Topological Elements of Geodesic Spaces

Definition. Let (X,ρ) be a metric space. The *length* of a path $\gamma : [a,b] \to X$ is

$$L(\gamma) = L_\rho(\gamma) = \sup\left\{ \sum_{i=1}^{n} \rho(\gamma(t_{i-1}),\gamma(t_i)) \ : \ n \geq 1 \text{ and } a = t_0 \leq t_1 \leq \cdots \leq t_n = b \right\}.$$

The path γ is *rectifiable* if $L(\gamma) < \infty$. (X,ρ) is *rectifiable* if every pair of points of X is joined by a rectifiable path in X. (X,ρ) is *locally rectifiable* if for every $\varepsilon > 0$, every point of X has a neighborhood in which any two points are joined by a path in X of length $< \varepsilon$.

Exercise 1. Let (X,ρ) is a metric space. Prove that if $\gamma : [a,b] \to X$ is a path in X and $a = t_0 \leq t_1 \leq \cdots \leq t_n = b$, then $L(\gamma) = \sum_{i=1}^{n} L(\gamma|[t_{i-1},t_i])$.

Exercise 2. Prove that a connected locally rectifiable metric space is rectifiable.

Lemma 1. Let (X,ρ) be a rectifiable metric space. Define $\sigma : X\times X \to [0,\infty)$ by

$$\sigma(x,y) = \inf\{ L_\rho(\gamma) : \gamma \text{ is a path in X joining x to y} \}.$$

Then:

a) σ is a metric on X;

b) $L_\sigma = L_\rho$ and, hence, $\sigma(x,y) = \inf\{L_\sigma(\gamma) : \gamma$ is a path in X joining x to y $\}$;

c) if ρ is a complete metric, then so is σ; and

d) σ is equivalent to ρ if and only if (X,ρ) is locally rectifiable.

Exercise 3. Prove a).

Proof of b). First note that for any path γ in X joining x to y, $\rho(x,y) \leq L_\rho(\gamma)$. Hence, $\rho \leq \sigma$. Hence, $L_\rho(\gamma) \leq L_\sigma(\gamma)$ for any path γ in X. Now if $\gamma : [a,b] \to X$ is a path and $a = t_0 \leq t_1 \leq \cdots \leq t_n = b$, then

$$\sum_{i=1}^{n} \sigma(\gamma(t_{i-1}),\gamma(t_i)) \leq \sum_{i=1}^{n} L_\rho(\gamma|[t_{i-1},t_i]) = L_\rho(\gamma).$$

Taking the supremum over all partitions $a = t_0 \le t_1 \le \cdots \le t_n = b$, we see that $L_\sigma(\gamma) \le L_\rho(\gamma)$. □

Proof of c). Let $\{x_n\}$ be a Cauchy sequence in (X,σ). Since $\rho \le \sigma$, then $\{x_n\}$ is Cauchy in (X,ρ). So $\{x_n\}$ converges to a point z in (X,ρ). Since $\{x_n\}$ is Cauchy in (X,σ), then $\{x_n\}$ has a subsequence $\{y_k\}$ such that $\sigma(y_{k+1},y_k) < 2^{-k}$. Hence, for each $k \ge 1$, there is a path γ_k in X joining y_{k+1} to y_k such that $L_\rho(\gamma_k) < 2^{-k}$. Since we can linearly reparametrize γ_k without changing its length, we can assume that the domain of γ_k is $[2^{-(k+1)},2^{-k}]$ with $\gamma_k(2^{-(k+1)}) = y_{k+1}$ and $\gamma_k(2^{-k}) = y_k$. Now a path $\gamma : [0,1] \to (X,\rho)$ is defined by $\gamma(0) = z$ and $\gamma|[2^{-(k+1)},2^{-k}] = \gamma_k$. γ is continuous at 0 because $\{y_k\}$ converges to z in (X,ρ), γ_k joins y_{k+1} to y_k, and $\rho\text{-diam}(\mathrm{im}(\gamma_k)) \le L_\rho(\gamma_k) < 2^{-k}$. Now $\sigma(z,y_k) \le$

$$L_\rho(\gamma|[0,2^{-k}]) = \sum_{i=k}^{\infty} L_\rho(\gamma_k) < \sum_{i=k}^{\infty} 2^{-k} = 2^{-k+1}. \text{ (Verify that } L_\rho(\gamma|[0,2^{-k}]) = \sum_{i=k}^{\infty} L_\rho(\gamma_k).)$$

Hence, $\{y_k\}$ converges to z in (X,σ). Since $\{y_k\}$ is a subsequence of $\{x_n\}$ and $\{x_n\}$ is Cauchy in (X,σ), it follows that $\{x_n\}$ converges to z in (X,σ). □

Exercise 4. Prove d).

Definition. Let (X,ρ) be a rectifiable metric space. ρ is a *path metric* and (X,ρ) is a *path metric space* if $\rho(x,y) = \inf\{ L(\gamma) : \gamma \text{ is a path in X joining x to y} \}$ for all $x, y \in X$.

Corollary 2. A connected (complete) metric space has an equivalent (complete) path metric if and only if it is locally rectifiable.

Definition. Let X be a topological space, and let (Y,ρ) be a metric space. A map $f : X \to Y$ is ρ-*bounded* if $\rho\text{-diam}(f(X)) < \infty$. Let $C_\rho(X,Y)$ denote the set of all ρ-bounded maps from X to Y. We always endow $C_\rho(X,Y)$ with the *supremum metric* σ_ρ defined by $\sigma_\rho(f,g) = \sup\{ \rho(f(x),g(x)) : x \in X \}$. Recall that if ρ is a complete metric, then so is σ_ρ.

Theorem 3: An Ascoli-Arzela Theorem. Let X be a separable topological space and let (Y,ρ) be a complete metric space. Then a subset F of $C_\rho(X,Y)$ has compact closure in $C_\rho(X,Y)$ if F has the following two properties.

a) There is a dense countable subset D of X such that for every $x \in D$, $\{ f(x) : f \in F \}$ has compact closure in Y.

b) For every $\varepsilon > 0$, every point of X has a neighborhood U such that $\rho\text{-diam}(f(U)) < \varepsilon$ for each $f \in F$.

Remark. We abbreviate property a) by saying that the *F-image* of every point of the dense countable set D has compact closure in Y. Property b) is usually abbreviated by saying that F is *equicontinuous*.

Proof. Let $D = \{ x_n : n \geq 1 \}$. Let $\{f_k\}$ be a sequence in F. We will prove that some subsequence of $\{f_k\}$ converges to a point of $C_\rho(X,Y)$. We will exploit the fact that for each $x \in D$, each sequence in $\{ f(x) : f \in F \}$ has a converging subsequence which, in turn, has subsequences that converge at any prescribed rate. For $n = 1, 2, 3, \cdots$, we inductively construct sequences $\{k_i^n\}_{i\geq 1}$ of positive integers such that for $n \geq 1$,

a) $\{k_i^{n+1}\}_{i\geq 1}$ is a subsequence of $\{k_i^n\}_{i\geq 1}$, and

b) ρ-diam $(\{f_{k_i^n}(x_n)\}_{i\geq 1}) < 1/n$, and ρ-diam $(\{f_{k_j^n}(x_n)\}_{j\geq i}) < 1/i$ for $i \geq 1$.

Set $m_n = k_n^n$ and $n \geq 1$. Then for $i \geq n \geq 1$, $\{m_j\}_{j\geq i}$ is a subsequence of $\{k_j^n\}_{j\geq i}$. It follows that for $n \geq 1$ and $i \geq 1$, ρ-diam $(\{f_{m_j}(x_n)\}_{j\geq i}) < 1/i$.

We assert that for any $x \in X$ and $i \geq 1$, ρ-diam $(\{f_{m_j}(x)\}_{j\geq i}) \leq 3/i$. Let $x \in X$ and $i \geq 1$. x has a neighborhood U such that ρ-diam$(f(U)) < 1/i$ for each $f \in F$. There is an $n \geq 1$ such that $x_n \in U$. Then for $j, k \geq i$,

$$\begin{aligned} \rho (f_{m_j}(x), f_{m_k}(x)) &\leq \rho (f_{m_j}(x), f_{m_j}(x_n)) + \rho (f_{m_j}(x_n), f_{m_k}(x_n)) + \rho (f_{m_k}(x_n), f_{m_k}(x)) \\ &< \rho\text{-diam}(f_{m_j}(U)) + \rho\text{-diam} (\{f_{m_r}(x_n)\}_{r\geq i}) + \rho\text{-diam}(f_{m_k}(U)) < 3/i. \end{aligned}$$

This proves the assertion. It follows that $\sigma_\rho(f_{m_j}, f_{m_k}) \leq 3/i$ for $j, k \geq i$. Hence, $\{f_{m_i}\}_{i\geq 1}$ is a Cauchy sequence in $(C_\rho(X,Y), \sigma_\rho)$. Therefore, $\{f_{m_i}\}_{i\geq 1}$ converges in $C_\rho(X,Y)$. This proves cl(F) is compact. □

Theorem 4. If (X,ρ) is a locally compact complete path metric space, then for every $x \in X$ and every $r > 0$, $B(x,r) = \{ y \in X : \rho(x,y) \leq r \}$ is compact.

Remark. The completeness of the metric ρ is necessary here. For observe that $\mathbb{R}^2 - \{(0,0)\}$ with the Euclidean metric inherited from $\mathbb{R}^2$ is a locally compact path metric space in which $B((1,0),2)$ is non-compact.

Lemma 5. Let (X,ρ) be a path metric space. If $x, y \in X$ and $\rho(x,y) < r + s$ where $r > 0$ and $s > 0$, then there is a $z \in X$ such that $\rho(x,z) < r$ and $\rho(z,y) < s$.

Proof of Lemma 5. Let $0 < \varepsilon < \min\left\{ r, s, \frac{r + s - \rho(x,y)}{2} \right\}$. There is a path $\gamma : [a,b] \to X$ joining x to y such that $L(\gamma) < \rho(x,y) + \varepsilon$. The choice of ε insures that $\rho(x,y) - s + \varepsilon < \min\{ \rho(x,y), r - \varepsilon \}$ and $0 < \min\{ \rho(x,y), r - \varepsilon \} \leq \rho(x,y)$. Since $\rho(x,\gamma(a)) = 0$ and $\rho(x,\gamma(b)) = \rho(x,y)$, then there is a $t \in [a,b]$ such that $\rho(x,\gamma(t)) = \min\{ \rho(x,y), r - \varepsilon \}$. Set $z = \gamma(t)$. Then $\rho(x,y) - s + \varepsilon < \rho(x,z) < r$. So $\rho(x,y) - s + \varepsilon + \rho(z,y) < \rho(x,z) + \rho(z,y) \leq L(\gamma) < \rho(x,y) + \varepsilon$. Hence, $\rho(z,y) < s$. □

Proof of Theorem 4. Let $x \in X$. Since X is locally compact, there is an $s > 0$ such that $B(x,s)$ is compact. Hence, $B(x,r)$ is compact whenever $0 < r \leq s$. Assume $B(x,r)$ is non-compact for some $r > 0$, and set $t = \inf\{ r > 0 : B(x,r) \text{ is non-compact} \}$. Then $s \leq t$.

First we will prove that $B(x,t)$ is compact. Let $\{y_n\}$ be a sequence in $B(x,t)$. Since $B(x,t)$ is a closed subset of X, it suffices to find a subsequence of $\{y_n\}$ that converges in X. We will achieve this through an application of Theorem 3.

For each $n \geq 1$ and each $k \geq 1$, since $\rho(x,y_n) \leq t < (t - 1/k) + 2/k$, then Lemma 5 provides a point $z_n^k \in X$ such that $\rho(x,z_n^k) < t - 1/k$ and $\rho(z_n^k,y_n) < 2/k$.

Let W denote the subspace $\{0\} \cup \{ 1/k : k \geq 1 \}$ of $\mathbb{R}$. For each $n \geq 1$, define the function $f_n : W \to X$ by $f_n(0) = y_n$ and $f_n(1/k) = z_n^k$ for $k \geq 1$. Set $F = \{ f_n : n \geq 1 \}$. Each f_n is continuous because $\{f_n(1/k)\}_{k\geq 1} = \{z_n^k\}_{k\geq 1}$ converges to $f_n(0) = y_n$. Each f_n is ρ-bounded because $f_n(W) \subset B(x,t)$. Hence, $F \subset C_\rho(W,X)$.

We now verify that F satisfies the hypotheses of Theorem 3. $\{ 1/k : k \geq 1 \}$ is a countable dense subset of W. For each $k \geq 1$, the set $\{ f_n(1/k) : n \geq 1 \} = \{ z_n^k : n \geq 1 \}$ has compact closure because it lies in the compactum $B(x, t - 1/k)$. Hence, the F-image of each point of $\{ 1/k : k \geq 1 \}$ has compact closure in X. For any $\varepsilon > 0$ and $k \geq 1$, $\{1/k\}$ is a neighborhood of $1/k$ in W such that $\rho\text{-diam}(f_n\{1/k\}) = \rho\text{-diam}(\{z_n^k\}) = 0 < \varepsilon$ for all $n \geq 1$. For $\varepsilon > 0$, $U = W \cap [0,\varepsilon/4)$ is a neighborhood of 0 in W such that $\rho\text{-diam}(f_n(U)) \leq \varepsilon$ for all $n \geq 1$. To see this, note that if $k \geq 1$ such that $1/k \in U$, then $\rho(f_n(0),f_n(1/k)) = \rho(y_n,z_n^k) < 2/k < \varepsilon/2$. Hence, if $j, k \geq 1$ such that $1/j$ and $1/k \in U$, then $\rho(f_n(1/j),f_n(1/k)) \leq \rho(f_n(1/j),f_n(0)) + \rho(f_n(0),f_n(1/k)) < \varepsilon$. Hence, F is equicontinuous. We conclude that F satisfies the hypotheses of Theorem 3.

Theorem 3 implies that F has compact closure in $C_\rho(W,X)$. Hence, $\{f_n\}$ has a subsequence $\{f_{n_i}\}$ that converges in $C_\rho(W,X)$ to a map $g : W \to X$. Since $\rho(y_{n_i},g(0)) = \rho(f_{n_i}(0),g(0)) \leq \sigma_\rho(f_{n_i},g)$, then $\{y_{n_i}\}$ converges to $g(0)$ in X. Thus, $\{y_n\}$ has a converging subsequence. It follows that $B(x,t)$ is compact.

Since X is locally compact and B(x,t) is compact, then B(x,t)is covered by finitely many open sets with compact closure. Their union is an open set U with compact closure which contains B(x,t). Since B(x,t) is compact, there is a $\delta > 0$ such that U contains the δ-neighborhood of every point of B(x,t). (Otherwise, there is a sequence $\{y_n\}$ in B(x,t) and a sequence $\{z_n\}$ in X–U such that $\rho(y_n,z_n) \to 0$. Since B(x,t) is compact, we can assume (after passing to a subsequence) that $\{y_n\}$ converges to a point $y \in B(x,t)$. Then $\{z_n\}$ converges to y and $y \in U$. It follows that $\{z_n\}$ eventually enters U. ↯) Let V denote the union of the δ-neighborhoods of all the points of B(x,t). Then $B(x,t) \subset V \subset U$. So V has compact closure.

We now prove that $B(x, t + \delta/2) \subset V$. (This is not necessarily true in an arbitrary metric space because there may be a point within $t + \delta/2$ of x which is not near any point of B(x,t). (Find an example of such a space.) However, it is true in a path metric space.) Let $y \in B(x, t + \delta/2)$. Since $\rho(x,y) < t + \delta$, Lemma 5 provides a point $z \in X$ such that $\rho(x,z) < t$ and $\rho(z,y) < \delta$. Then clearly $z \in B(x,t)$ and $y \in V$. Thus, $B(x, t + \delta/2) \subset V$. Since cl(V) is compact, so is $B(x, t + \delta/2)$. It follows that B(x,r) is compact for $0 < r \le t + \delta/2$. This contradicts our initial choice of t as inf { r > 0 : B(x,r) is non-compact }. We conclude that B(x,r) is compact for all r > 0. □

Definition. Let (X,ρ) be a metric space. A path $\gamma : [a,b] \to X$ is a *geodesic* if $\rho(\gamma(s),\gamma(t)) = |s - t|$ for all $s, t \in [a,b]$. ρ is a *geodesic metric* and (X,ρ) is a *geodesic metric space* if every pair of points in X is joined by a geodesic.

Exercise 5. a) Let (X,ρ) be a metric space. Prove that if $\gamma : [a,b] \to X$ is a geodesic, then $L(\gamma|[s,t]) = \rho(\gamma(s),\gamma(t)) = |s - t|$ for all $s, t \in [a,b]$.

b) Prove that every geodesic metric is a path metric.

Theorem 6. Every locally compact complete path metric space is a geodesic metric space.

Remark. Again the completeness of the metric is necessary here. Indeed, $\mathbb{R}^2 - \{(0,0)\}$ is a locally compact path metric space which is not a geodesic space, because there is no geodesic in $\mathbb{R}^2 - \{(0,0)\}$ joining (-1,0) to (1,0).

Definition. Let (X,ρ) be a metric space. A path $\gamma : [c,d] \to X$ is a *reparametrization* of a path $\beta : [a,b] \to X$ if there is non-decreasing onto map $\phi : [c,d] \to [a,b]$ such that $\gamma = \beta \circ \phi$. ("$\phi$ is non-decreasing" means that $\phi(s) \le \phi(t)$ whenever $c \le s \le t \le d$.)

Exercise 7. Let (X,ρ) be a metric space. Prove that if a path $\beta : [c,d] \to X$ is a reparametrization of a path $\gamma : [a,b] \to X$, then $L(\beta) = L(\gamma)$.

Definition. Let (X,ρ) be a metric space. A path $\gamma : [a,b] \to X$ is *constant speed* if there is a non-negative real number ζ, called the *speed* of γ, such that $L(\gamma|[s,t]) = \zeta(t-s)$ whenever $a \le s \le t \le b$.

Exercise 8. Let (X,ρ) be a metric space.

a) Prove that every constant speed path is rectifiable.

b) Prove that if $\gamma : [a,b] \to X$ is a constant speed path, then the speed of γ equals $\frac{L(\gamma)}{b-a}$.

c) Prove that every geodesic is a constant speed path of speed 1.

Lemma 7. Let (X,ρ) be a metric space and let $[c,d]$ be an interval in $\mathbb{R}$. Then every rectifiable path in X is the reparametrization of a constant speed path with domain $[c,d]$.

Definition. Let (X,ρ) be a metric space. For each rectifiable curve $\gamma : [a,b] \to X$, define the function $\Lambda_\gamma : [a,b] \to [0,\infty)$ by $\Lambda_\gamma(t) = L(\gamma|[a,t])$.

Lemma 8. Let (X,ρ) be a metric space. If $\gamma : [a,b] \to X$ is a rectifiable curve, then $\Lambda_\gamma : [a,b] \to [0,\infty)$ is continuous.

Proof of Lemma 8. Let $\varepsilon > 0$. There is a partition $a = t_0 < t_1 < \cdots < t_n = b$ of $[a,b]$ such that $\sum_{i=1}^{n} \rho(\gamma(t_{i-1}),\gamma(t_i)) > L(\gamma) - \varepsilon/3$. Since refining this partition only increceases the sum, we can further assume that $\rho(\gamma(t_{i-1}),\gamma(t_i)) < \varepsilon/3$ for $1 \le i \le n$. Then for $1 \le j < n$,

$$L(\gamma|[a,t_{j-1}]) + L(\gamma|[t_{j-1},t_{j+1}]) + L(\gamma|[t_{j+1},b]) - \varepsilon/3 \;=\; L(\gamma) - \varepsilon/3 \;<$$

$$\sum_{i=1}^{j-1} \rho(\gamma(t_{i-1}),\gamma(t_i)) + \rho(\gamma(t_{j-1}),\gamma(t_j)) + \rho(\gamma(t_j),\gamma(t_{j+1})) + \sum_{i=j+2}^{n} \rho(\gamma(t_{i-1}),\gamma(t_i)) \;\le$$

$$L(\gamma|[a,t_{j-1}]) + 2\varepsilon/3 + L(\gamma|[t_{j+1},b]).$$

Hence, $L(\gamma|[t_{j-1},t_{j+1}]) < \varepsilon$. Let $0 < \delta < \min\{ t_i - t_{i-1} : 1 \le i \le n \}$. If $a \le s \le t \le b$ and $|s - t| < \delta$, then $s, t \in [t_{j-1},t_{j+1}]$ for some j, $1 \le j < n$. Hence, $|\Lambda_\gamma(s) - \Lambda_\gamma(t)| = L(\gamma|[a,t]) - L(\gamma|[a,s]) = L(\gamma|[s,t]) \le L(\gamma|[t_{j-1},s]) + L(\gamma|[s,t]) + L(\gamma|[t,t_{j+1}]) = L(\gamma|[t_{j-1},t_{j+1}]) < \varepsilon$. This establishes the continuity of Λ_γ. □

Proof of Lemma 7. Let $\gamma : [a,b] \to X$ be a rectifiable path. $\Lambda_\gamma(a) = L(\gamma|[a,a]) = 0$ and $\Lambda_\gamma(b) = L(\gamma|[a,b]) = L(\gamma)$. Since $\Lambda_\gamma(t) = L(\gamma|[a,t]) = L(\gamma|[a,s]) + L(\gamma|[s,t]) \ge L(\gamma|[a,s]) = \Lambda_\gamma(s)$ for $a \le s \le t \le b$, then Λ_γ is non-decreasing. Hence, Λ_γ maps $[a,b]$ onto $[0,L(\gamma)]$.

If $a \le s \le t \le b$ and $\Lambda_\gamma(s) = \Lambda_\gamma(t)$, then $\rho(\gamma(s),\gamma(t)) \le L(\gamma|[s,t]) = L(\gamma|[a,t]) - L(\gamma|[a,s]) = \Lambda_\gamma(t) - \Lambda_\gamma(s) = 0$. So $\gamma(s) = \gamma(t)$. Hence, if $t \in [0,L(\gamma)]$, then $\gamma(\Lambda_\gamma^{-1}(t))$ is a single point. Hence, a function $\beta : [0,L(\gamma)] \to X$ is defined by $\beta(t) = \gamma(\Lambda_\gamma^{-1}(t))$. Then clearly $\beta \circ \Lambda_\gamma = \gamma$.

We now prove that β is continuous. Let C be a closed subset C of X. Then $\beta^{-1}(C) = (\gamma \circ \Lambda_\gamma^{-1})^{-1}(C) = \Lambda_\gamma(\gamma^{-1}(C))$. $\gamma^{-1}(C)$ is a closed and, hence, compact subset of $[a,b]$. Therefore, $\Lambda_\gamma(\gamma^{-1}(C)) = \beta^{-1}(C)$ is a compact and, hence, closed subset of $[0,L(\gamma)]$.

So γ is a reparametrization of β.

Let $0 \le s \le t \le L(\gamma)$. Then $s = \Lambda_\gamma(s')$ and $t = \Lambda_\gamma(t')$ where $a \le s' \le t' \le b$. Hence, $\Lambda_\gamma|[s',t']$ is a non-decreasing map of $[s',t']$ onto $[s,t]$ such that $(\beta|[s,t]) \circ (\Lambda_\gamma|[s',t']) = \gamma|[s',t']$. So $\gamma|[s',t']$ is a reparametrization of $\beta|[s,t]$. Hence, $L(\beta|[s,t]) = L(\gamma|[s',t']) = L(\gamma|[a,t']) - L(\gamma|[a,s']) = \Lambda_\gamma(t') - \Lambda_\gamma(s') = t - s$. If follows that β is a constant speed path of speed 1.

Define the affine homeomorphism $\phi : [0,L(\gamma)] \to [c,d]$ by $\phi(t) = \left(\frac{d-c}{L(\gamma)}\right)t + c$. Since $\Lambda_\gamma : [a,b] \to [0,L(\gamma)]$ and $\phi : [0,L(\gamma)] \to [c,d]$ are non-decreasing onto maps, then so is $\phi \circ \Lambda_\gamma : [a,b] \to [c,d]$. Define the path $\alpha : [c,d] \to X$ by $\alpha = \beta \circ \phi^{-1}$. Then $\alpha \circ \phi = \beta$ and $\alpha \circ (\phi \circ \Lambda_\gamma) = \beta \circ \Lambda_\gamma = \gamma$. So γ is a reparametrization of α. Let $c \le s \le t \le d$. Set $s' = \phi^{-1}(s) = \left(\frac{L(\gamma)}{d-c}\right)(s - c)$ and set $t' = \phi^{-1}(t) = \left(\frac{L(\gamma)}{d-c}\right)(t - c)$. $\phi|[s',t']$ is a non-decreasing map of $[s',t']$ onto $[s,t]$ such that $(\alpha|[s,t]) \circ (\phi|[s',t']) = \beta|[s',t']$. So $\beta|[s',t']$ is a reparametrization of

$\alpha|[s,t]$. Hence, $L(\alpha|[s,t]) = L(\beta|[s',t']) = t' - s' = \left(\frac{L(\gamma)}{d-c}\right)(t-c) - \left(\frac{L(\gamma)}{d-c}\right)(s-c) = \left(\frac{L(\gamma)}{d-c}\right)(t-s)$. Therefore, α is a constant speed path with domain $[c,d]$. □

Proof of Theorem 6. Let (X,ρ) be a locally compact complete path metric space. Let $x, y \in X$. We must construct a geodesic joining x to y.

Since $\rho(x,y) = \inf\{ L(\gamma) : \gamma$ is a path in X joining x to y $\}$, then there is a sequence $\{\gamma_i\}$ of paths in X joining x to y such that $L(\gamma_i) < \rho(x,y) + {}^1/_i$. Since reparametrization doesn't change path length, then by Lemma 7 we can assume that each γ_i is a constant speed path with domain $[0,1]$. Set $G = \{\gamma_i\}$. Then $G \subset C_\rho([0,1],X)$.

We will now apply Theorem 3 to G. If $t \in [0,1]$, then $\rho(x,\gamma_i(t)) = \rho(\gamma_i(0),\gamma_i(t)) \leq L(\gamma_i|[0,t]) \leq L(\gamma_i) < \rho(x,y) + 1$ for $i \geq 1$. Hence, for each $t \in [0,1]$, $\{ \gamma_i(t) : i \geq 1 \} \subset B(x,\rho(x,y) + 1)$. Since $B(x,\rho(x,y) + 1)$ is compact by Theorem 4, then $\{ \gamma_i(t) : i \geq 1 \}$ has compact closure for each $t \in [0,1]$. Thus, the G-image of every point of $[0,1]$ has compact closure in X. Next let $\varepsilon > 0$ and suppose $s \in [0,1]$. Set $\delta = \dfrac{\varepsilon}{2(\rho(x,y) + 1)}$ and set $U = (s-\delta,s+\delta) \cap [0,1]$. U is a neighborhood of s in $[0,1]$. Suppose $t, t' \in U$ such that $t \leq t'$. Let $i \geq 1$. Exercise 8.b) implies that the speed of γ_i is $L(\gamma_i)$. Hence, $\rho(\gamma_i(t),\gamma_i(t')) \leq L(\gamma_i|[t,t']) = L(\gamma_i)(t' - t) < 2\delta(\rho(x,y) + 1) = \varepsilon$. Thus, $\rho\text{-diam}(\gamma_i(U)) \leq \varepsilon$ for every $i \geq 1$. Hence, G is equicontinuous.

Theorem 3 now implies that G has compact closure in $C_\rho([0,1],X)$. Hence, after passing to a subsequence if necessary, we can assume that $\{\gamma_i\}$ converges to a path $\beta : [0,1] \to X$ in $C_\rho([0,1],X)$. Hence, $\{\gamma_i(0)\}_{i\geq 1} = \{x\}$ converges to $\beta(0)$ and $\{\gamma_i(1)\}_{i\geq 1} = \{y\}$ converges to $\beta(1)$. So $\beta(0) = x$ and $\beta(1) = y$; i.e., β joins x to y.

Lemma 9. Let (X,ρ) be a metric space. If $\gamma_i : [a,b] \to X$, $i \geq 1$, and $\beta : [a,b] \to X$ are paths such that $\{\gamma_i\}$ converges to β in $C_\rho([a,b],X)$, then $L(\beta) \leq \liminf L(\gamma_i)$.

Proof of Lemma 9. Let $a = t_0 < t_1 < \cdots < t_n = b$ be a partition of $[a,b]$. Suppose $\varepsilon > 0$. Then there is a $k \geq 1$ such that $\sigma_\rho(\beta,\gamma_j) < {}^\varepsilon/_{2n}$ for $j \geq k$. Then for $j \geq k$ and $1 \leq i \leq n$, $\rho(\beta(t_{i-1}),\beta(t_i)) \leq \rho(\beta(t_{i-1}),\gamma_j(t_{i-1})) + \rho(\gamma_j(t_{i-1}),\gamma_j(t_i)) + \rho(\gamma_j(t_i),\beta(t_i)) \leq 2\sigma_\rho(\beta,\gamma_j) + \rho(\gamma_j(t_{i-1}),\gamma_j(t_i))$

$< \varepsilon/n + \rho(\gamma_j(t_{i-1}),\gamma_j(t_i))$. Hence, for $j \geq k$, $\sum_{i=1}^{n} \rho(\beta(t_{i-1}),\beta(t_i)) < \varepsilon + L(\gamma_j)$. This proves that for every $\varepsilon > 0$, there is a $k \geq 1$ such that $\sum_{i=1}^{n} \rho(\beta(t_{i-1}),\beta(t_i)) - \varepsilon \leq \inf\{ L(\gamma_j) : j \geq k \}$.

Consequently, $\sum_{i=1}^{n} \rho(\beta(t_{i-1}),\beta(t_i)) \leq \liminf L(\gamma_j)$. It follows that $L(\beta) \leq \liminf L(\gamma_j)$. □

Continuation of the Proof of Theorem 6. Lemma 9 implies that $L(\beta) \leq \liminf L(\gamma_j) \leq \rho(x,y)$. On the other hand, $\rho(x,y) = \rho(\beta(0),\beta(1)) \leq L(\beta)$. Thus, $L(\beta) = \rho(x,y)$.

Lemma 7 implies that β is the reparametrization of a constant speed path $\alpha : [0,\rho(x,y)] \to X$. So $\alpha(0) = \beta(0) = x$ and $\alpha(\rho(x,y)) = \beta(1) = y$. Thus, α joins x to y. Also $L(\alpha) = L(\beta) = \rho(x,y)$ by Exercise 7. Moreover, according to Exercise 8. b), the speed of α is $L(\alpha)/\rho(x,y) = L(\beta)/\rho(x,y) = 1$. The following Lemma establishes that α is a geodesic. □

Lemma 10. Let x and y be points of a metric space (X,ρ). If $\gamma : [a,b] \to X$ is a constant speed path of speed 1 joining x to y and $L(\gamma) = \rho(x,y)$, then γ is a geodesic.

Proof. Since γ is speed 1, then $\rho(x,y) = L(\gamma) = b - a$, and $\rho(\gamma(s),\gamma(t)) \leq L(\gamma|[s,t]) = t - s$ for $a \leq s \leq t \leq b$. Let $a \leq s \leq t \leq b$. Set $x' = \gamma(s)$ and $y' = \gamma(t)$. Then each of the three terms in the sum

$$S = [(s-a) - \rho(\gamma(a),\gamma(s))] + [(t-s) - \rho(\gamma(s),\gamma(t))] + [(b-t) - \rho(\gamma(t),\gamma(b))] =$$

$$[(s-a) - \rho(x,x')] + [(t-s) - \rho(x',y')] + [(b-t) - \rho(y',y)]$$

is non-negative. So $S \geq 0$. On the other hand, $S = (b-a) - [\rho(x,x') + \rho(x',y') + \rho(y',y)] = \rho(x,y) - [\rho(x,x') + \rho(x',y') + \rho(y',y)] \leq 0$ because $\rho(x,y) \leq \rho(x,x') + \rho(x',y') + \rho(y',y)$. Thus, $S = 0$. Hence, each of the three terms in S equals 0. In particular, $(t-s) - \rho(\gamma(s),\gamma(t)) = 0$. Hence, $\rho(\gamma(s),\gamma(t)) = t - s$. This proves γ is a geodesic. □

Definition. Let (X,ρ) be a metric space. If $\gamma : [a,b] \to X$ is a path, let the path $\gamma^1 : [0,1] \to X$ be the reparametrization of γ defined by $\gamma^1(t) = \gamma((1-t)a + tb)$, and call γ^1 the *unit interval reparametrization* of γ.

Observation. Let (X,ρ) be a metric space. Suppose $\gamma : [a,b] \to X$ and $\beta : [c,d] \to X$ are paths, $\varepsilon > 0$, and $\sigma_\rho(\gamma^1,\beta^1) < \varepsilon$ in $C_\rho([0,1],X)$. Then $\rho(\gamma((1\text{-}t)a+tb), \beta((1\text{-}t)a+tb)) < \varepsilon$ for $0 \le t \le 1$. In other words, γ^1 and β^1 being close in $C_\rho([0,1],X)$ implies that γ and β are in some sense close paths.

Lemma 11. Let x and y be points of a metric space (X,ρ).
a) If $\gamma : [a,b] \to X$ is a geodesic joining x to y, then γ^1 is a constant speed path of speed $\rho(x,y)$ and $\rho(\gamma^1(s),\gamma^1(t)) = \rho(x,y)(t-s)$ for $0 \le s \le t \le 1$.
b) If $\beta : [0,1] \to X$ is a path joining x to y such that $\rho(\beta(s),\beta(t)) = \rho(x,y)(t-s)$ for $0 \le s \le t \le 1$, then β is the unit interval reparametrization of a geodesic which joins x to y.
c) If $\gamma : [a,b] \to X$ and $\beta : [c,d] \to X$ are geodesics and γ is a reparametrization of β, then $\gamma^1 = \beta^1$.

Proof. a) First note that $\rho(x,y) = \rho(\gamma(a),\gamma(b)) = b - a$ because γ is a geodesic. Let $0 \le s \le t \le 1$. Set $s' = (1-s)a + sb$ and $t' = (1-t)a + tb$. Then $\gamma^1(s) = \gamma(s')$ and $\gamma^1(t) = \gamma(t')$. Since γ is a geodesic, then $\rho(\gamma^1(s),\gamma^1(t)) = \rho(\gamma(s'),\gamma(t')) = t' - s' = (b-a)(t-s) = \rho(x,y)(t-s)$. Furthermore, since $u \to (1-u)a + ub$ is a non-decreasing map from $[s,t]$ onto $[s',t']$, then $\gamma^1|[s,t]$ is a reparametrization of $\gamma|[s',t']$. So by Exercise 5.a), $L(\gamma^1|[s,t]) = L(\gamma|[s',t']) = \rho(\gamma(s'),\gamma(t')) = t' - s' = (b-a)(t-s) = \rho(x,y)(t-s)$. Thus, γ^1 is a constant speed path of speed $\rho(x,y)$.

b) Set $\rho = \rho(x,y)$. Define $\gamma : [0,\rho] \to X$ by $\gamma(t) = \beta(t/\rho)$. Then for $0 \le s \le t \le \rho$, $\rho(\gamma(s),\gamma(t)) = \rho(\beta(s/\rho),\beta(t/\rho)) = \rho(t/\rho - s/\rho) = t - s$. So γ is a geodesic. Since $\beta(t) = \gamma((1-t)0 + t\rho)$, then β is the unit interval reparametrization of γ.

c) $\gamma = \beta\circ\phi$ for some non-decreasing onto map $\phi : [a,b] \to [c,d]$. For $t \in [a,b]$, $t - a = \rho(\gamma(a),\gamma(t)) = \rho(\beta(\phi(a)),\beta(\phi(t)) = \rho(\beta(c),\beta(\phi(t)) = \phi(t) - c$. Hence, $\phi(t) = t - a + c$. Thus, $d = \phi(b) = b - a + c$. So $d - c = b - a$. Now for $0 \le t \le 1$, $\gamma^1(t) = \gamma(\,(1-t)a + tb\,) = \beta(\phi(\,(1-t)a + tb)\,) = \beta(\,((1-t)a + tb) - a + c\,) = \beta(\,t(b-a) + c\,) = \beta(\,t(d-c) + c\,) = \beta(\,(1-t)c + td\,) = \beta^1(t)$. Hence, $\gamma^1 = \beta^1$. □

Remark. Let x and y be points of a metric space (X,ρ). It follows from Lemma 11.c) that if there is only one geodesic *up to reparametrization* joining x to y in X, then there is *exactly one* unit interval reparametrization of a geodesic that joins x to y in X.

Theorem 12. Let (X,ρ) be a metric space. Suppose U is a subset of $X \times X$ such that for every $(x,y) \in U$, there is only one geodesic up to reparametrization joining x to y in X. For each $(x,y) \in U$, let $\gamma_{x,y}$ denote the unique unit interval reparametrization of a geodesic which joins x to y in X. Then the function $(x,y) \to \gamma_{x,y} : U \to C_\rho([0,1],X)$ is continuous.

Proof. Let $\{ (x_i,y_i) \}_{i\geq 1}$ be a sequence in U which converges to the point $(x_0,y_0) \in U$. Set $\gamma_i = \gamma_{x_i,y_i}$ for $i \geq 0$. We must prove $\{\gamma_i\}_{i\geq 1}$ converges to γ_0. Set $G = \{ \gamma_i : i \geq 1 \}$. Then $G \subset C_\rho([0,1],X)$. We will apply Theorem 3 to G.

Since $\{x_i\}_{i\geq 0}$ and $\{y_i\}_{i\geq 0}$ are converging sequences together with their limits, they are bounded sets. So there is a $B > 0$ such that $\rho\text{-diam}(\{x_i\}_{i\geq 0}) \leq B$ and $\rho\text{-diam}(\{y_i\}_{i\geq 0}) \leq B$. Lemma 11.a) implies that for $i \geq 0$ and $0 \leq s \leq t \leq 1$, $\rho(\gamma_i(s),\gamma_i(t)) = \rho(x_i,y_i)(t-s)$. Hence, for $t \in [0,1]$ and $i \geq 1$, $\rho(x_0,\gamma_i(t)) \leq \rho(x_0,x_i) + \rho(x_i,\gamma_i(t)) = \rho(x_0,x_i) + \rho(\gamma_i(0),\gamma_i(t)) = \rho(x_0,x_i) + \rho(x_i,y_i)t \leq \rho(x_0,x_i) + \rho(x_i,y_i) \leq \rho(x_0,x_i) + \rho(x_i,x_0) + \rho(x_0,y_0) + \rho(y_0,y_i) \leq \rho(x_0,y_0) + 3B$. It follows that $\gamma_i([0,1]) \subset B(x_0,\rho(x_0,y_0) + 3B)$ for $i \geq 1$. Since $B(x_0,\rho(x_0,y_0) + 3B)$ is compact by Theorem 4, we conclude that $\{ \gamma_i(t) : i \geq 1 \}$ has compact closure in X for each $t \in [0,1]$. In other words, the G-image of every point of [0,1] has compact closure in X.

Now let $\varepsilon > 0$ and suppose $s \in [0,1]$. Set $\delta = \dfrac{\varepsilon}{2(\rho(x_0,y_0) + 2B)}$ and set $U = (s-\delta,s+\delta) \cap [0,1]$. U is a neighborhood of s in [0,1]. Suppose $t, t' \in U$ such that $t \leq t'$. Let $i \geq 1$. Then $\rho(\gamma_i(t),\gamma_i(t')) = \rho(x_i,y_i)(t-t') \leq 2\delta(\rho(x_i,x_0) + \rho(x_0,y_0) + \rho(y_0,y_i)) \leq 2\delta(\rho(x_0,y_0) + 2B) = \varepsilon$. Thus, $\rho\text{-diam}(\gamma_i(U)) \leq \varepsilon$ for every $i \geq 1$. Hence, G is equicontinuous.

Theorem 3 now implies that G has compact closure in $C_\rho([0,1],X)$. Hence, a subsequence $\{\gamma_{n_i}\}$ of $\{\gamma_i\}$ converges to a path $\beta : [0,1] \to X$ in $C_\rho([0,1],X)$. Therefore, $\{\gamma_{n_i}(0)\} = \{x_{n_i}\}$ converges to $\beta(0)$, and $\{\gamma_{n_i}(1)\} = \{y_{n_i}\}$ converges to $\beta(1)$. Thus, $\beta(0) = x_0$ and $\beta(1) = y_0$. So β joins x_0 to y_0.

Let $0 \leq s \leq t \leq 1$. Then $\{\gamma_{n_i}(s)\}$ converges to $\beta(s)$, and $\{\gamma_{n_i}(t)\}$ converges to $\beta(t)$. Since the metric $\rho : X \times X \to [0,\infty)$ is continuous, we conclude that $\rho(\beta(s),\beta(t)) = \lim_{i\to\infty} \rho(\gamma_{n_i}(s),\gamma_{n_i}(t)) = \lim_{i\to\infty} \rho(x_{n_i},y_{n_i})(t-s) = \rho(x_0,y_0)(t-s)$. It follows from Lemma 11.b) that β is the unit interval parametrization of a geodesic joining x_0 to y_0. By hypothesis, this forces $\beta = \gamma_0$. We conclude that the subsequence $\{\gamma_{n_i}\}$ converges to γ_0 in $C_\rho([0,1],X)$.

Suppose that the full sequence $\{\gamma_i\}$ doesn't converge to γ_0 in $C_\rho([0,1],X)$. Then there is a $\delta > 0$ and a subsequence $\{\gamma_{s_i}\}$ of $\{\gamma_i\}$ such that $\sigma_\rho(\gamma_{s_i},\gamma_0) \geq \delta$ for $i \geq 1$. But since $\{(x_{s_i},y_{s_i})\}_{i\geq 1}$ converges to (x_0,y_0) in U, then the preceding argument proves that some subsequence of $\{\gamma_{s_i}\}$ converges to γ_0 in $C_\rho([0,1],X)$. We have reached a contradiction. We are forced to conclude that $\{\gamma_i\}$ converges to γ_0. □