From *Proceedings of the Fifth Annual Western Workshop in Geometric Topology*
(Colorado Springs, 1988)

Two applications of topology to physics

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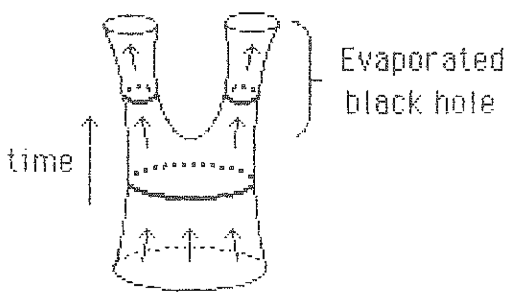
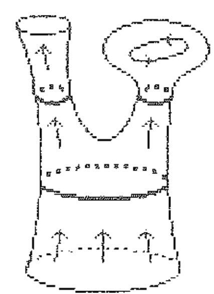
I. Stephen Hawking has predicted that black holes can "evaporate" and disappear from the universe. From the point of view of the universe as a 4-manifold with a preferred time direction, one possible explanation of the disappearance of a black hole is as an increase in the number of components in the cross-sections of the universe transverse to the time direction as time increases. (See Figure 1.)



the universe
(also $\mathbb{CP}^2$-(3 points))

Figure 1



$\mathbb{CP}^2 \# \mathbb{RP}^4$-(2 points)

Figure 2

The universe is a spacetime; i.e., a smooth 4-manifold with a Lorentz metric (a semi-Riemannian metric with signature: $-+++$). John Friedman of the Physics Department at the University of Wisconsin, Milwaukee asked for explicit examples of spacetimes which exhibit such behavior. In other words, he asked for specific examples of spacetimes in which the number of components in the cross-sections transverse to the time direction increases with increasing time?

It is simple to discover such examples based on the observation that a smooth manifold admits a Lorentz metric if it admits a non-zero vector field. (The vectors point in the direction of increasing time.) This observation reduces the question to an exercise in using the Poincaré-Hopf Index Theorem.

$\mathbb{C}P^2$ has Euler characteristic 3. Hence, it admits a vector field with 3 zeroes which can be chosen to be one source and two sinks. (Both sources and sinks have index +1 in dimension 4.) Thus, $\mathbb{C}P^2$−(3 points) has a Lorentz metric in which the number of components in the cross-sections transverse to the time direction changes from 1 to 2. Furthermore, a typical section with 2 components separates $\mathbb{C}P^2$−(3 points) into three noncompact pieces. (See Figure 1 again.)

$\mathbb{R}P^4$ has Euler characteristic 1 and, therefore, admits a vector field with one source (and some closed trajectories). Hence, $\mathbb{C}P^2 \# \mathbb{R}P^4$ has a vector field with one source and one sink. Thus, $\mathbb{C}P^2 \# \mathbb{R}P^4$−(2 points) has a Lorentz metric in which the number of components in the cross-sections transverse to the time direction changes from 1 to 2. Furthermore, a typical section with 2 components separates $\mathbb{C}P^2 \# \mathbb{R}P^4$−(2 points) into two noncompact pieces and one *compact* piece containing *closed* time-like particle paths. (See Figure 2.)

II. At the 1988 Spring Topology Conference in Gainesville, Florida, Otto Laback, an Austrian physicist, posed the following question. Given that we can directly observe only certain subsets of $\mathbb{R}^n$ (such as smoothly embedded 1-manifolds corresponding to particle paths), what possible topologies on $\mathbb{R}^n$ are compatible with the usual topology on physically observable subsets? To make this more precise, for a collection $\mathcal{S}$ of subsets of $\mathbb{R}^n$, let

$\mathcal{T}_{\mathcal{S}}$ = $\{ U \subset \mathbb{R}^n : U \cap S$ is a relatively open subset of S for each $S \in \mathcal{S} \}$
= the largest topology on $\mathbb{R}^n$ which induces the standard topology on each element of $\mathcal{S}$,

and let

$\mathcal{H}_{\mathcal{S}}$ = the homeomorphism group of $\mathbb{R}^n$ with the topology $\mathcal{T}_{\mathcal{S}}$.

Then we reformulate Laback's question as follows:

Question. For which collections $\mathcal{S}$ of subsets of $\mathbb{R}^n$ is $\mathfrak{T}_{\mathcal{S}}$ the standard topology on $\mathbb{R}^n$, and is $\mathfrak{H}_{\mathcal{S}}$ the standard homeomorphism group of $\mathbb{R}^n$?

We answer this question for two different choices of $\mathcal{S}$.

Theorem 1. *If $\mathcal{S}$ is a collection of subsets of $\mathbb{R}^n$ which contains all C^1 embedded 1-manifolds, then $\mathfrak{T}_{\mathcal{S}}$ is the standard topology on $\mathbb{R}^n$ and $\mathfrak{H}_{\mathcal{S}}$ is the standard homeomorphism group of $\mathbb{R}^n$.*

A subset S of $\mathbb{R}^n$ is a *smooth set* if for each $p \in S$, there is a neighborhood U of p in $\mathbb{R}^n$, there is an $r \geq 1$, and there is a C^{r+1} map $f : U \to \mathbb{R}$ such that $f^{-1}(0) = S \cap U$ and f has a non-zero partial derivative of order $\leq r$ at each point of U. For example, every C^2 embedded submanifold of $\mathbb{R}^n$ is a smooth set, and the zero set of every non-zero polynomial is a smooth set.

Theorem 2. *If $\mathcal{S}$ is the collection of all smooth subsets of $\mathbb{R}^n$, then $\mathfrak{T}_{\mathcal{S}}$ is strictly larger that the standard topology on $\mathbb{R}^n$, and $\mathfrak{H}_{\mathcal{S}}$ neither contains nor is contained in the standard homeomorphism group of $\mathbb{R}^n$.*

The following lemma is the key to the proof of Theorem 2.

Lemma 1. *There is a tame arc A in $\mathbb{R}^n$ with endpoint 0 such that for every smooth subset S of $\mathbb{R}^n$, if $0 \in S$, then $0 \notin \mathrm{cl}(S \cap (A - \{0\}))$.*

Then $A-\{0\}$ is a closed subset of $\mathbb{R}^n$ with respect to the topology $\mathfrak{T}_{\mathcal{S}}$, but $A-\{0\}$ is not closed in the standard topology on $\mathbb{R}^n$. Furthermore, the standard homeomorphism of $\mathbb{R}^n$ which carries a straight line segment onto A is not continuous under the topology $\mathfrak{T}_{\mathcal{S}}$, and, hence, is not an element of $\mathfrak{H}_{\mathcal{S}}$. (A slight strengthening of this lemma is used to produce an element of $\mathfrak{H}_{\mathcal{S}}$ which is not a standard homeomorphism.)

To produce the arc A, we use the following notation. Set $\omega = \{0,1,2,\cdots\}$. For $a = (a_1,\cdots,a_n) \in \omega^n$, set

$$\|a\| = \Sigma a_i,$$

$$a! = \Pi(a_i!),$$

$$x^a = \Pi x_i^{a_i} \quad \text{for } x = (x_1,\cdots,x_n) \in \mathbb{R}^n,$$

$$f^{(a)}(p) = \left(\frac{\partial}{\partial x_1}\right)^{a_1}\left(\frac{\partial}{\partial x_2}\right)^{a_2}\cdots\left(\frac{\partial}{\partial x_n}\right)^{a_n} f(p) \qquad \text{for } f : U \to \mathbb{R},\ U \text{ an open}$$

subset of $\mathbb{R}^n$, and $p \in U$.

Let $r \geq 1$, and let U be an open subset of $\mathbb{R}^n$. A function $f : U \to \mathbb{R}$ is *of class* C^r if for each $a \in \omega^n$ with $\|a\| \leq r$, $f^{(a)}(p)$ exists and is continuous at every $p \in U$. Let $C^r(U)$ denote the collection of all functions from U to $\mathbb{R}$ which are of class C^r.

For $r \geq 1$, U an open subset of $\mathbb{R}^n$, $f \in C^r(U)$, and $p \in U$, the *degree* r *Taylor polynomial of* f *at* p is

$$T_p^r f(x) = \sum_{\substack{a \in \omega^n \\ \|a\| \leq r}} \frac{1}{a!} f^{(a)}(p)\, x^a \qquad \text{for } x \in \mathbb{R}^n.$$

We can now state

A Version of Taylor's Theorem. *Let* $r \geq 1$, *let* U *be an open subset of* $\mathbb{R}^n$, *let* $f \in C^{r+1}(U)$, *and let* $p \in U$. *If* $x \in \mathbb{R}^n$ *such that* U *contains the straight line segment from* p *to* p+x, *then there is a* $\theta \in (0,1)$ *such that*

$$f(p+x) = T_p^r f(x) + \sum_{\substack{a \in \omega^n \\ \|a\| = r+1}} \frac{1}{a!} f^{(a)}(p+\theta x)\, x^a .$$

We observe that a subset S of $\mathbb{R}^n$ is a smooth if for each $p \in S$, there is a neighborhood U of p in $\mathbb{R}^n$, there is an $r \geq 1$, and there is an $f \in C^{r+1}(U)$ such that $f^{-1}(0) = S \cap U$ and $T^r{}_q f \neq 0$ for each $q \in U$.

To prove Lemma 1, we impose a linear order $<$ on ω^n as follows. For $a, b \in \omega^n$, we declare that $a < b$ if either

$\|a\| < \|b\|$

or

$\|a\| = \|b\|$ and there is a k such that $1 \leq k \leq n$, $a_i = b_i$ for $1 \leq i < k$, and $a_k < b_k$.

To prove Lemma 1, we also need

Lemma 2. *There is an embedding* $\varphi = (\varphi_1,\cdots,\varphi_n) : [0,1] \to \mathbb{R}^n$ *such that if* $a, b \in \omega^n$ *and* $b < a$, *then*

$$\lim_{t \to 0} \frac{\varphi^a(t)}{\varphi^b(t)} = 0 .$$

(*Here* $\varphi^a(t) = \Pi(\varphi_i(t))^{a_i}$ *for* $a = (a_1,\cdots,a_n) \in \omega^n$.)

Proof of Lemma 2. First define $\psi : [0,1] \to [0,1]$ by $\psi(0) = 0$ and $\psi(t) = \ln 2/(\ln 2 - \ln t)$ for $0 < t \leq 1$. Then L'Hospital's Rule implies that for any $r \geq 0$, $t/(\psi(t))^r \to 0$ as $t \to 0$. For $1 \leq i \leq n$, set $\psi_i = \psi\circ\cdots\circ\psi$ (the i-fold composition of ψ with itself). Then for any $r \geq 0$, $\psi_i(t)/(\psi_{i+1}(t))^r \to 0$ as $t \to 0$, for $1 \leq i \leq n$. Finally, set $\varphi_i(t) = t\psi_i(t)$ for $1 \leq i \leq n$. □

Proof of Lemma 1. Set $A = \varphi[0,1]$. Suppose S is a smooth subset of $\mathbb{R}^n$ and $0 \in S$. Then there is a neighborhood U of 0 in $\mathbb{R}^n$, there is an r ≥ 1, and there is an $f \in C^{r+1}(U)$ such that $f^{-1}(0) = S \cap U$ and $T^r{}_p f \neq 0$ for each $p \in U$. We shall show that $0 \notin \mathrm{cl}(S \cap (A - \{0\}))$. For assume $S \cap (A - \{0\})$ contains a sequence that converges to 0. We shall argue that $T^r{}_0 f = 0$, and thereby reach a contradiction.

By our assumption, there is a sequence $\{t_i\}$ in $(0,1]$ converging to 0 such that for each $i \geq 1$, $\varphi(t_i) \in S \cap U$ and U contains the straight line segment from 0 to $\varphi(t_i)$. According to Taylor's Theorem, for every $i \geq 1$, there is a $\theta_i \in (0,1)$ such that

$$0 = f(\varphi(t_i)) = T_0^r f(\varphi(t_i)) + \sum_{\substack{a \in \omega^n \\ \|a\| = r+1}} \frac{1}{a!} f^{(a)}(\theta_i \varphi(t_i))\, \varphi^a(t_i) .$$

$f^{(0)}(0) = f(0) = 0$, because $0 \in S$. Now suppose $b \in \omega^n$, $\|b\| \leq r$ and $f^{(a)}(0) = 0$ for every $a \in \omega^n$ such that $a < b$. We shall argue that $f^{(b)}(0) = 0$. It will then follow that $T^r{}_0 f = 0$.

For $i \geq 1$: if $0 \leq s \leq r$, set $x_{s,i} = 0$; and if $s = r+1$, set $x_{s,i} = \theta_i \varphi(t_i)$. Then the above Taylor formula becomes

$$0 = \sum_{\substack{a \in \omega^n \\ \|a\| \leq r+1 \\ b \leq a}} \frac{1}{a!} f^{(a)}(x_{\|a\|,i})\, \varphi^a(t_i) .$$

Divide this equation by $\varphi^b(t_i)$ and let $i \to \infty$. Then according to Lemma 2, we are left with $(1/b!)f^{(b)}(0) = 0$. □