AN ALTERNATIVE PROOF OF M. BROWN'S THEOREM ON INVERSE SEQUENCES OF NEAR HOMEOMORPHISMS

by Fredric D. Ancel

Abstract. Theorem 4 of [B] is an interesting and useful result about inverse sequences of near homeomorphisms. We present a short alternative proof of this theorem. We thank Bob Daverman for a suggestion which has led to a slicker exposition.

Let (X, ρ) and (Y, σ) be compact metric spaces. Let $\mathfrak{M}(X, Y)$ denote the space of maps from X to Y endowed with the compact-open topology. A complete metric $\tilde{\sigma}$ on $\mathfrak{M}(X, Y)$ is defined by $\tilde{\sigma}(f, g) = \sup\{\sigma(f(x), g(x) : x \in X\}$. A map from X to Y is a *near homeomorphism* if it belongs to the closure of the set of homeomorphisms in $\mathfrak{M}(X, Y)$. Let $\varepsilon > 0$. A map $f : X \to Y$ is an ε-map if ρ-diam$(f^{-1}(y)) < \varepsilon$ for every $y \in Y$. Let $\mathfrak{M}_\varepsilon(X, Y)$ denote the set of all ε-maps in $\mathfrak{M}(X, Y)$. We shall use the following two basic facts.

Lemma 1. *Let X, Y and Z be compact metric spaces. Then composition $(f, g) \mapsto g \circ f : \mathfrak{M}(X, Y) \times \mathfrak{M}(Y, Z) \to \mathfrak{M}(X, Z)$ is continuous.*

This is easily proved using the uniform continuity of maps from Y to Z. One immediate consequence is that the composition of near homeomorphisms is a near homeomorphism.

Lemma 2. *Let (X, ρ) and (Y, σ) be compact metric spaces. For each $\varepsilon > 0$, $\mathfrak{M}_\varepsilon(X, Y)$ is an open subset of $\mathfrak{M}(X, Y)$.*

Proof. Let $f \in \mathfrak{M}_\varepsilon(X, Y)$. Set $\delta = (1/2) \inf\{\sigma(f(x), f(z)) : x, z \in X \text{ and } \rho(x, z) \geq \varepsilon\}$. Then $\delta > 0$. Let $g \in \mathfrak{M}(X, Y)$ such that $\tilde{\sigma}(f, g) < \delta$. We assert that $g \in \mathfrak{M}_\varepsilon(X, Y)$. If $x, z \in X$ and $g(x) = g(z)$, then $\sigma(f(x), f(z)) \leq \sigma(f(x), g(x)) + \sigma(g(z), f(z)) < 2\delta$. This makes $\rho(x, z) < \varepsilon$, and proves our assertion. □

A Theorem of M. Brown ([B], Theorem 4). *Suppose $X_1 \xleftarrow{f_1} X_2 \xleftarrow{f_2} \cdots$ is an inverse sequence of compact metric spaces and near homeomorphisms. If X_∞ is its inverse limit, then each $f_{\infty,k} : X_\infty \to X_k$ is a near homeomorphism.*

Proof. Recall that $X_\infty = \{x \in \prod X_k : f_k(x_{k+1}) = x_k \text{ for each } k \geq 1\}$, and that each $f_{\infty,k} : X_\infty \to X_k$ is simply projection: $f_{\infty,k}(x) = x_k$ for $x \in X_\infty$. Let ρ_k be a metric on X_k such that ρ_k-diam$(X_k) < 1/k$. Then a metric ρ_∞ on X_∞ is defined by $\rho_\infty(x, z) = \sup\{\rho_k(x_k, z_k) : k \geq 1\}$ for

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$x, z \in X_\infty$. It follows that each $f_{\infty,k} : X_\infty$ is a $(1/k)$-map. Indeed, if $x, z \in X_\infty$ and $f_{\infty,k}(x) = f_{\infty,k}(z)$, then $x_j = z_j$ for $1 \le j \le k$; so $\rho_\infty(x, z) = \sup\{\rho_j(x_j, z_j) : j > k\} < 1/k$.

It suffices to prove that $f_{\infty,1} : X_\infty \to X_1$ is a near homeomorphism. To this end, let $\mathcal{F}$ denote the closure in $\mathfrak{M}(X_\infty, X_1)$ of the set of all maps of the form $h \circ f_{\infty,k}$ where $k \ge 1$ and $h : X_k \to X_1$ is a homeomorphism. Then $f_{\infty,1} \in \mathcal{F}$, and $\tilde{\rho}_1$ restricts to a complete metric on $\mathcal{F}$. We shall complete the proof by arguing that $\mathcal{F}$ has a dense subset consisting of homeomorphisms.

We remark that a map between compact metric spaces is onto, if it is a limit of onto maps. Hence, each of the near homeomorphisms $f_k : X_{k+1} \to X_k$ is onto. It follows that each $f_{\infty,k} : X_\infty \to X_k$ is onto. Consequently, each element of $\mathcal{F}$ is onto.

Let $\varepsilon > 0$. Set $\mathcal{F}(\varepsilon) = \mathcal{F} \cap \mathfrak{M}_\varepsilon(X, Y)$. Lemma 2 implies that $\mathcal{F}(\varepsilon)$ is an open subset of $\mathcal{F}$. We shall argue that $\mathcal{F}(\varepsilon)$ is a dense subset of $\mathcal{F}$. It suffices to show that if $k \ge 1$ and $h : X_k \to X_1$ is a homeomorphism then the δ-neighborhood of $h \circ f_{\infty,k}$ contains an element of $\mathcal{F}(\varepsilon)$ for every $\delta > 0$. Choose $j > k$ so that $1/j \le \varepsilon$. Notice that $h \circ f_{\infty,k} = h \circ f_k \circ \cdots \circ f_{j-1} \circ f_{\infty,j}$. We apply Lemma 1 twice: first to conclude that $f_k \circ \cdots f_{j-1}$ is a near homeomorphism; and second to see that we can choose a homeomorphism $g : X_j \to X_k$ so close to $f_k \circ \cdots \circ f_{j-1}$ that $\tilde{\rho}_1(h \circ f_{\infty,k}, h \circ g \circ f_{\infty,j}) < \delta$. Since $f_{\infty,j}$ is a $(1/j)$-map, then $h \circ g \circ f_{\infty,j} \in \mathcal{F}(\varepsilon)$.

The Baire Category Theorem implies that $\mathcal{H} = \cap\{\mathcal{F}(1/k) : k \ge 1\}$ is a dense subset of $\mathcal{F}$. Since each element of $\mathcal{H}$ is one-to-one, and each element of $\mathcal{F}$ is onto, then $\mathcal{H}$ is a dense set of homeomorphisms in $\mathcal{F}$. □

Reference

[B]. M. Brown, Some applications of an approximation theorem for inverse limits, *Proc. Amer. Math. Soc.* 11 (1960), 478–481.

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