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The Orbifold Notation for Two-Dimensional Groups[1]

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This paper gives a detailed introduction to the orbifold notation for two-dimensional (2-D) symmetry groups. It discusses the correspondence between properties of orbifolds and symmetries in the original surface. The problem of determining a group *in situ* is addressed. Elementary proofs of the classification of the Euclidean and spherical 2-D symmetry groups are presented.

KEY WORDS: Symmetry; group; notation; classification; two-dimensional.

PRELIMINARIES

Introduction

The finite subgroups of the three-dimensional (3-D) orthogonal group have been enumerated by several authors, using several different methods (see [3]). The 17 plane crystallographic groups have also been enumerated by Polya and Niggli, who used a distinct method analogous to that used 30 years earlier for the harder problem of enumerating the 219 space groups.

The aim of this paper is to describe a uniform method that enumerates all these groups and also the analogous ones in the hyperbolic plane, notable examples of which are the symmetry groups of Escher's four "Circle Limit" pictures. However, in this paper, we concentrate on the spherical and Euclidean cases.

The method is due to Macbeath [4] who studied groups of Möbius transformations and it has been elevated by Thurston to a general method for studying the geometry of manifolds [1,6]. We shall use Thurston's "orbifold" language, but point out that our "orbifold symbol" is just an elegant form of Macbeath's "signature."

We make extensive use of this method [2] and give short "fibrifold names" to all 3-D crystallographic space groups.

The Orbifold Concept

What the above 2-D groups have in common is that they act discretely on surfaces of constant curvature, namely, the sphere for the orthogonal groups, the Euclidean plane for the 17 crystallographic groups, and the hyperbolic plane for the non-Euclidean crystallographic groups. To cover all three cases we shall speak merely of "the surface."

The *orbifold* of such a group is "the surface divided by the group": that is to say, the quotient topological space whose points are the orbits under the group. (We can regard "orbifold" as an abbreviation of "orbit-manifold.") The *orbit* of a point p under a group G is the set of all images of p under elements of G.

Our *orbifold symbol*

$$\circ \cdots \circ ABC \cdots * ab \cdots c * \alpha\beta \cdots \times \cdots \times$$

indicates the features of the orbifold. Here the letters represent numbers: these numbers together with the symbols $\circ$, $*$, and $\times$ we call the *characters* of the orbifold symbol.

We can freely permute the numbers A, B, C that represent gyrations and also the parts $*ab\cdots c$, $*\alpha\beta, \ldots,$ that represent boundaries, and cyclically permute the numbers a, b, c that represent corners on any given boundary. Finally, we can always reverse the cyclic orders for all boundaries simultaneously and individually if an $\times$ character is present.

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We shall now explain the meanings of the different parts of our symbol. Like all connected 2-D manifolds, the orbifold can be obtained from a sphere by possibly punching some holes so as to yield boundary curves (indicated by $*$) and maybe adjoining a number of handles ($\circ$) or crosscaps ($\times$). However, an orbifold is slightly more than a topological manifold, because it inherits a metric from the original surface, which means, in particular, that angles are defined on it.

Numbers $a, b, \ldots, c$ added after a star indicate *corner points*, that is, points on the corresponding boundary curve at which the angles are $\pi/a, \pi/b, \ldots, \pi/c$. Finally, numbers $A, B, C\cdots$ not after any star represent *cone points*, that is, nonboundary points at which the total angles are $2\pi/A, 2\pi/B, 2\pi/C\cdots$. (We usually print these numbers in a slightly larger font.)

The orbifold idea is the most powerful way to achieve a conceptual understanding of these groups and, in particular, it trivializes their enumeration. However, it is also important to be able to find the group of a particular pattern *in situ* without needing to visualize its orbifold. We do this by studying those structures in the original surface that correspond to important features of the orbifold.

Our explanations will have the following form. We start from a property of the orbifold, then describe its correlate in the original surface, and provide a way to recognize and indicate this on a figure. In other words, the actions we perform are determined by considering the orbifold, but, for convenience, we actually perform them on the original surface.

ORBIFOLD BOUNDARIES AND KALEIDOSCOPES

The symmetry group of a finite physical object necessarily preserves some sphere, for example, one centered at the object's center of gravity. The symmetry group of the table shown in Fig. 1 acts on the sphere drawn around it. Most points of the sphere are in orbits of size 4, like $\{x, x', x'', x'''\}$; however, points that lie on either of the

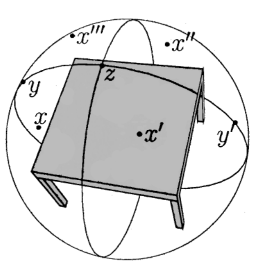

Fig. 1. The symmetries of a table act on a sphere drawn around it.

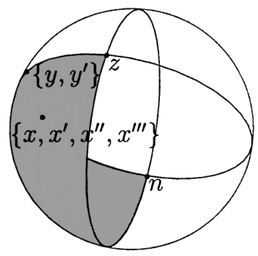

Fig. 2. The orbifold associated with the symmetry group of a table.

planes of symmetry belong to orbits of size 2 (such as $\{y, y'\}$) and, finally, there are two orbits of size 1, corresponding to the zenith (z) and nadir (n).

The orbifold is found by collapsing each orbit to a point—in this case, it is isomorphic to the quarter of a sphere shown in Fig. 2 (like the peel from one quarter of an orange.) We see that it has a boundary.

Orbifold Boundaries

The boundary of this orbifold is a curve with two *corner points* of angle $\pi/2$, symbolized $*22$. In general, we write $*ab\cdots c$ for a boundary curve that has corner points with angles $\pi/a, \pi/b, \ldots, \pi/c$ in that order, around it, but is otherwise smooth. Obviously, this is the same as $*b\cdots ca$; only the *cyclic* order matters.[5]

A boundary curve of an orbifold may have any number of corners from 0 upward. An object, such as the chair in Fig. 3 that just has bilateral symmetry, has a hemispherical orbifold as depicted in Fig. 4. Since this boundary has no corners, we denote it by $*$ (followed by no numbers).

On the other hand, the nine reflecting planes of a cube (Fig. 5) cut the sphere into 48 triangular regions, whose angles are $\pi/4, \pi/3, \pi/2$. Thus, the orbifold here has symbol $*432$.

Recognizing Kaleidoscopes

It is important to recognize what corresponds to a boundary and its corners on the original surface.

The boundary symbol $*ab\cdots c$ is also called a *kaleidoscope symbol* because its preimage in the surface is a set of mirror lines that meet in sets of $a, b, \ldots, c$ at their crossing points. (For example, the kaleidoscope $*22$ of

[5]In the case that an orientable orifold has several boundary curves, the corners on these should be listed in the cyclic orders induced from some fixed orientation. Changing this orientation reverses the cyclic orders for all boundary curves.

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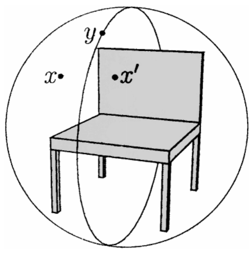

Fig. 3. The symmetries of a chair.

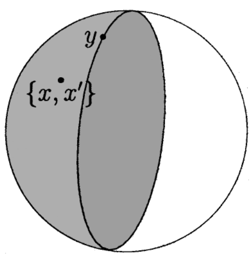

Fig. 4. The orbifold associated with the symmetry group of a chair.

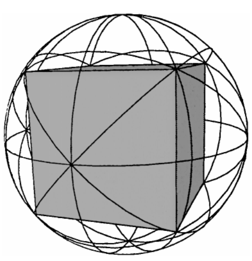

Fig. 5. The symmetry group of a cube consists of reflections in nine different planes.

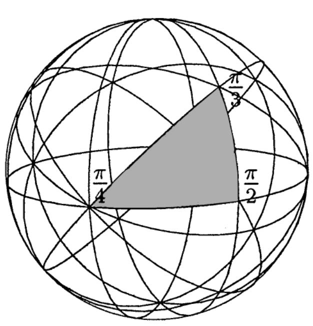

Fig. 6. The orbifold ∗432 associated with the symmetry group of the cube.

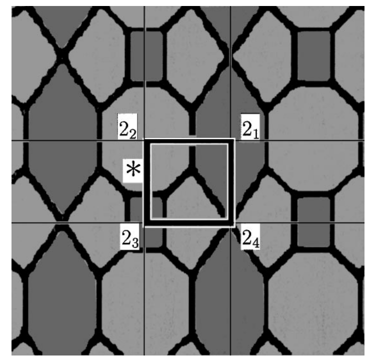

Fig. 7. Kaleidoscope of type ∗2222.

the table in Fig. 1 consists of two great circles that meet at both the zenith and nadir.)

The kaleidoscope of the chair is just a great circle, which has no point on two or more mirror lines and so has symbol ∗, while that of the cube (Fig. 5) has type ∗432; corresponding to the fact that its nine great circles meet in sets of 4, 3, and 2 at various points (Fig. 6).

Marking Kaleidoscopes

For a kaleidoscope, we draw a heavy line marked ∗ over just enough mirror segments to define the orbifold boundary, then mark each corner on this with the number of mirrors through it, possibly with a subscript to distinguish between different types of corner. For example, Fig. 7 has a kaleidoscope of type ∗2222; the four types of corners are labeled 2_1, 2_2, 2_3, and 2_4.

The black-and-white brick wall of Fig. 8 looks very different, but has the same group ∗2222, where now 2_1 is the center of a white brick, 2_2 is between two white bricks, 2_3 is the center of a black brick, and 2_4 is between two black bricks.

CONE POINTS AND GYRATION POINTS

Cone Points

The typical point on the surface is fixed only by the identity element. At such points, the orbifold looks locally exactly like the original surface. This is not so for boundary points of the orbifold, because the corresponding points on the original surface are fixed by reflections. There is only one other type of singular point that an orbifold can have: the *cone point,* which comes about when a point on the surface is fixed by a nontrivial rotation, but no reflection. The *order A* of the cone point is the largest order of any

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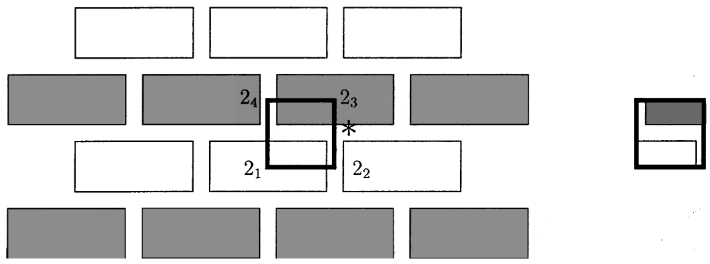

Fig. 8. A black-and-white brick wall and the orbifold associated with its symmetry group.

such rotation—the angle around it will then be $2\pi/A$. We indicate a cone point in the orbifold symbol by writing its order A (in a large font) before any boundary symbol $*ab\cdots c$.

What corresponds to this on the original surface?

Gyration Points

Figure 9 is obtained from Fig. 8 by making all the bricks have the same color. Now there is a rotation of order 2 around the center of the square outlined by the mirrors. A nontrivial rotation like this around a point that does *not* lie on a mirror line, we call a *gyration* and the corresponding point a *gyration point*. The fact that the center does not lie on a mirror is important, since it makes the corresponding point of the orbifold a cone point rather than a corner point.

The orbifold for Fig. 8 was a square whose four corners corresponded to 2_1, 2_2, 2_3, and 2_4. However, in Fig. 9 the new gyration interchanges 2_1 with 2_3 and 2_2 with 2_4. Correspondingly, the orbifold of Fig. 9 is obtained from the intermediate figure by identifying opposite points. With a paper model, this can actually be done by tearing a path from the boundary to the center and then coiling the paper to double thickness.

Since this orbifold has one order 2 cone point and a boundary with two order 2 corner points, its orbifold symbol is $2{*}22$.

Marking Gyration Points

The *order* of a gyration point is the largest order of any rotation that fixes it. As in Fig. 10, we indicate a representative of each type of gyration point by a heavy spot marked with its order (usually in a larger font than that used for corner points). Once again we can use subscripts to distinguish between different types of gyration points with the same order.

THE GLOBAL TOPOLOGY OF AN ORBIFOLD

We have now described everything about the orbifold that can be discovered by analyzing the locality of a single point. What remains is its 2-D topology. Any 2-D manifold (perhaps with boundary) may be obtained from a sphere (possibly perforated) by adding either handles or crosscaps. We discuss two examples.

The orbifold of Fig. 11a is a torus. We have outlined a fundamental region by joining the centers of four equivalent parallelograms. The torus is obtained from this in the usual way by identifying opposite sides (see Fig. 11b). Topologically, a torus can be obtained from a sphere by adjoining a handle ($\circ$), so the orbifold symbol for Fig. 11c is $\circ$.

For Fig. 12 the orbifold is a Möbius strip, obtained by rolling up (with a twist) the strip outlined by two vertical lines. Topologically, a Möbius strip can be obtained from

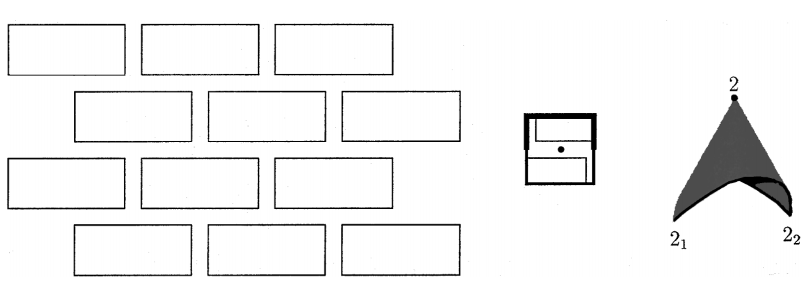

Fig. 9. A brick wall. The corresponding orbifold shown on the right is obtained from the intermediate figure by identifying opposite points.

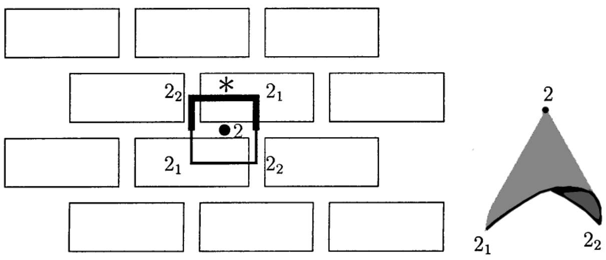

Fig. 10. The markings on the brick wall indicate the connection between the symmetries of the wall and the corresponding orbifold.

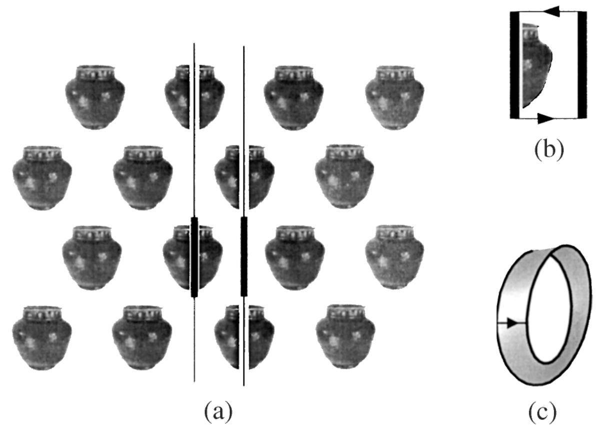

Fig. 12. This periodic system of vases (a) gives rise to the orbifold $*\times$ (b), which is topologically a Möbius strip (c).

a disk or sphere with one hole ($*$), by adjoining a crosscap ($\times$), so the orbifold symbol for Fig. 12 is $*\times$.

Recognizing and Marking the Global Topology

A crosscap makes the orbifold nonorientable. Thus, the presence of at least one crosscap can be detected on the original surface by finding a path from a place in the motif to a mirror image copy of itself that does not pass through a mirror line. Since this kind of "mirrorless" reflection of motif is rather paradoxical, we shall call it a *miracle cross* (for "mirrorless crossing"), or just a *miracle*.

We indicate each miracle in our figures by a dotted line marked with a cross (see Fig. 13a). Note that Fig. 12 contains both ordinary reflections and mirrorless ones, corresponding to its orbifold symbol $*\times$.

A handle in the orbifold corresponds to another kind of repetition of motif. It forces the existence of two independent, homologically nontrivial paths on the orbifold, as in Fig. 11c. These correspond to two paths from a part of the motif to two nonreflected images of itself. These repetitions of motif have the property that they cannot be explained by mirrors, gyrations, or miracles. We call this a *wonderful wandering*, or just a *wonder*, and indicate it in our illustrations by drawing the corresponding pair of dotted paths accompanied by a ring (a "wonder-ring"!), as in Fig. 13b.

We have shown that we can detect the presence of miracles and wonders by examination of the original surface. It is rather hard to count them and for this we recommend the reader construct the orbifold.

Fortunately, only one of the spherical and Euclidean groups has more than one wonder or miracle, illustrated in Fig. 14a. The orbifold is a Klein bottle obtained by identifying the sides of the indicated rectangle (Fig. 14b). Topologically, a Klein bottle can be obtained from a sphere by adding two crosscaps, so, indeed, there must be two miracles here (Fig. 14c), corresponding to the orbifold symbol $\times\times$.

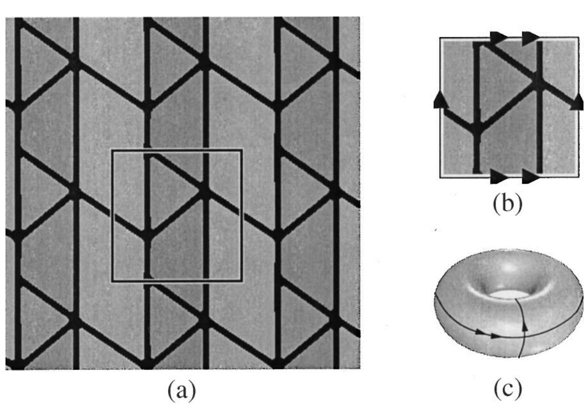

Fig. 11. This periodic tiling (a) gives rise to the orbifold $\circ$ (b), which is topologically a torus (c).

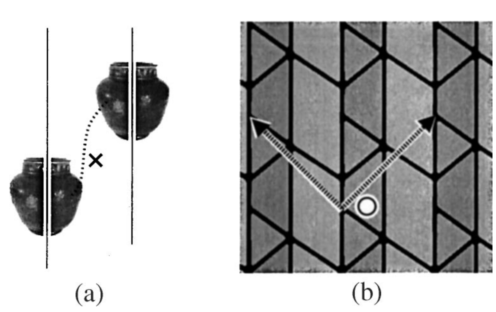

Fig. 13. (a) A "miracle" accounts for the nonvertical repetition in Fig. 12, whereas (b) a "wonder" accounts for the repetitions of motif in Fig. 11.

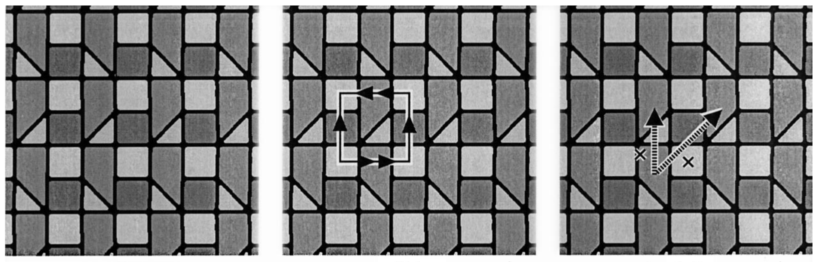

Fig. 14. A tiling (a) with the orbifold (b) and two different "miracles" (c) that generate the group $\times\times$.

THE ORBIFOLD SYMBOL

We indicate the type of an orbifold by juxtaposing the symbols for the handles, cone points, boundaries (with corners), and crosscaps from which it is made. Thurston shows that this symbol determines the orbifold as a constant curvature surface up to isotopy (i.e., continuous variation). We shall not prove this, in general, here, since the Euclidean and spherical cases are so simple that not much proof is required.

For example the orbifold of $*pqr$ is a triangle with angles π/p, π/q, and π/r, and it is easy to see that this is unique up to scale, in a space that is Euclidean, spherical or hyperbolic accordingly, as the sum of these angles equals π, or is greater or smaller than π.

Again, that of $*2222$ is a quadrilateral with four right angles, which must be a rectangle in the Euclidean plane, so the group $*2222$ is generated by the reflections in the sides of the rectangle. Since any one rectangle can be continuously deformed into any other, any two groups of type $*2222$ are isotopic.

The orbifold of a group of type $*ab$ is a two-sided figure with angles π/a and π/b. In a constant-curvature space, the only possibility is the lune bounded by two great circles on the sphere, in which both the angles are equal. Thus, $*ab$ can only exist when $a = b$.

THE DEFECT FORMULA

How can we find the order g of a group G from its orbifold symbol Q? The answer, when finite, is given by the remarkable *defect formula*:

$$\frac{2}{g} = 2 - \sum \text{defect(s)} = \text{ch}(Q)$$

summed over all the characters of the orbifold symbol, where these defects are tabulated in Table I.

Let us explain how this comes about. The right-hand side of the defect formula is the *orbifold Euler characteristic* ch (Q) of the orbifold Q. It can be obtained from the usual formula

$$\text{ch}(Q) = v - e + f$$

where v, e, and f are the numbers of vertices, edges, and faces of a map drawn on Q, provided these are suitably chosen. Because these numbers are often fractional, ch(Q) is also called the *fractional Euler characteristic* of Q.

In our figures, we will enlarge the vertices and edges of maps of surfaces and orbifolds into discs and strips so that we can see how they break up. Thus, Fig. 16 shows the map formed by the vertices, edges, and faces of the brick of Fig. 15.

On the sphere it has $V = 8$ vertices, $E = 12$ edges, and $F = 6$ faces, agreeing with the fact that the Euler characteristic of the sphere is $2 = 8 - 12 + 6$. However, the orbifold Q here is just one eighth of the sphere (Fig. 17) and for it the corresponding numbers are $v = 1$, $e = 3/2$, $f = 3/4$, and so the orbifold Euler characteristic ch$(Q) = 1 - 3/2 + 3/4 = 1/4 = 2/8$. Applying the defect formula to the orbifold symbol $*222$ gives the same value $2 - (1 + 1/4 + 1/4 + 1/4) = 1/4 = 2/8$. We

Table I. Defects Associated with Different Characters of an Orbifold[a]

Char	Defect	Char	Defect
$\circ$	2	$*$ or $\times$	1
2	1/2	2	1/4
3	2/3	3	1/3
4	3/4	4	3/8
5	4/5	5	2/5
6	5/6	6	5/12
N	$(N-1)/N$	N	$(N-1)/2N$
∞	1	∞	1/2

[a] The larger font numbers on the left are those not following any $*$.

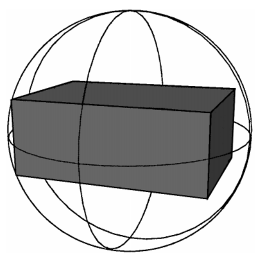

Fig. 15. The symmetries of a brick.

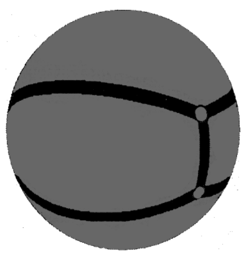

Fig. 16. The map formed by the vertices, edges, and faces of a brick.

Fig. 17. The orbifold for the symmetry group of a brick is one eighth of the sphere.

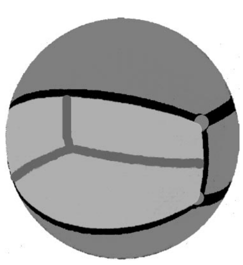

Fig. 18. Introducing a hole (whose boundary has no corner) decreases the Euler characteristic by 1.

prefer to write the Euler characteristics in the form $2/g$, since then the denominator is the group order.

The maps on an orbifold are images of those maps on the sphere that are invariant under the group G. Then, obviously, the counts of vertices, edges, and faces of such orbifold maps are

$$v = \frac{V}{g}, \quad e = \frac{E}{g}, \quad \text{and } f = \frac{F}{g}$$

and so the orbifold Euler characteristic will be:

$$\mathrm{ch}(Q) = v - e + f = \frac{V - E + F}{g} = \frac{2}{g}$$

where V, E, and F are the corresponding counts on the sphere. Obviously, we can evaluate ch(Q) as $v - e + f$ for *any* map on the orbifold.

We shall use this principle to show how the characteristic changes as we modify the orbifold. As usual, it implies that adding a handle or crosscap reduces the Euler characteristic by 2 or 1, respectively.

Introducing a hole $*$ (whose boundary has no corner) decreases ch (Q) by 1. To see this, take the hole to be a face of the map (Fig. 18). This face and the adjacent n vertices and n edges contribute $n - n + 1$ to $v - e + f$ before the modification, but $n/2 - n/2 + 0$ after, a decrease of 1.

Again, changing an interior point P from an ordinary point to an A-fold cone point decreases ch(Q) by $1 - 1/A = (A - 1)/A$, because we can suppose P is a vertex of the map, when its contributions to v before and after the change are 1 and $1/A$.

Similarly, changing an ordinary boundary point P (contributing $1/2$ to v) to an a-fold corner point (contributing $1/2a$) reduces the characteristic by $1/2 - 1/2a = (a - 1)/2a$.

We have now proved that for the orbifold Q of a finite group of order g, we do indeed have

$$\mathrm{ch}(Q) = \frac{2}{g} = 2 - \sum \text{defect(s)}$$

These are the orbifolds of positive characteristic.

Those of characteristic 0 are precisely the Euclidean groups, since, for them, the orbifold is a quotient of a torus, which has ordinary Euler characteristic 0. Those of negative characteristic correspond to groups acting in the hyperbolic plane. This follows from the universal interpretation of the orbifold Euler characteristic as the integrated Gaussian curvature over the orbifold, divided by 2π.

ENUMERATING THE GROUPS

In enumerating the Q for which ch(Q) has a given sign, it suffices in the first instance to list only those that

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consist of one or more $*$ followed only by digits since the following "demotions" preserve the sign of $\mathrm{ch}(Q)$. In the first demotion, $AB\cdots C$ must be the entire orbifold symbol, and the characteristic is halved. The other three preserve the characteristic and may be performed locally.

$$\begin{array}{ccc}
*AB\cdots C & \begin{array}{c}\leftarrow\text{demote}-\\ -\text{promote}\rightarrow\end{array} & AB\cdots C\\
(\text{final})\ * & \begin{array}{c}\leftarrow\text{demote}-\\ -\text{promote}\rightarrow\end{array} & \times\\
** & \begin{array}{c}\leftarrow\text{demote}-\\ -\text{promote}\rightarrow\end{array} & \circ\\
AA & \begin{array}{c}\leftarrow\text{demote}-\\ -\text{promote}\rightarrow\end{array} & A
\end{array}$$

The Euclidean Plane Crystallographic Groups

After these demotions, a case with just one $*$ has the form: $*ab\cdots c$, for which

$$\mathrm{ch}(Q) = 2 - 1 - \frac{a-1}{2a} - \frac{b-1}{2b} - \cdots - \frac{c-1}{2c}$$

For a Euclidean group, this must be 0, and so we must solve:

$$\frac{a-1}{a} + \frac{b-1}{b} + \cdots + \frac{c-1}{c} = 2$$

The solutions are: (6, 3, 2), (4, 4, 2), (3, 3, 3), and (2, 2, 2, 2). If two characters $*$ are present, there can be no further character, since they already have total defect 2. This leads to the five cases on the left in Table II: they promote to give the cases that follow them.

This must surely be the simplest and most conceptual enumeration of these 17 groups. However, it should be supplemented by a proof that for each of the candidate symbols there is just one group up to isotopy. This is easily proved for each particular case, for instance, it is clear that any group of type $*2222$ must be generated by reflections in the four sides of a rectangle; since this rectangle can be continuously transformed into any other, all such groups are isotopic. Similarly, a group of type $4*2$ is generated by reflections in sides of a square, together with the order 4 gyration about the center of that square: again, all such groups are isotopic.

A general proof, which applies also to the hyperbolic case, can be found in the literature; we omit further details.

Table II. The 17 Plane Crystallographic Groups

$*632$			632
$*442$	$4*2$		442
$*333$	$3*3$		333
$*2222$	$2*22$	$22*$	2222
$**$	$*\times$	$\times\times$	$\circ$

The Spherical Groups

We find these by enumerating the candidate symbols Q for which $\mathrm{ch}(Q)$ is positive, starting with those of form $*ab\cdots c$, which correspond to the solutions of

$$\frac{a-1}{a} + \frac{b-1}{a} + \cdots + \frac{c-1}{c} < 2$$

namely: (5, 3, 2), (4, 3, 2), (3, 3, 2), (2, 2, n), and (m, n), where we allow m or n to be 1.

However, we shall see in a moment that the last case only corresponds to a group when $m = n$. This gives rise to the cases on the left in Table III: once again the remaining cases on any given line are obtained by promotion.

The Bad Orbifolds

Once again the argument should be supplemented by a discussion of the existence and uniqueness of the corresponding groups. Most cases easily follow from the fact that there exists a spherical triangle (unique up to isometry) with angles of the form $\pi/a, \pi/b$, and π/c, just if the sum of these angles exceeds π.

For an arbitrary orbifold symbol, in general, the existence and uniqueness up to isotopy is proved by dissecting the orbifold into triangles. However, in the case $*mn$ ($m, n \geq 1$) the relevant polygon is a two-sided one (see Fig. 19) with angles π/m and π/n. Obviously, this can only be the spherical lune bounded by two great circles, for which the two angles must be equal, so that $m = n$. The same must hold for mn, since any such group could necessarily be extended to $*mn$ by adjoining a reflection through the two corresponding gyration points.

The symbols $*mn$ and mn ($m \neq n$), together with the particular cases $*m$ and m ($m > 1$) that arise by putting $n = 1$, are the only ones that do not correspond to groups.

Table III. The Spherical Groups[a]

$*532$			532
$*432$			432
$*332$	$3*2$		332
$*22n$	$2*n$		$22n$
$*nn$	$n*$	$n\times$	nn

[a]There are seven particular groups and seven infinite series controlled by a parameter $n \geq 1$. (When $n = 1$, it is customarily omitted from the symbol.)

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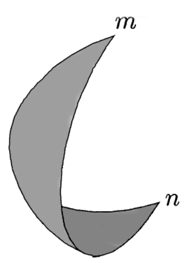

Fig. 19. For the symbol $*mn$, the orbifold is a spherical lune bounded by two great circles, so that $m = n$.

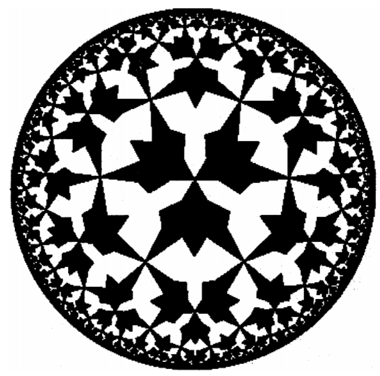

Fig. 20. Inspired by Escher's Circle Limit IV, this tilings has symmetry group $4{*}3$.

The Seven Frieze Groups

So far we have tacitly assumed that the numbers in our orbifold symbols are finite, which corresponds to the compactness of the orbifold. The frieze groups correspond to cases when this condition is violated: they correspond to the orbifold symbols Q that contain the character ∞ and have $\mathrm{ch}(Q) = 0$. The enumeration is easy (Table IV); it turns out that these groups are obtained by putting $n = \infty$ in Table III.

Groups in the Hyperbolic Plane

Escher's Circle Limit pictures are really in the hyperbolic plane. For example, the angels and devils of his Circle Limit IV ([5] p. 296) form a picture with symmetry group $4{*}3$, if we ignore the fact that every fourth figure is facing away from us (and the artist's monogram) (see Fig. 20).

Although such groups are not our main concern here, we should point out that one of the great strengths of the orbifold method is that makes them just as easy to handle as the Euclidean and spherical cases—they correspond precisely to the orbifold symbols Q, for which $\mathrm{ch}\,(Q) < 0$, the orbifold being compact just if ∞ is not mentioned.

The orbifold notation helps to understand the many relationships between these groups. For example, passing to a subgroup of index i multiplies the characteristic by i.

The subtler properties of Escher's pictures often hint at such relationships. For example, if we do take account of the fact that some of the angels and devils are facing away from us in Circle Limit IV, the group of that picture drops to $*3333$, of index 4 in $4{*}3$. This agrees with the orbifold Euler characteristics:

$$\text{for } 4{*}3: \quad 2 - \frac{3}{4} - 1 - \frac{1}{3} = -\frac{1}{12}$$

$$\text{for } {*}3333: \quad 2 - 1 - \frac{1}{3} - \frac{1}{3} - \frac{1}{3} - \frac{1}{3} = -\frac{1}{3}$$

Table IV. The Seven Frieze Groups

$*22\infty$	$2{*}\infty$		22∞
$*\infty\infty$	$\infty *$	$\infty\times$	$\infty\infty$

GENERATORS AND RELATIONS FOR TWO-DIMENSIONAL GROUPS

It is a well-known principle that if a simply-connected manifold is divided by a group G to obtain another manifold, then the fundamental group of the quotient manifold is isomorphic to G. What happens is that a path from the base point to itself in the quotient manifold lifts to a path in the original manifold that might not return to the base point, in which case it corresponds to a nontrivial element of G.

This principle applies also when the quotient space is a more general orbifold, except that some care is required for the definitions. The important point is that a path that bounces off a mirror boundary in the orbifold should be lifted to a path that goes through the corresponding mirror in the original surface.

Figure 21 shows the paths in the orbifold whose lifts are the generators for the corresponding group. We chose

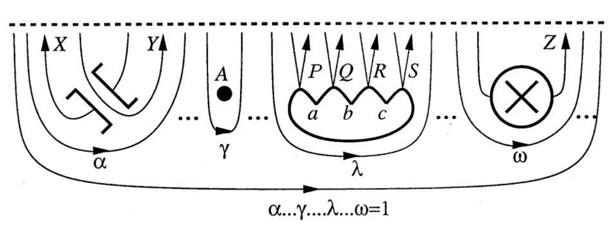

Fig. 21. Paths in the orbifold that lift to generators of the corresponding group.

a base point in the upper half plane and for each of the features

$$\circ \cdots A \cdots *abc \cdots \times$$

we have one Greek generator corresponding to a path that circumnavigates that feature in the positive direction, and maybe some Latin generators.

For a $\circ$ symbol, represented in the figure by a bridge, the two Latin generators X, Y are homology generators for the handle so formed. They satisfy the relations:

$$X^{-1}Y^{-1}XY = [X, Y] = \alpha$$

For a gyration symbol A, there is no Latin generator, but the corresponding Greek generator γ satisfies the relation

$$\gamma^A = 1$$

For a mirror boundary with n corners, there are $n + 1$ Latin generators $P, Q, \ldots, S$ corresponding to paths that bounce off the boundary and are separated by the corners. These correspond to reflections in the group that satisfy the relations.

$$1 = P^2 = (PQ)^a = Q^2 = (QR)^b = R^2 = (RS)^c$$
$$= S^2 \text{and } \lambda^{-1}P\lambda = S$$

Finally, for a crosscap $\times$, represented in our figure by a cross inside a circle whose opposite points are to be identified, the Latin generator Z corresponds to a path "through" the crosscap and satisfies the relation:

$$Z^2 = \omega$$

A complete presentation for the group is obtained by combining the generators and relations that we have described for each feature, and adjoining the *global relation*:

$$\alpha \cdots \gamma \cdots \lambda \cdots \omega = 1$$

which asserts that the product of all Greek generators is 1.

Table AI. The Abstract Structure of Spherical Groups[a]

Order	← Families of groups → nn	n×	n*	*nn	22n	2*n	*22n		
1	1								
2	22	×	*	*	22				
3	33								
4	44	2×	2*	*22	222	2*	*22		
5	55								
6	66	3×	3*	*33	223				
7	77								
8	88	4×	4*	*44	224	2*2	*222		
9	99								
10	10 10	5×	5*	*55	225				
11	11 11								
12	12 12	6×	6*	*66	226	2*3	*223	332	
24	24 24	12×	12*	*12 12	22 12	2*6	*226	432 *332	3*2
36	36 36	18×	15*	*18 18	22 18	2*9	*229		
48	48 48	24×	24*	*24 24	22 24	2*12	*22 12		*432
60	60 60	30×	30*	*30 30	22 30	2*15	*22 15	532	
120	120 120	60×	60*	*60 60	22 60	2*30	*22 30		*532
	C		2×C		D		2×D	P	2×P
	← Group structures →								

[a] Unseparated groups on the same line are isomorphic. The group structures are cyclic (C), dihedral (D) and polyhedral (P), or the direct products ($2 \times C$, $2 \times D$, $2 \times P$) of these with a group of order 2. The structures of the polyhedral groups 332, 432, and 532 are the alternating and symmetric groups A_4, S_4, and S_5.

We propose the following notation for this set of generators and relations:

$$\circ^{XY} \cdots A^{\gamma} \cdots {}^{\lambda} *^{P}\ a^{Q} b^{R} c^{S} \cdots \times^{Z}$$

APPENDIX: ISOMORPHISMS BETWEEN THE SPHERICAL GROUPS

Occasionally, two of the infinite series contain the same group. For example, when $a = 1$ and $b = 2$ we have:

$$22a = bb, \qquad *22a = *bb, \qquad 2*a = b* \qquad \text{and}$$

$$*aa = a*$$

But also, two different spherical groups can be isomorphic as abstract groups. For example, since all groups of order two are abstractly isomorphic, we have $\times \cong 22 \cong *$.

All these matters are displayed in Table AI. The groups of any given order occupy one line and groups are abstractly isomorphic just if they are not separated by a dividing line.

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