A Topological Proof Of The Primitive Element Theorem, The Omegatator Of A Free Group

A subset B of a free group F is a basis for F if for every group G and every function $\beta: B \to G$, there is a unique homomorphism $\varphi: F \to G$ $\ni$ $\varphi | B = \beta$.

An element x of a free group F is a primitive element of F if there is a basis for F which contains x.

Using covering space techniques, we prove:

The Primitive Element Theorem: If $F_0 \supset F_1 \supset F_2 \supset \cdots$ is a decreasing sequence of free groups $\ni$ no primitive element of F_{i-1} belongs to F_i for $i = 1, 2, 3, \ldots$, then $\bigcap_{i=0}^{\infty} F_i = \{1\}$.

The commutator subgroup of a group G is the subgroup of G generated by $\{x y x^{-1} y^{-1} : x, y \in G\}$. If $G_0 \supset G_1 \supset G_2 \supset \cdots$ is the sequence of subgroups of G defined by $G_0 = G$ and G_i is the commutator subgroup of G_{i-1} for $i = 1, 2, 3, \ldots$, then $\omega(G) = \bigcap_{i=0}^{\infty} G_i$ is the omegatator subgroup of G.

The mod 2 commutator subgroup of a group G is the subgroup of G generated by $\{x^2 : x \in G\}$. If $G_0 \supset G_1 \supset G_2 \supset \cdots$ is the sequence of subgroups of G defined by $G_0 = G$ and G_i is the mod 2 commutator subgroup of G_{i-1} for $i = 1, 2, 3, \ldots$, then $\omega_2(G) = \bigcap_{i=0}^{\infty} G_i$ is the mod 2 omegatator subgroup of G.

We shall obtain a general corollary to the Primitive Element Theorem which contains the result that if F is a free group, then $\omega(F) = \{1\}$ and $\omega_2(F) = \{1\}$.

The following fact is central to our proof of the Primitive Element Theorem.

A Covering Space Lemma: Suppose
(a) for $i = 0, 1, 2, \ldots, \omega$, X_i is a path connected, locally path connected regular space and $b_i \in X_i$;
(b) for $0 \le j < i \le \omega$, $p_{ij} : (X_i, b_i) \to (X_j, b_j)$ is a covering projection such that for $0 \le k < j < i \le \omega$, $p_{ik} = p_{jk} \circ p_{ij}$
(hence for $0 \le k < j < i \le \omega$, $p_{ik*}\pi_1(X_i, b_i) \subset p_{jk*}\pi_1(X_j, b_j) \subset \pi_1(X_k, b_k)$);
(c) $p_{\omega 0*}\pi_1(X_\omega, b_\omega) = \bigcap_{0 \le i < \omega} p_{i0*}\pi_1(X_i, b_i)$.
If K is a compact subset of X_ω, then $\exists\, n$ such that $n < \omega$ and for $i \ge n$, $p_{\omega i}|K : K \to X_i$ is one-to-one.

Proof: Let K be a compact subset of X_ω.

For a path connected open subset U of X_0, let $Q(U)$ denote the collection of path components of $p_{\omega 0}^{-1}(U)$. Since X_ω is locally path connected, the elements of $Q(U)$ are path connected open subsets of X_ω. A path connected open subset U of X_0 is evenly covered by $p_{\omega 0}$ if $\forall\, W \in Q(U)$, $p_{\omega 0}|W : W \to U$ is a homeomorphism.

Claim (1): If U is a path connected open subset of X_0 which is evenly covered by $p_{\omega 0}$, V is a path connected open subset of X_0 and $\bar{V} \subset U$, then V is evenly covered by $p_{\omega 0}$ and the number of elements of $Q(V)$ which intersect K is finite.

For every $W \in Q(U)$, let $W_* = (p_{\omega 0}|W)^{-1}(V)$. Then $Q(V) = \{W_* : W \in Q(U)\}$ and for every $W \in Q(U)$, $p_{\omega 0}|W_* : W_* \to V$ is a homeomorphism. This shows V is evenly covered by $p_{\omega 0}$.

Furthermore, for every $W \in Q(U)$, $\overline{W_*} = (p_{\omega 0}|W)^{-1}(\bar{V}) \subset W$, and $p_{\omega 0}^{-1}(\bar{V}) = \bigcup \{\overline{W_*} : W \in Q(U)\}$. Since $p_{\omega 0}^{-1}(\bar{V})$ is a closed subset of X_ω, then $K \cap p_{\omega 0}^{-1}(\bar{V})$ is compact. $\{W \in Q(U) : \overline{W_*} \cap K \neq \emptyset\}$ is an open cover of $K \cap p_{\omega 0}^{-1}(\bar{V})$, no proper subset of which covers $K \cap p_{\omega 0}^{-1}(\bar{V})$. Hence $\{W \in Q(U) : \overline{W_*} \cap K \neq \emptyset\}$ must be finite. Therefore $\{W \in Q(U) : W_* \cap K \neq \emptyset\}$ must be finite; so only finitely many elements of $Q(V)$ intersect K. ∎

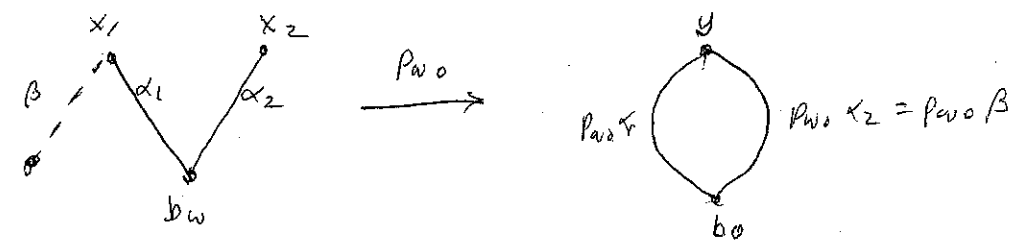
Claim (2): There is a collection $\mathcal{V}$ of path connected open subsets of X_0 which covers X_0 $\ni$ $\forall V \in \mathcal{V}$, V is evenly covered by $p_{\omega 0}$ and the number of elements of $Q(V)$ which intersect K is finite.

(a) and (b) imply that every point x of X_0 has a path connected open nbhd U which is evenly covered by $p_{\omega 0}$. Since X_0 is locally path connected and regular, then there is a path connected open nbhd V of x $\ni$ $\bar{V} \subset U$. Then by Claim (1), V is evenly covered by $p_{\omega 0}$ and the number of elements of $Q(V)$ which intersect K is finite. ∎

Claim (3): If V is a path connected open subsets of X_0 which is evenly covered by $p_{\omega 0}$, $W_1, W_2 \in Q(V)$ and $W_1 \neq W_2$, then $\exists n \ni n<\omega$ and for $i \geq n$, $p_{\omega i}(W_1) \cap p_{\omega i}(W_2) = \emptyset$.

Let $y \in V$, $x_1 \in W_1 \cap p_{\omega 0}^{-1}(y)$ and $x_2 \in W_2 \cap p_{\omega 0}^{-1}(y)$. We shall prove $\exists n \ni n<\omega$ and for $i \geq n$, $p_{\omega i}(x_1) \neq p_{\omega i}(x_2)$. It will then follow that for $i \geq n$, $p_{\omega i}(W_1) \cap p_{\omega i}(W_2) = \emptyset$.

For $i=1,2$, let $\alpha_i : (I,0,1) \to (X_\omega, b_\omega, x_i)$ be a path. Let $\beta : (I,1) \to (X_\omega, x_1)$ be a path $\ni$ $p_{\omega 0}\beta = p_{\omega 0}\alpha_2$.



Then $\beta(0) \neq b_\omega$ (for $\beta(0) = b_\omega \Rightarrow \beta = \alpha_2 \Rightarrow x_1 = \beta(1) = \alpha_2(1) = x_2$).

Since $p_{\omega 0}(\alpha_1 \cdot \beta^{-1})$ is a loop in in X_0 at b_0, then $[p_{\omega 0}(\alpha_1 \cdot \beta^{-1})] \in \pi_1(X_0, b_0)$. Since the lift $\alpha_1 \cdot \beta^{-1}$ of $p_{\omega 0}(\alpha_1 \cdot \beta^{-1})$ to a path in X_ω starting at b_ω is not a loop, then $[p_{\omega 0}(\alpha_1 \cdot \beta^{-1})] \notin p_{\omega 0 *}\pi_1(X_\omega, b_\omega)$. Then (b) and (c) imply $\exists n \ni n<\omega$ and for $i \geq n$, $[p_{\omega 0}(\alpha_1 \cdot \beta^{-1})] \notin p_{i0*}\pi_1(X_i, b_i)$.

Hence for $i \geq n$, the lift $p_{\omega i}(\alpha_1 \cdot \beta^{-1})$ of $p_{\omega 0}(\alpha_1 \cdot \beta^{-1})$ to a path ~~in~~ X_i starting at b_i is not a loop. It follows that $p_{\omega i}(x_1) \neq p_{\omega i}(x_2)$; for $p_{\omega i}(x_1) = p_{\omega i}(x_2) \Rightarrow p_{\omega i}\beta = p_{\omega i}\alpha_2$ (because $p_{\omega i}\beta$ and $p_{\omega i}\alpha_2$ are both lifts of $p_{\omega 0}\beta = p_{\omega 0}\alpha_2$ and $p_{\omega i}\beta(1) = p_{\omega i}(x_1) = p_{\omega i}(x_2) = p_{\omega i}\alpha_2(1)$) $\Rightarrow p_{\omega i}(\alpha_1 \cdot \beta^{-1}) = (p_{\omega i}\alpha_1) \cdot (p_{\omega i}\alpha_2)^{-1}$ $\Rightarrow p_{\omega i}(\alpha_1 \cdot \beta^{-1})$ is a loop — a contradiction.

Now let $i \geq n$ and suppose $p_{\omega i}(W_1) \cap p_{\omega i}(W_2) \neq \emptyset$. Let $\bar{x}_1 \in W_1$ and $\bar{x}_2 \in W_2 \ni p_{\omega i}(\bar{x}_1) = p_{\omega i}(\bar{x}_2)$. Then $p_{\omega o}(\bar{x}_1) = p_{\omega o}(\bar{x}_2)$, because $p_{\omega o} = p_{io} \circ p_{\omega i}$. Let $\bar{y} = p_{\omega o}(\bar{x}_1) = p_{\omega o}(\bar{x}_2)$. Let $\gamma_1 : (I, 0, 1) \to (W_1, x_1, \bar{x}_1)$ be a path. Then $p_{\omega o} \circ \gamma_1 : (I, 0, 1) \to (V, y, \bar{y})$ is a path. Since $p_{\omega o} | W_2 : (W_2, x_2, \bar{x}_2) \to (V, y, \bar{y})$ is a homeomorphism, then $\gamma_2 = (p_{\omega o}|W_2)^{-1} \circ p_{\omega o} \circ \gamma_1$ defines a path $\gamma_2 : (I, 0, 1) \to (W_2, x_2, \bar{x}_2) \ni p_{\omega o}\gamma_1 = p_{\omega o}\gamma_2$. Since $p_{\omega i}\gamma_1$ and $p_{\omega i}\gamma_2$ are both lifts of $p_{\omega o}\gamma_1 = p_{\omega o}\gamma_2$ and $p_{\omega i}\gamma_1(1) = p_{\omega i}(\bar{x}_1) = p_{\omega i}(\bar{x}_2) = p_{\omega i}\gamma_2(1)$, then $p_{\omega i}\gamma_1 = p_{\omega i}\gamma_2$. Consequently $p_{\omega i}(x_1) = p_{\omega i}\gamma_1(0) = p_{\omega i}\gamma_2(0) = p_{\omega i}(x_2)$, a contradiction. This proves $p_{\omega i}(W_1) \cap p_{\omega i}(W_2) = \emptyset$. ∎

Claim (2) guarantees the existence of a collection $\mathcal{V}$ of path connected open subsets of X_0 which covers $X_0 \ni \forall V \in \mathcal{V}$, V is evenly covered by $p_{\omega o}$ and the number of elements of $Q(V)$ which intersects K is finite. Since $p_{\omega o}(K)$ is compact, there is a finite subset $\{V_1, V_2, \ldots, V_q\}$ of $\mathcal{V}$ which covers $p_{\omega o}(K)$. For $j = 1, 2, \ldots, q$, there is a finite number, r_j, of elements of $Q(V_j)$ which intersect K; say $\{W \in Q(V_j) : W \cap K \neq \emptyset\} = \{W_{j1}, W_{j2}, \ldots, W_{jr_j}\}$ where for $1 \leq k < l \leq r_j$, $W_{jk} \neq W_{jl}$.

Now for $j = 1, 2, \ldots, q$ and $1 \le k < l \le r_j$, claim (3) guarantees the existence of an $n(j,k,l) < \omega \ni$ for $i \ge n(j,k,l)$, $p_{\omega i}(W_{jk}) \cap p_{\omega i}(W_{jl}) = \emptyset$.

Let $n = \max\{n(j,k,l) : j = 1, 2, \ldots, q$ and $1 \le k < l \le r_j\}$.
Then $n < \omega$.

Claim (4): For $i \ge n$, $p_{\omega i}|K : K \to X_i$ is one-to-one.

Suppose $i \ge n$, $x_1, x_2 \in K$ and $p_{\omega i}(x_1) = p_{\omega i}(x_2)$. We shall prove $x_1 = x_2$. $p_{\omega 0}(x_1) = p_{\omega 0}(x_2)$ because $p_{\omega 0} = p_{i0} \circ p_{\omega i}$. Since $p_{\omega 0}(x_1) = p_{\omega 0}(x_2) \in p_{\omega 0}(K)$, then $\exists\, j \in \{1, 2, \ldots, q\} \ni$ $p_{\omega 0}(x_1) = p_{\omega 0}(x_2) \in V_j$. Hence $\exists\, k, l \in \{1, 2, \ldots, r_j\} \ni$ $x_1 \in W_{jk}$ and $x_2 \in W_{jl}$. Assume $k \le l$. If $k < l$, then the fact that $i \ge n \ge n(j,k,l)$ implies $p_{\omega i}(W_{jk}) \cap p_{\omega i}(W_{jl}) = \emptyset$, which is impossible if $p_{\omega i}(x_1) = p_{\omega i}(x_2)$. So $k = l$. Then the fact that $p_{\omega 0}|W_{jk}$ is a homeomorphism implies $x_1 = x_2$. ∎

This proves the Covering Space Lemma.

We now give the proof of

<u>The Primitive Element Theorem</u>: If $F_0 \supset F_1 \supset F_2 \supset \cdots$ is a decreasing sequence of free groups $\ni$ no primitive element of F_{i-1} belongs to F_i for $i = 1, 2, 3, \ldots$, then $\bigcap_{i=0}^{\infty} F_i = \{1\}$.

<u>Proof</u>: Let $F_\omega = \bigcap_{i=0}^{\infty} F_i$. $\exists$ a graph Γ_0 with $b_0 \in \Gamma_0$ $\ni$ $\pi_1(\Gamma_0, b_0) = F_0$. Furthermore for $0 < i \le \omega$, $\exists$ covering spaces $p_{i0} : (\Gamma_i, b_i) \to (\Gamma_0, b_0)$ $\ni$ $p_{i0*}\pi_1(\Gamma_i, b_i) = F_i$. For $0 < j < i \le \omega$, since $F_i \subset F_j$, then $\exists$ a covering projection $p_{ij} : (\Gamma_i, b_i) \to (\Gamma_j, b_j)$ $\ni$ the diagram

$$\begin{tikzcd}
(\Gamma_i, b_i) \arrow[rr, "p_{ij}"] \arrow[dr, "p_{i0}"'] & & (\Gamma_j, b_j) \arrow[dl, "p_{j0}"] \\
& (\Gamma_0, b_0) &
\end{tikzcd}$$

commutes. The unique lifting theorem implies that of $0 \le k < j < i \le \omega$, the diagram

$$\begin{tikzcd}
(\Gamma_i, b_i) \arrow[rr, "p_{ij}"] \arrow[dr, "p_{ik}"'] & & (\Gamma_j, b_j) \arrow[dl, "p_{jk}"] \\
& (\Gamma_k, b_k) &
\end{tikzcd}$$

commutes; i.e., $p_{ik} = p_{jk} \circ p_{ij}$.

For $0 \le i \le \omega$, Γ_i is a graph. Suppose Γ_0 has a triangulation in which b_0 is a vertex. Then for $0 < i \le \omega$, p_{i0} induces a unique triangulation on Γ_i $\ni$ $p_{i0} : (\Gamma_i, b_i) \to (\Gamma_0, b_0)$ is a non-degenerate simplicial map. Consequently for $0 \le j < i \le \omega$, $p_{ij} : (\Gamma_i, b_i) \to (\Gamma_j, b_j)$ is a non-degenerate simplicial map.

Let T_ω be a maximal tree in the triangulation of Γ_ω containing b_ω. We shall prove $T_\omega = \Gamma_\omega$, whence $\pi_1(\Gamma_\omega, b_\omega) = \{1\}$,

from which it follows that $F_\omega = p_{\omega 0 *}\pi_1(\Gamma_\omega, b_\omega) = \{1\}$.
So suppose $T_\omega \neq \Gamma_\omega$ and A is a 1-simplex of Γ_ω which does not lie in T_ω. Then $A \cap T_\omega = \partial A = \{a_0, a_1\}$. Let $\alpha : (I, 0, 1) \to (A, a_0, a_1)$ be an arc and for $i = 0, 1$, let $\beta_i : (I, 0, 1) \to (T_\omega, b_\omega, a_i)$ be an arc. Let $T = \beta_0(I) \cup \beta_1(I)$ and $K = T \cup \alpha(I)$. Then T is a finite tree in the triangulation of T_ω; in fact, T is either a triod whose three endpoints are b_ω, a_0 and a_1, or an arc whose endpoints are a_0 and a_1. K is a compact subcomplex of Γ_ω which contains a non-degenerate cycle, and which, therefore, can never lie in a tree.

By the Covering Space Lemma, $\exists\, n \ni n < \omega$ and for $i \geq n$, $p_{\omega i}|K : K \to \Gamma_i$ is one-to-one. Let $i \geq n$. Then $p_{\omega i}(T)$ is a finite tree in the triangulation of Γ_i and $p_{\omega i}(K)$ is a compact subcomplex of Γ_i which can never lie in a tree. Extend $p_{\omega i}(T)$ to a maximal tree T_i in the triangulation of Γ_i. Then $p_{\omega i}(A)$ is a simplex of the triangulation of Γ_i which cannot lie in T_i. So $p_{\omega i}(A) \cap T_i = p_{\omega i}(\partial A) = p_i\{a_0, a_1\}$. It follows that $[p_{\omega i}(\beta_0 \cdot \alpha \cdot \beta_1^{-1})]$ is a primitive element of $\pi_1(\Gamma_i, b_i)$. Since $p_{i0*} : \pi_1(\Gamma_i, b_i) \to F_i$ is an isomorphism, then $p_{i0*}[p_{\omega i}(\beta_0 \cdot \alpha \cdot \beta_1^{-1})]$ is a primitive element of F_i. But $p_{i0*}[p_{\omega i}(\beta_0 \cdot \alpha \cdot \beta_1^{-1})] = [p_{\omega 0}(\beta_0 \cdot \alpha \cdot \beta_1^{-1})]$. So we have proved that for $i \geq n$, $[p_{\omega 0}(\beta_0 \cdot \alpha \cdot \beta_1^{-1})]$ is a primitive element of F_i. This contradicts the hypothesis of the theorem. So $T_\omega = \Gamma_\omega$. ∎

Notice that in the proof we show that $[p\circ(\beta_0\cdot\alpha\cdot\beta_i^{-1})]$ is a primitive element of F_i for all $i \geq n$. So apparently we can weaken the hypothesis of the Primitive Element Theorem as follows: <u>If $F_0 \supset F_1 \supset F_2 \supset \cdots$ is a decreasing sequence of free groups $\ni$ no element of F_{i-1} which is primitive in F_{i-1} is also primitive in F_i for $i = 1, 2, 3, \ldots$, then $\bigcap_{i=0}^{\infty} F_i = \{1\}$</u>. In actuality, the two hypotheses are equivalent, as is shown by the following.

<u>Proposition</u>: <u>If $F_0 \supset F_1$ are free groups, x is a primitive element of F_0 and $x \in F_1$, then x is a primitive element of F_1.</u>

<u>Proof</u>: $\exists$ a graph Γ_0, a vertex $b_0 \in \Gamma_0$, a maximal tree T_0 in the triangulation of Γ_0 containing b_0, a 1-simplex A of the triangulation of Γ_0 which is not contained in T_0 so that $A \cap T_0 = \partial A = \{a_0, a_1\}$, and arcs $\alpha: (I, 0, 1) \to (A, a_0, a_1)$ and $\beta_i: (I, 0, 1) \to (T_0, b_0, a_i)$ for $i = 0, 1$ $\ni$ $\pi_1(\Gamma_0, b_0) = F_0$ and $[\beta_0\cdot\alpha\cdot\beta_1^{-1}] = x$. Furthermore, $\exists$ a covering space $p: (\Gamma_1, b_1) \to (\Gamma_0, b_0)$ $\ni$ $p_* \pi_1(\Gamma_1, b_1) = F_1$. Then Γ_1 is a graph, and the triangulation of Γ_0 lifts to a unique triangulation of Γ_1 $\ni$ p is a nondegenerate simplicial map.

Let $T = \beta_0(I) \cup \beta_1(I)$. Then T is a finite tree in the triangulation of Γ_0; in fact T is a triod whose three endpoints are b_0, a_0 and a_1. $T \cup A$ is a finite subcomplex of Γ_0 which contains a non-degenerate cycle, and which, therefore, can never lie in a tree. Clearly $\pi_1(T \cup A, b_0)$ is an infinite cyclic group generated by $[\beta_0 \cdot \alpha \cdot \beta_1^{-1}]$. Let $i: T \cup A \subset \Gamma_0$. Then $i_* \pi_1(T \cup A, b_0) = (x)$, the subgroup of F_0 generated by x. Since $(x) \subset F_1$, then $\exists$ a lift $j: (T \cup A, b_0) \to (\Gamma_1, b_1)$ of $i: T \cup A \subset \Gamma_0$; so $p \circ j = i$ and j is a simplicial imbedding.

Then $j(T)$ is a finite tree in the triangulation of Γ_1 and $j(T \cup A)$ is a finite subcomplex of Γ_1 which can never lie in a tree. Extend $j(T)$ to a maximal tree T_1 in the triangulation of Γ_1. Then $j(A)$ is a simplex of the triangulation of Γ_1 which cannot lie in T_1. So $j(A) \cap T_1 = j(\partial A) = j\{a_0, a_1\}$. It follows that $[j(\beta_0 \cdot \alpha \cdot \beta_1^{-1})]$ is a primitive element of $\pi_1(\Gamma_1, b_1)$. Since $p_*: \pi_1(\Gamma_1, b_1) \to F_1$ is an isomorphism, then $p_*[j(\beta_0 \cdot \alpha \cdot \beta_1^{-1})]$ is a primitive element of F_1. But $p_*[j(\beta_0 \cdot \alpha \cdot \beta_1^{-1})] = [\beta_0 \cdot \alpha \cdot \beta_1^{-1}] = x$. So x is a primitive element of F_1. ▩

We now make an application of the Primitive Element Theorem. Let $\mathcal{G}$ be the category of groups and homomorphisms. We call a covariant functor $\tau: \mathcal{G} \to \mathcal{G}$ an <u>algebraic</u> functor if

(a) for every group G, $\tau(G)$ is a subgroup of G, and

(b) for every homomorphism $\varphi: G \to H$ of groups G and H, $\varphi\tau(G) \subset \tau(H)$ and $\tau(\varphi) = \varphi \mid \tau(G)$.

If, in addition, τ satisfies

(c) $\tau(\mathbb{Z})$ is a <u>proper</u> subgroup of $\mathbb{Z}$,

then we call τ a <u>proper</u> algebraic functor.

<u>Proposition:</u> <u>If $\tau: \mathcal{G} \to \mathcal{G}$ is an algebraic functor, then for every group G, $\tau(G)$ is a normal subgroup of G.</u>

<u>Proof:</u> For $y \in G$, let $\varphi_y: G \to G$ be the automorphism of G obtained by conjugating by y; i.e., $\varphi_y(x) = yxy^{-1}$. Since $\varphi_y \tau(G) \subset \tau(G)$, then $y\tau(G)y^{-1} \subset \tau(G)$, for $y \in G$. ∎

If $\tau: \mathcal{G} \to \mathcal{G}$ is a functor, then for $i = 1, 2, 3, \ldots$ define the functor $\tau^i: \mathcal{G} \to \mathcal{G}$ by $\tau^1 = \tau$ and $\tau^{i+1} = \tau \circ \tau^i$ for $i = 1, 2, 3, \ldots$. It is easy to see that if τ is a (proper) algebraic functor, then so is τ^i for $i = 1, 2, 3, \ldots$.

If $\tau: \mathcal{G} \to \mathcal{G}$ is a (proper) algebraic functor, define a functor $\tau^\omega: \mathcal{G} \to \mathcal{G}$ as follows:

(a) for every group G, $\tau^\omega(G) = \bigcap_{i=1}^{\infty} \tau^i(G)$, and

(b) for every homomorphism $\varphi: G \to H$ of groups G and H, since $\varphi\tau^i(G) \subset \tau^i(H)$ for $i = 1, 2, 3, \dots$, then $\varphi\tau^\omega(G) \subset \tau^\omega(H)$, and we can define $\tau^\omega(\varphi): \tau^\omega(G) \to \tau^\omega(H)$ by $\tau^\omega(\varphi) = \varphi / \tau^\omega(G)$.

Clearly $\tau^\omega: \mathcal{G} \to \mathcal{G}$ is a (proper) algebraic functor.

A general class of examples of algebraic functors is described as follows. Let $\mathcal{W}$ be a set of words in the variables $x_1, x_2, x_3, \dots$; i.e., an element of $\mathcal{W}$ is an expression $w = w(x_1, x_2, x_3, \dots)$ of the form "$x_{i_1}^{n_1} x_{i_2}^{n_2} \cdots x_{i_k}^{n_k}$" where $k, i_1, i_2, \dots, i_k \in \{1, 2, 3, \dots\}$ and $n_1, n_2, \dots, n_k \in \mathbb{Z}$. For a group G, define $\tau(G)$ to be the subgroup of G generated by the subset $\{w(x_1, x_2, x_3, \dots) : w \in \mathcal{W} \text{ and } x_1, x_2, x_3, \dots \in G\}$.
For a homomorphism $\varphi: G \to H$, if $w \in \mathcal{W}$ and $x_1, x_2, x_3, \dots \in G$, then $\varphi(w(x_1, x_2, x_3, \dots)) = w(\varphi(x_1), \varphi(x_2), \varphi(x_3), \dots)$; so $\varphi\tau(G) \subset \tau(H)$, and we can define $\tau(\varphi) = \varphi | \tau(G)$. Thus τ is an algebraic functor. Obviously τ is proper if and only if $1 \notin \tau(\mathbb{Z})$.

We now present a familiar instance of this class of algebraic functor. For a group G, let $\tau_*(G)$ be the <u>commutator subgroup</u> of G; i.e., the subgroup of G generated by the subset $\{xyx^{-1}y^{-1} : x, y \in G\}$. Then τ_* is a proper algebraic functor of the type just described. The proper algebraic functor $(\tau_*)^\omega$ is usually denoted simply by ω; and for a group G, $\omega(G)$ is called the <u>omegatator subgroup</u> of G.

Another instance of this class of algebraic functors is obtained as follows. Let p be a positive integer > 1. For a group G, let $\tau_p(G)$ be the subgroup of G generated by the subset $\{x^p : x \in G\}$. Then τ_p is a proper algebraic functor of the type described above. Let us denote the proper algebraic functor $(\tau_p)^\omega$ simply by ω_p. For a group G, if $x, y \in G$, then $xyx^{-1}y^{-1} = (xy)^2(y^{-1}x^{-1}y)^2(y^{-1})^2$; so we see that <u>$\tau_*(G) \subset \tau_2(G)$</u> and, hence, that <u>$\omega(G) \subset \omega_2(G)$</u>. For this reason $\tau_2(G)$ is called the <u>mod 2 commutator subgroup</u> of G and $\omega_2(G)$ is called the <u>mod 2 omegatator subgroup</u> of G.

<u>Lemma</u>: <u>If $\tau : \mathcal{G} \to \mathcal{G}$ is a proper algebraic functor and F is a free group, then no primitive element of F belongs to $\tau(F)$.</u>

<u>Proof</u>: Suppose x is a primitive element of F and B is a basis for F containing x. Define the function $\beta : B \to \mathbb{Z}$ by $\beta(b) = 1$ for all $b \in \mathbb{Z}$. Then there is a

homomorphism $\varphi: F \to \mathbb{Z}_1 \ni \varphi|B = \beta$; so $\varphi(x) = 1$. Since τ is a proper algebraic functor, $\tau(\mathbb{Z}_1)$ is a proper subgroup of $\mathbb{Z}_1$, so that $1 \notin \tau(\mathbb{Z})$. Thus $\varphi\tau(F) \subset \tau(\mathbb{Z}_1)$ but $\varphi(x) \notin \tau(\mathbb{Z})$, proving $x \notin \tau(F)$. ∎

<u>Corollary: If $\tau: \mathcal{G} \to \mathcal{G}$ is a proper algebraic functor and F is a free group, then $\tau^{\omega}(F) = \{1\}$.</u>

<u>Proof:</u> Define a decreasing sequence $F_0 \supset F_1 \supset F_2 \supset \cdots$ of free subgroups of F by $F_0 = F$ and $F_i = \tau(F_{i-1}) = \tau^i(F)$ for $i = 1, 2, 3, \ldots$. The preceding lemma implies that no primitive element of F_{i-1} belongs to F_i for $i = 1, 2, 3, \ldots$. Now the Primitive Element Theorem implies that $\tau^{\omega}(F) = \bigcap_{i \geq 0}^{\infty} F_i = \{1\}$. ∎

<u>Corollary: If F is a free group, then $\omega(F) = \{1\}$ and $\omega_2(F) = \{1\}$.</u>