2-Dimensional Axiomatic Geometry Revisited

Abstract: We sketch a pedagogically motivated development of 2-dimensional axiomatic geometry that uses isometries and proves not just consistency of the theory but categoricity as well.

Let X be a metric space of diameter $D > 0$. A *line* in X is a subset that is isometric to either the real line or a circle of circumference 2D (if D is finite). A *segment* is a compact connected subset of a line of diameter $< D$. A subset is *convex* if it contains a segment whenever it contains the segment's endpoints.

X is a 2-dimensional geometry if it satisfies the following 3 axioms:

INCIDENCE: Two points (of distance $< D$) lie on a (unique) line.

SEPARATION: The complement of every line is the union of two disjoint nonempty convex sets called the sides of the line, and any segment with its endpoints on different sides of the line intersects the line.

REFLECTION: For every line, there is an isometry that fixes the points of the line and interchanges the two sides of the line.

We sketch a proof that every 2-dimensional geometry is isometric to either a 2-sphere of positive radius, the Euclidean plane, or a hyperbolic plane of negative curvature. In spirit the proof comes from Bolyai and Lobachevsky. The author has rediscovered some parts on his own and other parts in old texts.

Motivations:

- Approach geometry as a study of metric spaces a la Gromov.
- Include isometries in the axioms so that congruence has a simple universal definition and the Side-Angle-Side Principle is a theorem.
- Study spherical, Euclidean and hyperbolic geometry simultaneously.
- Prove *categoricity* (not just the independence of the Parallel Postulate which is the usual goal of undergraduate axiomatic geometry courses).

Let X be a metric space of diameter $D \leq \infty$.

Notation. For $P, Q \in X$, let *PQ* denote the distance between P and Q.

Definition. A subset L of X is a *line* if L has a *parametrization* $f : \mathbf{R} \to L$.
$f : \mathbf{R} \to L$ is a *parametrization* if it is a surjective function that satisfies

- $f(s)f(t) = | s - t |$ when $| s - t | \leq D$, and
- $f(t + 2D) = f(t)$.

Definition. If $s < t < s + D$, then $f([s,t])$ is a *segment with endpoints* $f(s)$ *and* $f(t)$.

Definition. A subset U of X is *convex* if a segment lies in U whenever its endpoints lie in U.

Definition. A metric space X of diameter $D \leq \infty$ is a *2-dimensional geometry* if it satisfies the following three axioms.

The Incidence Axiom: Any two points $P, Q \in X$ lie on a line in X, and the line is unique if $0 < PQ < D$.

The Separation Axiom: For every line $L \subset X$, $X - L = U \cup V$ where U and V are non-empty disjoint convex sets, and every segment with one endpoint in U and the other in V intersects L. (U and V are called *opposite sides of* L.)

The Reflection Axiom: For every line L with opposite sides U and V, there is an isometry $r : X \to X$ such that $r|L = id_L$, $r(U) = V$ and $r(V) = U$. (r is called a *reflection in* L.)

Special Features of the $D < \infty$ Case.

Definition. Points P and $Q \in X$ are *antipodal* if $PQ = D$.

Definition. For a point $P \in X$ and a line $L \subset X$, if $PQ = D/2$ for each $Q \in L$, then P is called a *pole of* L and L is called an *equator of* P.

Consequences of the Incidence and Separation Axioms ($D < \infty$):

- Every point has a unique antipode, and the antipodal map is an isometry.
- Any two distinct lines meet in a pair of antipodal points.
- Every line has at most two poles which are antipodal and lie on opposite sides of the line.
- Every point has at most one equator.

Consequences of the Incidence, Separation and Reflection Axioms ($D < \infty$).

- Every line has two poles and every point has an equator.

Basic Theorems in the General Case ($D \le \infty$).

Collinearity Criterion: Let P, Q and R $\in$ X be points. P, Q and R are collinear if and only if either

- one of the three numbers PQ, QR, PR equals the sum of the other two, or
- PQ + QR + PR = 2D.

Notation. For each line $L \subset X$, let $U_L = X - \{$ poles of L $\}$. ($U_L = X$ if $D = \infty$.)

Projection Theorem: Let $L \subset X$ be a line. Then there is a function $\pi_L : U_L \to L$ that satisfies the following two conditions.

- For each $P \in U_L$, $P\pi_L(P) < PQ$ for every $Q \in L - \pi_L(P)$.
- If $r : X \to X$ is a reflection in the line L, then for each $P \in U_L$, $\pi_L(P)$ lies on the segment $\overline{Pr(P)}$ joining P to r(P).

Theorem. The reflection in a line is unique.

Notation. If $L \subset X$ is a line, let r_L denote the unique reflection in L.

Definition. If L, M $\subset$ X are lines such that $L \neq M$ and $r_L(M) = M$, then we say L *is perpendicular to* M and write $L \perp M$.

Theorem. If L, M $\subset$ X are lines and $L \perp M$, then $M \perp L$.

Theorem. If L, M and N $\subset$ X are lines such that $L \perp M$, $L \perp N$ and $M \neq N$, then $M \cap N$ = either $\varnothing$ (if $D = \infty$) or the set of poles of L (if $D < \infty$).

Theorem. $\pi_L : U_L \to L$ is continuous.

Theorem. $\pi_L|(U_L - L) : (U_L - L) \to L$ is surjective.

Corollary. Let $L \subset X$ be a line. Then for every point $P \in L$, there is a line through P that is perpendicular to L.

Theorem. Let P and Q $\in$ X be points. If $P \neq Q$, then $\{ R \in X : PR = QR \}$ is a line in X that is perpendicular to any line that contains P and Q.

The Three Reflections Theorem. Every isometry of X is the composition of three or fewer reflections.

Definition. Subsets A and B of X are *congruent* if there is an isometry $f : X \to X$ such that $f(A) = B$.

The Side-Angle-Side Theorem. Triangles ΔPQR and $\Delta P'Q'R'$ are congruent if the segments $\overline{PQ}$ and $\overline{PR}$ and the angle $\overrightarrow{PQ} \cup \overrightarrow{PR}$ are congruent to the segments $\overline{P'Q'}$ and $\overline{P'R'}$ and the angle $\overrightarrow{P'Q'} \cup \overrightarrow{P'R'}$, respectively.

Theorem. There is a unique angle measure function on X.

Definition. Two lines in X are *parallel* if they are disjoint.

Definition.

The Spherical Parallel Postulate. For every point $P \in X$ and every line $L \subset X$ such that $P \notin L$, then there is no line through P that is parallel to L.

The Euclidean Parallel Postulate. For every point $P \in X$ and every line $L \subset X$ such that $P \notin L$, then there a unique line through P that is parallel to L.

The Hyperbolic Parallel Postulate. For every point $P \in X$ and every line $L \subset X$ such that $P \notin L$, then there is more than one line through P that is parallel to L.

Theorem. One of the parallel postulates holds in X.

Equivalence Theorem 1. The following four statements are equivalent.
- The Spherical Parallel Postulate holds.
- The sum of the angle measures of every triangle is greater than π.
- The sum of the angle measures of every convex quadrilateral is greater than 2π.
- $D < \infty$.

Equivalence Theorem 2. The following three statements are equivalent.
- The Euclidean Parallel Postulate holds.
- The sum of the angle measures of every triangle is equal to π.
- The sum of the angle measures of every convex quadrilateral is equal to 2π.

Furthermore, each of these three statements implies $D = \infty$.

Equivalence Theorem 2. The following three statements are equivalent.
- The Hyperbolic Parallel Postulate holds.
- The sum of the angle measures of every triangle is less than π.
- The sum of the angle measures of every convex quadrilateral is less than 2π.

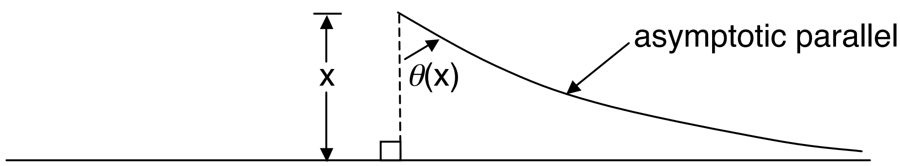
Furthermore, each of these three statements implies $D = \infty$.

The Categoricity Theorem: X is isometric to either a sphere of some positive radius, the Euclidean plane, or a hyperbolic plane of some negative curvature.

The proof of the Categoricity Theorem is motivated by the synthetic proof of the **Bolyai-Lobachevsky Formula:**

$$\tan\left(\frac{\theta(x)}{2}\right) = e^{x/k}$$

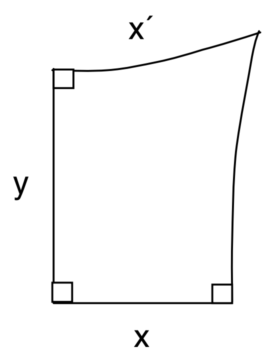
where $\theta(x)$ is the *angle of parallelism*.



The Outline of the Proof of the Categoricity Theorem in the Hyperbolic Case.

Goal: Establish the **(Hyperbolic) Law of Cosines**.

Step 1: Consider the Lambert Quadrilateral



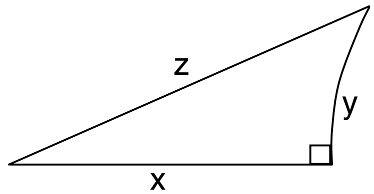
Define the function $\varphi : [0,\infty) \to [0,\infty)$ by

$$\varphi(0) = 1 \text{ and } \varphi(y) = \lim_{x \to 0}\left(\frac{x'}{x}\right) \text{ for } y > 0.$$

It can then be proved "synthetically" (i.e., from the axioms) that:
- φ is continuous,
- φ is non-constant,
- $\varphi(0) = 1 \leq \varphi(y)$ for all $y > 0$, and
- $\varphi(y + z) + \varphi(y - z) = 2\varphi(y)\varphi(z)$.

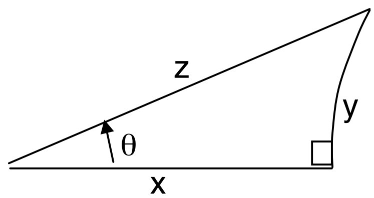
Then there is a constant $k < 0$ such that $\varphi(y) = \cosh\left(\frac{y}{k}\right)$. (The previous formula implies this equation holds on a dense set. Since φ is continuous it holds everywhere.)

Step 2. Prove the **(Hyperbolic) Pythagorean Theorem**:



$$\cosh\left(\frac{z}{k}\right) = \cosh\left(\frac{x}{k}\right)\cosh\left(\frac{y}{k}\right).$$

Step 3. Consider the right triangle



Define the function C : $[0, \pi/2] \to [0,\infty]$ by

$$C(0) = 1 \ \text{ and } \ C(\theta) = \lim_{x \to 0}\left(\frac{x}{z}\right) \text{ for } \theta > 0.$$

Then using the (Hyperbolic) Pythagorean Theorem, the following formula can be proved:

$$C(\theta) = \frac{\tanh\left(\frac{x}{k}\right)}{\tanh\left(\frac{z}{k}\right)}.$$

It can also be proved that:

- C is continuous,
- $C(\pi/2) = 0$,
- $C(\theta + \psi) + C((\theta - \psi) = 2C(\theta)C(\psi)$.

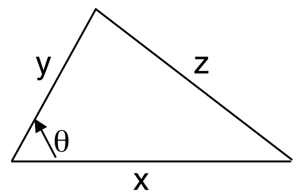
Then $C(\theta) = \cos\theta$. (The previous formula implies this equation holds on a dense set. Since C is continuous it holds everywhere.)

Hence,
$$\boldsymbol{\cos(\theta) = \frac{\tanh\left(\frac{x}{k}\right)}{\tanh\left(\frac{z}{k}\right)}}$$

Step 4: The **(Hyperbolic) Pythagorean Theorem**

plus the formula $\boldsymbol{\cos(\theta) = \dfrac{\tanh\left(\frac{x}{k}\right)}{\tanh\left(\frac{z}{k}\right)}}$

together imply the **(Hyperbolic) Law of Cosines**:



$$\cosh\left(\frac{z}{k}\right) = \cosh\left(\frac{x}{k}\right)\cosh\left(\frac{y}{k}\right) + \sinh\left(\frac{x}{k}\right)\sinh\left(\frac{y}{k}\right)\cos(\theta).$$

Step 5: Define an "exponential map" from a model of the curvature k hyperbolic plane onto X. Since the (Hyperbolic) Law of Cosines is valid in both the model of the curvature k hyperbolic plane and in X, then this exponential map is an isometry.

(From talk at Tallahassee AMS meeting in March, 2004.)