A NEW PROOF THAT $\mathcal{N}_3 = 0$

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In [5], C. Rourke gives a brief clever proof of the classical result of V.A. Rokhlin [4] that every closed orientable 3-manifold bounds a compact orientable 4-manifold (i.e., $\Omega_3 = 0$). The non-orientable version of Rohklin's theorem, originally proven by R. Thom [6], guarantees that every closed 3-manifold (possibly non-orientable) bounds a compact 4-manifold ($\mathcal{N}_3 = 0$). In this presentation, we indicate how Rourke's approach can be extended to give a short proof of this latter theorem.

In [5], $\Omega_3 = 0$ is deduced as a corollary of a stronger theorem (proven earlier in [7] and [2]) that every closed orientable 3-manifold can be reduced to S^3 by a finite number of elementary Dehn surgeries. Here "elementary" means that a meridian of the attached solid torus is identified with a curve in the boundary of the removed solid torus that is homotopic to the core of the removed solid torus. Then $\Omega_3 = 0$ follows from the observation that any two closed orientable 3-manifolds which differ by an elementary Dehn surgery cobound a compact orientable 4-manifold.

Similarly we can deduce that $\mathcal{N}_3 = 0$ from a stronger theorem (first proven in [3]) about the reducibility by surgery of every non-orientable 3-manifold to a simple model. In the non-orientable situation the simple model which replaces S^3 is the non-orientable 2-sphere bundle over S^1, which we denote $\mathbb{T}$. Our basic theorem is:

THEOREM *Every closed non-orientable 3-manifold can be reduced to* $\mathbb{T}$ *by a finite number of elementary Dehn surgeries.*

Since $\mathbb{T}$ bounds the non-orientable B^3 bundle over S^1, and since any two closed 3-manifolds (orientable or not) which differ by an elementary Dehn surgery cobound a compact 4-manifold, we have:

COROLLARY. ($\mathcal{N}_3 = 0$) *Every closed 3-manifold bounds a compact 4-manifold.*

We describe how to extend Rourke's techniques to give an elementary proof of the above Theorem . As in [5], we will use an induction argument based on a complexity assigned to Heegaard diagrams.

Suppose $M = H_1 \cup H_2$ is a Heegaard splitting of a non-orientable 3-manifold M. Then H_1 and H_2 are non-orientable handlebodies meeting along a non-orientable surface S. If the H_i's are of genus *n*, then *S* has Euler characteristic *2-2n*, and we will call *S* a *non-orientable surface of genus n*. A set of *n* disjoint 2-sided (i.e., having an annular regular neighborhood) simple closed curves on S whose complement is a punctured disk is called a *complete system of curves on S*. (Every non-orientable surface of genus n has a complete system of curves.) It is easy to see that if *X* and Y are complete systems of curves on *S* with the property that each element of *X* bounds a disk in H_1 and each element of Y bounds a disk in H_2, then *M* is completely determined by *S*, *X* and Y. We then call *S(X,Y)* a *Heegard diagram for M*. Moreover, any Heegaard diagram, *S(X,Y)*, uniquely determines a 3-manifold which we will denote *M(X,Y)*.

A 2-sided curve *x* on a non-orientale surface *S* is called *exceptional* if *S - x* is orientable, otherwise it is called *ordinary*. A complete system of curves on *S* is called uniform if it contains only ordinary curves, or if genus(*S*) = 1. It is easy to see that every non-orientable surface of genus n has a uniform complete system of curves. Note that a genus 1 complete system necessarily contains a single exceptional curve. A Heegaard diagram *S(X,Y)* will be called a *uniform* if both X and Y are uniform systems.

REMARK. The assumption of "2-sidedness" for all curves used in a Heegaard diagram is of utmost importance. While this property is automatic for a curve on an orientable surface, the situation is much different for non-orientable surfaces. On the other hand, our preference for ordinary curves evolved during our work on this problem. Use of uniform Heegard diagrams substantially simplified our original proof. Much of the work done in proving the our theorem is aimed at securing these properties when choosing new curves (see for example the lemma below).

To a uniform Heegaard diagram $S(X,Y)$, where S is non-orientable, assign a complexity $\mathcal{E}(X,Y) = (n,k)$ where $n = genus(S)$ and $k = \min\{|x \cap y| : x \in X, y \in Y\}$. Note that since S is non-orientable, then $n \geq 1$. Our proof is by induction on the complexity of these uniform Heegaard diagrams under the lexicographic ordering.

While many facts about surfaces and 3-manifolds must be verified to give a complete proof of the theorem, the key is the following:

LEMMA *Suppose* x *and* y *are two non-separating 2-sided curves on a non-orientable genus* n *surface* S *and that* x *meets* y *transversally. Let* $|x \cap y|$ *denote the number of intersection points.*

(a) *If* $|x \cap y| = 0$ *and both* x *and* y *are ordinary, then there is a (necessarily ordinary) non-separating 2-sided curve* z *on* S *which meets each of* x *and* y *transversally in a single point.*

(b) *If* $|x \cap y| > 1$, *then there is a non-separating 2-sided curve* z *on* S *with* $|x \cap z| < |x \cap y|$ *and* $|y \cap z| < |x \cap y|$. *Moreover, if* x *and* y *are ordinary then* z *can be chosen to be ordinary.*

Proof of this lemma requires careful examination of approximately ten different cases. Its complete proof as well as the remaining ingredients in the proof of the main theorem can be found in [1]

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