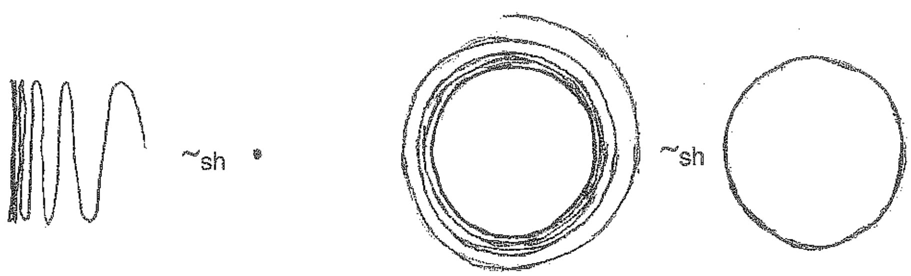
On the Cell-like Equivalence of CAT(0) Group Boundaries

by F. Ancel, C. Guilbault[1] and J. Wilson

Roughly speaking, two metric compacta X and Y are *shape equivalent,* denoted $X \sim_{sh} Y$, if when X and Y are embedded in the Hilbert cube $[0,1]^{\infty}$, they have homotopy equivalent neighborhood sequences mod index shifts. (See [C] page 39, for example, for the precise definition.)

Examples.



Definition. A metric compactum is a *cell-like set* if it is shape equivalent to a point. A function $f : X \to Y$ between metric compact is *cell-like* if $f^{-1}(y)$ is a cell-like set for every $y \in Y$.

Definition. Two finite dimensional metric compacta X and Y are *cell-like equivalent,* denoted $X \sim_{ce} Y$, if there is a finite "zigzag" sequence of finite dimensional metric compacta and cell-like maps

$$\begin{tikzcd}[column sep=small] & Z_1 \arrow[dl] \arrow[dr] & & Z_3 \arrow[dl] \arrow[dr] & & {} \arrow[dl] & \cdots & {} \arrow[dr] & & Z_{2n+1} \arrow[dl] \arrow[dr] & \\ X & & Z_2 & & Z_4 & & & & Z_{2n} & & Y \end{tikzcd}$$

joining X to Y.

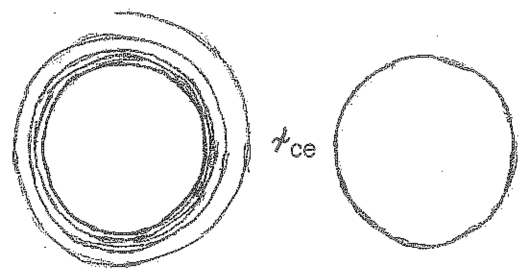
Fact: [Sh] For finite dimensional metric compacta, $X \sim_{ce} Y \Rightarrow X \sim_{sh} Y$.

Remark. This implication is false without the assumption of finite dimensionality for X, Y and all the Z_n's in the definition of "cell-like equivalent". [T]

[1] The second author wishes to acknowledge the support of the National Science Foundation.

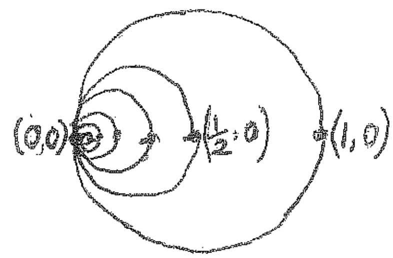
Also, the converse of this implication is false:

Example. [F]



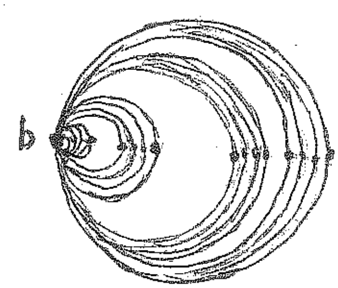
Definition. If p and q are distinct points in the Euclidean plane $\mathbb{R}^2$, let [p,q] denote the straight line segment joining them and let C[p,q] denote the unique circle with diameter [p,q]. The *Hawaiian earring* is the space

$$\mathcal{H} = \bigcup_{n \geq 1} C[(0,0),(1/n,0)]$$



More generally, if X is any compact totally disconnected infinite subset of the x axis $\mathbb{R} \times \{0\}$ and $b \in X$, any space homeomorphic to

$$\mathcal{H}(X,b) = \bigcup \{ C[b,p] : p \in X \}$$



is called a *generalized Hawaiian earring.*

Lemma 1. If X is any compact totally disconnected infinite subset of the x axis and $b \in X$, then $\mathcal{H}(X,b) \sim_{ce} \mathcal{H}$.

Proof. $\mathcal{H}(X,b) \swarrow \Sigma(X) \swarrow C(X) \cup J \searrow \mathcal{H}$,

where:

- $\Sigma(X)$ denotes the suspension of X and the cell-like map $\Sigma(X) \to \mathcal{H}(X,b)$ is the quotient map obtained from $\Sigma(X)$ by crushing the suspension arc through the point b to a point.

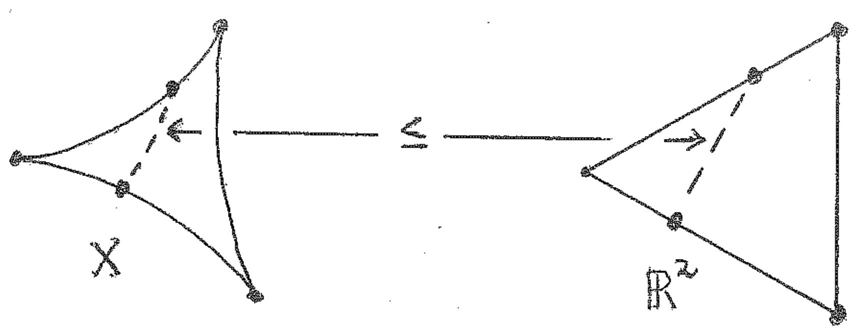
- J is the shortest closed interval in the x axis containing the set X, C(X) is the cone over the set X from a vertex point lying above the x axis in $\mathbb{R}^2$, and the cell-like map $C(X) \cup J \to \Sigma(X)$ is the quotient map obtained from $C(X) \cup J$ by crushing J to a point.
- The cell-like map frm $C(X) \cup J \to \mathcal{H}$ is the quotient map obtained from $C(X) \cup J$ by crushing C(X) to a point. (The resulting quotient space is the wedge of a decreasing sequence of circles. This space is homeomorphic to $\mathcal{H}$.) □

Results like this are found in [DV].

Geometric group theory originated in the work of M. Dehn who in 1912 proved that the fundamental group of a surface of genus ≥ 2 has a solvable word problem. His proof is geometric. It exploits the fact that the universal cover of a surface of genus ≥ 2 can be identified with the hyperbolic plane so that the fundamental group of the surface acts by isometry on the hyperbolic plane. In the 1980's, M. Gromov achieved a sweeping generalization of Dehn's work by introducing and studying hyperbolic groups. These are groups that act by isometry on metric spaces which have negative curvature in a very general sense. A further generalization to groups that act by isometry on non-positively curved (or CAT(0)) metric spaces has led to the study of CAT(0) groups.

Definition. X is a *CAT(0) space* if:

- X is a *geodesic* metric space. (For any two points p and $q \in X$, if $d = d(p,q)$, then there is an embedding $e : [0,d] \to X$ such that $e(0) = p$, $e(d) = q$, and $d(e(s),e(t)) = |s-t|$ for all $s, t \in [0,d]$.)
- X is a *proper* metric space. (For every $p \in X$ and every $r > 0$, the closed metric ball $B_r(p) = \{ q \in X : d(p,q) \leq r \}$ is compact.)
- Distances in geodesic triangles in X are dominated by distances in comparison triangles in $\mathbb{R}^2$.

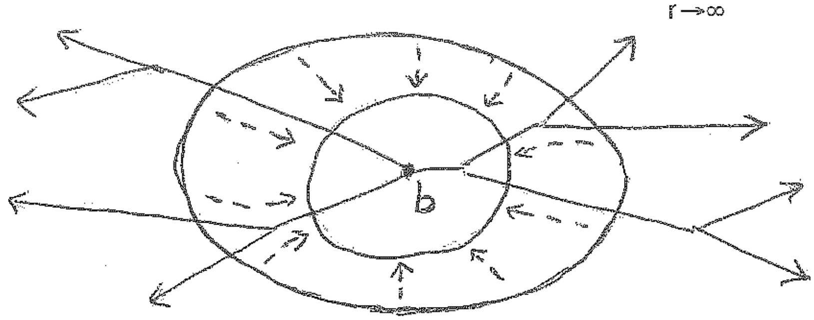


The concept of a CAT(0) space comes from the work of A. Alexandrov in the 1950's, although the terminology is Gromov's. The book [BH] is a comprehensive source.

Definition. A space X is *hyperbolic* if it as a proper geodesic metric space and there is a $\delta > 0$ such that all geodesic triangles in X are δ-slim. A geodesic triangle T in a metric space X is *δ-slim* if every non-vertex point p of T is distance $< \delta$ from a point on one of the two sides of T that doesn't contain p.

Definition. Let X be a CAT(0) space and choose a basepoint $b \in X$. Then X has a *visual boundary* denoted ∂X:

$$\partial X = \{ \text{geodesic rays in X emanating from b} \} = \varprojlim_{r \to \infty} \partial B_r(b).$$



Remark. We regard geodesic rays as functions from $[0,\infty)$ into X, and we put the compact-open topology on ∂X. The CAT(0) property implies that ∂X is independent of the choice of basepoint. Alternatively, the CAT(0) property implies that for $0 < r < s$, a geodesic joining b to a point of $\partial B_s(b)$ intersects $\partial B_r(b)$ at a unique point. This observation gives rise to the "geodesic retraction" $g_{s,r} : \partial B_s(b) \to \partial B_r(b)$. Hence, we obtain an inverse system $\{ \partial B_r(b), g_{r,s} \}$ which has an inverse limit $\varprojlim_{r \to \infty} \partial B_r(b)$. This inverse limit is also homeomorphic to ∂X.

Remark. The visual boundary of a hyperbolic space is defined similarly. See [BH], page 427, for details.

Definition. A group G is a *CAT(0) group* if G acts geometrically on a CAT(0) space. The action of a group G on a metric space X is *geometric* if the action is:

- properly discontinuous (For every compact subset C of X, $\{ g \in G : C \cap g(C) \neq \varnothing \}$ is finite.)
- cocompact (The orbit space X/G is compact.), and
- by isometry.

Definition. A group is *hyperbolic* if it acts geometrically on a hyperbolic space.

Remark. If a CAT(0) (or hyperbolic) group G acts geometrically on a CAT(0) (or hyperbolic) space X, then the action of G on X extends to an action of G on ∂X (because the elements of G are isometries of X, and isometries of X carry geodesic rays to geodesic rays).

Definition. If a CAT(0) (or hyperbolic) group G acts geometrically on a CAT(0) (or hyperbolic) space X, then the visual boundary ∂X of X is called a *boundary* of G.

Remark. There is an intimate connection between certain algebraic properties of a CAT(0) group G and certain topological properties of the boundaries of G. For example:

- If ∂G is any boundary of G, then c-dim$_{\mathbb{Z}}$ G = dim(∂G) + 1. [BM], [B]
- If G is a one-ended CAT(0) group, ∂G is any boundary of G, and ∂G has a global cut point, then G contains an infinite torsion subgroup that fixes the global cut point. [Sw]

Theorem. (Gromov) If G is a hyperbolic group, then all the boundaries of G are equivariantly homeomorphic. ([G], page 189.)

Remark. The use of "equivariant" makes sense here because the action of G on X extends to an action of G on ∂X.

Example. The *Croke-Kleiner group* is the group

$$\Gamma = (a, b, c, d \mid [a,b] = [b,c] = [c,d] = 1).$$

(In other words, Γ is the free group generated by a, b, c and d modulo the smallest normal subgroup containing the three commutators $[a,b] = aba^{-1}b^{-1}$, $[b,c] = bcb^{-1}c^{-1}$ and $[c,d] = cdc^{-1}d^{-1}$.) Γ is a CAT(0) group with non-homeomorphic boundaries. [CK]

Theorem. [B] If G is a CAT(0) group, then all the boundaries of G are shape equivalent.

Question. [B] Are all the boundaries of a given CAT(0) group cell-like equivalent?

Digression. Within the discipline of the philosophy of science, an oft quoted example of an apparent scientific law is the statement "All crows are black". (The question raised in conjunction with this law is whether a non-black non-crow (e.g., a white rabbit) should be regarded as supporting evidence for this law.) Regardless of whether this statement is truly a scientific law, there is an assertion which at this moment in time may be regarded as a truth:

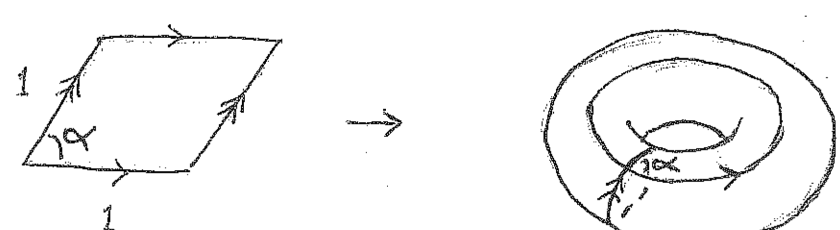
All known crows are black.

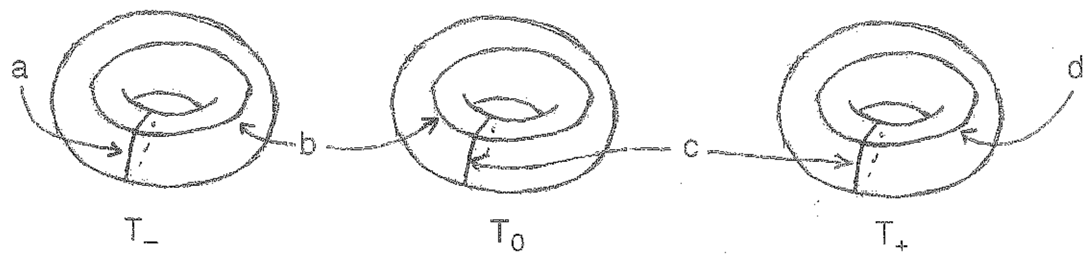
Encouraged by this example of a scientific truth, we state the principal result of this article.

Theorem 1. (F. Ancel, C. Guilbault, J. Wilson) All known boundaries of the Croke-Kleiner group are cell-like equivalent - - - to the Hawaiian earring $\mathcal{H}$.

More precisely: In [CK] an infinite family of CAT(0) spaces $X(\alpha)$, $0 < \alpha \leq \pi/2$, is described with the property that the Croke-Kleiner group Γ acts geometrically on each of them, and the their visual boundaries are not all homeomorphic. At the moment this is being written, the visual boundaries of the spaces $X(\alpha)$, $0 < \alpha \leq \pi/2$, are the only know boundaries of the Croke-Kleiner group Γ. We prove that for each $\alpha \in (0,\pi/2]$, the visual boundary of $X(\alpha)$ is cell-like equivalent to the Hawaiian earring $\mathcal{H}$.

Sketch of proof of Theorem 1. We first describe some basic features of the CAT(0) spaces $X(\alpha)$. Let $0 < \alpha \leq \pi/2$. $X(\alpha)$ is the universal cover of a space $Y(\alpha)$ that is the union of three 2-dimensional tori T_-, T_0, and T_+. Each of these tori is obtained by isometrically identifying opposite edges of a parallelogram in which each side has edge length 1 and the angles between the sides are α and $\pi - \alpha$. Thus, the fundamental group of each torus is generated by two closed geodesics of length 1 that intersect in a single point where the angle between them is α. To form the space $Y(\alpha)$ from the three tori, let *b* and *c* denote the two length 1 closed geodesic π_1-generators on T_0, identify b with one of the length 1 closed geodesic π_1-generators on T_-, and identify c with one of the length 1 closed geodesic π_1-generators on T_+. If we let *a* denote the other (non-identified) length 1 closed geodesic π_1-generator on T_-, and we let *d* denote the other (non-identified) length 1 closed geodesic π_1-generator on T_+, then clearly $\pi_1(Y(\alpha)) \approx \Gamma$.





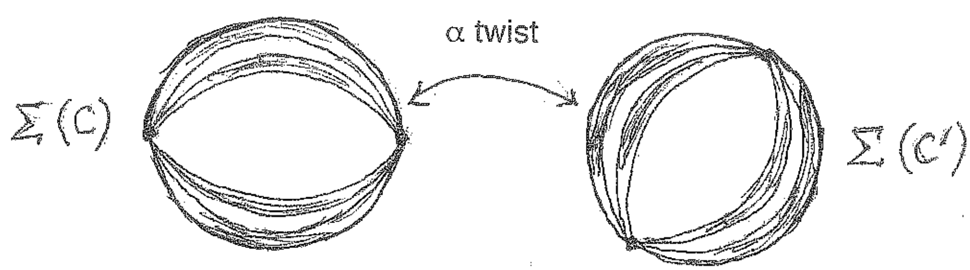
$$Y(\alpha) = T_- \underset{b}{\cup} T_0 \underset{c}{\cup} T_+$$

We let $X(\alpha)$ be the universal cover of $Y(\alpha)$, and we lift the geometry of $Y(\alpha)$ to $X(\alpha)$ by declaring the length of a path in $X(\alpha)$ to be the same as the length of its image in $Y(\alpha)$. This makes $X(\alpha)$ a CAT(0) space, and it makes Γ act geometrically on $X(\alpha)$.

It is useful to organize the structure of $X(\alpha)$ into "blocks". There are two types of blocks in $X(\alpha)$: –blocks and +blocks. Each component in $X(\alpha)$ of the inverse image of $T_- \underset{b}{\cup} T_0$ under the covering map $X(\alpha) \to Y(\alpha)$ is called a *–block,* and each component in $X(\alpha)$ of the inverse image of $T_0 \underset{c}{\cup} T_+$ under the covering map $X(\alpha) \to Y(\alpha)$ is called a *+block.* Thus, each –block is a universal cover of $T_- \underset{b}{\cup} T_0$ and each +block is a universal cover of $T_0 \underset{c}{\cup} T_+$. Since the spaces $T_- \underset{b}{\cup} T_0$ and $T_0 \underset{c}{\cup} T_+$ are homeomorphic to $S^1 \times 8$ (where 8 denotes a topological figure 8), then each block is homeomorphic to $\mathbb{R} \times \mathbb{T}$ (where $\mathbb{T}$ denotes an infinite 4-valent tree = the universal cover of 8).

In $X(\alpha)$, distinct –blocks are disjoint and distinct +blocks are disjoint. A –block and a +block may be disjoint, or they may intersect in a 2-dimensional plane (called a *wall*) that is a component of the inverse image of T_0 under the covering map $X(\alpha) \to Y(\alpha)$. It is helpful to encode the intersection pattern of the blocks in a 1-complex called the *nerve.* The vertices of the nerve correspond to the blocks in $X(\alpha)$, and two vertices of the nerve are connected by an edge if and only if the two corresponding blocks share a common wall. The nerve is a tree in which each vertex has $\aleph_0$ edges emanating from it. If the vertex at one end of an edge corresponds to a –block, then the vertex at the other end must correspond to a +block, and vice versa.

Since each block B is homeomorphic to $\mathbb{R} \times \mathbb{T}$, then its visual boundary ∂B is homeomorphic to the suspension of a Cantor set $\sum(C)$. The visual boundary of each block is embedded as a subset of the visual boundary $\partial X(\alpha)$ of $X(\alpha)$. If two blocks B and B´ share a common wall W (and, therefore, represent adjacent vertices in the nerve), then their visual boundaries $\partial B = \sum(C)$ and $\partial B' = \sum(C')$ intersect in a circle which is the visual boundary ∂W of that common wall. (C and C´ are Cantor sets.) The suspensions $\sum(C)$ and $\sum(C')$ are glued together along the common circle ∂W with a twist through the angle α. In other words, the suspension points of $\sum(C)$ appear as diametrically opposed *poles* on the circle ∂W, as do the suspension points of $\sum(C')$, and the angle between the north pole of $\sum(C)$ and the north pole of $\sum(C')$ is α. (α is also the angle between the south pole of $\sum(C)$ and the south pole of $\sum(C')$.) The union of the visual boundaries of all the blocks is a dense subset of the entire visual boundary of $X(\alpha)$.



By analyzing the positions and limit properties of the poles in $\partial X(\alpha)$, one can distinguish visual boundaries $\partial X(\alpha)$ and $\partial X(\beta)$ for certain values of α and β. The first result along these lines was:

Theorem. [CK] If $0 < \alpha < \pi/2$, then $\partial X(\alpha)$ is not homeomorphic to $\partial X(\pi/2)$.

Thus the Croke-Kleiner group Γ has at least two distinct boundaries.

More recently, we have established:

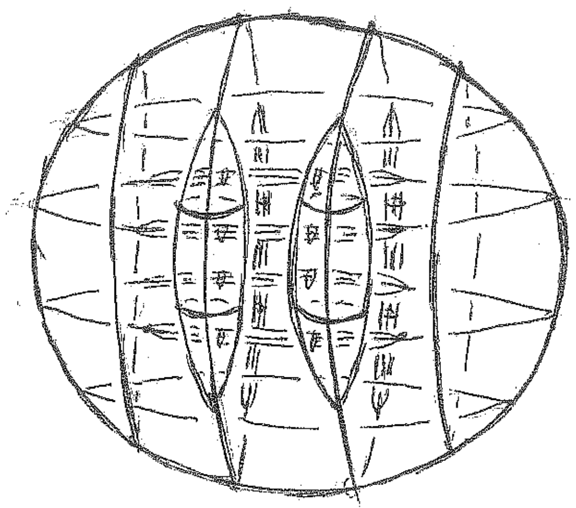
Theorem. [AW] If $0 < \alpha < \pi/2n \leq \beta \leq \pi/2$ for some positive interger n, then $\partial X(\alpha)$ is not homeomorphic to $\partial X(\beta)$.

Thus, the Croke-Kleiner group Γ has at least $\aleph_0$ distinct boundaries.

To prove that each visual boundary $\partial X(\alpha)$ is cell-like equivalent to the Hawaiian earring $\mathcal{H}$, it is useful to consider the inverse limit representation of $\partial X(\alpha)$. To this end, we choose a basepoint b of $X(\alpha)$. (For technical reasons, we choose b so that it doesn't lie in any wall of $X(\alpha)$.) For $r > 0$, let S_r denote the sphere of radius r centered at b in $X(\alpha)$. Then for $0 < r < r'$, there is a geodesic retraction from $S_{r'}$ to S_r. Choose an increasing sequence of positive real numbers converging to ∞: $0 < r_1 < r_2 < r_3 < \ldots$ For each $n \geq 1$, let $g_n : S_{r_{n+1}} \to S_{r_n}$ denote the geodesic retraction. Then $\partial X(\alpha)$ is homeomorphic to the inverse limit

$$\lim_{\substack{\longleftarrow \\ n \to \infty}} \{S_{r_n}, g_n\}.$$

Each S_r is a finite 1-complex that is a union of circles. Here is a picture of a typical S_r for $\alpha = \pi/2$ and $r = 2.13$.



To analyze the inverse system $\{S_{r_n}, g_n\}$, we expand to the inverse sequence

$$\begin{tikzcd}
S_{r_1} & T_1 \arrow[l] & S_{r_2} \arrow[l] \arrow[ll, bend right, "g_1"'] & T_2 \arrow[l] \arrow[ll, bend left, "h_1"] & S_{r_3} \arrow[l] \arrow[ll, bend right, "g_2"'] & T_3 \arrow[l] \arrow[ll, bend left, "h_2"] & \ldots \arrow[l] \arrow[ll, bend right, "g_3"']
\end{tikzcd}$$

by interpolating a second inverse sequence $\{T_n, h_n\}$. (Thus, $\lim\limits_{\substack{\longleftarrow \\ n \to \infty}} \{T_n, h_n\} \cong$

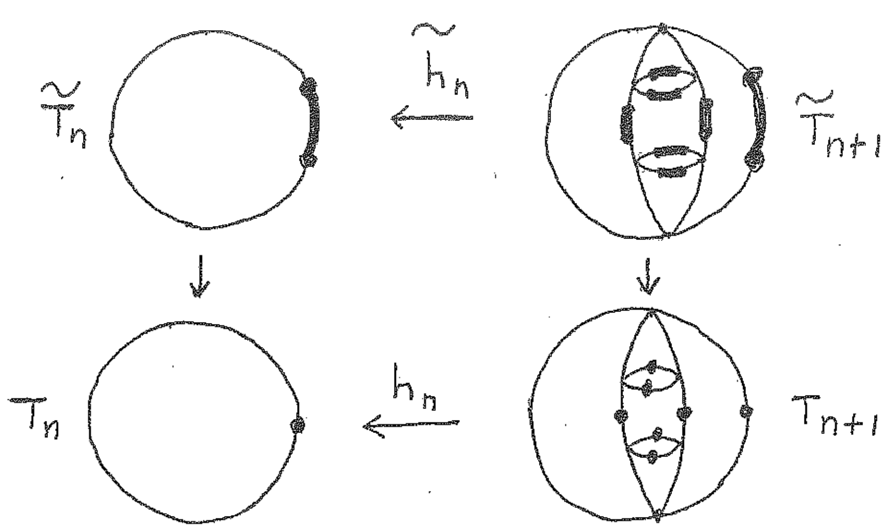
$\lim\limits_{\substack{\longleftarrow \\ n\to\infty}} \{S_{r_n}, g_n\} \cong \partial X(\alpha)$.) Furthermore, we choose the inverse sequence $\{T_n, h_n\}$ so that it has the following properties. For each $n \geq 1$:

- T_n is a finite 1- complex.
- There is a finite subset A_n of T_n such that $T_n - A_n$ is contractible, and for each $x \in A_n$, $\varnothing \neq h_n^{-1}(x) \subset A_{n+1}$.
- $\lim\limits_{n\to\infty}$ cardinality $(A_n) = \infty$.
- No point of A_n is an "essential vertex" of T_n. (In other words, each point of A_n has a neighborhood in T_n that is homeomorphic to $\mathbb{R}$.)
- h_n locally separates its image at each point of $h_n^{-1}(A_n)$. (In other words, for every $x \in h_n^{-1}(A_n)$, for each arc neighborhood U of h(x) in T_n, there is an arc neighborhood V of x in T_{n+1} such that $h(V) \subset U$ and h maps the two components of $V - \{x\}$ into distinct components of $U - \{h_n(x)\}$.)

Now, to finish the proof of Theorem 1 (i.e., that $\partial X(\alpha) \sim_{ce} \mathcal{H}$), it suffices to establish:

Lemma 2. $\lim\limits_{\substack{\longleftarrow \\ n\to\infty}} \{T_n, h_n\} \sim_{ce} \mathcal{H}$.

Outline of proof of Lemma 2. In each T_n, "blow up" each point of A_n to an arc.



If done carefully, this process "blows up" the entire inverse sequence $\{T_n, h_n\}$ to an inverse sequence $\{\tilde{T}_n, \tilde{h}_n\}$ with inverse limit $\tilde{T}_\infty$, giving rise to an infinite commutative diagram;

$$\begin{tikzcd}
\tilde{T}_1 \arrow[d] & \tilde{T}_2 \arrow[l, "\tilde{h}_1"'] \arrow[d] & \tilde{T}_3 \arrow[l, "\tilde{h}_2"'] \arrow[d] & \arrow[l, "\tilde{h}_3"'] \cdots & \tilde{T}_\infty \arrow[d] \\
T_1 & T_2 \arrow[l, "h_1"] & T_3 \arrow[l, "h_2"] & \arrow[l, "h_3"] \cdots & \partial X(\alpha)
\end{tikzcd}$$

For each $n \geq 1$, $\tilde{T}_n$ is the union of finitely many arcs (the "blow ups" of the points of A_n) and a compact contractible set P_n (the closure of the inverse image of $T_n - A_n$ under the blow up map). Passing to the inverse limit, we see that $\tilde{T}_\infty$ is the union of a Cantor set's worth of arcs and a cell-like set P_∞. (P_∞ is the inverse limit of the sequence of contractible sets $\{P_n\}$.) Furthermore, since the blow up maps $\tilde{T}_n \to T_n$ are cell-like and converge to the map $\tilde{T}_\infty \to \partial X(\alpha)$, then the map $\tilde{T}_\infty \to \partial X(\alpha)$ is cell-like.

Now consider quotient space $\tilde{T}_\infty/P_\infty$. Clearly, $\tilde{T}_\infty/P_\infty$ is a Cantor set's worth of arcs with all of their endpoints identified to a single point. Because of the way that $\tilde{T}_\infty$ arises as an inverse limit, one can see that $\tilde{T}_\infty/P_\infty$ is, in fact, a generalized Hawaiian earring $\mathcal{H}(C,b)$ where C is a Cantor set. Also, since P_∞ is a cell-like set, then the quotient map $\tilde{T}_\infty \to \tilde{T}_\infty/P_\infty$ is cell-like. Thus, we have cell-like maps

$$\partial X(\alpha) \leftarrow \tilde{T}_\infty \rightarrow \tilde{T}_\infty/P_\infty \cong \mathcal{H}(C,b).$$

Hence, $\partial X(\alpha) \sim_{ce} \mathcal{H}(C,b)$.

In addition, Lemma 1 implies $\mathcal{H}(C,b) \sim_{ce} \mathcal{H}$. We conclude that $\partial X(\alpha) \sim_{ce} \mathcal{H}$. □

Questions.

Question 1) Are the known boundaries of the Croke-Kleiner group Γ (i.e., the $\partial X(\alpha)$'s for $0 < \alpha \leq \pi/2$) the only boundaries of Γ?

Question 2) Assuming the answer to Question 1 is "no": are all the boundaries of Γ cell-like equivalent to the Hawaiian earring $\mathcal{H}$?

Definition. Suppose that a group G acts on the finite dimensional metric compacta X and Y. We say that X and Y are *equivariantly cell-like equivalent* if there is a finite sequence $Z_1, Z_2, \ldots, Z_{2n+1}$ of finite dimensional metric compacta on which G acts and a "zigzag" sequence

$$\begin{tikzcd}
& Z_1 \arrow[dl] \arrow[dr] & & Z_3 \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & Z_{2n+1} \arrow[dl] \arrow[dr] & \\
X & & Z_2 & & Z_4 & & Z_{2n} & & Y
\end{tikzcd}$$

of equivariant cell-like maps.

Recall: If a CAT(0) group G acts geometrically on a CAT(0) space X, then the action of G on X extends to an action of G on ∂X.

Question 3) Are any two boundaries of the Croke-Kleiner group Γ equivariantly cell-like equivalent?

Definition. Suppose that a CAT(0) group G acts geometrically on CAT(0) spaces X and Y. We say that ∂X and ∂Y are *equivariantly cell-like equivalent through boundaries of G* if there is a finite sequence $Z_1, Z_2, \ldots, Z_{2n+1}$ of CAT(0) spaces on which G acts geometrically and a "zigzag" sequence

$$\begin{tikzcd}
& \partial Z_1 \arrow[dl] \arrow[dr] & & \partial Z_3 \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & \partial Z_{2n+1} \arrow[dl] \arrow[dr] & \\
\partial X & & \partial Z_2 & & \partial Z_4 & & \partial Z_{2n} & & \partial Y
\end{tikzcd}$$

of equivariant cell-like maps.

Question 4) Are any two boundaries of the Croke-Kleiner group Γ equivariantly cell-like equivalent through boundaries of Γ?

The next two questions are refinements of Question 4.

Question 5) If the Croke-Kleiner group Γ acts geometrically on the CAT(0) spaces X and Y, then does Γ act geometrically on a CAT(0) space Z so that there are equivariant cell-like maps

$$\begin{tikzcd} & \partial Z \arrow[dl] \arrow[dr] & \\ \partial X & & \partial Y \end{tikzcd}$$

Question 6) Is there a *maximal* Croke-Kleiner group boundary? In other words, does the Croke-Kleiner group Γ act geometrically on a CAT(0) space Z with the property that if Γ acts geometrically on any other CAT(0) space X, then there is an equivariant cell-like map $\partial Z \to \partial X$?

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