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★**Classical tessellations and three-manifolds.**
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Springer-Verlag, Berlin, 1987. xviii+230 *pp.* $35.00. *ISBN* 3-540-15291-1

Symmetry exists in nature and art. The rotational symmetry of a simple daisy must have at one time or another stirred geometric thoughts in the least mathematical mind. On a higher scale, the florets of *Helianthus maximus* naturally arrange themselves into two sets of logarithmic spirals with opposite sense of coiling. Indeed it is possible to argue with some force that mathematics is the study of pattern—the common theme that links symmetry in nature with our rational attempts at understanding.

It is entirely appropriate that the author of this beautiful book comes from the country which produced the intricate wall decorations of the Alhambra in Granada. (Incidentally, for the benefit of Hispanophiles, this book produces photographic evidence once and for all that all 17 plane symmetry patterns appear in the Alhambra.)

So what do 3-manifolds have to do with tessellations? At first sight there does not seem to be much of a connection, but consider the following example: Let M be the set of tilings of $\mathbf{R}^2$ by square tiles of unit side. Then an element of M is determined by a point z in $\mathbf{R}^2$ as a vertex of a tile and the angle which an edge makes with the x-axis. But the point of $\mathbf{R}^2$ is only well defined modulo unit translations and the angle is only well defined modulo $\pi/2$. So M is the quotient torus of the action on T^2 with monodromy $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. So M is a 3-manifold: but there is more! Consider the circle action on points of M (tilings) as follows: If M is a tiling and α is an angle let $\alpha\cdot m$ denote the tiling obtained from M by rotating the plane about the origin 0 through an angle α. Since this action has no fixed points M is decomposed as a disjoint union of circle fibres corresponding to the orbits under the S^1 action. The orbits return after a complete turn except when 0 lies at a vertex, at the centre of a tile or at the centre of an edge. So M has a Seifert fibre structure with three exceptional fibres of order 4, 4 and 2.

The contents of the book given by chapter titles are (1) S^1-bundles over surfaces. (2) Manifolds of tessellations in the Euclidean plane. (3) Manifolds of spherical tessellations. (4) Seifert manifolds. (5) Manifolds of hyperbolic tessellations. There are also two appendices: (A) on orbifolds and (B) on the hyperbolic plane. The book is written in a readable style with many examples and clear diagrams together with three pages of colour photographs—mineral crystals and tessellations from the Alhambra.

The reviewer greatly enjoyed reading this book and has only a few criticisms—there are only a few misprints, Figure 16 on page 64 is not clear, at least to me, and Exercise 1, page 76, is loosely worded since $\tilde{S}2222$ has the presentation $\{z,a,b,c,d|a^2=b^2=c^2=d^2=z, abcd=z^2\}$.

For those mathematicians with interests in this area this book will make a valuable contribution to their library.

Roger Fenn