The Planar Crystallographic Groups Represented at the Alhambra

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Abstract

The geometric patterns that cover the walls, ceilings and floors of the Alhambra in Granada, Spain have been the subject of study by many mathematical researchers ever since 1944, when E. Müller first reported the number of distinct planar crystallographic groups found there. In subsequent years, the count has varied, sometimes due to inconsistencies in the ways in which the patterns were chosen and analyzed. As such, contradictions and controversy still surround the issue. After a brief introduction, this paper will provide the criteria used by this author in an on-site visit in 2003 in order to catalog different crystallographic groups represented by patterns in the Alhambra. Each group identified is illustrated by a representative pattern.

Introduction

The Alhambra (whose name comes from the Arabic, *Al Qal'a al-Hamra*, meaning "red fortress") in Granada, Spain is currently that country's most popular tourist site. Many of the edifices there (that were constructed over earlier buildings dating back to Roman, and perhaps even earlier, times) evolved over a period of 260 years, from 1232 until the last of the Moors were expelled from Spain in1492. The elaborate geometric tilings and patterns that completely cover most of the walls, ceilings and floors of these buildings are of special interest to those concerned with classifying the patterns based on the symmetries they possess. Since there are 17 possible planar symmetry groups of periodic patterns, known as the crystallographic ("wallpaper") groups, a natural question that arises for contemporary mathematicians is "How many of these symmetry groups are represented by patterns in the Alhambra?" Many researchers have sought to answer this question, but contradictions and controversy still surround the issue, sometimes due to inconsistencies in the ways in which the patterns were chosen.

In her 1944 Ph.D. thesis, E. Müller reported that she had found 11 of the 17 planar crystallographic groups represented at the Alhambra [1]. In 1986, B. Grünbaum, Z. Grünbaum and G. C. Shephard [2] reported that they had found examples of at most 13 different symmetry groups and that examples of *p*2, *pg*, *pgg* and *p*3*m*1 patterns were missing. A year later, R. Perez-Gomez and J. M. Montesinos [3] reported that they had found these missing ones at the Alhambra. However, in the June/July 2006 issue of the *Notices* of the AMS, B. Grünbaum challenged these findings and again asked, *What Symmetry Groups are Present in the Alhambra?* [4]. He called for classification in "a consistent and well-explained manner, and following explicit criteria" when answering this question. This paper will offer one possible answer by providing the criteria used by the author in classifying symmetry groups of Alhambra patterns. Photos by the author will illustrate each of the symmetry groups discovered.

Of the 17 possible symmetry groups for uncolored periodic planar patterns, some cultures prefer certain ones and intuitively recognize these as being "right." And, it is true that all 17 of the symmetry groups are represented in Islamic ornament found throughout the world. But that is not to say that all 17 symmetry groups are represented in any one limited locale or attributed to any particular time period, such as at the Alhambra, whose ornament dates to the 14th through 15th centuries.

It should also be noted that there is no indication that the Moorish or *Mudéjar* artisans of the day made any conscious efforts to classify the geometric Islamic patterns found at the Alhambra according to their symmetries. So why would we be interested in doing this now? Many feel strongly that this is a legitimate avenue to explore, including MIT art historian W. K. Chorbachi who wrote:

> The importance of group theory and its notational system for Islamic art lies in the fact that it provides a tool for exact cataloging of the infinite number of geometric designs used in Islamic art. It is also helpful as an analytical tool in recognizing the symmetry used within a design. … Moreover, it provides a precise language and terminology by which those who are interested in these patterns can communicate precisely with each other about these patterns. All this might seem redundant to the scientists who have been involved in the study of symmetry, yet, for the art historians, it is still an unacknowledged tool [5].

The Criteria Used to Classify the Patterns

During a month-long trip to *Andalucia* in 2003, the author spent several days examining and photographing as many of the periodic patterns at the Alhambra (from walls, ceilings and floors) that were accessible, including *alicatado* (mosaic tilings), *yeserías* (plasterwork), brickwork and paintings. All of the patterns were analyzed as they exist today, ignoring small imperfections such as minor inaccuracies in the cut tiles, or variations due to human error in laying out the pattern, or damage due to weathering, and so on. To keep the analysis straightforward and uncomplicated, all of the patterns were considered to be uncolored, thus allowing the focus to be solely on the underlying skeletal structure. Color-preserving symmetry groups were avoided, since

> "the approach to multicolor ornaments via color symmetries appears to be inadequate in many respects and inappropriate to describe practical examples. It seems that new mathematical tools for the understanding and classification of such patterns will have to be developed." [2].

Any existing interlacings or medallions containing Arabic inscriptions were taken into account, thus resulting in limited reflection or rotational symmetries of some ornamental patterns. Each pattern had to cover an area of least three feet by three feet (approximately one square meter), and appear to be *meant* to be a planar wall covering, and not just a "wide" frieze pattern or a small portion within a larger design. The pattern also had to include at least two repeats in each of two linearly independent directions. Each pattern was also analyzed as if the motif repeats indefinitely in all directions, and in tilings, the grout lines were considered to be infinitesimally thin.

A Quick Review of Symmetry Terminology

Recall that there are four **isometries** (distance-preserving transformations) of the plane: translations, rotations, reflections, and glide-reflections. A translation is determined by a vector that shows the direction and distance that all points are moved. A rotation is determined by a center and an angle; all points are rotated about that center through the given angle. A reflection is determined by a line (called a mirror line) across which all points are reflected. A glide-reflection is a composite transformation determined by a line (called the glide line) and a translation whose vector (called a glide vector) is parallel to the glide line: all points are reflected in the glide line and then translated by the glide vector. A figure in the plane is **symmetric** if there are isometries that move some points of the object to new positions, but leave the appearance of the figure unchanged (**invariant**). The isometries that leave a figure invariant are called **symmetries** of the figure. The **symmetry group** of a figure is the collection of all symmetries of the figure. A repeating pattern in the plane is called **periodic** if it has translation symmetries in two linearly independent directions, and there is a minimum-length translation vector in each of these directions. It is well-known that there are only 17 different symmetry groups of periodic planar patterns, and these are also known as **wallpaper** (or crystallographic) groups. A standard notation for symmetry groups of periodic planar patterns was established by the International Union of Crystallography, and we will identify symmetry groups by the short form of this notation. Explanations of this notation can be found in [6] and [7].

Questions Posed and Answered for Each Pattern

To determine which symmetry groups are represented by patterns in the Alhambra, the author posed some of the following questions as patterns were examined.

Is there reflection symmetry?
If so, are there mirror lines in more than one direction?
If so, what is the angle between adjacent mirror lines?
Is there rotational symmetry?
If so, what is the smallest angle of rotation?
Are there rotation centers not on mirror lines?
Is there glide reflection symmetry?
Do the glide lines coincide with mirror lines?
Are there glide lines off the mirrors?

Charts and algorithms for identifying the symmetry group of a periodic planar pattern using questions such as these can be found in [6] and [7].

Patterns with No Rotational Symmetry

The four types of patterns with no rotational symmetry are denoted ***p*1**, ***pm, cm***, and ***pg***.

The first pattern (**Figure 1)** is classified as ***p*1,** since the calligraphy within the pattern "breaks" its apparent mirror symmetry of this pattern. The pattern has only translation symmetry. This pattern may be found as part of a *yeso* (plaster) wall covering in the *Cuarto des los Leones* (Court of the Lions), which was constructed between 1362 and 1391 by Muhammad V during the second part of his reign (after 1367), considered to be the Golden Age of *Nasrid* rule.

The second pattern (**Figure 2)** is classified as ***pm*** since it has mirror lines and glide lines only in one (vertical) direction, and all glide lines are on mirror lines. This incised *yeso* wall covering may be found in the hallway that leads from the *Patio de Cuarto Dorado* (Gold Courtyard) into the *Patio de los Arrayanes* (Court of the Myrtles).

The third pattern (**Figure 3**) is a tiling known as the *ktaf u darj* ("shoulder and step") and occurs frequently in Spain and Morocco [8]. It is classified as ***cm*** since it has (vertical) mirror lines that split the tiles, and glide lines halfway between adjacent mirror lines. This tiling incised in plaster covers an entire wall in the *Sala de los Reyes* (Hall of Kings).

Figure 1. *A* ***p*1** *pattern*

Figure 2. *A* ***pm*** *pattern*

Figure 3. *A* ***cm*** *pattern*

The fourth pattern we would like to classify as ***pg,*** taking into account the over-under relationships of the curving bands. The pattern (**Figure 4a**) is painted on *yeso* on the inside façade of the *Puerta del Vino* (Wine Gate) one of the oldest buildings in the Alhambra, built during the time of Sultan Muhammad III (1302 – 1309) and decorated during Muhammad V's second reign. The photo shows most of what remains, which is only a small portion of the original pattern; how the pattern continues in both vertical and horizontal directions must be inferred. The over-under relationships are apparent in the photo for two motifs in the top row and for one motif in the second row. The "over" bands in one direction in the top row, become "under" bands in that same direction in the second row. **Figure 4b** is the author's depiction of an extended portion of the pattern, constructed using *The Geometer's Sketchpad* [9]. This depiction assumes that the over-under relationships that can be clearly seen in the photo continue both vertically and horizontally, with all the "over" bands in one direction in each row, reversing the over-under relationship from row to row. This inferred pattern has parallel glide lines, two of which are shown. It has no mirror lines due to the overlapping bands.

Figure 4a. *A possible* ***pg*** *pattern*

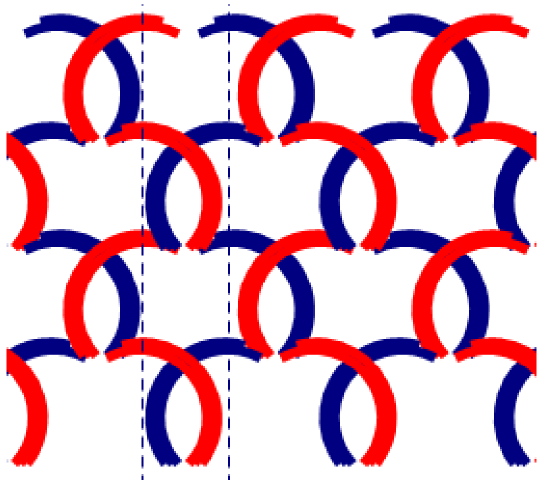

Figure 4b. *An extended version of the pattern in Figure 4a; Two adjacent glide lines are shown (dashed); this pattern has pg symmetry group.*

Two-fold Rotational Symmetry

The five types of patterns with two-fold rotational symmetry are denoted ***p*2**, ***pmm***, ***cmm***, ***pgg***, and ***pmg***.

The pattern in **Figure 5** is part of a dado (lower wall mosaic) in the *Sala de los Reyes* (Hall of Kings). It is classified as ***pmm*** since it has mirror lines in vertical and horizontal directions, and all two-fold rotation centers occur at the intersections of mirror lines.

Figure 5. *A **pmm** pattern*

The pattern in **Figure 6** is classified as ***cmm*** since it has horizontal and vertical mirror lines that split each tile through its center and two-fold rotation centers where four tiles touch. These centers are not on mirror lines. Note that the white "bone" shapes are substantially larger than the colored ones; there are two distinct shapes of tiles. This mosaic may be found as part of a dado (lower wall tiling) in the *Salon de los Embajadores* (Hall of the Ambassadors).

The paving in **Figure 7** is classified as ***pgg*** since it has glide lines in two perpendicular directions, but no mirror lines. Glide lines make 45^{o} angles with edges of the bricks. The two-fold rotation centers are not on the glide lines; they are at the centers of rectangles formed by the grid of glide lines. This tiling can be found as a brick floor tiling in the *Mirador de Linderaja* (the Queen's Overlook Window) adjacent to the *Sala de Dos Hermanas* (Hall of the Two Sisters).

Figure 6. *A* ***cmm*** *pattern*

Figure 7. *A* ***pgg*** *pattern*

Figure 8. *A* ***p*****3** *pattern*

Three-fold Rotational Symmetry

The three types of patterns with three-fold rotational symmetry are denoted ***p*****3**, ***p*****31*m*,** and ***p*****3*m*1.**

The pattern in **Figure 8** is classified as ***p*****3** since it has only three-fold rotation symmetry, with the rotation centers located at the centers of the stars, the point where three motifs almost touch, and the centers of the white hexagons formed by three “V” edges of motifs. This mosaic tiling may be found as part of a dado (lower wall tiling) in a recess in the *Patio de los Arrayanes* (Court of the Myrtles).

The pattern in **Figure 9** is classified as ***p*****31*m*** since it has mirror lines inclined at 60 degrees to one another (in three distinct directions), and has three-fold rotation centers not on mirror lines. The pattern in **Figure 9a**, very faintly visible, is painted on *yeso* and may be found under the vaulted ceiling of the *Puerta del Vino* (Wine Gate). The same pattern, incised in plaster (**Figure 9b**) may also be found the hallway that leads from the *Patio de Cuarto Dorado* (Gold Courtyard) into the *Patio de los Arrayanes* (Court of the Myrtles).

Figure 9a. *A* ***p*****31*m*** *pattern*

Figure 9b. *The same* ***p*****31*m*** *pattern*

Four-fold Rotational Symmetry

The three types of patterns with four-fold rotational symmetry are denoted ***p*****4**, ***p*****4*m*,** and ***p*****4*g*.**

The first pattern (**Figure 10)** in this group is classified as ***p*4** since it has only four-fold rotation centers and two-fold rotation centers. The four-fold centers are at the centers of the stars and at the meeting points of four L-curves that make a swastika. The two-fold rotation centers are midway between identical adjacent four-fold centers. This pattern may be found as part of a *yeso* wall covering in the *Cuarto des los Leones* (Court of the Lions).

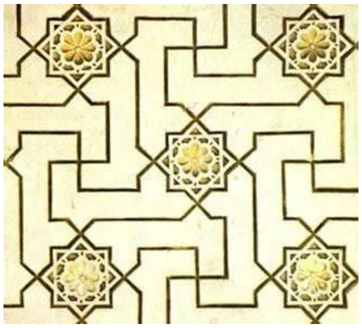

Figure 10. *A* ***p*4** *pattern*

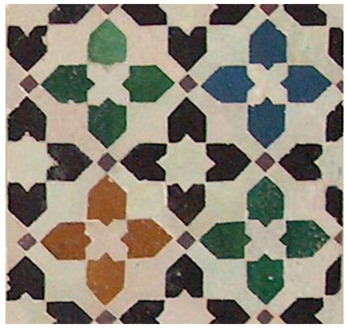

Figure 11. *A* ***p*4*m*** *pattern*

Figure 12. *A* ***p*4*g*** *pattern*

The second pattern (**Figure 11**) in this group is classified as ***p*4*m*** since it has mirror lines inclined at 45 degrees to one another (in four distinct directions) and all rotation centers lie on mirror lines. This mosaic tiling may be found as part of a dado (lower wall tiling) in the *Mirador de Linderaja* (the Queen's Overlook Window) adjacent to the *Sala de Dos Hermanas* (Hall of the Two Sisters).

The last pattern (**Figure 12**) in this group is classified as ***p*4*g*** since it has mirror lines in two perpendicular directions (these cut through the centers of the motifs). The four-fold rotation centers do not lie on mirror lines; they are located at the centers of the small squares made up of four triangles. This mosaic tiling may be found as part of a dado (lower wall tiling) in the *Mexuar.*

Six-fold Rotational Symmetry

The two types of patterns with six-fold rotational symmetry are denoted ***p*6** and ***p*6*m***.

The pattern in **Figure 13** shows six-fold rotational symmetry and has no mirror lines, so is classified as ***p*6.** There are also two-fold and three-fold rotation centers. This mosaic tiling may be found covering an entire wall in the *Sala de los Reyes* (Hall of Kings).

Figure 13. *A* ***p*6** *pattern*

Figure 14. *A* ***p*6*m*** *pattern*

The last pattern (**Figure 14**) is classified as ***p*6*m*** since it has both six-fold rotation centers and mirror lines. Six mirror lines meet at the six-fold rotation centers and are inclined at 30° to one another (in six distinct directions). There are also two-fold and three-fold rotation centers; these also are located at intersections of mirror lines. This mosaic tiling may be found as part of a dado (lower wall tiling) in the Alhambra Museum and also found reproduced in the *Sala de los Reyes* (Hall of Kings).

Conclusion

This author was able to locate patterns representing 14 of the 17 groups, with only the ***p*2, *pmg*** and ***p*3*m*1** symmetry groups so far undiscovered. A ***p*2** pattern would have two-fold rotation centers and no reflection or glide reflection symmetry. A ***pmg*** pattern would have parallel mirror lines in one direction, glide lines perpendicular to the mirror lines, and all two-fold rotation centers on glide lines. A ***p*3*m*1** pattern would have mirror lines in three distinct directions with all rotation centers located at intersections of the mirror lines. Examples of these three pattern types might have been missed by the author or might only be found in chambers of the Alhambra to which the author did not have access. The Alhambra Museum, which houses objects excavated from the Nasrid palace complex, displays a 14th century hexagonal tile containing a design, that when replicated, yields a ***p*3*m*1** pattern. But since it exists as a single tile and not a planar tiling, it does not meet our criteria for inclusion.

Acknowledgements

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