# SYMMETRY GROUPS IN THE ALHAMBRA

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Abstract

*The question of the presence of the seventeen symmetry crystallographic groups in the mosaics and other ornaments of the Arabic palace of the Alhambra (Spain) seems to be not yet settled. We provide new evidences supporting that the answer should be positive.*

## 1. INTRODUCTION

The study of the group of movements of the plane, and of some of their subgroups, the so-called plane symmetry groups, has besides a theoretical interest a practical one, to generate new ornamentations as well to classify the existent ones.

The Islamic art is a clear example of the use of the geometric ornamentation in all its manifestations.

Different researchers have raised the question: Are the 17 crystallographic groups represented in the arabic tiles of the Spanish palace of the Alhambra? Surprisingly after many years the question seems still open.

Some authors like E. Müller or B. Grünbaum [3], have answer that question, providing partial results that show the presence of 13 or 14 of such groups. J.M. Montesinos [5] provides photographic evidence of the presence of the 17 groups, but this was questioned by others authors; see by instance R. Fenn [2] and B. Grünbaum [4]. So Grümbaum argue that "There is no explanation as to what is the size or extent of an ornament that is sufficient to accept it as a representative of a certain group", "Several of the ornaments shown are deteriorated to such an extent that is impossible to see the pattern" and "Do we count the symmetries of the underlying tiles, without taking into account the colors of the tiles, or do we insist on color-preserving symmetries?".

We believe that the objections of Grünbaum to the work of Montesinos about the sizes and deterioration of some of the ornamentations do not have importance; such physical defects are unavoidable in a building that has more than six centuries.

We present here new evidences proving that the 17 crystallographic groups can really be found in the Alhambra. In our study we have considered the colors and not only the form of the mosaics. We also give a dynamical generation of each of the groups.

The structure of the paper is as follow: Section 2 reviews the mathematical concepts and properties of crystallographic groups and gives an algorithm allowing classifying them; a detailed study of those topics can be found in [1] and [5]. The main Section 3 shows pictures of the ornaments of Alhambra were the different groups appear. For each of them we study their generation from the basic motif that defines it, showing its fundamental region, and/or its fundamental parallelogram.

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## 2. CRYSTALLOGRAPHIC GROUPS

Let us briefly recall some mathematical concepts that will be used in the following:
We consider the plane $R^2$ endowed with the ordinary Euclidean affine structure. An **isometry or movement** of the plan is an application of the plane onto itself that preserves distances, i.e.

$$f: R^2 \longrightarrow R^2, \text{ such that for each couple of points } P, Q \in R^2$$

$$d(f(P), f(Q)) = d(P,Q)$$

As a consequence $f$ also preserves angles.
If $g$ is another isometry in the plane, the composite (product) of $f$ and $g$, $f \circ g$, is also an isometry, in effect:

$$d(f(g(P)), f(g(Q))) = d(g(P),g(Q)) = d(P,Q)$$

The first equality follows because $f$ is an isometry and the second because $g$ is another. The set $\mathcal{M}$ of all the isometries of the plane, with the given operation of composition of applications have then a group structure. The group $\mathcal{M}$ is called the **Symmetry Group** of the plane and its elements movements.

Fixed an orientation in the plane, there are two types of plane isometries: **directs**, which preserve the orientation, and **indirects**, which reverse the orientation.

On the other hand we have the following classification of the plane isometries (see [1] and [5]):

Identity, all points of the plane are fixed points.

**Reflection**, indirect isometry with a pointwise invariant line, the axis of reflection.

**Rotation**, direct isometry with only a fixed points, the centre of rotation.

**Translation**, direct isometry without fixed points.

**Glide reflection**, indirect isometry without fixed points.

Let $\mathcal{T}$ be the set of translations of the plane, $\mathcal{T}$ is an abelian group, subgroup of the group of isometries $\mathcal{M}$.

It is worth recalling the Cartan-Diendonné theorem [1] that says:

"Every isometry is a product of at most three reflections".

As a Corollary the reflections are a set of generators of the group $\mathcal{M}$.

### 2.1 Conjugation map

For each isometry of the plane, $\alpha \in \mathcal{M}$, we can define the mapping, $\Phi: \mathcal{M} \longrightarrow \mathcal{M}$,

$\Phi(\rho) = \rho^{\alpha} = \alpha \circ \rho \circ \alpha^{-1}$, called the conjugation map by $\alpha$.

The isometry $\rho^{\alpha}$ is an isometry of the same type (direct or indirect) that $\rho$. Indeed:

1) If $\rho = t_{\boldsymbol{u}}$, is a translation with vector $\boldsymbol{u}$, $\rho^{\alpha} = t_{\alpha(\boldsymbol{u})}$, is a translation with vector $\alpha(\boldsymbol{u})$.

For each point in the plane P, $\rho(P) = t_{\boldsymbol{u}}(P) = P + \boldsymbol{u}$, while

$\rho^{\alpha}(P) = \alpha \circ \rho \circ \alpha^{-1}(P) = \alpha(\alpha^{-1}(P) + \boldsymbol{u}) = P + \alpha(\boldsymbol{u})$,

i.e. $\rho^{\alpha} = t_{\alpha(\boldsymbol{u})}$, is the translation with vector $\alpha$ ***(u).***

2) If $\rho = g_{C,\vartheta}$ is a rotation of centre O and angle $\vartheta$, $\rho^{\alpha}$ is a rotation of centre $\alpha$ (O) and angle $\vartheta$:

$\rho$ is a direct isometry with a fixed point O. Therefore $\rho^{\alpha}$ is a direct isometry, we see that has a fixed point, $\alpha$ (O):

$$\rho^{\alpha}(\alpha(O)) = \alpha \circ \rho \circ \alpha^{-1}(\alpha(O)) = \alpha(\rho(O)) = \alpha(O).$$

i.e., $\rho^{\alpha}$, is a rotation of centre $\alpha$ (O) and angle $\beta$, equal to the angle formed by the vectors $\overrightarrow{\alpha(O)R}$ and $\overrightarrow{\alpha(O)\rho^{\alpha}(R)}$, for any point, R of the plane.

Since a movement preserves angles, $\beta$ should coincides with the angle between vectors $\overrightarrow{O\alpha^{-1}(R)}$ and $\overrightarrow{O\rho(\alpha^{-1}(R))}$, which is precisely the rotation angle $\vartheta$ of $\rho$.

3) If $\rho$ is a reflection in line $r$, $\rho^{\alpha}$ is reflection in line $\alpha(r)$.

The isometry $\rho^{\alpha}$, is an indirect isometry, as $\rho$. All points on the line $\alpha(r)$ are fixed points of $\rho^{\alpha}$, let P be any point on $r$, $\rho(P) = P$.

$$\rho^{\alpha}(\alpha(P)) = \alpha(\rho(P)) = \alpha(P).$$

$\rho^{\alpha}$ is a inverse isometry with one pointwise invariant line, $\alpha(r)$. Therefore $\rho^{\alpha}$ is a reflection in line $\alpha(r)$.

4) If $\rho = s_r^a = t_a \circ s_r$ is a glide reflection with axis *r* and vector ***a***, the vector ***a*** in the direction of the line *r*, $\rho^\alpha$ is a glide reflection with axis $\alpha(r)$ and vector $\alpha$ (***a***) in the direction of $\alpha\ (r)$. In fact:

$\rho^\alpha = \alpha \circ ( t_{\boldsymbol{a}} \circ s_{\boldsymbol{r}}) \circ \alpha^{-1} = (\alpha \circ t_{\boldsymbol{a}} \circ \alpha^{-1}) \circ ( \alpha \circ s_{\boldsymbol{r}} \circ \alpha^{-1}) = t_{\boldsymbol{a}}^{\alpha} \circ s_{\boldsymbol{r}}^{\alpha} = t_{\alpha(\boldsymbol{a})} \circ s_{\alpha(\boldsymbol{r})}$.

**Symmetry of a figure**

Definition.- A **figure** F in $\mathbf{R}^2$ is a nonempty set of points in the plane. A **symmetry** of the figure F, is an isometry of the plan that carry F onto itself. The set of all symmetries of the figure F is a group under the composition application, the symmetry group of the F.

The symmetry group of a figure F is a subgroup of the group of isometries of the plane.

A **motif** is a figure F which only has identity in its symmetry group.

**2.2 The seventeen plane symmetry groups**

A **crystallographic group** is a subgroup $G$ of the group $\mathcal{M}$ of isometries of the plane, such that their intersection with the group of translations $\mathcal{T}$ is:

$$\mathcal{T}_2 = G \cap \mathcal{T} = \{ t_{n\boldsymbol{a}} \circ t_{m\boldsymbol{b}} = t_{n\boldsymbol{a}+m\boldsymbol{b}} \ / < a, b > = \mathrm{R}^2, \ n, m \in \mathbf{Z} \}$$

$\mathcal{T}_2$ is a free abelian group of rank two, i.e. *G* contains translations in two independent directions.

A mathematical analysis of these groups shows that there are exactly seventeen different types of plane symmetry groups. ([1], [5], [6])

**Lattices**

We can always choose two generators of $\mathcal{T}_2$, $t_{\boldsymbol{a}}$ and $t_{\boldsymbol{b}}$ such that the set $\{t_{\boldsymbol{a}}, t_{\boldsymbol{b}}\}$ is a reduced set of generators, i.e., let $t_{\boldsymbol{a}}$ be a shortest non-identity translation ($\boldsymbol{a} \neq \mathbf{0}$ and the norm or length of vector ***a*** is minimal among all the translations of $\mathcal{T}_2$) and let $t_{\boldsymbol{b}}$ be a translation, such that the vector ***b*** has a minimum norm among the translations of $\mathcal{T}_2$, with vector not collinear with the vector ***a***, hence $\| \boldsymbol{a} \| \leq \| \boldsymbol{b} \|$. The vectors *a* and *b* generate the plane, $< a, b > = \mathbf{R}^2$.

When choosing a point in the plane O, the group $\mathcal{T}_2$ determines a lattice $\mathcal{L}$, formed by the set of points,

$$\mathcal{C} = \{t(\mathrm{O}) \mathbin{/} t = t_{n\boldsymbol{a}+m\boldsymbol{b}} \in \mathcal{T}_2\},$$

Or in other words the orbit of O under the action of $\mathcal{T}_2$. If the set of generators of $\mathcal{T}_2$ is reduced we obtain the **fundamental parallelogram** or unit cell: O, O + $t_{\boldsymbol{a}}$ (O), O + $t_{\boldsymbol{b}}$ (O) and O + $t_{\boldsymbol{a}+\boldsymbol{b}}$ (O) of the lattice.

**Remark**: The set of vectors $\{\overrightarrow{\mathrm{O}t(\mathrm{O})} \mathbin{/} t = t_{n\boldsymbol{a}+m\boldsymbol{b}} \in \mathcal{T}_2\}$, as a subset of $\mathrm{R}^2$ is independent of the choice of the point O.

### Rotations in Crystallographic groups. Crystallographic restriction

The order of a rotation σ is the least natural number n such that $\sigma^n = I$. The Crystallographic Restriction show that, if $G$ is a crystallographic group and $\sigma \in G$ is a rotation, the order of the rotation must be n = 1, 2, 3, 4 or 6, see [1].

We call **crystallographic group of type n**, $G_n$, a crystallographic group which contain rotations of order at most n, so there are groups of type 1, 2, 3, 4 and 6.

If a translation $t$ and a rotation $\sigma = g_{\mathrm{O},\,\alpha}$ are in $G_n$ , the conjugate of σ by $t$, is another rotation in $G_n$, $\sigma^t = g_{t(\mathrm{O}),\,\alpha} \in G_n$,. The group $G_n$ contains all the rotations with center in every vertex of the lattice and order n.

If $t_{\boldsymbol{a}}$ and $\sigma \in G_n$, the conjugate of $t_{\boldsymbol{a}}$ by σ, $t^{\sigma}_{\boldsymbol{a}} = t_{\sigma(\boldsymbol{a})} \in G_n$, i.e. the translation of vector $\sigma(\boldsymbol{a})$ belongs to the crystallographic group $G_n$, which for some values of n determines the shape of the lattice. Thus we have:

The lattice in the groups of type 4 is a square lattice. In fact, if σ is a rotation with center O and order 4, and we take the point O as the basis for the lattice and as the reduced set of generators $\{t_{\boldsymbol{a}}, t_{\boldsymbol{b}}\}$, with $\boldsymbol{b} = \sigma(\boldsymbol{a})$, the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal and with equal norm: $\|\boldsymbol{a}\| = \|\boldsymbol{b}\|$, and both vectors determine the square lattice.

The parallelogram fundamental in a lattice for the groups of type 3 would have the shape of two juxtaposed equilateral triangles, i.e. a diagonal of the fundamental parallelogram divides it into two equilateral triangles.

If σ is a rotation of centre O and order 3 and $t_{\boldsymbol{a}}$ the translation with vector $\boldsymbol{a}$ are in the group, the translation of vector $\sigma(\boldsymbol{a})$ is also in the group $G_3$. The composite translation, translation of vector $\boldsymbol{b} = \boldsymbol{a} + \sigma(\boldsymbol{a})$, is obviously also in the group.

We take the point O as the basis for the lattice and the reduced set of generators $\{t_{\boldsymbol{a}}, t_{\boldsymbol{b}}\}$, with $\boldsymbol{b} = \boldsymbol{a} + \sigma(\boldsymbol{a})$. The vectors $\boldsymbol{a}$ and $\sigma(\boldsymbol{a})$ have the same norm and the angle determining for them is $2\pi/3$ and so they determine a rhombus, with the vector $\boldsymbol{b} = \boldsymbol{a} + \sigma(\boldsymbol{a})$ a diagonal of the rhombus, that divides it into two equilateral triangles. Thus we call this lattice a triangular lattice.

The groups of type 6 have triangular lattice, in fact, if $\sigma$ is a rotation of centre O and order 6, we take the point O as the basis for the lattice and reduced set of generators $\{t_{\boldsymbol{a}}, t_{\boldsymbol{b}}\}$, with $\boldsymbol{b} = \sigma\ (\boldsymbol{a})$**.** The vectors $\boldsymbol{a}$ and $\boldsymbol{b} = \sigma\ (\boldsymbol{a})$ have the same norm and the angle determining for them is $\pi/3$, and their fundamental parallelogram define a rhombus, one of whose diagonals divides it into two equilateral triangles, such a triangle is called **fundamental region**.

Note that if the group contains a rotation $\sigma$ of order 6, also contains its square $\sigma^2$, which is a rotation of order three, and as direct application of the above, the lattice is triangular.

Each vertex of the lattice of a group $\mathcal{G}_n$ is a center of rotation of order n. This is because the conjugate of a rotation $\sigma$ with centre O and order n, by the translation $t_{n\boldsymbol{a}+m\boldsymbol{b}} \in \mathcal{T}_2,$ is a rotation of centre $t_{n\,\boldsymbol{a}+m\,\boldsymbol{b}}\,(O) = O + n\,\boldsymbol{a} + m\,\boldsymbol{b}$ and order n.

As said before there are exactly seventeen different crystallographic groups in the plane. Chosen a basic motif F in the plane it is possible to generate a so called wallpaper group: if F is a motif and $G$ is one of the seventeen symmetry group, the union of all images $\alpha$(F) with $\alpha$ in $G$ is the wallpaper.

**Classification algorithm of the crystallographic groups**

Given a candidate we must, in the first place, to check that it is really a crystallographic group, identifying the translations in two independent directions. Then, we identify the rotation of maximum order n, contained in the group, i.e. the smallest degree rotations and, as we saw above, these orders should be n = 1, 2, 3, 4 or 6. This gives us a first information about the type of the group in question, depending on the value of n. Then we can use the algorithm given by the tables below:

<table>
<tr><td colspan="4">The maximum order of the rotations is 1</td></tr>
<tr><td colspan="4">Is there an axis reflection?</td></tr>
<tr><td colspan="2">YES</td><td colspan="2">NO</td></tr>
<tr><td colspan="2">Is there a glide reflection in an axis which is not a reflection axis?</td><td colspan="2">Is there a glide reflection?</td></tr>
<tr><td>YES</td><td>NO</td><td>YES</td><td>NO</td></tr>
<tr><td>cm</td><td>pm</td><td>pg</td><td>p1</td></tr>
</table>

<table>
<tr><td colspan="3">The maximum order of the rotations is 3</td></tr>
<tr><td colspan="3">Is there an axis reflection?</td></tr>
<tr><td colspan="2">YES</td><td>NO</td></tr>
<tr><td colspan="2">Are all rotation centres on reflection axes?</td><td rowspan="3">p3</td></tr>
<tr><td>YES</td><td>NO</td></tr>
<tr><td>p3m1</td><td>p31m</td></tr>
</table>

<table>
<tr><td colspan="2">The maximum order of the rotations is 6</td></tr>
<tr><td colspan="2">Is there an axis reflection?</td></tr>
<tr><td>YES</td><td>NO</td></tr>
<tr><td>p6m</td><td>p6</td></tr>
</table>

<table>
<tr><td colspan="5">The maximum order of the rotations is 2</td></tr>
<tr><td colspan="5">Is there an axis reflection?</td></tr>
<tr><td colspan="3">YES</td><td colspan="2">NO</td></tr>
<tr><td colspan="3">Are there two reflections in perpendicular axis?</td><td colspan="2">Is there a glide reflection?</td></tr>
<tr><td colspan="2">YES</td><td rowspan="2">NO</td><td rowspan="2">YES</td><td rowspan="2">NO</td></tr>
<tr><td colspan="2">Are all rotation centres on reflection axes?</td></tr>
<tr><td>YES</td><td>NO</td><td rowspan="2">pmg</td><td rowspan="2">pgg</td><td rowspan="2">p2</td></tr>
<tr><td>pmm</td><td>cmm</td></tr>
</table>

<table>
<tr><td colspan="3">The maximum order of the rotations is 4</td></tr>
<tr><td colspan="3">Is there an axis reflection?</td></tr>
<tr><td colspan="2">YES</td><td>NO</td></tr>
<tr><td colspan="2">Are all rotation centres on reflection axes?</td><td rowspan="3">p4</td></tr>
<tr><td>YES</td><td>NO</td></tr>
<tr><td>p4m</td><td>p4g</td></tr>
</table>

## 3. **THE SEVENTEEN PLANE SYMMETRY GROUPS IN THE ALHAMBRA (GRANADA, SPAIN)**

We present pictures of ornaments locate in the Alhambra showing each of the seventeen crystallographic groups. We classified these groups in five types, according to their order of rotation. We will use the following symbols in the representations of fundamental parallelograms:

| The maximum order of the rotations | Symbol |
|---|---|
| 2 | ◇ |
| 3 | △ |
| 4 | □ |
| 6 | ⬡ |
| axis of reflection | |
| axis of glide reflection | |

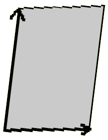
### 3.1. Crystallographic group of TYPE 1

Let $G_1$ be the crystallographic groups that do not contain proper rotations. There are four non-isomorphic such groups:

1. **Group *p*1**: containing only translations,

$$p1 = < t_{\boldsymbol{a}} , t_{\boldsymbol{b}} >$$

The lattice is generated by a parallelogram of sides $\| \boldsymbol{a} \|$ y $\| \boldsymbol{b} \|$.



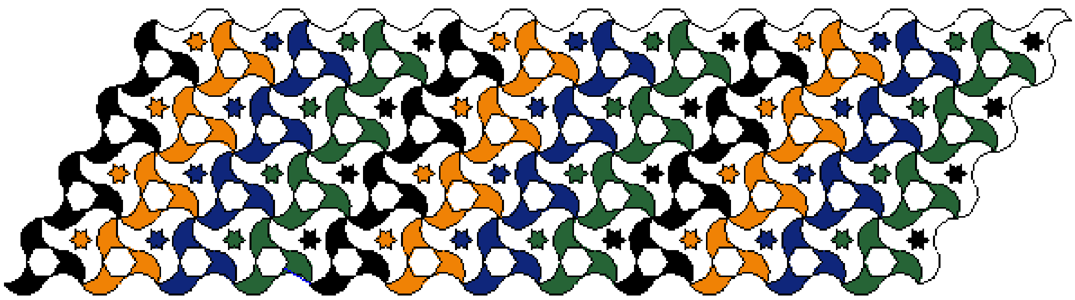
Fundamental parallelogram

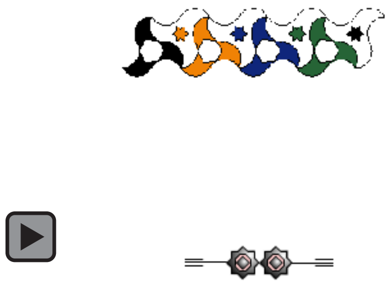
In the *Patio de los Arrayanes* we find this group which containing translations in two directions, but no rotations or reflections or glide reflections.

Palacio de Comares. Patio. Patio de los Arrayanes.

| The maximum order of the rotations is 1 | | | |
|---|---|---|---|
| Is there an axis reflection? | | | |
| YES | | NO | |
| Is there a glide reflection in an axis which is not a reflection axis? | | Is there a glide reflection? | |
| YES | NO | YES | NO |
| cm | pm | pg | p1 |

Fundamental parallelogram





Group $p1$

2) Groups that only contain translations and reflections. The reflections can only be in parallel directions, because the group has not proper rotations.

Let $s_{r,}$ be a reflection in line $r$. Conjugate reflections of $s_r$ by the translations $t_{\boldsymbol{u}}$ of the group, must belong to the group. They are reflections in axes parallel to axis $r$, and at distances that are multiples of the $\| \boldsymbol{u} \|$.

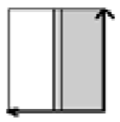
If $s_r$ is a reflection of the group, we have two possibilities:

2. a) **Group *pm***: The vector $\boldsymbol{a}$ is in the direction of the line $r$, $s_r(\boldsymbol{a}) = \boldsymbol{a}$.

The conjugates of $s_{r,}$ by $t_{\mathrm{m}\boldsymbol{b}}$ are reflections in lines $t_{\mathrm{m}\boldsymbol{b}}(r)$. The axes of these reflections are lines parallel to line $r$, to distance m$\| \boldsymbol{b} \|$ of $r$, then we can take the vector $\boldsymbol{b}$ as being perpendicular to vector $\boldsymbol{a}$ and so $s_r(\boldsymbol{b}) = -\boldsymbol{b}$. Therefore the group contains reflections with axes in the direction of one vector of translation and perpendicular to the other vector of translation, so:

$$pm = < t_{\boldsymbol{a}}, t_{\boldsymbol{b}}, s_r \,/\, s_r(\boldsymbol{a}) = \boldsymbol{a}, \text{ and } s_r(\boldsymbol{b}) = -\boldsymbol{b} >$$

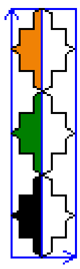
A fundamental parallelogram for the lattice of the translations of this group is then a rectangle. And one can be chosen that is split by an axis of reflection so that one of the half rectangles forms a fundamental region for the symmetry group.



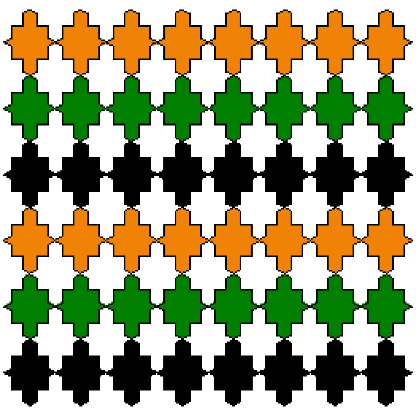
Fundamental parallelogram

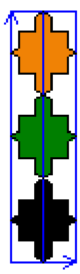
Pieza del Museo de la Alhambra R. 1375

| **The maximum order of the rotations is 1** | | | |
|---|---|---|---|
| Is there an axis reflection? | | | |
| YES | | NO | |
| Is there a glide reflection in an axis which is not a reflection axis? | | Is there a glide reflection? | |
| YES | NO | YES | NO |
| cm | pm | pg | p1 |

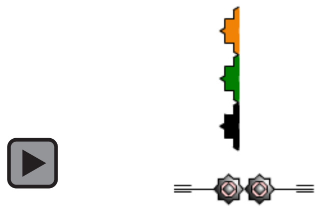


Basic motif





Fundamental parallelogram

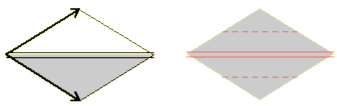


Group *p*m

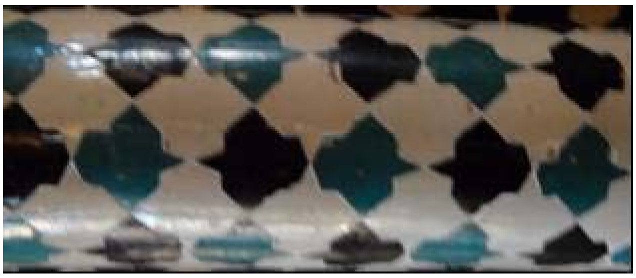
2. b) ***Group cm***: The vector $\boldsymbol{a}$ is not in the direction of the line $r$. In this case we can take the vector $\boldsymbol{b} = s_r(\boldsymbol{a})$, $\| \boldsymbol{a} \| = \| \boldsymbol{b} \|$.

$$cm = < t_{\boldsymbol{a}} , t_{\boldsymbol{b}}, s_r \,/\, s_r(\boldsymbol{a}) = \boldsymbol{b}, \text{ y } s_r(\boldsymbol{b}) = \boldsymbol{a} >$$

The fundamental parallelogram for the translation group is then a rhombus, and a fundamental region for the symmetry group is half the rhombus. The group contains translations and reflections that are not in the direction of the vectors generating the lattice:

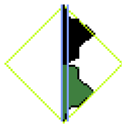


Fundamental paralellogram

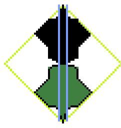


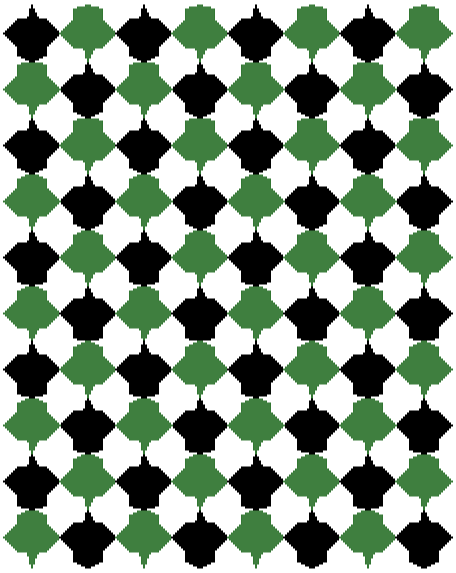
Salón de Comares

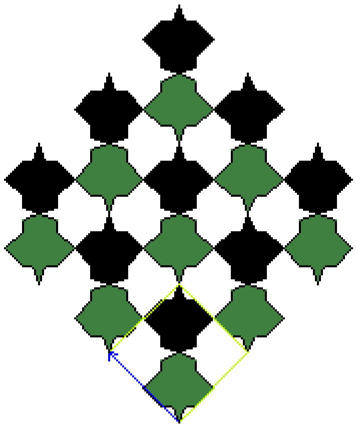
| The maximum order of the rotations is 1 | | | |
|---|---|---|---|
| Is there an axis reflection? | | | |
| YES | | NO | |
| Is there a glide reflection in an axis which is not a reflection axis? | | Is there a glide reflection? | |
| YES | NO | YES | NO |
| cm | pm | pg | p1 |



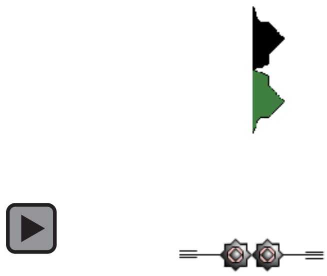
Basic motif







Fundamental parallelogram



Group *cm*

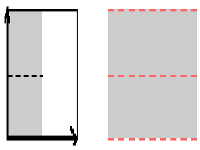
3) **Group *pg***: The group contains translations and glide reflections $s_r^{\boldsymbol{u}}$, but it does not contain rotations or reflections.

The axes of the glide reflections must be parallels, because the group contains no rotation. On the other hand, $s_r^{\boldsymbol{u}} \circ s_r^{\boldsymbol{u}} = t_{2\boldsymbol{u}}$, and so we can take one the generating translations as $t_{\boldsymbol{a}}$, with $\boldsymbol{a} = 2\boldsymbol{u}$.[1]

The axes of glide reflections are in the direction of one vector of translation.

$$pg = < t_{\boldsymbol{a}} , t_{\boldsymbol{b}}, s_r / s_r^{\boldsymbol{a}/2} >.$$

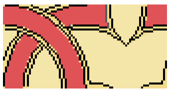
A fundamental parallelogram for the lattice of translations is a rectangle. And we can choose it such that is split by an axis of a glide reflection so that one of the half rectangles forms a fundamental region for the symmetry group.



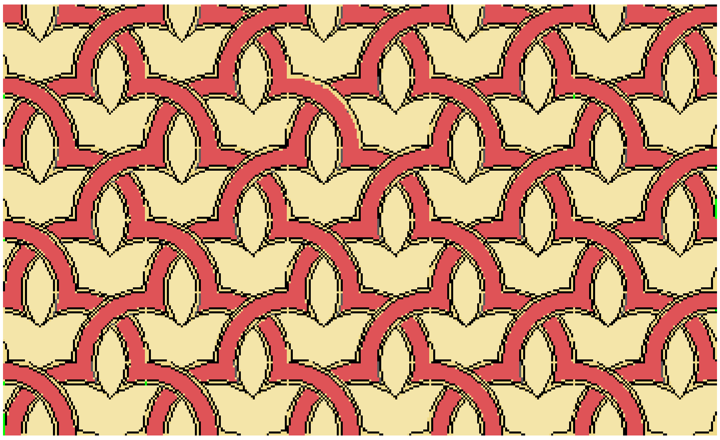
Fundamental parallelogram

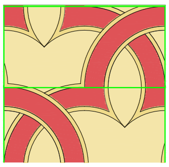
Puerta del Vino

| **The maximum order of the rotations is 1** | | | |
|---|---|---|---|
| Is there an axis reflection? | | | |
| YES | | NO | |
| Is there a glide reflection in an axis which is not a reflection axis? | | Is there a glide reflection? | |
| YES | NO | YES | NO |
| cm | pm | pg | p1 |



Basic Motif





Fundamental parallelogram

Group *pg*

### 3.2 Crystallographic group TYPE 2

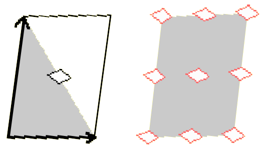
Let $G_2$ be the crystallographic groups that contain rotations of order 2.

If $g_{O,\pi}$ is a rotation of order 2 and $t_{\boldsymbol{u}}$ a translation in the group, the composite application, $t_{\boldsymbol{u}} \circ g_{O,\pi} = g_{O',\pi}$, is a rotation of order 2 with centre the point O' = O + $\boldsymbol{u}$ /2. As a result, the fundamental parallelogram is a parallelogram whose vertices, midpoints of the sides and midpoint of the parallelogram are centres of rotations of the group. There are five non-isomorphic groups of type 2:

1. **Group *p*2**: It contains translations and rotations of order 2, but  not indirect isometries.

$$p2 = < t_{\boldsymbol{a}} , t_{\boldsymbol{b}} , g_{O,\pi} >$$

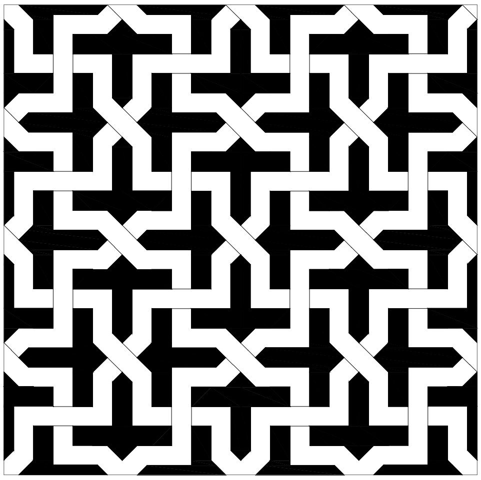
The fundamental region for the symmetry group is a half of a fundamental paralellogram for the translation group.

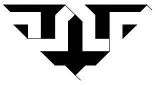


Fundamental paralellogram

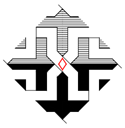
Museo de la Alhambra

<table>
<tr><td colspan="5">The maximum order of the rotations is 2</td></tr>
<tr><td colspan="5">Is there an axis reflection?</td></tr>
<tr><td colspan="3">YES</td><td colspan="2">NO</td></tr>
<tr><td colspan="3">Are there two reflections in perpendicular axis?</td><td colspan="2">Is there a glide reflection?</td></tr>
<tr><td colspan="2">YES<br>Are all rotation centres on reflection axes?</td><td>NO</td><td>YES</td><td>NO</td></tr>
<tr><td>YES</td><td>NO</td><td rowspan="2">pmg</td><td rowspan="2">pgg</td><td rowspan="2">p2</td></tr>
<tr><td>pmm</td><td>cmm</td></tr>
</table>





Basic motif.



Fundamental parallelogram

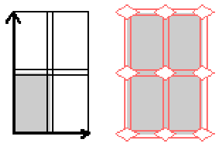
Group *p2*

2. The group contains at least one reflection $s_r$. There are two possibilities: that either the axis of reflection passes through a centre of rotation or not.

2. a) If O $\in$ $r$, the composite application of the rotation $g_{O,\pi}$ and the reflection $s_r$, is a reflection, $g_{O,\pi} \circ s_r = s_m$, in line $m$, where $m$ is the line through O and perpendicular to $r$. Two other possibilities can occur:

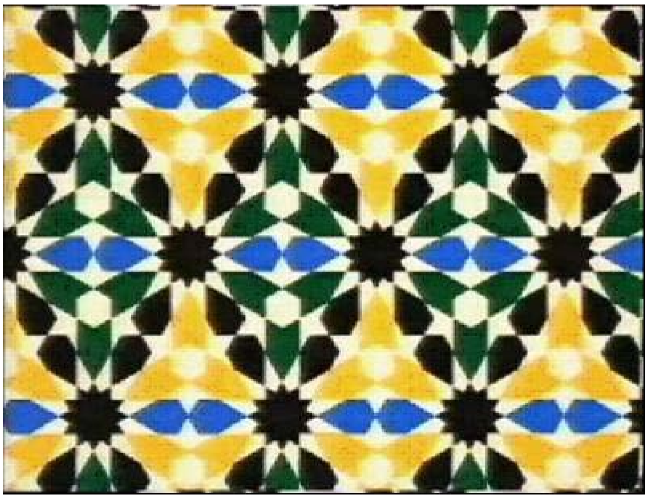
2.a.1) **Group *pmm***: The axis of reflection is in the direction of the vector $\boldsymbol{a}$, i.e. $\boldsymbol{a} \in r$, and $s_r(\boldsymbol{a}) = \boldsymbol{a}$. The translation of vector $s_r(\boldsymbol{b})$ belongs to the group (conjugate of $t_b$ by $s_r$), and also the translation with vector $\boldsymbol{c} = \boldsymbol{b} - s_r(\boldsymbol{b})$. The vector $\boldsymbol{c}$ is perpendicular to the vector $\boldsymbol{a}$, which allows us to take as a fundamental parallelogram a rectangle, whose sides are axes of reflection, so that a fundamental region for the symmetry group can be chosen as a quarter of the fundamental rectangle. [1]

$$pmm = < t_a , t_b , g_{O,\pi}, s_r(\boldsymbol{a}) = \boldsymbol{a}, \text{ y } s_r(\boldsymbol{b}) = -\boldsymbol{b} >$$



Fundamental parallelogram

The lines parallel to the translation vector through of the rotation centres are axes of reflection.



Palacio de los Leones. Sala de los Reyes

<table>
<tr><td colspan="5">The maximum order of the rotations is 2</td></tr>
<tr><td colspan="5">Is there an axis reflection?</td></tr>
<tr><td colspan="3">YES</td><td colspan="2">NO</td></tr>
<tr><td colspan="3">Are there two reflections in perpendicular axis?</td><td colspan="2">Is there a glide reflection?</td></tr>
<tr><td colspan="2">YES<br>Are all rotation centres on reflection axes?</td><td>NO</td><td>YES</td><td>NO</td></tr>
<tr><td>YES</td><td>NO</td><td rowspan="2">pmg</td><td rowspan="2">pgg</td><td rowspan="2">p2</td></tr>
<tr><td>pmm</td><td>cmm</td></tr>
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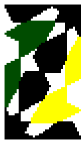
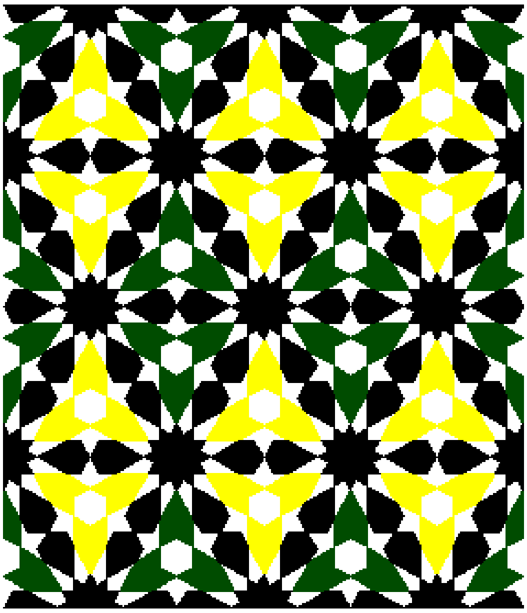
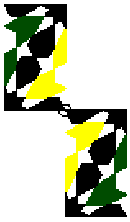
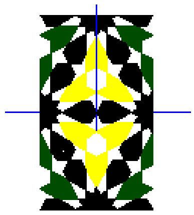


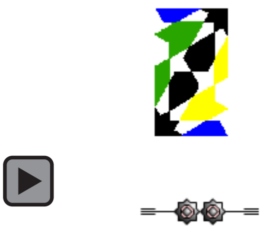
Basic motif







Fundamental parallelogram

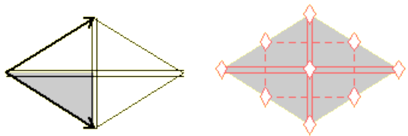


Group *pmm*

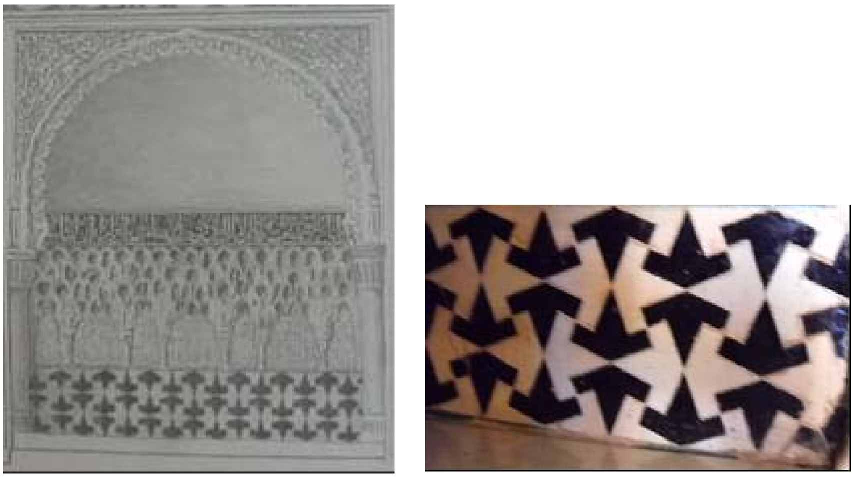
2.a.2) **Group *cmm***: The axis of reflection is not in the direction of the vector ***a***. If we take as vector ***b*** = $s_r$ (***a***), ||***a***|| = ||***b***||, the fundamental parallelogram is a rhombus with sides ***a*** and ***b***.

The diagonals of the rhombus are perpendicular axes of reflection, which pass through centres of rotation. Not all the centres of the rotations are on the reflection axes. Therefore a quarter of the fundamental parallelogram is a fundamental region for the lattice of this symmetry group.

$$cmm = < t_{\boldsymbol{a}},\ t_{\boldsymbol{b}},\ g_{\,\mathrm{O},\,\pi},\ s_r(\boldsymbol{a}) = \boldsymbol{b},\ \mathrm{y}\ s_r(\boldsymbol{b}) = \boldsymbol{a} >$$



Fundamental parallelogram



Palacio de Comares. Taca a la entrada del Salón de Comares

<table>
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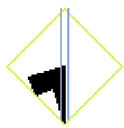
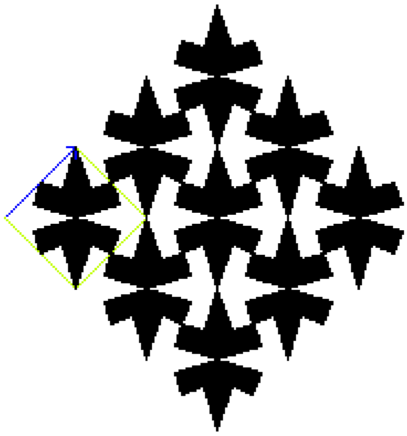
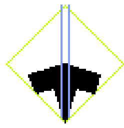
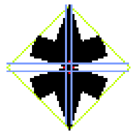
<tr><td colspan="5">Is there an axis reflection?</td></tr>
<tr><td colspan="3">YES</td><td colspan="2">NO</td></tr>
<tr><td colspan="3">Are there two reflections in perpendicular axis?</td><td colspan="2">Is there a glide reflection?</td></tr>
<tr><td colspan="2">YES<br>Are all rotation centres on reflection axes?</td><td>NO</td><td>YES</td><td>NO</td></tr>
<tr><td>YES</td><td>NO</td><td rowspan="2">pmg</td><td rowspan="2">pgg</td><td rowspan="2">p2</td></tr>
<tr><td>pmm</td><td>cmm</td></tr>
</table>



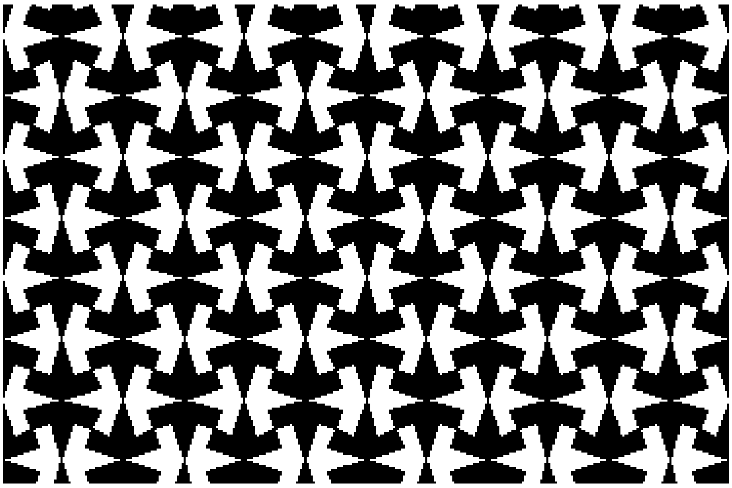
Basic motif







Fundamental parallelogram



Group *cmm*

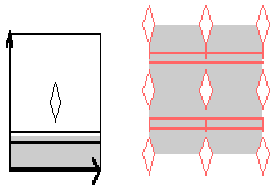
2. b) **Group *pmg***: If $O \notin r$, the centres of rotations are not in the axes of reflection.

If the axis of reflection is in the direction of ***a***, $\boldsymbol{a} \in r$, we can take the vector ***b*** orthogonal to ***a***, as noted above.

The composite application $g_{O,\pi} \circ s_r = s_m{}^{\boldsymbol{u}}$, of the rotation $g_{O,\pi}$ with the reflection $s_r$, is a glide reflection of axis $m$ and vector $\boldsymbol{u} = \boldsymbol{b}/4$, where $m$ is a line through O and perpendicular to $r$.[1]

The group contains reflections of axes that don't pass through the centres of rotation, and contains glide reflections. The fundamental parallelogram is a rectangle and a quarter of this rectangle is a fundamental region for the lattice of the symmetry group.

$$pmg = < t_{\boldsymbol{a}}, t_{\boldsymbol{b}}, g_{O,\pi}, s_r(\boldsymbol{a}) = \boldsymbol{a} >$$

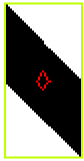


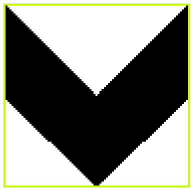
Fundamental parallelogram

Fuente del patio del Cuarto Dorado

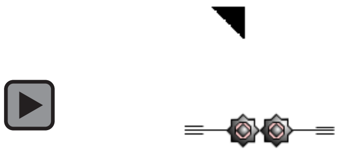
<table>
<tr><td colspan="5">The maximum order of the rotations is 2</td></tr>
<tr><td colspan="5">Is there an axis reflection?</td></tr>
<tr><td colspan="3">YES</td><td colspan="2">NO</td></tr>
<tr><td colspan="3">Are there two reflections in perpendicular axis?</td><td colspan="2">Is there a glide reflection?</td></tr>
<tr><td colspan="2">YES<br>Are all rotation centres on reflection axes?</td><td>NO</td><td>YES</td><td>NO</td></tr>
<tr><td>YES</td><td>NO</td><td rowspan="2">pmg</td><td rowspan="2">pgg</td><td rowspan="2">p2</td></tr>
<tr><td>pmm</td><td>cmm</td></tr>
</table>

Basic motif





Fundamental parallelogram



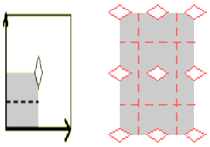
Group *pmg*

3) **Group *pgg***: It contains glide reflections but does not contain reflections.

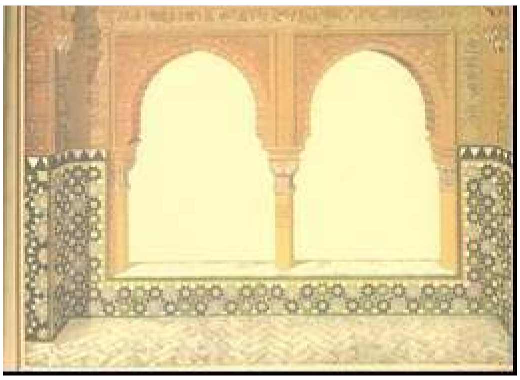
The group contains glide reflections with perpendicular axes that do not pass through the centres of rotation, the group is *pgg*:

$$pgg = < t_{\boldsymbol{a}} , t_{\boldsymbol{b}} , g_{\mathrm{O}, \pi}, s_r^{\boldsymbol{a}/2} >$$

and the fundamental parallelogram is a rectangle, and a quarter of this is a fundamental region for its lattice.



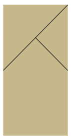
Fundamental paralellogram



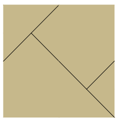
Palacio de los Leones. Mirador de Lindaraja

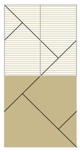
Solería de la Sala de los ajimeces

| The maximum order of the rotations is 2 | | | | |
|---|---|---|---|---|
| Is there an axis reflection? | | | | |
| YES | | | NO | |
| Are there two reflections in perpendicular axis? | | | Is there a glide reflection? | |
| YES Are all rotation centres on reflection axes? | | NO | YES | NO |
| YES | NO | pmg | pgg | p2 |
| pmm | cmm | | | |

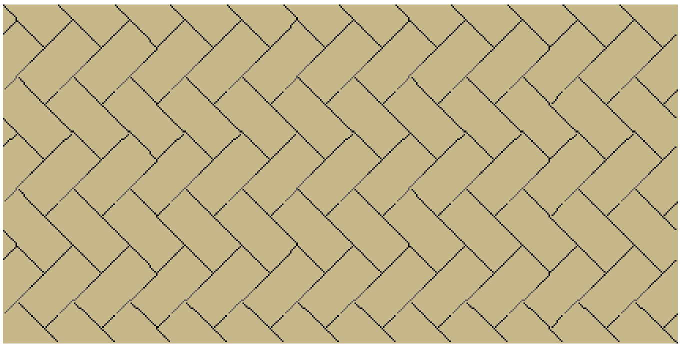


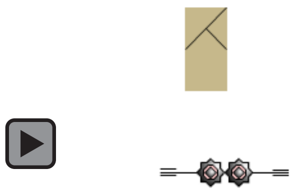
Basic motif





Fundamental parallelogram





Group *pgg*

### 3.3. Crystallographic group TYPE 4.

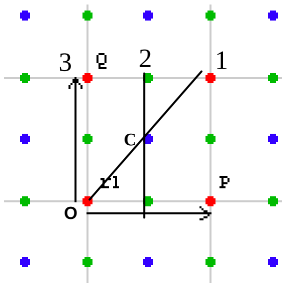
Let $G_4$ be the crystallographic groups that contain rotations of order 4.

As was seen in Section 2, the lattice of groups of type 4 is square. The sides of the square are $\boldsymbol{a}$ and $\boldsymbol{b}$, $\boldsymbol{b} = g_{O,\,\pi/2}(\boldsymbol{a})$, $||\boldsymbol{a}|| = ||\boldsymbol{b}||$.

The vertices of the square are centres of rotation of order 4.

If we compose the translation of vector $\boldsymbol{a}$, $t_{\boldsymbol{a}}$ with the rotation $g_{O,\,\pi/2}$, previously expressed as a composition of reflections, we have:

$$t_{\boldsymbol{a}} \circ g_{O,\,\pi/2} = (s_2 \circ s_3) \circ (s_3 \circ s_1) = s_2 \circ s_1 = g_{C,\pi/2}$$




Then the centre of the square is a centre of rotation of order 4.

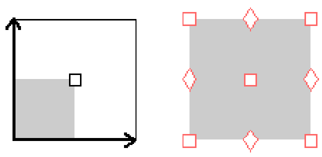
If we compose a rotation of order 4, about O and angle $\pi/2$, with itself, we get a new rotation with the centre O and angle $\pi$, of order 2. Therefore the vertices of the square and its centre are centres of rotation of order 2 and the midpoints of the sides of the square are too.

There are three non-isomorphic groups of order 4.

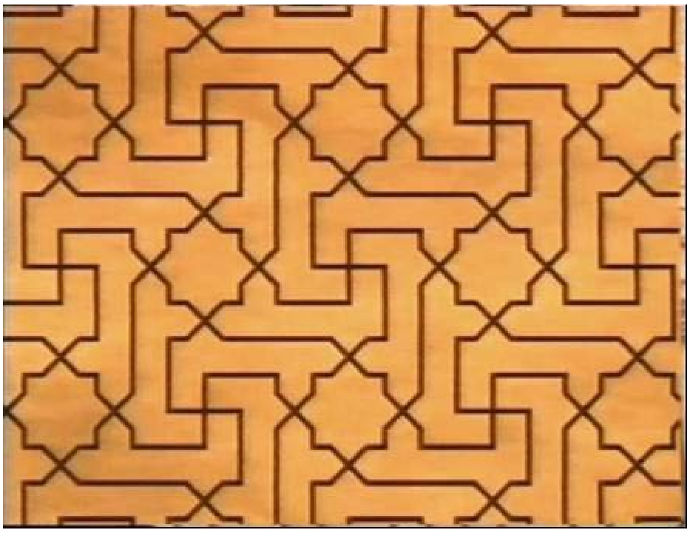
1) **Group *p*4:** It contains translations and rotations of order 4.

$$p4 = \langle t_{\boldsymbol{a}}, t_{\boldsymbol{b}}, g_{O,\pi/2} \rangle$$

The fundamental parallelogram is a square, and a quarter of it is a fundamental region for the symmetry group.

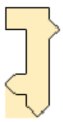


Fundamental parallelogram

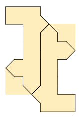


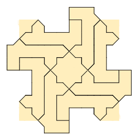
Patio de los Leones.  Galería junto a la Sala de los Mocárabes

| The maximum order of the rotations is 4 | | |
|---|---|---|
| Is there an axis reflection? | | |
| YES | | NO |
| Are all rotation centres on reflection axes? | | p4 |
| YES | NO | |
| p4m | pg | |

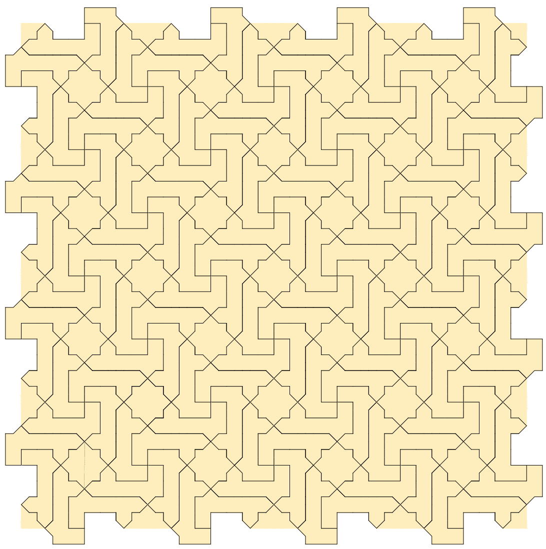


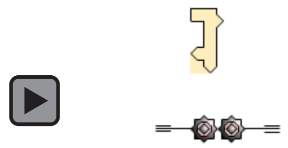
Basic motif





Fundamental parallelogram





Group *p4*

2) If the group contains an indirect isometry, there might be two cases

2.a) **Group *p4m***: It contains at least one reflection $s_r$**.**

If the axis of reflection *r* has the direction of vector ***a*** and passes through the centre of rotation O, the composite application of the rotation $\sigma = g_{O,\pi/2}$ with the reflection $s_r$ is a reflection of the axis *d*, *d* is a line through O and forms an angle of $\pi/4$ with the line *r*: $g_{O,\pi/2} \circ s_r = s_d$

The application $\sigma^2 = g_{O,\pi}$ is in the group and is a rotation about O and order 2, and the composite application

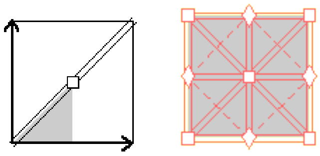
$$\sigma^2 \circ s_r = g_{O,\pi} \circ s_r = s_m$$

is a reflection in line *m*, where *m* is a line perpendicular to *r*, in the direction of vector ***b.***

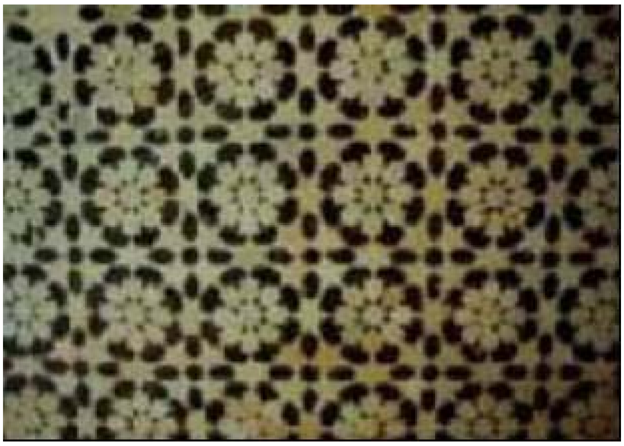
The sides, the diagonals and straight lines connecting the midpoints of the sides of the squares of the lattice are axes of reflection. The symmetry group is:

$$p4m = < t_{\boldsymbol{a}} , t_{\boldsymbol{b}} , g_{O,\pi/2} , s_r, s_r(\boldsymbol{a}) = \boldsymbol{a}, \text{ y } s_r(\boldsymbol{b}) = - \boldsymbol{b} >$$

The fundamental parallelogram is a square, and an eighth of it, a triangle, is a fundamental region of symmetry group.



Fundamental parallelogram

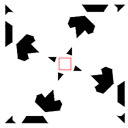


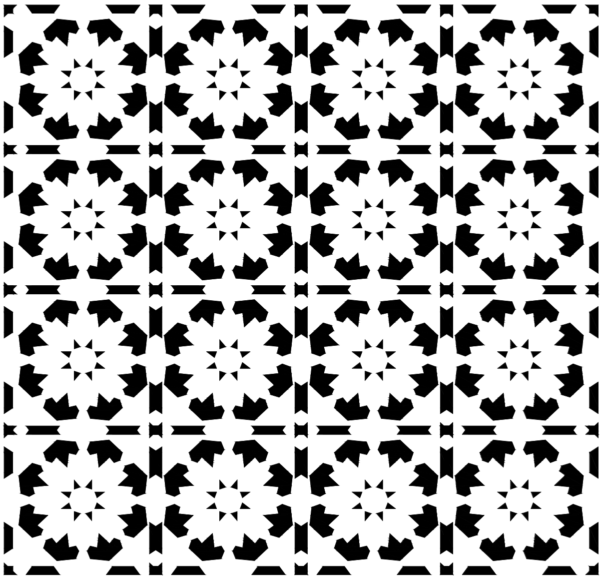
Torre de las Infantas

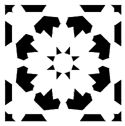
Salón de Comares

<table>
<tr><td colspan="3">The maximum order of the rotations is 4</td></tr>
<tr><td colspan="3">Is there an axis reflection?</td></tr>
<tr><td colspan="2">YES</td><td>NO</td></tr>
<tr><td colspan="2">Are all rotation centres on reflection axes?</td><td rowspan="3">p4</td></tr>
<tr><td>YES</td><td>NO</td></tr>
<tr><td>p4m</td><td>pg</td></tr>
</table>

Basic motif







Fundamental parallelogram

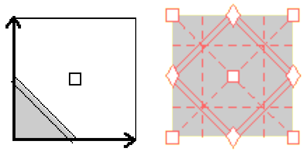
Group *p4m*

2.b) **Group *p4g***: It has no reflection with axis in the direction of the vector ***a***, but it contains a glide reflection $\rho$ of axis in the direction of the vector ***a***, then $\rho^2 = t_{\boldsymbol{a}}$. We can take $\rho = s_r^{\boldsymbol{a}/2}$, with $r$ line in the direction of vector ***a***. The group contains reflections; which axes of reflections do not pass through the centre of rotation of order 4.

$$p4g = < t_{\boldsymbol{a}} , t_{\boldsymbol{b}} , g_{\,\mathrm{O},\, \pi/2}, s_r^{\boldsymbol{a}/2} / \; >$$

The fundamental parallelogram is a square and an eighth of it, a triangle, is a fundamental region of the lattice.

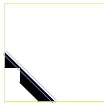
.

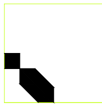


Fundamental parallelogram

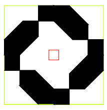
Palacio de Comares. Salón de Comares

| The maximum order of the rotations is 4 | | |
|---|---|---|
| Is there an axis reflection? | | |
| YES | | NO |
| Are all rotation centres on reflection axes? | | p4 |
| YES | NO | |
| p4m | pg | |

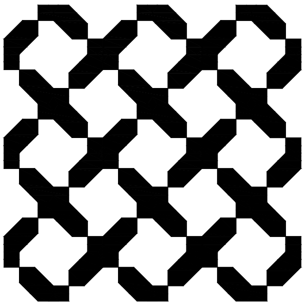


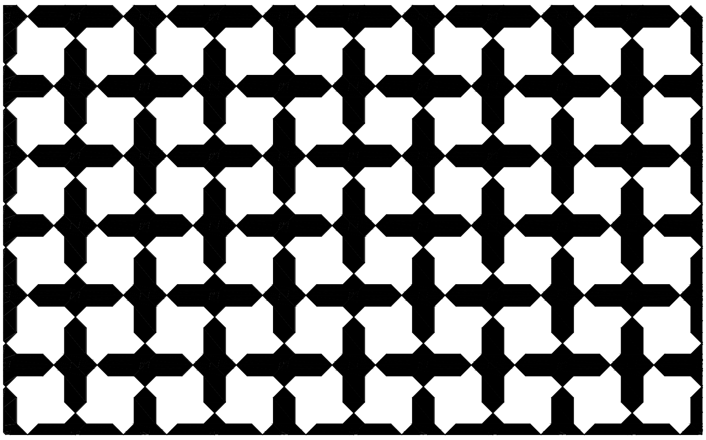


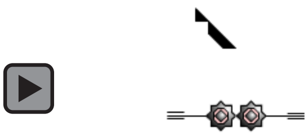
Basic motif



Fundamental parallelogram





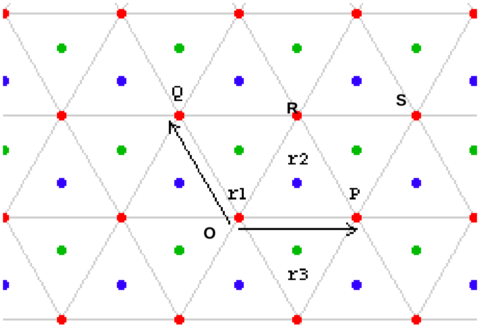


Group *p4g*

### 3.4. Crystallographic group TYPE 3.

Let $G_3$ be the crystallographic groups that contain rotations of order 3.

As seen in Section 2 the lattice of this groups of type 3 is triangular.




The rotation of centre O and $2\pi/3$, transforms P into Q. Let $\boldsymbol{a} = \overrightarrow{OP}$, and let $\boldsymbol{c} = \overrightarrow{OQ}$, we take $\boldsymbol{b} = \boldsymbol{a} + \boldsymbol{c} = \overrightarrow{OR}$. The fundamental parallelogram OPSR is formed by the union of equilateral triangles OPR and PSR.

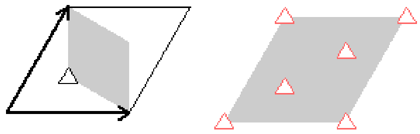
The vertices and the centres of the triangles are centres of rotation of $2\pi/3$ and $4\pi/3$.

The number of crystallographic groups of type 3 is three:

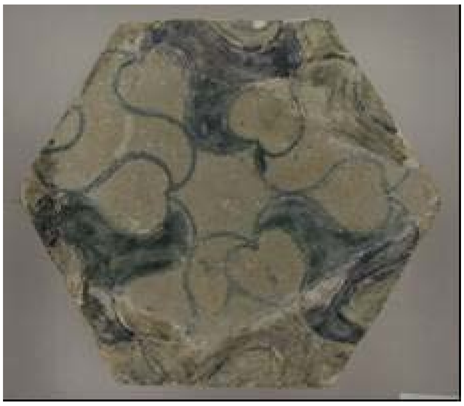
1) **Group *p*3:** It only contains translations and rotations of order 3.

$$p3 = \langle t_{\boldsymbol{a}} , t_{\boldsymbol{b}} , g_{O, 2\pi/3} \rangle$$

The fundamental parallelogram is

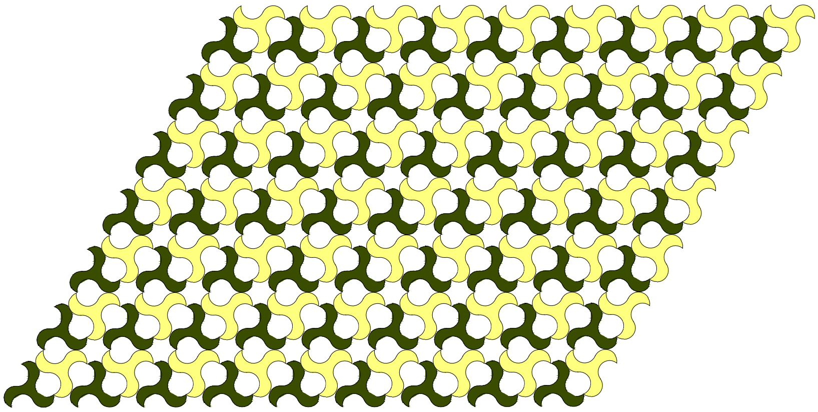


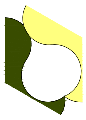
Fundamental paralellogram



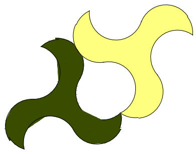
Museo de la Alhambra

| The maximum order of the rotations is 3 | | |
|---|---|---|
| Is there an axis reflection? | | |
| YES | | NO |
| Are all rotation centres on reflection axes? | | p3 |
| YES | NO | |
| p3m1 | p31m | |

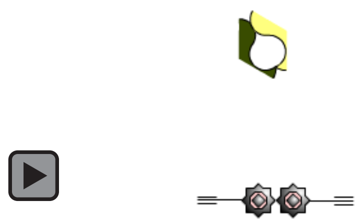




Basic motif



Fundamental region

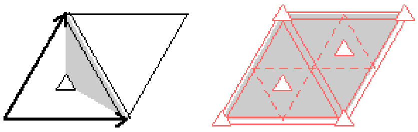


Group *p3*

2) If the group contains an indirect isometry, it must contain a reflection. There are two possibilities:

2-a) **Group *p*31*m***: The direction of the axis of reflection coincides with the direction of the vector ***a***, which implies that the group also contains reflections on the directions of the vectors ***b*** and ***b*** – ***a***. These axes cut two to two in points that are centres of rotation. Thus the group contains axes of reflections corresponding to the sides of the equilateral triangles that form the fundamental region.

$$p31m = < t_{\boldsymbol{a}} , t_{\boldsymbol{b}} , g_{\,O, 2\pi/3}, s_r \,/\, s_r(\boldsymbol{a}) = \boldsymbol{a} >$$



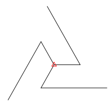
Fundamental parallelogram

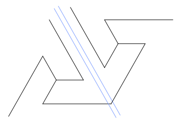
We note that not all centres of rotations are in the axes of reflection.

Puerta del Vino

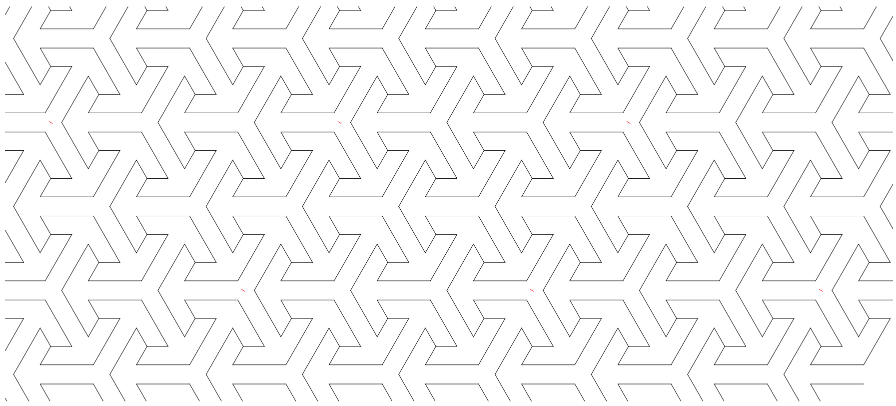
| | | |
|---|---|---|
| The maximum order of the rotations is 3 | | |
| Is there an axis reflection? | | |
| YES | | NO |
| Are all rotation centres on reflection axes? | | p3 |
| YES | NO | |
| p3m1 | p31m | |

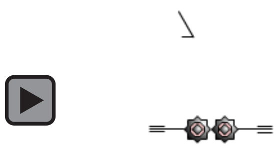
Basic motif





Fundamental parallelogram

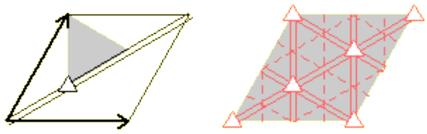




Group *p31m*

2-b) **Group *p*3*m*1**: The group contains no reflections in the direction of the vector $\boldsymbol{a}$. Let $s_r$ be a reflection and the vector $\boldsymbol{b} = s_r(\boldsymbol{a})$, so the direction of the line $r$ is $\boldsymbol{a} + \boldsymbol{b}$. The group contains reflections in the lines corresponding to the heights of the equilateral triangles that form the fundamental parallelogram.

$$p3m1 = < t_{\boldsymbol{a}}, t_{\boldsymbol{b}}, g_{O,2\pi/3}, s_r \,/\, s_r(\boldsymbol{a}) = \boldsymbol{b}, s_r(\boldsymbol{b}) = \boldsymbol{a} >$$

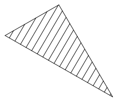


Fundamental parallelogram

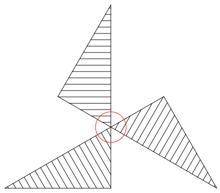
In this case all the centres of rotation are in lines of reflection.

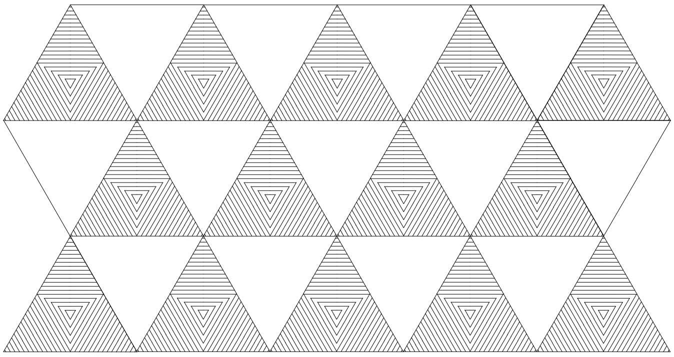
Palacio de los Leones. Arco entre la sala de Abencerrajes y el Patio de los Leones
Remark that the group *p*3*m*1 appears inside of the petal.

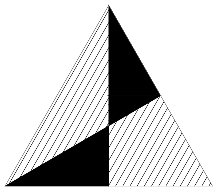
<table>
<tr><td colspan="3">The maximum order of the rotations is 3</td></tr>
<tr><td colspan="3">Is there an axis reflection?</td></tr>
<tr><td colspan="2">YES</td><td>NO</td></tr>
<tr><td colspan="2">Are all rotation centres on reflection axes?</td><td rowspan="3">p3</td></tr>
<tr><td>YES</td><td>NO</td></tr>
<tr><td>p3m1</td><td>p31m</td></tr>
</table>



Basic motif







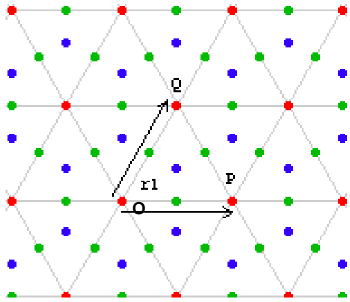
Fundamental region

Group *p3m1*

### 3.5. Crystallographic group TYPE 6.

Let $G_6$ be the crystallographic groups that contain rotations of order 6.

As seen in Section 2 the lattice of these groups of type 6 is triangular.



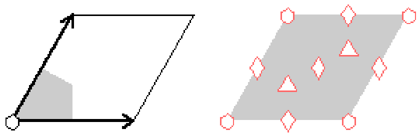

The centres of rotation $\sigma$ of order 6 are also centres of rotation $\sigma^2$ of order 3, and centres of rotation $\sigma^3$ of order 2. The vertices of the equilateral triangles are centres of rotation of orders 6, 3 and 2. The centres of triangles are centres of rotation of order 3, as seen in the groups of type 3 and the midpoints of the sides of the triangles are centres of rotation of order 2, as seen in groups of type 2.

The fundamental region is triangular.

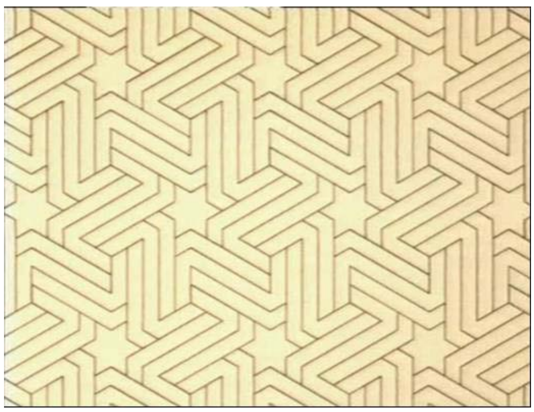
There are two non-isomorphic groups of order 6.

1) **Group *p6***: It only contains translations and rotations of order 6.

$$p6 = < t_{\boldsymbol{a}} , t_{\boldsymbol{b}} , \sigma = g_{O, \pi/3} \ / \ \sigma(\boldsymbol{a}) = \boldsymbol{b} >$$

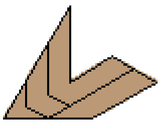


Fundamental parallelogram

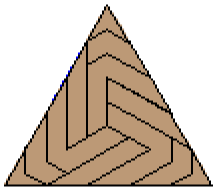


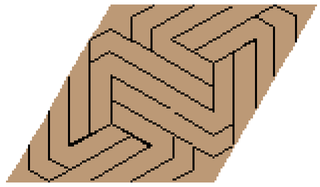
Palacio del Partal

| The maximum order of the rotations is 6 | |
|---|---|
| Is there an axis reflection? | |
| YES | NO |
| p6m | p6 |

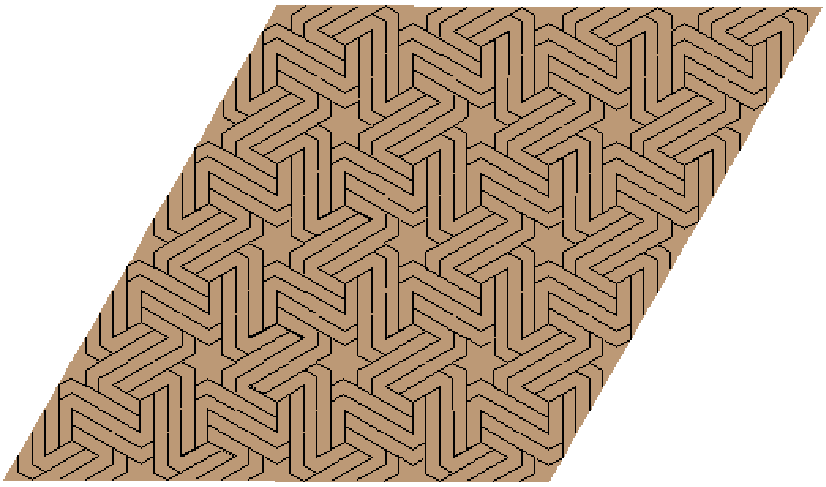


Basic motif





Fundamental parallelogram

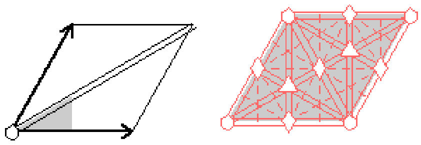


Group *p6*

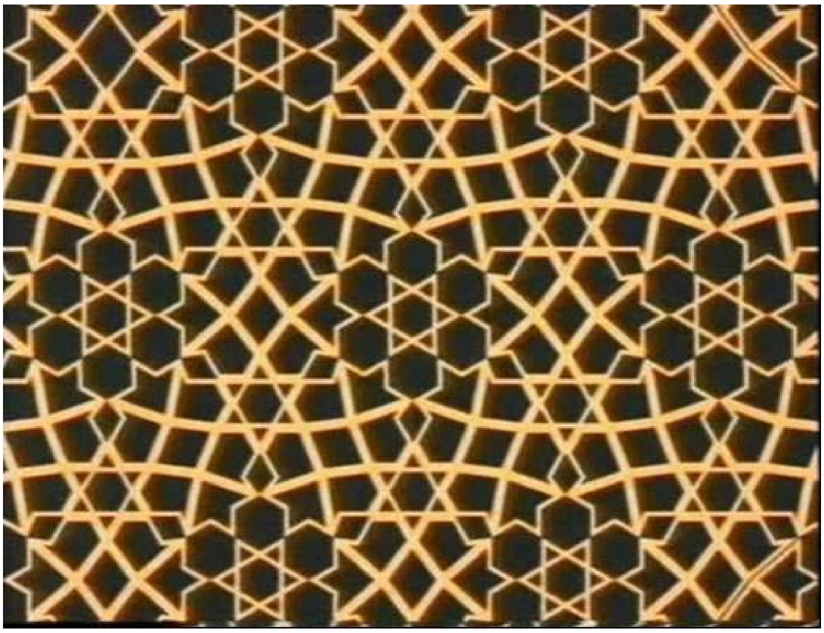
2) **Group *p*6*m***: It contains an indirect isometry, so as seen for the groups of type 3, it should also contains a reflection $s_r$. The composite application $\sigma \circ s_r$ is another reflection in line $r'$, where the line $r'$ forms an angle of $\pi/6$ with $r$. The composition of the two reflections $s_r \circ s_{r'}$ is a rotation about the point intersection of the two axes and angle $\pi/3$.

Therefore for each centre of rotation of order 6 pass six axes of reflection, forming between each two of them angles $\pi/6$.

$$p6m = < t_{\boldsymbol{a}}, t_{\boldsymbol{b}}, \sigma = g_{O,\pi/3}, s_r \,/\, \sigma(\boldsymbol{a}) = \boldsymbol{b}, s_r(\boldsymbol{a}) = \boldsymbol{a} >.$$

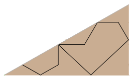


Fundamental parallelogram

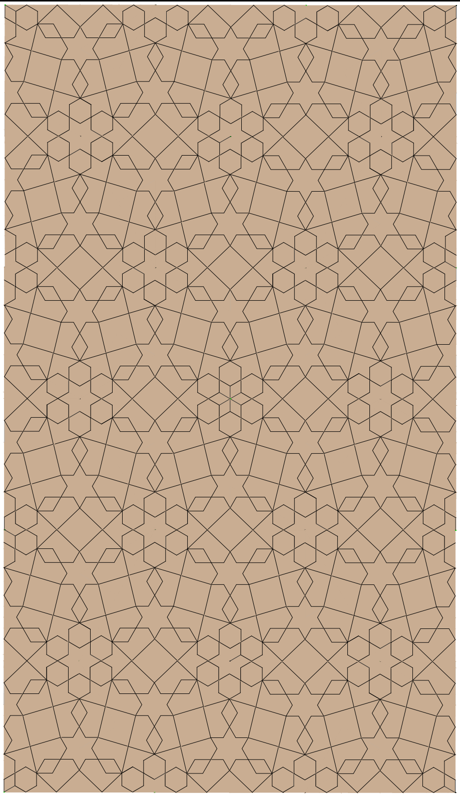


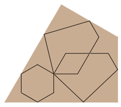
Ventana Del Patio de los Arrayanes

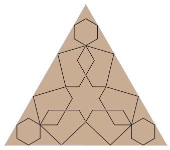
| The maximum order of the rotations is 6 | |
|---|---|
| Is there an axis reflection? | |
| YES | NO |
| p6m | p6 |

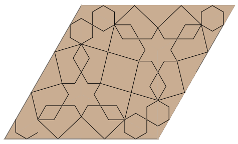


Basic motif

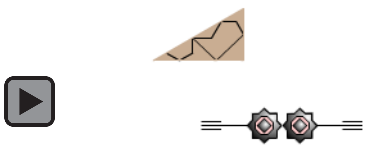








Fundamental parallelogram



Group *p6m*

AGRADECIMENTOS

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