A Counterexample to a Geometric General Position Conjecture

Let $\dim_H(X)$ denote the Hausdorff dimension of a subset X of $\mathbb{R}^n$, and for $X \subset \mathbb{R}^n$ and $p \in \mathbb{R}^n$, let $X+p = \{x+p : x \in X\}$.

The following conjecture is false.

Geometric General Position Conjecture. If X and Y are compacta in $\mathbb{R}^n$ such that $\dim_H(X) + \dim_H(Y) < n$, then for every $\varepsilon > 0$, there is a $p \in \mathbb{R}^n$ such that $|p| < \varepsilon$ and $(X+p)\cap Y = \emptyset$.

Example. There are Cantor sets X and Y in $[0,1]$ such that $\dim_H(X) = \dim_H(Y) = 0$ and $(X+t)\cap Y \neq \emptyset$ for every $t \in [-1,1]$.

Proof. For each $t \in \mathbb{R}$, let L_t denote the line $\{(x,y) \in \mathbb{R}^2 : y = x+t\}$. Observe that the statement "$(X+t)\cap Y \neq \emptyset$ for every $t \in [-1,1]$" is equivalent to the statement "$L_t \cap (X\times Y) \neq \emptyset$ for every $t \in [-1,1]$". Our construction of X and Y is aimed at satisfying the latter statement.

First we inductively construct two sequences $\{m_i\}$ and $\{n_i\}$ of positive integers and two sequences $\{a_i\}$ and $\{b_i\}$ of positive real numbers satisfying the following conditions. $m_1 = n_1 = 1 = a_1 = b_1$. Furthermore, for $i \geq 2$,

a) $m_i > (a_{i-1}/b_{i-1})+1$,
b) $a_i = (m_1 \cdots m_i)^{-1}$,
c) $n_i > (b_{i-1}/a_i)+1$, and
d) $b_i = (n_1 \cdots n_i)^{-1}$.

Observe that for $i \geq 2$, m_i and $n_i \geq 2$, a_i and $b_i \leq 2^{-i}$, $m_i a_i < a_{i-1}$, and $n_i b_i < b_{i-1}$.

Next we inductive construct two decreasing sequences of compacta $X_1 \supset X_2 \supset X_3 \supset \cdots$ and $Y_1 \supset Y_2 \supset Y_3 \supset \cdots$ in $[0,1]$ satisfying the following conditions. $X_1 = [0,1]$. Furthermore, if $i \geq 2$, then

a) X_i has exactly $m_1 \cdots m_i$ components, each of which is a closed interval of length a_i; and
b) if I is a component of X_{i-1}, then I contains exactly m_i components of X_i,

and $I-X_i$ has exactly m_i-1 components, each of which is an open interval of length $(a_{i-1} - m_ia_i)/(m_i-1)$. (Here we use the fact that $m_ia_i < a_{i-1}$.) The Y_i's are described similarly in terms of the n_i's and b_i's.

Now set $X = \bigcap_{i\geq 1}X_i$ and $Y = \bigcap_{i\geq 1}Y_i$. Then X and Y are Cantor sets in $[0,1]$.

Let $k \geq 1$. We now estimate the $(1/k)$-dimensional Hausdorff measure of X. Observe that for each $i \geq k$, $X \subset X_i$, X_i is the union of $m_1 \cdots m_i$ disjoint closed intervals of length a_i, and $m_1 \cdots m_i(a_i^{1/i}) = 1$. Also observe that for $i \geq k$, $m_1 \cdots m_i(a_i^{1/k}) \leq m_1 \cdots m_i(a_i^{1/i}) = 1$. It follows that the $(1/k)$-dimensional Hausdorff measure of X is $\leq 1 < \infty$. Hence, $\dim_H(X) \leq 1/k$. We conclude that $\dim_H(X) = 0$. Similarly $\dim_H(Y) = 0$.

Clearly, $X_1 \times Y_1 \supset X_2 \times Y_2 \supset X_3 \times Y_3 \supset \cdots$, and $X \times Y = \bigcap_{i\geq 1}X_i \times Y_i$. Let $t \in [-1,1]$. We prove $L_t \cap (X \times Y) \neq \emptyset$, by inductively demonstrating that for each $i \geq 1$, $L_t \cap (X_i \times Y_i) \neq \emptyset$.

We begin by observing that $((1-t)/2,(1+t)/2) \in L_t \cap [0,1]^2 = L_t \cap (X_1 \times Y_1)$.

Let $i \geq 2$ and inductively assume $L_t \cap (X_{i-1} \times Y_{i-1}) \neq \emptyset$. Then there is a component I of X_{i-1} and a component J of Y_{i-1} such that $L_t \cap (I \times J) \neq \emptyset$. We assert that $L_t \cap ((I \cap X_i) \times J) \neq \emptyset$. For if not, then $L_t \cap (I \times J)$ lies in $(I-X_i) \times J$. So $L_t \cap (I \times J)$ is contained in a rectangle whose base is one of the open interval components of $I-X_i$ and whose height is J. Since the open interval components of $I-X_i$ are of length $(a_{i-1} - m_ia_i)/(m_i-1)$, J is of length b_{i-1}, and L_t has slope 1, then

$$b_{i-1}/[(a_{i-1} - m_ia_i)/(m_i-1)] < 1.$$

Hence,

$$m_ib_{i-1} - b_{i-1} < a_{i-1} - m_ia_i.$$

So,

$$m_ib_{i-1} < m_ib_{i-1} + m_ia_i < a_{i-1} + b_{i-1}.$$

Therefore,

$$m_i < (a_{i-1}/b_{i-1}) + 1,$$

a contradiction. Our assertion follows: $L_t \cap ((I \cap X_i) \times J) \neq \emptyset$. So there is a component I' of X_i such that $L_t \cap (I' \times J) \neq \emptyset$.

Next we assert that $L_t \cap (I' \times (J \cap Y_i)) \neq \emptyset$. For if not, then $L_t \cap (I' \times J)$ lies in $I' \times (J - Y_i)$. So $L_t \cap (I' \times J)$ is contained in a rectangle whose base is I' and whose height is one of the open interval components of $J - Y_i$. Since I' is of length a_i, the open interval components of $J - Y_i$ are of length $(b_{i-1} - n_i b_i)/(n_i - 1)$, and L_t has slope 1, then

$$1 < [(b_{i-1} - n_i b_i)/(n_i - 1)]/a_i.$$

Hence,

$$n_i a_i - a_i < b_{i-1} - n_i b_i.$$

So,

$$n_i a_i < n_i a_i + n_i b_i < a_i + b_{i-1}.$$

Therefore,

$$n_i < (b_{i-1}/a_i) + 1,$$

a contradiction. Our assertion follows: $L_t \cap (I' \times (J \cap Y_i)) \neq \emptyset$.

Since $I' \subset X_i$, we have prove that $L_t \cap (X_i \times Y_i) \neq \emptyset$.

Now $L_t \cap (X \times Y) \neq \emptyset$ follows by induction. Hence, there is a point $(x,y) \in L_t \cap (X \times Y)$. So $y = x + t$, $x \in X$ and $y \in Y$. Thus, $y \in (X+t) \cap Y$. This shows $(X+t) \cap Y \neq \emptyset$ for every $t \in [-1,1]$. □

Observation. $Z = X \cup Y$ is a Cantor set in $[0,1]$ such that $\dim_H(Z) = 0$ and $(Z+t) \cap Z \neq \emptyset$ for every $t \in [-1,1]$.