Covers of Aspherical Manifolds with Geometric Fundamental Groups

FREDRIC D. ANCEL CRAIG R. GUILBAULT

September 1, 1997

ABSTRACT. In this note we describe results which imply that closed aspherical n-manifolds ($n > 4$) having isomorphic fundamental groups which are either word hyperbolic or $CAT(0)$ have homeomorphic universal covers. This may be viewed as progress towards a weak version of the Borel Conjecture.

1. INTRODUCTION

One of the most famous open problems in geometric topology is the following:

The Borel Conjecture. If P and Q are closed aspherical manifolds with isomorphic fundamental groups, then they are homeomorphic.

Since a solution to the Borel Conjecture has been so illusive, we suggest the following.

A Weak Borel Conjecture. If P and Q are closed aspherical manifolds with isomorphic fundamental groups, then their universal covers are homeomorphic.

In this note we describe results which imply some special cases of the latter conjecture. In particular, we have:

Theorem 1. *Let P^n and Q^n be closed aspherical n-manifolds ($n > 4$) with isomorphic fundamental groups. If this group is word hyperbolic or CAT(0), then P^n and Q^n have homeomorphic universal covers.*

This result is obtained by combining results from geometric group theory (see Lemma 3) with the following:

Theorem 2. *Suppose M^n and N^n are contractible open n-manifolds ($n > 4$) which admit $\mathcal{Z}$-compactifications having homeomorphic $\mathcal{Z}$-boundaries. Then M^n and N^n are homeomorphic.*

Definitions of the above terminology and descriptions of the relevant geometric group theory are contained in Section 2. A quick sketch of the proof of Theorem 2 is given in Section 3. For a thorough presentation of this work the reader should consult [AG].

2. DEFINITIONS AND EXAMPLES

A closed subset A of a compact ANR X is a *$\mathcal{Z}$-set* if any of the following equivalent conditions is satisfied:

- There is a homotopy $H : X \times I \to X$ with $H_0 = id_X$ and $H_t(X) \cap A = \emptyset$ for all $t > 0$.
- For every $\varepsilon > 0$ there is an ε-homotopy $K : X \times I \to X$ with $K_0 = id_X$ and $K_1(X) \subset X \backslash A$.
- For every $\varepsilon > 0$ there is a map $f : X \to X$ which is ε-close to the identity with $f(X) \subset X \backslash A$.
- For every open set U of X, $U \backslash A \hookrightarrow U$ is a homotopy equivalence.

Let Y be a noncompact ANR. A *$\mathcal{Z}$-compactification* of Y is a compact ANR $\widehat{Y}$ containing Y as an open subset and having the property that $\widehat{Y} - Y$ is a $\mathcal{Z}$-set in $\widehat{Y}$. In this case we call $\widehat{Y} - Y$ a *$\mathcal{Z}$-boundary* for Y and denote it $\partial_{\mathcal{Z}} Y$. Note that Y may admit many different $\mathcal{Z}$-boundaries, hence $\partial_{\mathcal{Z}} Y$ is not well defined unless the $\mathcal{Z}$-compactification is specified.

For our purposes, the key facts from geometric group theory may be summarized in the following result borrowed from [Be] (see Lemma 1.4). A more thorough development including other useful references may be found in Section 3 of [AG].

Lemma 3. *Let P be an aspherical manifold with word hyperbolic or CAT(0) fundamental group. Then the universal cover of P admits a $\mathcal{Z}$-compactification. Moreover, if Q is another aspherical manifold having fundamental group isomorphic to that of P, then the universal coverings of P and Q admit $\mathcal{Z}$-compactifications with homeomorphic $\mathcal{Z}$-boundaries.*

3. SKETCH OF THE PROOF OF THEOREM 2

The first major ingredient in the proof of Theorem 2 is a gluing theorem which is interesting in its own right.

Theorem 4. *Let $\widehat{M^n}$ and $\widehat{N^n}$ be $\mathcal{Z}$-compactifications of open n-manifolds ($n > 4$) and $h : \partial_{\mathcal{Z}} \widehat{M^n} \to \partial_{\mathcal{Z}} \widehat{N^n}$ be a homeomorphism. Then $\widehat{M^n} \cup_h \widehat{N^n}$ is an n-manifold. If M^n and N^n are contractible, then $\widehat{M^n} \cup_h \widehat{N^n} \approx S^n$.*

The proof of the above result is rather technical. The idea is to show that $\widehat{M^n} \cup_h \widehat{N^n}$ is a resolvable ANR homology manifold which satisfies the disjoint disks property. It then follows from Edwards' Cell-Like Approximation Theorem that $M^n \cup_h \widehat{N^n}$ is a manifold.

The last part of Theorem 4 is obtained by an application of the high dimensional Poincaré Conjecture.

To prove Theorem 2, identify $\widehat{M^n} \cup_h \widehat{N^n}$ with S^n viewed as the boundary of an $(n+1)$-ball, B^{n+1}; let Z denote the common copy of $\partial_Z \widehat{M^n} = \partial_Z \widehat{N^n}$ in S^n; and let $W^{n+1} = B^{n+1} \backslash Z$ (see Figure 1). Using properties of $\mathcal{Z}$-sets, one may show that (W^{n+1}, M^n, N^n) is a proper h-cobordism. Next, using the algebraic machinery developed in [Si] and [CS] it can be shown that the inclusion $M^n \hookrightarrow W^{n+1}$ is an (infinite) simple homotopy equivalence. An application of Siebenmann's proper s-cobordism theorem then guarantees that $W^{n+1} \approx M^n \times [0,1]$, completing the proof.

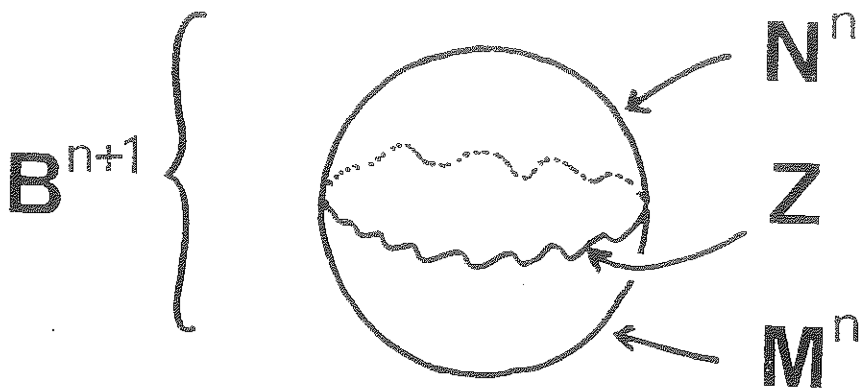

Figure 1.

REFERENCES

[AG] F.D. Ancel and C.R. Guilbault, *Z-compactifications of open manifolds*, preprint.

[Be] M. Bestvina, *Local homology properties of boundaries of groups*, Michigan Math. J., 43 (1996), 123-139.

[CS] T.A. Chapman and L.C. Siebenmann, *Finding a boundary for a Hilbert cube manifold*, Acta Math. 137 (1976), 171-208.

[Si] L.C. Siebenmann, *Infinite simple homotopy types*, Indag. Math. 32 (1970), 479-495.