of the Moorish ornaments from the Alhambra (in Granada, Spain) shows that their symmetry groups belong to 13 different crystallographic (wallpaper) classes; this corrects several earlier enumerations and claims. The four classes of wallpaper groups missing in Alhambra (pg, p2, pgg, p3m1) have not been found in other Moorish ornaments, either. But the classification of repeating patterns by their symmetry groups is in many cases not really appropriate—account should be taken of the coloring of the patterns, of their interlace characteristics, etc. This leads to a variety of "symmetry groups", not all of which have been fully investigated. Moreover, the "global" approach to repeating ornaments is only of limited applicability, since it does not correspond to the way of thinking of the artisans involved, and does not cover all the possibilities of "local" order. The proper mathematical tools for the study of such structures which are only "locally orderly" remain to be developed.

The idea of investigating the ornaments and decorations of various cultures by consideration of their symmetry groups appears to have originated with Pólya[22]. This mathematically motivated approach, which was in sharp contrast to the earlier descriptive methods, gained wider recognition through the influential book of Speiser[27]. The earlier methods, which are exemplified by such works as those of Jones[18], Bourgoin[4], Day[6], Grasset[9], and many others, relied largely on the analysis of ornaments by considering the character of their motifs. By contrast, the mathematical approach depends on the symmetries of the design considered as a whole, and is to be found in the publications of Müller[21], Shepard[25], Weyl[29], Garrido[8], Shubnikov & Koptsik[26], Washburn[28] and others. Among mathematically inclined investigators the newer, quantitative way of looking at the physical evidence has become universally adopted; however, many practitioners of the decorative arts and their analysis for ethnographic, anthropological, archeological and other purposes still use almost exclusively the descriptive method.

The present investigation arose from a desire to clarify and settle contradictory statements regarding the symmetry groups of the Moorish ornaments that are to be found in the Alhambra, in Granada (Spain). The background facts are the following. It is well known that there are 17 classes of symmetry groups of planar ornaments which repeat in at least two nonparallel directions; these are known as the (classes of) wallpaper (or crystallographic plane) groups. In Fig. 1 are shown examples of patterns, with a very simple motif, of each of the 17 classes of wallpaper groups. In an early application of group-theoretic methods to the analysis of historic ornaments, Müller[21] examined the patterns and tilings in the Alhambra and found that there are 11 different groups present. In contrast to her findings, Coxeter[5] states that 13 wallpaper groups are represented there,‡ while the number 17 was claimed by others (see, for example, Belov[2], Fejes Tóth[7, p. 43], Martin[20, p. 111]). So when an opportunity to visit the Alhambra was provided by a Guggenheim Fellowship, an on-the-spot investigation to settle these conflicting claims was undertaken. The results of this investigation, as well as comments and observations which arose in this connection, form the topics of the present paper.

The bottom line is relatively easy to draw. After a reasonably thorough examination of the Alhambra, 13 different wallpaper groups were identified among the symmetry groups of

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‡In correspondence during 1981 between Professors H. S. M. Coxeter, J. J. Burckhardt and one of the authors (B.G.), the source of the number 13 was traced to a misunderstanding of some statements in Müller[21]. The same conclusion was reached independently by Professor D. W. Crowe.

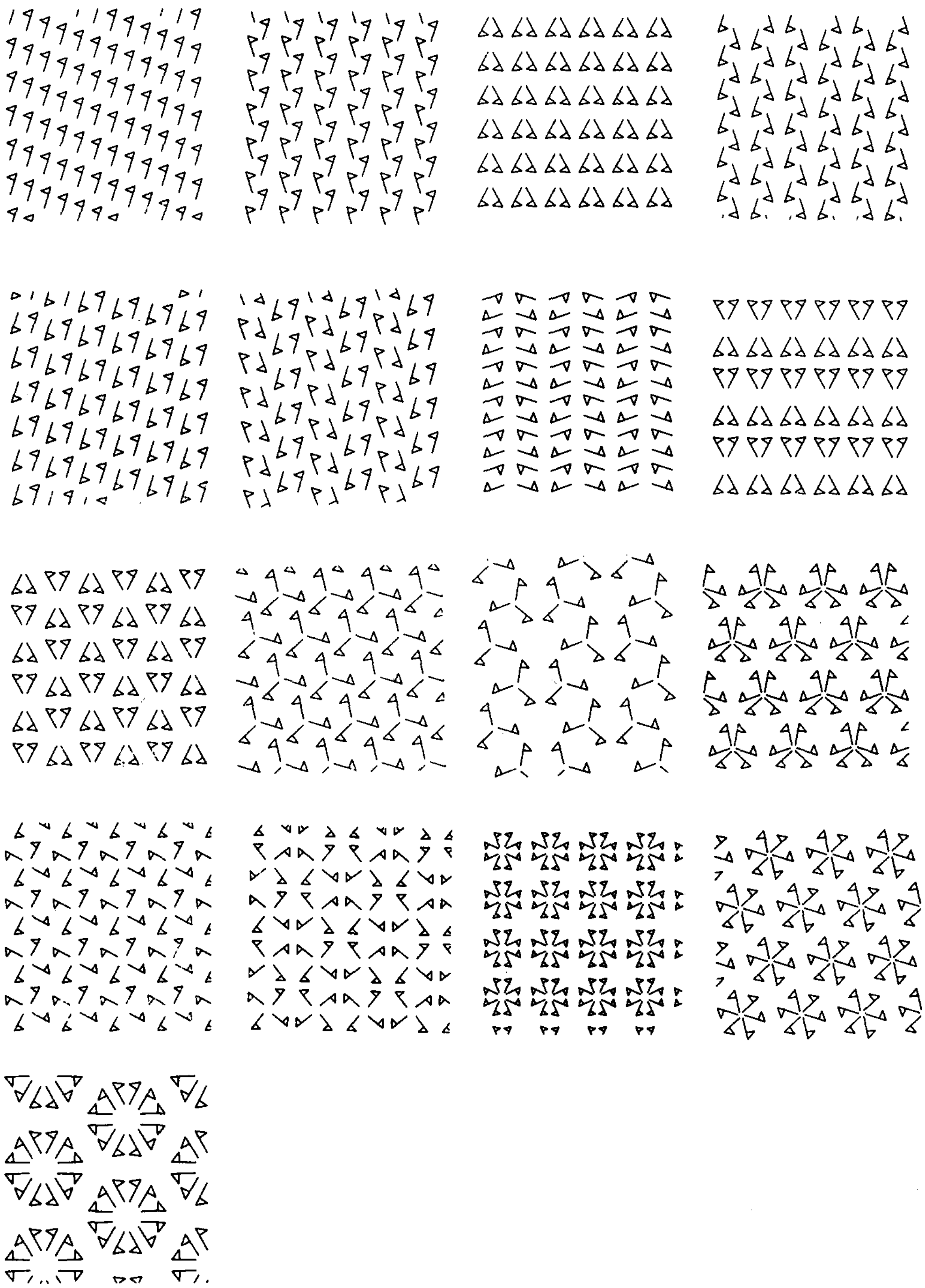

Fig. 1. Patterns in which the common motif is a small "flag", which can exemplify the 17 classes of wallpaper groups.

the ornaments found there. Actually, Müller seems to have missed only one group present in the Alhambra palace proper (namely the group pm, of which the example is given in Fig. 6 below). The other group that she missed seems not to be represented in the palace, but only in the Museum of Alhambra, and it is not clear whether this museum was accessible to her. The four groups which have not been found in the Alhambra (pg, p2, pgg, p3m1) do not appear to be represented in other Moorish artifacts either (though naturally, a really systematic examination of the enormous volume of extant materials may turn up some or all of them). It is of interest to note that two of these four groups have been located in Toledo (Spain) in buildings approximately contemporaneous with some of those in the Alhambra—one (p2, see Fig. 7) in a church, and the other (p3m1, see Fig. 8) in a synagogue. On the other hand, it seems that the groups pg and pgg fail to be represented not only in Moorish decorations, but in Islamic ornaments in general.

The attempt to determine which of the wallpaper groups are present in the Alhambra turned out to be rather interesting regardless of the numerical answer obtained, since the obvious (and even the correct) answer is not necessarily the best or the most appropriate. The material forces one to consider mathematical questions which would probably not have arisen otherwise.

Among the first serious difficulties that one encounters is to decide what is it that one is counting; for many of the ornaments there are several different yet reasonable ways in which they can be considered symmetric, and these lead to different values for the numbers of groups found. Some examples will clarify this.

(i) To begin with, there is the symmetry group in the most immediate sense: we look at the ornament, exactly as it is (except that we imagine it to be continued indefinitely in all directions and we ignore minor variations that are due to practical considerations). We then ask what isometries (rigid motions) map the ornament precisely onto itself, under preservation of all its properties. For example, in the tilings from the Alhambra shown in Fig. 2, the symmetries include reflections in the vertical and in the horizontal lines through the centers of the colored (brown or green) tiles, but not reflections in lines through the centers of the black tiles because such reflections would map brown tiles onto green ones, and so fail to be symmetries; this symmetry group is pmm.

(ii) Next, there is the symmetry group of the *underlying uncolored* ornament—the coloring of the tiles is disregarded, and the isometries that map the resulting uncolored ornament onto itself are considered. In the example shown in Fig. 2—where the underlying tiling consists of the horizontal and vertical "dogbones"—this leads to many symmetries additional to those in (i); among them are reflections in vertical and in horizontal lines through the centers of all the tiles, 90° rotations about the meeting points of quadruplets of tiles, etc. (these form the group p4g).

As additional examples we consider the tilings shown in Figs. 3 and 4. In the first the coloring destroys most of the symmetries and the symmetry group is p1; however, the underlying uncolored tiling has many rotational symmetries and its symmetry group is p3. Figure 4 shows an analogous situation, except that here the modifications to the underlying tiling consist not in coloring the tiles, but in adding inscriptions or other designs to some of them. The tiling with inscriptions has symmetry group p1; ignoring the inscriptions we find that the underlying tiling has symmetry group p6m.

(iii) Then there are the *color-symmetry* groups: instead of ignoring the colors in multicolor ornaments, or of considering the colors as unrelated to each other, we consider *color symmetries*, that is, isometries which map the underlying ornament onto itself *coupled* with *consistent* permutations of colors, so that the combination maps the colored pattern onto itself.

For example, the tiling with asymmetric trefoils shown in Fig. 5 admits as rotational symmetries only 120° turns about the centers of the trefoils and the points where six trefoils meet. The underlying tiling admits 60° turns about the latter points; these can be made into color symmetries by agreeing to accompany each 60° rotation by an interchange of colors. In the tiling shown in Fig. 6 some—but not all—symmetries of the underlying uncolored tiling can be made into color symmetries by suitable choices of permutations of the colors. By contrast, every symmetry of the uncolored tiling underlying the three-colored tiling shown in Fig. 7 can be made into a color symmetry by coupling it with a suitable permutation of colors. (Colored patterns with this property are called "perfectly colored"; see Grünbaum & Shephard[12,14] or Senechal[24] for details of the known results.) In the four-color tiling in the center of Fig. 8 blue, gray and black tiles are equivalent under color symmetries of the tiling, but the white tiles are not equivalent to them.

(iv) Many ornaments in Moorish art (as in the art of many other cultures) are *interlace patterns* (for example, Figs. 9 and 10 can be interpreted as showing interlaced polygons). Then one can consider either the group of symmetries [as in (i) above] or that of the "underlying" pattern formed by the "overlapping polygons," in which we disregard the fact that certain portions of the polygons (which may be reasonably assumed to exist) are "hidden" by parts of other polygons. For purposes of discussion, we consider the simpler interlace pattern of squares shown in Fig. 11(a), and its underlying pattern shown in Fig. 11(b). Due to the way

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Fig. 2. A colored tiling from the Alhambra. Its symmetry group is pmm, and the symmetry group of the underlying tiling is p4g.

Fig. 3. Another colored tiling from the Alhambra. All symmetries of the underlying tiling (except some of the translational symmetries) are eliminated by the coloring; the colored tiling has symmetry group p1, while the underlying tiling has symmetry group p3.

Fig. 4. A stucco wall decoration from the Alhambra. With the inscriptions and designs in the star-shaped regions, the only symmetries are translations and the symmetry group is p1. The underlying ornament, in which the inscriptions and designs are disregarded, has symmetry group p6m.

Fig. 5. A two-color tiling from the Alcazar in Sevilla (Spain) in which the symmetry group is p3 and the underlying uncolored tiling has symmetry group p6. Every 60° rotational symmetry of the underlying tiling can be made into a color symmetry of the colored tiling by coupling it with an interchange of the two colors.

Fig. 6. A multicolored tiling from the Alhambra; for purposes of the current discussion we disregard the fact that the tiling is in part wrapped about a cylindrical column. The colored tiling has symmetry group pm (the only symmetries other than translations are reflections in vertical lines bisecting the white tiles), and the underlying uncolored tiling has symmetry group p4g. Among color symmetries of the tiling are reflections in the horizontal lines that bisect the tiles, as well as appropriate vertical translations coupled with cyclic permutations of the colors brown, green and blue; the 90° rotational symmetries of the underlying tiling cannot be made into color symmetries.

Fig. 7. A three-colored tiling on the floor of a church in Toledo (Spain) which has symmetry group p2—one of the wallpaper groups that seems not to have been used in Moorish ornaments. The underlying uncolored tiling has group p6m.

Fig. 8. A wall decoration from a synagogue in Toledo (Spain) with symmetry group p3m1, which is another of the wallpaper groups apparently missing in Moorish ornaments. The underlying tiling is the same as in Fig. 7.

Fig. 9. A polygonal interlace pattern from the Alcazaba in Malaga (Spain). The reflective symmetries of the underlying pattern are not symmetries of the interlace, but can be combined with layering interchanges to form layered symmetries.

Fig. 10. A pattern of interlaced regular octagons from the Alhambra; variants of this ornament occur frequently in Islamic art from many different countries. The symmetry group is p4, but reflections coupled with layering interchange are symmetries of the layered pattern.

Fig. 11. (a) A simple interlace formed by squares, which has no reflective symmetries (symmetry group is p4). (b) The underlying pattern of overlapping squares, which admits reflective symmetries.

Fig. 12. A colored interlace pattern from the Alhambra. Variants of this pattern are frequently found in Islamic art from many countries.

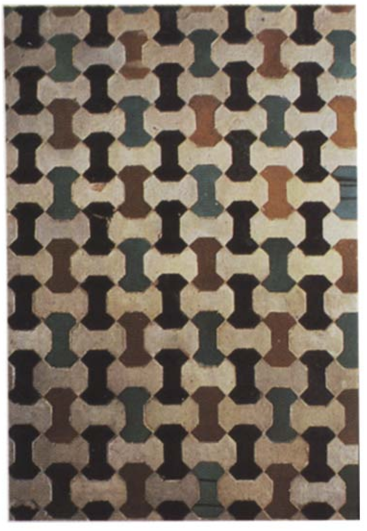


Fig. 2.

Fig. 3.

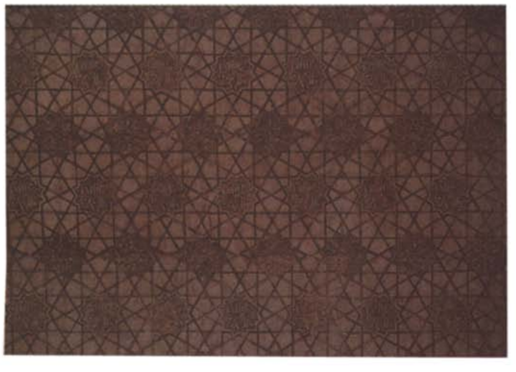


Fig. 4.

Fig. 5.

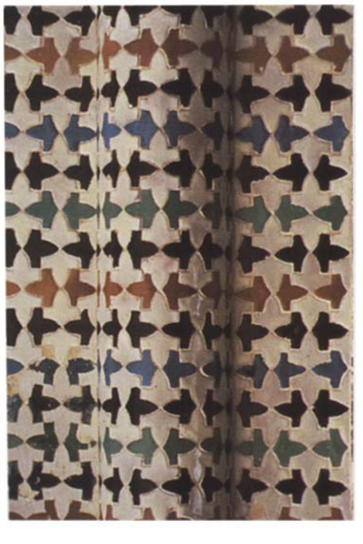


Fig. 6.

Fig. 7.

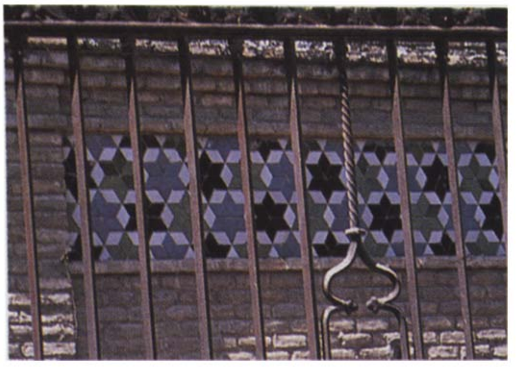
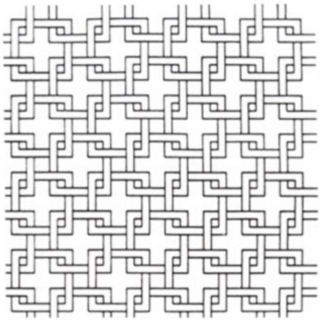


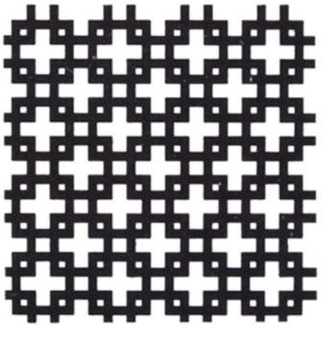
Fig. 8.

Fig. 9.

Fig. 10.



(a)



(b)

Fig. 11.

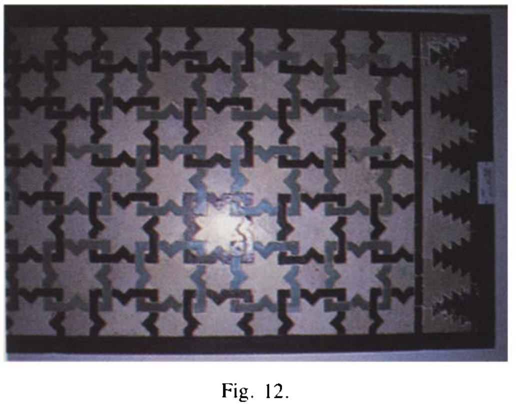


Fig. 12.

in which the squares are interlaced, the pattern in Fig. 11(a) clearly admits no reflections as symmetries (its symmetry group is p4), while the pattern in Fig. 11(b) has such symmetries (its symmetry group is p4m). However, just as with colored patterns, there is a third way of looking at this situation. A *layered symmetry* is an ordinary symmetry possibly combined with systematic interchanges of top and bottom layers at crossings, which maps the ornament onto itself. By using these the interlace pattern in Fig. 11(a) again acquires reflective symmetries—layered symmetries in which reflections are coupled with layer interchange. Similar considerations apply to the interlace patterns shown in Figs. 9 and 10.

(v) Finally, some ornaments are both colored and layered. For example, for the tiling from the Alhambra shown in Fig. 12, the complete set of symmetries arises only if we agree to consider isometries coupled with both color changes and interchanges of layers.

The above remarks show that in order to analyse Moorish patterns fully we have to consider not only the wallpaper groups [which are needed for the symmetry groups as described in (i) and (ii) above] but also groups which involve colors, or interlacing, or both. Some (but not all) of these kinds of groups have been studied. For example, it has been known for more than 50 years that, in analogy to the 17 classes of wallpaper groups, there are 46 classes of color-symmetry groups for two-colored patterns. (Here, as throughout the paper, we are discussing only the groups of ornaments periodic in at least two nonparallel directions; the "frieze groups" and finite groups can be treated similarly, but we do not consider these here.) During the last few years it has been shown that there are 23 classes of color groups for three colors, and that the corresponding numbers for four, five and six colors are 96, 14, and 90. (For the three-color result see Grünbaum[10] and Grünbaum & Shephard[13,14]; for the number of n-color groups for all $n \leq 60$ see Wieting[30]; a general survey with many references is given in Schwarzenberger[23].) Bearing these facts in mind it becomes obvious that not only are some (uncolored) symmetry groups missing in the Alhambra, but most of the color groups (even for only few colors), layered groups, and color-layered groups are also not represented. In fact, the more complicated of these groups have not even been enumerated so far! So the statement that "all symmetry groups were used by ancients", which is often repeated in connection with Moorish ornaments (as well as for those of the Egyptians) is no more than a myth! For further discussion and illustrations of this aspect see Grünbaum[11].

Yet another statement that gained wide acceptance through frequent repetition concerns the wealth of decorations present in the Alhambra—or in Islamic art in general, or in the art of ancient Egyptians, or Cretans, etc. Many authors state or imply that the variety of designs found in each of these cultures is overwhelming and boundless. But detailed study shows that no such assertions can be taken seriously. While the Morrish ornaments clearly exhibit a large number of designs, what is really surprising is how few of the many possibilities were utilized. Some examples should serve to illustrate how the number of possible designs of each kind mentioned above is astronomically large, even within bounds set by practical considerations, and that in consequence it was impossible to actually use even a moderate part of their number. One source of the tremendous abundance of possibilities lies in changing parameters which do not affect any of the symmetries of the pattern under consideration. For example, the nine interlace patterns shown in Fig. 13 are all the "same" in that they have the same layered symmetry group as the pattern in Fig. 11(a)—but their aesthetic and decorative effects are very distinct.

In many analyses of Moorish patterns it is mentioned with considerable awe that even just with regular octagons, or with regular dodecagons, their artisans knew how to create three or four distinct interesting patterns. Without wishing to detract from the skill involved, it is a fact that just expanding regular polygons centered at fixed positions leads to a theoretically infinite and practically very large number of designs that appear to be quite different (see the examples in Figs. 14 and 15). Actually, with almost any motif huge numbers of decorative patterns can be created, even if one insists that every copy of the motif plays the same role (that is, the symmetry group of the pattern acts transitively on the copies of the motif). In most cases there is still no complete enumeration of the possibilities, but it is known that in a reasonably detailed classification of patterns formed by nonoverlapping congruent circular disks there are 131 classes; for patterns formed by nonoverlapping congruent line segments there are more than 200 classes, and so on. In each case many of the patterns depend on several parameters which can be

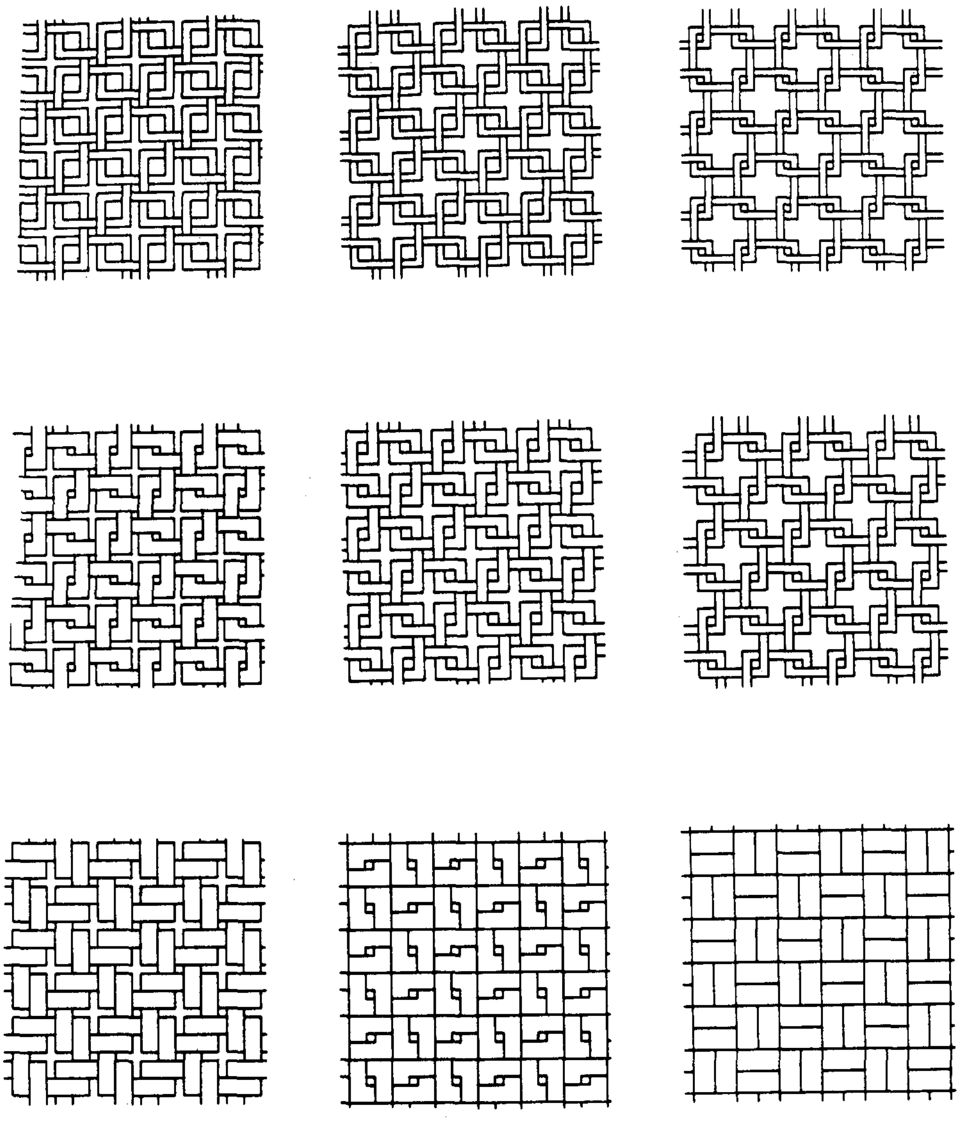


Fig. 13. A variety of interlace patterns formed by square motifs which differ only in relative distances and widths, and do not differ in their symmetry groups or their layered symmetry groups.

continuously varied. For results and references on this and related topics see Grünbaum and Shephard[13–17]. In view of these facts, it is clearly inappropriate to anticipate that any significant part of these patterns has been used in practice.

But even where it could be reasonably expected that inventiveness would produce appreciable variability, the actual material in existence presents near-monotony. An example, mentioned above, is that most of the color symmetry groups possible with even a few colors have never been used. An illustration of unused possibilities appears in Fig. 16, which shows three colorings which we devised for a relatively simple interlace with square motifs. We must also note the remarkable paucity evident in the layerings used in interlace patterns. Among thousands of interlace patterns from historical artifacts that we have seen, only two have crossings which depart from the simplest one-over-one-under variety—and even for these it is not clear whether they were really meant to exhibit a different crossing sequence or whether they arose through a mistake of the artisan or the recorder. (One of the two is a variant from Turkey of the ornament shown in Fig. 10; it was recorded by Aslanapa[1], and has also been reproduced in Grünbaum & Shephard[16]. In this variant the crossing pattern is not regular, and the symmetry group does not act transitively on the octagons.) An illustration in [16] shows a dozen variants of the crossings (from among the hundreds possible, even if equivalence of the motifs under the symmetry group is required) of the pattern of interlaced octagons shown in Fig. 10. No systematic

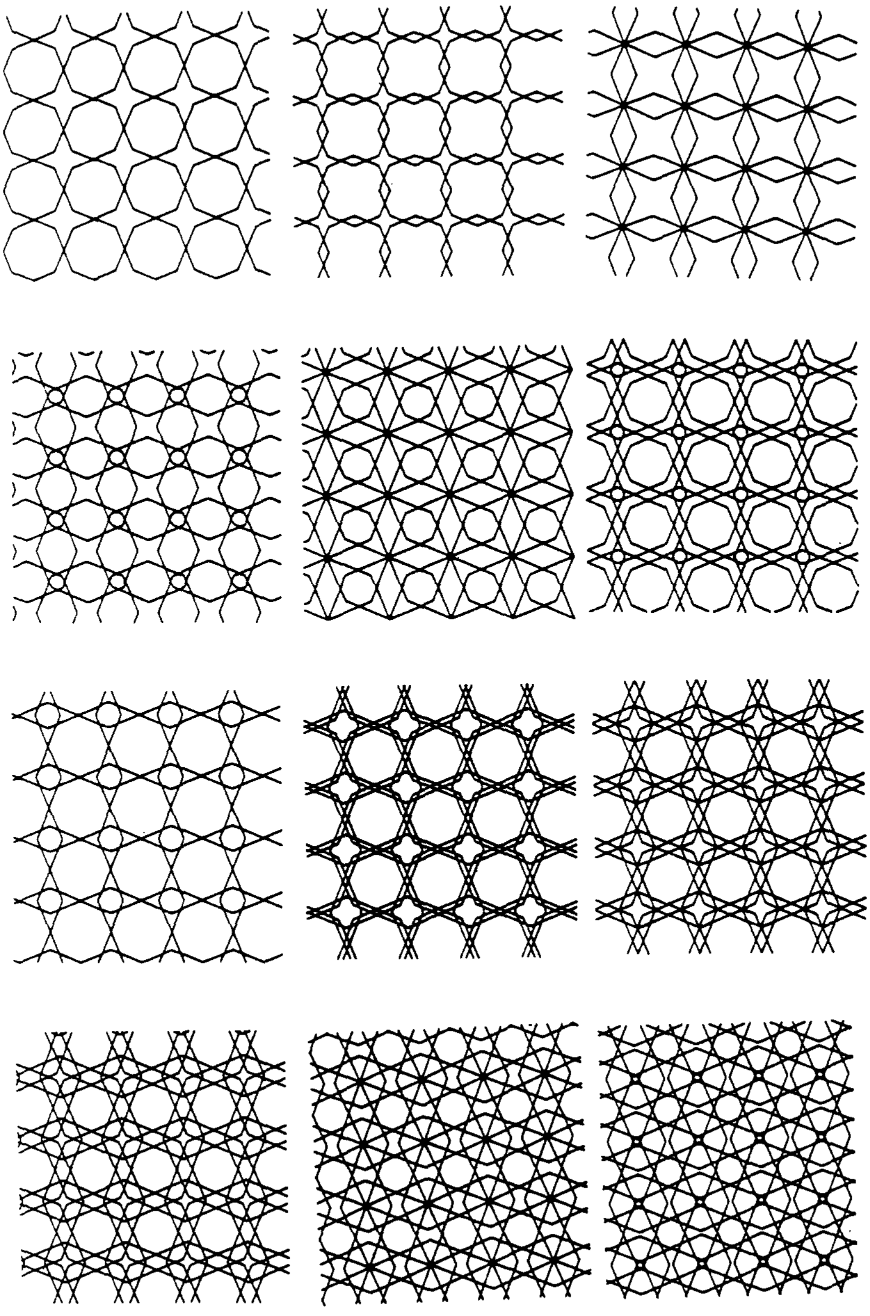


Fig. 14. A number of patterns formed by overlapping regular octagons; again the differences arise from different relative sizes, and not from changes in symmetry properties.

investigation of the possibilities seems to have been carried out so far. The lack of variety in crossing sequences is even more surprising in view of the fact that many different ones arise very naturally in the weaving of fabrics.

It should also be noted that in a certain sense all the preceding may be misdirecting us. We—mathematicians and some other scientists—may find it convenient and useful to interpret regularity of a pattern in terms of its group of symmetry (or color symmetry, etc.). In this way we can apply the results of algebra and other mathematical disciplines to the study of such patterns. However, it could be argued that this is not the concept of regularity that artisans (Moorish or any other) had in mind as they were creating their art. In fact, until a century or so ago, even to mathematicians regularity of mathematical objects had a completely different meaning. The difference between the two approaches is to a large degree the contrast of the *global* and *local* points of view. Mathematicians used to define regularity of objects such as

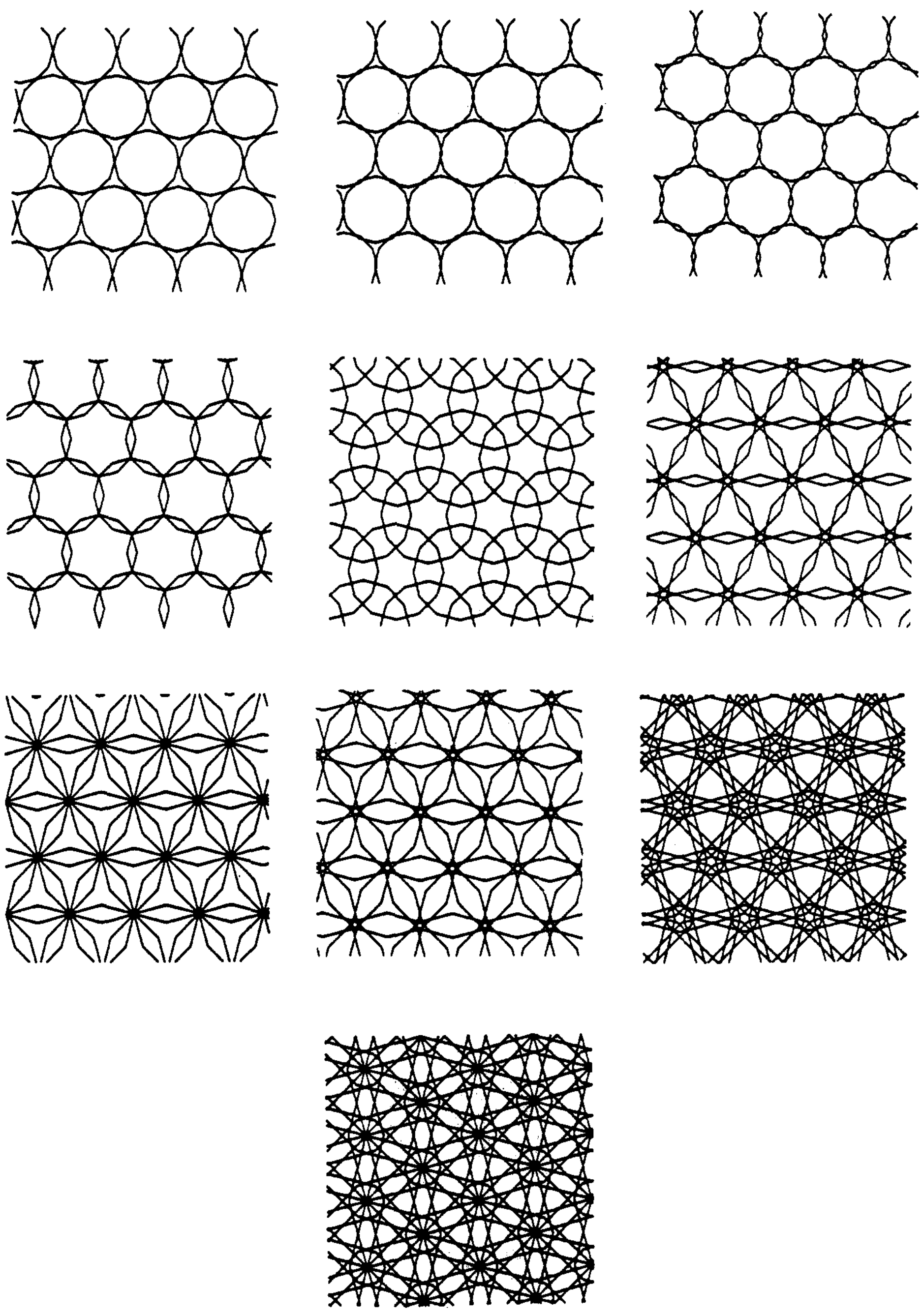
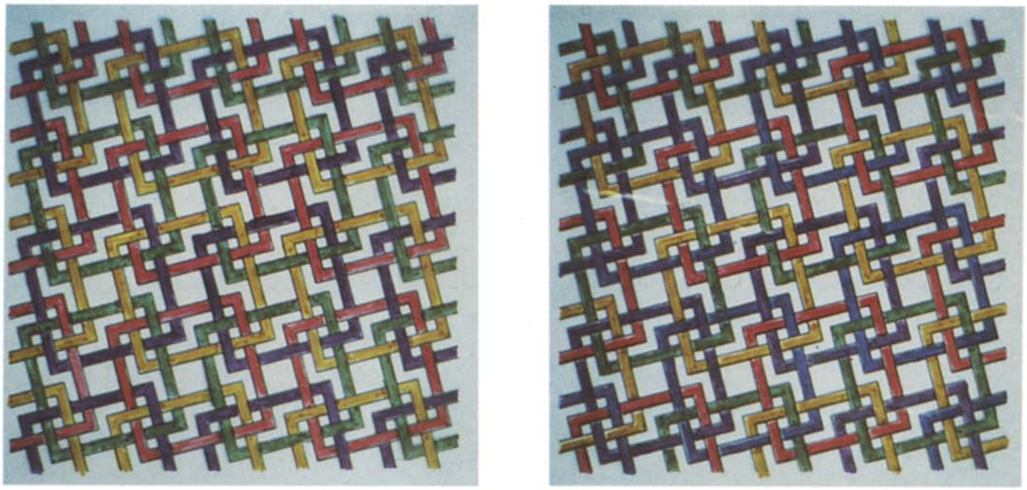


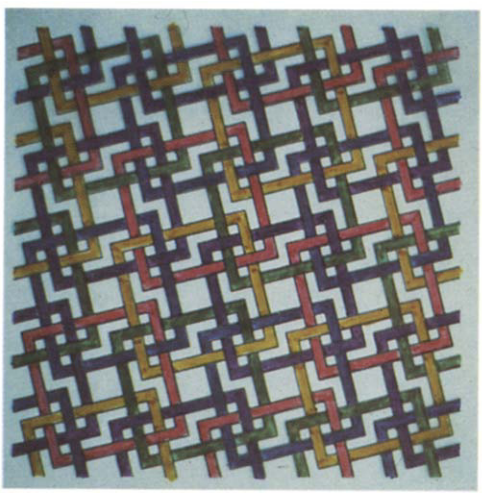
Fig. 15. A selection of patterns formed by regular dodecagons, which differ only in their sizes.

the Platonic polyhedra by requirements of congruent faces, equal angles, and other local properties; now it is customary to define regularity by the transitivity of the symmetry group on the set of flags. In the same way, it seems likely that the artisans meant to create ornaments in which each part is related to its immediate neighbors in some specific way (and not by attempting to obtain global symmetries of the infinitely extended design). We illustrate this remark with the simple tiling of squares and rectangles shown in Fig. 17. Every square tile touches two other square tiles and two rectangular tiles—one on its end and one on its side. Every rectangular tile touches four square tiles (two at its ends and two at its sides) and two rectangular tiles. From designs such as that shown in Fig. 2 it seems likely that a Moorish artist would consider this to be a perfectly legitimate pattern, yet it may be extended in such a way that its symmetry group contains only translations parallel to one direction! Of course "local" uniformity frequently leads to "global" symmetry, but it may well be that the former was the main objective



(a) (b)

Fig. 16. (a)–(c) A four-coloring and two five-colorings of an interlace pattern of squares.



(c)

Fig. 16.

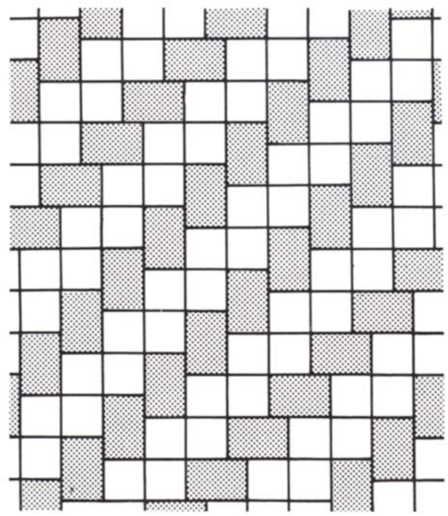


Fig. 17. A tiling by square and rectangular tiles; it is "locally" regular but is not regular in the "global" sense.

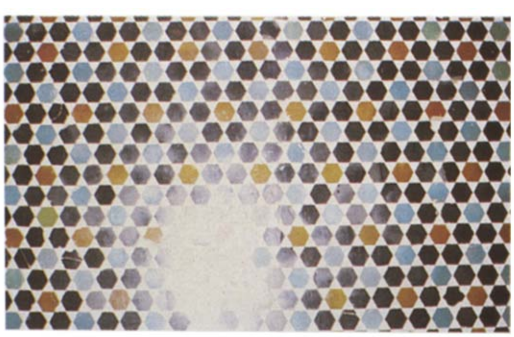


Fig. 18. A tiling from the Alcazar in Sevilla (Spain), in which the colors of the hexagons are apportioned in ratio 6:2:1.

while the latter was an accidental consequence. The interested reader may find more on this topic in [11]; the concept of "local regularity" or "orderliness" appears to be well worth investigating as a generalization of the approach to symmetry via groups. The latter may be needlessly restrictive and inappropriate in many contexts.

One last remark concerning colored patterns appears to be called for here. As mentioned earlier, very few of the groups of color symmetries are represented among Moorish and other ornaments. One reason for this situation seems to be that in many of the historical (and contemporary) multicolor decorations the various colors are not meant to play the same role. The different colors are apportioned to the copies of the motifs in unequal numbers; very frequently the proportions are 2:1:1, 4:2;1:1, 6:2:1, 6:3:1:1:1 or some similar ratios. For example, in Fig. 2 we have four white tiles for every two black ones and for each brown and green tile. In Fig. 6 the ratios are 6:3:1:1:1, while among the hexagons in Fig. 18 six black ones go with two blue and one brown. The mathematical theory of such colorings still awaits development.

We can summarize our conclusions as follows.

1. Only 13 of the 17 wallpaper groups occur as symmetry groups of ornaments in the Alhambra. The other four groups seem not to have been used by Moorish artists, or their use was very infrequent. In this context it is worth recalling that according to Makovicky & Makovicky[19], 6 of the 17 groups account for about 98% of the designs in the rich collection of Bourgoin[3]. But the importance of determining the number of groups used is greatly reduced when we appreciate that the mode of thinking in terms of symmetry groups was totally alien to the artisans (and mathematicians) of antiquity and the Middle Ages.

2. Despite the richness of the ornaments created by Moorish craftsmen, the designs actually used represent only an infinitesimally small fraction of those that are possible, even if allowance is made for the practical difficulties of realizing certain designs in ancient times. This applies to plain designs as well as to colored or interlaced ones.

3. The study of interlace patterns from a mathematical point of view is still in a most rudimentary stage, and needs considerable development before it can be applied to the description and classification of historical artifacts.

4. The various kinds of symmetry groups are useful in the description of many of the artifacts, but more general approaches (based on "adjacency relations" or other "local" criteria) are necessary for a better understanding of the ornaments and artwork, and of the ways their creators thought about them.

5. The approach to multicolor ornaments via color symmetries appears to be inadequate in many respects and inappropriate to describe practical examples. It seems that new mathematical tools for the understanding and classification of such patterns will have to be developed.

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