# ENGULFING THE TRACK OF A PROPER HOMOTOPY

by

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# 0. INTRODUCTION

Let $M^n$ be a boundaryless PL n-manifold. Suppose r is an integer with $0 \le r \le n-3$, $W \subset V \subset U$ are open subsets of $M^n$, and $\mathcal{T}$ is a collection of subsets of $M^n$. We say <u>(finite) r complexes in U can be pulled into V along $\mathcal{T}$ rel W</u> if whenever $P \supset Q$ are closed subpolyhedra of $M^n \ni$ $P \subset U$, $Q \subset W$, ($cl(P-Q)$ is compact,) $\dim Q \le n-3$, and $\dim cl(P-Q) \le r$, then there is a proper homotopy $\varphi : P \times I \to M^n \ni$ $\varphi(x,t) = x$ for every $x \in (P \times 0) \cup (Q \times I)$, $\varphi(P \times 1) \subset V$ and $\forall x \in P$, $\exists T \in \mathcal{T} \ni \varphi(x \times I) \subset T$.

Our principal aim is to present the following two engulfing theorems which are proved in section 5.

5.1. <u>A Simple Engulfing Theorem</u>:

<u>Hypothesis</u>: Let $M^n$ be a boundaryless PL n-manifold. Suppose r is an integer with $0 \le r \le n-3$, U is an open subset of $M^n$, and $\mathcal{T}$ is a collection of subsets of $M^n \ni$ finite r complexes in $M^n$ can be pulled into U along $\mathcal{T}$ rel U.

<u>Conclusion</u>: If $P \supset Q$ are closed subpolyhedra of $M^n \ni$ $Q \subset U$, $cl(P-Q)$ is compact, $\dim Q \le n-3$ and $\dim cl(P-Q) \le r$, then $\forall$ open neighborhood G of $1|M^n$ in $M^n \times M^n$, $\exists$ a compactly supported PL ambient isotopy h of $M^n$ which fixes $Q \ni h_1(U) \supset P$ and the h-track of each point of $M^n$ is either a singleton or lies in the G-neighborhood of the union of $\begin{Bmatrix} r+1 \text{ if } 0 \le r \le n-4 \\ r+2 \text{ if } r = n-3 \end{Bmatrix}$ elements of $\mathcal{T}$.

5.2 A Complicated-But-Useful Engulfing Theorem:

Hypothesis: Let $M^n$ be a boundaryless PL $n$-manifold. Suppose $r$ is an integer with $0 \le r \le n-3$,

$$V_{r+1} \subset V_r \subset \cdots \subset V_0 \subset U_r \subset \cdots \subset U_0 \subset U_{-1}$$

are open subsets of $M^n$, and for $i = 0, 1, 2, \ldots, r$, $\mathcal{J}_i$ is a collection of subsets of $U_{i-1}$ $\ni$ $i$-complexes in $U_i$ can be pulled into $V_i$ along $\mathcal{J}_i$ rel $V_{i+1}$.

Conclusion: If $P \supset Q$ are closed subpolyhedra of $M^n$ $\ni$ $P \subset U_r$, $Q \subset V_{r+1}$, $\dim Q \le n-3$ and $\dim cl(P-Q) \le r$, then $\forall$ open neighborhood $G$ of $1|M^n$ in $M^n \times M^n$, $\exists$ a PL ambient isotopy $h$ of $M^n$ which fixes $Q$ $\ni$ $h_1(V_0) \supset P$ and the $h$-track of each point of $M^n$ is either a singleton or lies in the $G$-neighborhood of the union of

$\begin{cases} \text{one element of each of } \mathcal{J}_0, \mathcal{J}_1, \ldots, \mathcal{J}_r \text{ if } 0 \le r \le n-4, \text{ and} \\ \text{one element of each of } \mathcal{J}_0, \mathcal{J}_1, \ldots, \mathcal{J}_{r-1} \text{ and two elements of } \mathcal{J}_r \text{ if } r = n-3. \end{cases}$

Moreover, if $cl(P-Q)$ is compact, then $h$ may be chosen to have compact support.

Both these engulfing theorems can be proved by induction on $r$ using three lemmas which we establish in sections 3 and 4. These three lemmas have the same hypothesis which we refer to as the "scenery".

## The "Scenery" For The Three Lemmas:

Let $M^n$ be a boundaryless PL $n$-manifold.
Suppose $r$ is an integer with $0 \le r \le n-3$.
Suppose $P \supset Q$ are closed subpolyhedra of $M^n$,
let $R = cl(P-Q)$, and suppose $\dim Q \le n-3$ and $\dim R \ge r$.
Let $X = (Q\times 0) \cup (R\times I)$ and let $Y = (Q\times 0)\cup[(Q\cap R)\times I]\cup(R\times 1)$.
Suppose $f: X \to M^n$ is a proper map $\ni f(x0) = x$ for every $x \in P$.
Let $\pi: X \to P$ and $\tau: X \to I$ denote the restriction to $X$ of the projections $P\times I \to P$ and $P\times I \to I$, respectively.

We now state the three lemmas.

3.3 <u>The Engulfing Lemma</u>: Given the scenery, suppose $A \supset B$ are closed subpolyhedra of $R$, $Z$ is a closed subpolyhedron of $X$, and $f$ is a PL map $\ni$
$(A\times I) \cap [(Q\cup R)\times I \cup Z] = B\times I$ (whence $B = A \cap [Q \cup \pi(Z)]$)
and $S(f|Y\cup Z\cup(A\times I)) \subset Z$. Let $U$ be an open neighborhood of $f(Y\cup Z)$ in $M^n$. Then $\forall$ open neighborhood $G$ of $1|M^n$ in $M^n\times M^n$, $\exists$ a PL ambient isotopy $h$ of $M^n$ which fixes $f(Y\cup Z) \ni$ $h_1(U) \supset f(Y\cup Z\cup(A\times I))$ and the $h$-track of each point of $M^n$ is either a singleton or is contained in $G(f(x\times I))$ for some $x \in A-B$.

4.1 <u>The Codimension $\ge 4$ Approximation Lemma</u>:
Given the scenery, suppose $r \le n-4$. Then $\forall$ open neighborhood $G$ of $1|M^n$ in $M^n\times M^n$, $\exists$ a proper PL map $g: X \to M^n$ and $\exists$ a closed subpolyhedron $Z$ of $X \ni g(x0) = x$ for every $x\in P$, $g \subset G\circ f$, $S(g)\subset Z$, $\pi^{-1}\pi(Z) = Z$ and $\dim Z < r$.

4.3 <u>The Codimension = 3 Approximation Theorem</u>:
Given the scenery, suppose $r = n-3$. Then $\forall$ open neighborhood $G$ of $1|M^n$ in $M^n \times M^n$, $\exists$ a proper PL map $g: X \to M^n$ and $\exists$ closed subpolyhedra $Z$ of $X$ and $A_1$ and $A_2$ of $R \ni g(x,0) = x$ for every $x \in P$, $g \subset G \circ f$, $\pi^{-1}\pi(Z) = Z$, $\dim Z < r$, $A_1 \cup A_2 = R$, $\pi(Z) \cap R \subset A_1$, $S(g | Y \cup (A_1 \times I)) \subset Z$, and $S(g \mid Y \cup (A_1 \times 0) \cup (A_2 \times I)) \subset (Y \cap Z) \cup (A_1 \times 0) \cup [(A_1 \cap A_2) \times I]$.

Sections 1 and 2 define the terms and state the topological and piecewise linear prerequisites necessary to understand the proofs of the three lemmas and the two engulfing theorems. Also, at the end of section two we describe the "Stallings' Stretch", a technique of stretching an open set along the join lines between ~~dual~~ subcomplexes of a complex, which is exploited in most applications of engulfing.

In section 6, we present several applications of the engulfing theorems: weak h-cobordism theorems, the cellularity criterion,

# 1. Topological Prerequisites

We begin with some notational conventions. Let $X$, $Y$ and $Z$ denote arbitrary sets. Define $\underline{1|X} = \{(x,x) \in X \times X : x \in X\}$.
If $R \subset X \times Y$ and $S \subset Y \times Z$, define
$\underline{S \circ R} = \{(x,z) \in X \times Z : \exists y \in Y \ni (x,y) \in R \text{ and } (y,z) \in S\}$.
If $R \subset X \times Y$, define $\underline{R^{-1}} = \{(y,x) \in Y \times X : (x,y) \in R\}$.
If $R \subset X \times Y$ and $A \subset X$, define $\underline{R(A)} = \{y \in Y : \exists x \in A \ni (x,y) \in R\}$.

1.1. Let $(X, \rho)$ be a metric space.
(a) If $\delta : X \to (0,\infty)$ is a continuous function, then
$\{(x,y) \in X \times X : \rho(x,y) < \delta(x)\}$ is an open neighborhood of $1|X$ in $X \times X$.
(b) $\forall$ open neighborhood $G$ of $1|X$ in $X \times X$, $\exists$ a continuous function $\delta : X \to (0,\infty) \ni \forall (x,y) \in X \times X$, if $\rho(x,y) < \delta(x)$, then $(x,y) \in G$.
(c) $\forall$ open cover $\mathcal{U}$ of $X$, $\exists$ an open neighborhood $G$ of $1|X$ in $X \times X \ni \forall x \in X$, $\exists U \in \mathcal{U} \ni G(x) \subset U$.

1.2. Let $(X, \rho)$ and $(Y, \sigma)$ be metric spaces, and suppose $f : X \to Y$ is a continuous function
(a) If $\delta : X \to (0,\infty)$ is a continuous function, then
$\{(x,y) \in X \times Y : \sigma(f(x),y) < \delta(x)\}$ is an open neighborhood of $f$ in $X \times Y$.
(b) $\forall$ open neighborhood $G$ of $f$ in $X \times Y$, $\exists$ a continuous function $\delta : X \to (0,\infty) \ni \forall (x,y) \in X \times Y$, if $\sigma(f(x),y) < \delta(x)$, then $(x,y) \in G$.

1.3 Let $(X,\rho)$ be a metric space. Then $\forall$ open neighborhood $G$ of $1\!\!1X$ in $X\times X$, $\exists$ an open neighborhood $H$ of $1\!\!1X$ in $X\times X \ni$ $H=H^{-1}$ and $H\circ H\subset G$.

1.4. Let $(X,\rho)$ be a metric space. If $\mathcal{U}$ is an open cover of $X$ and $r$ is a positive integer, then $\exists$ an open cover $\mathcal{V}$ of $X \ni$ if $V_0, V_1, \ldots, V_r \in \mathcal{V}$ and $V_{i-1}\cap V_i \neq \emptyset$ for $1\le i\le r$, then $\exists\, U\in\mathcal{U} \ni \bigcup_{i=0}^{r} V_i \subset U$.

Let $X$ be a topological space and let $G$ be an open neighborhood of $1\!\!1X$ in $X\times X$. For $A\subset X$, we call $G(A)$ the <u>G-neighborhood</u> of $A$ in $X$.

1.5. Let $X$ be a normal space. If $A\subset U\subset X$ where $A$ is a closed subset of $X$ and $U$ is an open subset of $X$, then $\exists$ an open neighborhood $G$ of $1\!\!1X$ in $X\times X \ni G(A)\subset U$.

1.6 If $X$ is a locally compact metric space, then $\exists$ an open neighborhood $G$ of $1\!\!1X$ in $X\times X \ni \forall$ compact subset $K$ of $X$, $\mathrm{cl}\, G(K)$ is also compact.

Let $X$ and $Y$ be topological spaces. A function $f: X\to Y$ is <u>proper</u> if $\forall$ compact subset $K$ of $Y$, $f^{-1}(K)$ is a compact subset of $X$.

1.7 Let $X$ and $Y$ be topological spaces and suppose $f: X\to Y$ is a continuous function.

(a) If $f$ is a closed map with compact point inverses, then $f$ is proper.

(b) Suppose $X$ and $Y$ are Hausdorff spaces and $Y$ is first countable; then if $f$ is proper, then $f$ is a closed map with compact point inverses.

1.8 Let $(X,\rho)$ and $(Y,\sigma)$ be metric spaces and suppose $f:X\to Y$ is a continuous function.

(a) If $H$ is an open neighborhood of $1|Y$ in $Y\times Y$, then $H\circ f$ is an open neighborhood of $f$ in $X\times Y$.

(b) If $f$ is proper, then $\forall$ open neighborhood $G$ of $f$ in $X\times Y$, $\exists$ an open neighborhood $H$ of $1|Y$ in $Y\times Y \ni H\circ f\subset G$.

1.9. <u>The Proper Approximation Theorem</u>. Let $(X,\rho)$ and $(Y,\sigma)$ be metric spaces and let $X$ be locally compact. Every proper continuous function $f:X\to Y$ has an open neighborhood $G$ in $X\times Y \ni$ if $g:X\to Y$ is a continuous function and $g\subset G$, then $g$ is proper.

1.10. Let $X$ and $Y$ be topological spaces and suppose $f:X\to Y$ is a continuous, closed, surjective function with compact point inverses. If $X$ is a separable metric space, then so is $Y$. [D, Theorem XI.5.2, page 235.]

# 2. Piecewise Linear Prerequisites

Unless otherwise noted, all references in this section are to Piecewise Linear Topology by J. F. P. Hudson, W.A. Benjamin, N.Y., 1969.

We assume the reader knows the meaning of the terms n-simplex, interior of an n-simplex, and boundary of an n-simplex for $n = 0, 1, 2, \ldots$. We recall the convention that the interior of a 0-simplex is the 0-simplex and the boundary of a 0-simplex is empty. We assume the reader knows the meaning of the term simplicial complex. We shall use the word complex to mean a finite-dimensional, countable, locally-finite simplicial complex lying in some Euclidean space. A polyhedron is the underlying point set of a complex. Let $K$ be a complex and let $L \subset K$. We let $|L| = \cup L$. For $i = 0, 1, 2, \ldots$, we let $L^i = \{\alpha \in L : \dim \alpha \leq i\}$. If $A \subset |K|$, we let $L|A = \{\alpha \in L : \alpha \subset A\}$. If $A$ and $B$ are subsets of some Euclidean space, we let $A * B = \{(1-t)x + ty : x \in A, y \in B \text{ and } 0 \leq t \leq 1\}$. If $\alpha \in K$, we let $\text{link}(\alpha, K) = \{\beta \in K : \alpha \cap \beta = \emptyset \text{ and } \alpha * \beta \in K\}$, and we let $\text{star}(\alpha, K) = \{\gamma \in K : \exists \beta \in K \ni \alpha \cup \gamma \subset \beta\}$

$= \{\gamma \in K : \exists \beta \in \text{link}(\alpha, K) \ni \gamma \subset \alpha * \beta\}$.

We assume the reader knows the meaning of the terms <u>subcomplex</u> of a complex and <u>subdivision</u> of a complex. A subcomplex $L$ of a complex $K$ is a <u>full</u> subcomplex of $K$, if every simplex of $K$ whose vertices lie in $L$ is itself an element of $L$. If $\alpha$ is a $k$-simplex in some Euclidean space with vertices $v_0, v_1, \dots, v_k$, then the <u>barycenter</u> of $\alpha$ is the point $\sum_{i=0}^{k} \frac{1}{k+1} v_i$. Let $K$ be a complex. For each $\alpha \in K$, let $v(\alpha)$ denote the barycenter of $\alpha$. The <u>first barycentric subdivision</u> of $K$ is the subdivision $K'$ of $K$ whose elements are precisely all simplices of the form $v(\alpha_0) * v(\alpha_1) * \dots * v(\alpha_k)$ where $\alpha_0, \alpha_1, \dots, \alpha_k \in K$ and $\alpha_0 \subsetneqq \alpha_1 \subsetneqq \dots \subsetneqq \alpha_k$ for $k = 0, 1, 2, \dots$.

For $l = 2, 3, 4, \dots$, the <u>$l^{th}$ barycentric subdivision</u> of $K$ is defined inductively to be the first barycentric subdivision of the $(l-1)$th barycentric subdivision of $K$. Observe that if $K \supset L$ are complexes and $K'$ is the first barycentric subdivision of $K$, then $K' \mid |L|$ is a full subcomplex of $K'$.

We presume the reader knows the meaning of the terms <u>PL cell</u>, <u>boundary of a PL cell</u>, <u>dimension</u> of a PL cell, <u>PL manifold</u>, <u>boundary</u> of a PL manifold, and <u>dimension</u> of a PL manifold.

2.1 Triangulating to refine an open cover: If $K$ is a complex and $\mathcal{U}$ is an open cover of $K$, then $\exists$ a subdivision $K'$ of $K \ni \forall \alpha \in K'$, $\exists\, U \in \mathcal{U} \ni |\mathrm{star}(\alpha, K')| \subset U$.
[This can be proved by a modification of the proof of Theorem 3.5, page 80.]

Suppose $X \supset Y$ are polyhedra and $Y$ is a closed subset of $X$. X collapses to Y by an elementary collapse, denoted $X \searrow^{e} Y$, if $\exists$ a PL cell $C \ni X = Y \cup C$, $Y \cap C$ is a PL cell in $\partial C$, and $\dim(Y \cap C) = \dim C - 1$. X collapses to Y, denoted $X \searrow Y$, if $\exists$ a finite sequence $Y = Y_0, Y_1, \ldots, Y_k = X$ of closed subpolyhedra of $X \ni Y_i \searrow^{e} Y_{i-1}$ for $i = 1, 2, \ldots, k$.

Suppose $K \supset L$ are complexes. K collapses simplicially to L by an elementary simplicial collapse, denote $K \searrow^{es} L$, if $\exists$ simplices $\alpha, \beta \in K \ni \alpha, \beta \notin L$, $K = L \cup \{\alpha, \beta\}$, $\beta \subset \alpha$ and $\dim \beta = \dim \alpha - 1$. K collapses simplicially to L, denoted $K \searrow^{s} L$, if $\exists$ a finite sequence $L = L_0, L_1, \ldots, L_k = K$ of subcomplexes of $K \ni K_i \searrow^{es} K_{i-1}$ for $i = 1, 2, \ldots, k$.

2.2. Triangulating a collapse: If $K \supset L$ are complexes and $|K| \searrow |L|$, then $\exists$ a subdivision $K'$ of $K \ni K' \searrow^{s} K' / |L|$. [Theorem 2.4, page 48.]

Let $X$ and $Y$ be polyhedra and let $f: X \to Y$ be a continuous function. $f$ is a piecewise linear (PL) map if whenever $K$ and $L$ are complexes triangulating $X$ and $Y$, respectively, there is a subdivision $K'$ of $K \ni f$ maps each simplex of $K'$ linearly <u>into</u> a simplex of $L$.

If $K$ and $L$ are complexes and $f: |K| \to |L|$ is a continuous function $\ni f$ maps each simplex of $K$ linearly <u>onto</u> a simplex of $L$, then we say $f$ is a <u>simplicial map</u> from $K$ to $L$. If, in addition, $f$ is bijective, we call $f$ a <u>simplicial isomorphism</u> from $K$ to $L$.

Let $X$ and $Y$ be metrizable spaces. Recall that a map $f: X \to Y$ is <u>proper</u> if $\forall$ compactum $K \subset Y$, $f^{-1}(K)$ is compact. Remember that $f$ is proper if and only if $f$ is a closed map with compact point inverses.

2.3 <u>Triangulating a proper PL map</u>:
If $K$ and $L$ are complexes and $f: |K| \to |L|$ is a proper PL map, then $\exists$ subdivisions $K'$ of $K$ and $L'$ of $L \ni f$ is a simplicial map from $K'$ to $L'$. [Theorem 3.6C, page 84.]

Let $X$ and $Y$ be topological spaces and let $f: X \to Y$ be a map. For $k = 2, 3, 4, \ldots$, we let

<u>$S_k(f)$</u> $= cl\{x \in X : f^{-1}f(x)$ contains at least $k$ points$\}$

and we let <u>$S(f)$</u> $= S_2(f)$.

2.4 If $X$ and $Y$ are polyhedra and $f: X \to Y$ is a proper PL map, then for $k = 2, 3, 4, \ldots$, $S_k(f)$ is a closed subpolyhedron of $X$. [Page 90.]

Let $K$ be a complex and let $M^n$ be a PL $n$-manifold. A map $f: |K| \to Y$ is <u>in general position with respect to $K$</u>, if

(a) $f: |K| \to M^n$ is a PL map,

(b) $f$ embeds each simplex of $K$ piecewise linearly in $M^n$, and

(c) if $\alpha_0, \alpha_1, \ldots, \alpha_k$ are distinct simplices of $K$, then
$\dim \left[ \bigcap_{i=0}^{k} f(\operatorname{int} \alpha_i) \right] \leq \sum_{i=0}^{k} \dim \alpha_i - kn$.

2.5 If $K$ is a complex, $M^n$ is a PL $n$-manifold and $f: |K| \to M^n$ is a PL map in general position with respect to $K$, then for $k = 2, 3, 4, \ldots$, $\dim S_k(f) \leq k \cdot \dim K - (k-1) \cdot n$.

2.6 <u>The General Position Approximation Theorem</u>: Let $K \supset L$ be complexes and let $M^n$ be a PL $n$-manifold with $\dim K \leq n$. Suppose $f: |K| \to M^n$ is a map $\ni f| |L| : |L| \to M^n$ is a PL embedding. Then $\forall$ open neighborhood $G$ of $f$ in $|K| \times M^n$, $\exists$ a PL map $g: |K| \to M^n \ni g| |L| = f| |L|$, $g \subset G$ and $\exists$ a subdivision $K'$ of $K \ni g$ is in general position with respect to $K'$. [This follows from Lemma 4.2, page 92, Lemma 4.4, page 95 and a modification of Lemma 4.7, page 99.]

2.7 Homeomorphisms via small vertex shifts: Suppose $K$ is a complex and $K_1$ is a subdivision of $K$. $\forall v \in K_1^0$, let $\alpha(v) \in K \ni v \in \operatorname{int} \alpha(v)$. Then every $v \in K_1^0$ has an open neighborhood $N_v$ in $|K| \ni$ the collection $\{N_v : v \in K_1^0\}$ has the following property: If $h^0 : K_1^0 \to |K|$ is a function $\ni h^0(v) \in N_v \cap \operatorname{int} \alpha(v)$ for every $v \in K_1^0$, then $\exists$ a subdivision $K_2$ of $K \ni$ $K_2^0 = h(K_1^0)$ and $h^0 : K_1^0 \to K_2^0$ induces a unique PL homeomorphism of $|K|$ which is a simplicial isomorphism from $K_1$ to $K_2$ and which maps each simplex of $K$ onto itself. [Folklore.]

2.8 If $X$ and $Y$ are polyhedra, then the projections $X \times Y \to X$ and $X \times Y \to Y$ are PL maps.

Throughout this paper we let $\underline{I} = [0,1]$. Let $X$ be a polyhedron and let $\pi : X \times I \to X$ and $\tau : X \times I \to X$ denote the projections. Suppose $h : X \times I \to X \times I$ is a map. For each $t \in I$, define $\underline{h_t} : X \to X$ by $h_t(x) = \pi h(x,t)$ for $x \in X$. We say $h$ is __level-preserving__ if $\tau \circ h = \tau$. We call $h$ a __PL ambient isotopy of X__ if $h$ is a level-preserving homeomorphism of $X \times I$ and $h_0 = 1|X$. For $A \subset X$. We say $h$ __fixes__ $A$ if $h|A \times I = 1|A \times I$. We say $h$ is __compactly supported__ or has __compact support__ if $h$ fixes the complement of a compact subset of $X$. For each $x \in X$, the set $\pi h(\{x\} \times I)$ is called the $h$-track of $x$.

Most applications of engulfing exploit Stallings' technique of stretching an open set along the join lines between dual subcomplexes of a complex. We formalize this technique in the following definition and theorem.

Let $K$ be a complex and let $L$ be a full subcomplex of $K$. We let $\underline{L^*} = \{\alpha \in K : \alpha \cap |L| = \emptyset\}$ and we call $L^*$ the subcomplex of $K$ which is dual to $L$. We observe that $L^*$ is a full subcomplex of $K$, and that $\forall \alpha \in K$, $\alpha \cap |L|$ and $\alpha \cap |L^*|$ are disjoint faces of $\alpha \ni \alpha = (\alpha \cap |L|) * (\alpha \cap |L^*|)$.

2.9 <u>The Stallings' Stretch</u>: Let $K$ be a complex, let $L$ be a full subcomplex of $K$, and let $L^* = \{\alpha \in K : \alpha \cap |L| = \emptyset\}$ (the subcomplex of $K$ which is dual to $L$). If $U$ and $U^*$ are open subsets of $|K| \ni |L| \subset U$ and $|L^*| \subset U^*$, then there is a PL ambient isotopy $h$ of $|K|$ which fixes $|L| \cup |L^*| \ni h_1(U) \cup U^* = |K|$ and $h_t(\alpha) = \alpha$ for every $\alpha \in K$ and $t \in I$. [A proof which provides a <u>topological</u> ambient isotopy is given in Lemma 8.2 of J.R Stallings, "On topologically unknotted spheres", Annals of Math 77 (1963), 490-503.]

Before commencing sections 3 and 4 and the proofs of "the three lemmas", we remind the reader of

The "Scenery" For Lemmas 3.3, 4.1 and 4.3 :

Let $M^n$ be a boundaryless PL $n$-manifold.

Suppose $r$ is an integer with $0 \le r \le n-3$.

Suppose $P \supset Q$ are closed subpolyhedra of $M^n$,

let $R = cl(P-Q)$, and suppose $\dim Q \le n-3$ and $\dim R = r$.

Let $X = (Q \times 0) \cup (R \times I)$ and let $Y = (Q \times 0) \cup [(Q \cap R) \times I] \cup (R \times 1)$.

Suppose $f : X \to M^n$ is a proper map $\ni$ $f(x,0) = x$ for every $x \in P$.

Let $\pi : X \to P$ and $\tau : X \to I$ denote the restriction to $X$ of the projections $P \times I \to P$ and $P \times I \to I$, respectivley.

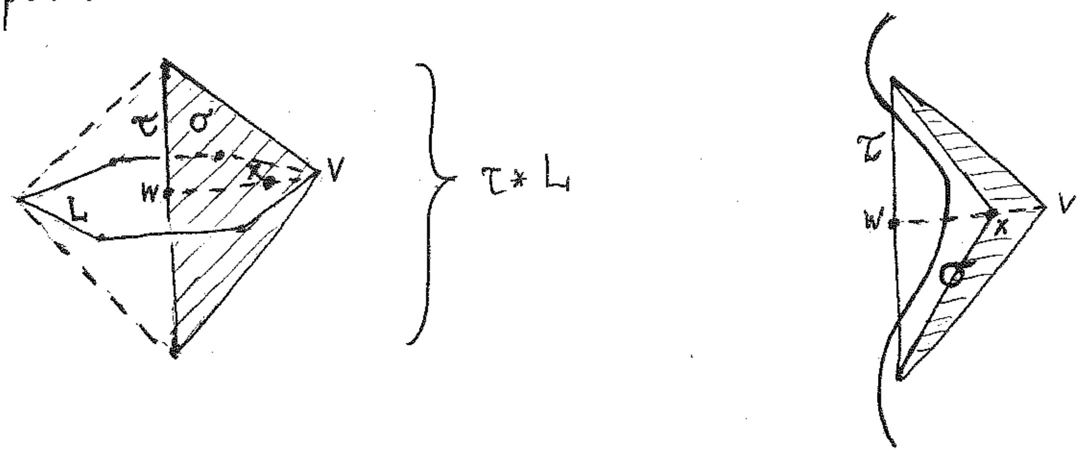
# 3. THE ENGULFING LEMMA

Our first proposition formulates the geometric heart of the engulfing process.

3.1. Engulfing the Track of a Collapse:
Suppose $A \supset B$ are closed subpolyhedra of a boundaryless PL $n$-manifold $M^n$ and $A \searrow B$. Let $U$ and $V$ be open subsets of $M^n \ni B \subset U$ and $cl_{M^n}(A-B) \subset V$. Then there is a PL ambient isotopy $h$ of $M^n$ which fixes $B \cup (M^n - V)$ such that $h_1(U) \supset A$.

Proof: 2.1 and 2.2 provide a complex $T$ which triangulates $M^n \ni$ subcomplexes of $T$ triangulate $A$ and $B$, $\bigcup\{\alpha \in T : \alpha \cap cl_{M^n}(A-B) \neq \emptyset\} \subset V$, and $T|A \overset{s}{\searrow} T|B$. By inducting on the number of elementary simplicial collapses required to collapse $T|A$ to $T|B$, we see that it suffices to consider the case in which $T|A \overset{es}{\searrow} T|B$. Thus $\exists\ \sigma, \tau \in T|A \ni \sigma, \tau \notin T|B$, $T|A = T|B \cup \{\sigma, \tau\}$, $\tau$ is a face of $\sigma$ and $\dim \tau = \dim \sigma - 1$. Hence $\exists$ a vertex $v$ of $\sigma \ni \sigma = v * \tau$; moreover, $A = B \cup \sigma$ and $B \cap \sigma = v * \partial\tau$.

Let $L = \text{link}(\tau, T)$. Then $L$ is a PL sphere, $\tau * L$ is a PL $n$-cell, $\tau * L \subset V$, $v \in L$ and $B \cap (\tau * L) \subset \partial(\tau * L)$. Let $w$ be the barycenter of $\tau$. Since $v * \partial\tau \subset B \subset U$, then there is a point $x \in v * w \ni x \neq v$ and $v * x * \partial\tau \,\dot{\subset}\, U$.




Since $L$ is a PL sphere and $x$ is an interior point of the PL cell $w * L$, then there is a PL ambient isotopy $g$ of $w * L$ which fixes $L$ such that $g(v * x) = v * w$. (We offer further evidence of the existence of $g$. Suppose $l = \dim L$ and $\varphi : \partial[-1,1]^{l+1} \to L$ is a PL homeomorphism. Let $q \in \partial[-1,1]^{l+1} \ni \varphi(q) = v$. Define the PL homeomorphism $\Phi : [-1,1]^{l+1} \to w * L$ by $\Phi(t,p) = (1-t)w + t\varphi(p)$ for $p \in \partial[-1,1]^{l+1}$, and $t \in I$. Then $\Phi | \partial[-1,1]^{l+1} = \varphi$, $\Phi(0) = w$ and $\exists\, s \in [0,1) \ni \Phi(sq) = x$. Define the PL ambient isotopy $G$ of $[-1,1]^{l+1}$ by specifying that for $t \in I$, $G_t$ is the conewise extension of the map which takes $sq$ to $(1-t)sq$ and which is the identity on $\partial[-1,1]^{\ell+1}$. Thus for $t\in I$, $p\in\partial[-1,1]^{\ell+1}$ and $u\in I$, $G_t((1-u)sq+up)=(1-u)(1-t)sq+up$. Hence $G$ is a PL ambient isotopy of $[-1,1]^{\ell+1}$ which fixes $\partial[-1,1]^{\ell+1}$ such that $G_1(sq*q)=[0*q]$. Finally define $g_t=\Phi\circ G_t\circ\Phi^{-1}$.)

Observe that $\tau * L=(w*L)*\partial\tau$.

Now define the PL ambient isotopy $h$ of $M^n$ by specifying that for $t\in I$, $h_t=1$ on $M^n-\operatorname{int}_{M^n}(\tau*L)$, and on $\tau*L$, $h_t$ is the join of $g_t: w*L\to w*L$, with $1|\partial\tau$. Thus for $t\in I$, $p\in w*L$, $q\in\partial\tau$ and $u\in I$, $h_t((1-u)p+uq)=(1-u)g_t(p)+uq$. Consequently, $h$ is a PL ambient isotopy of $M^n$ which fixes $M^n-\operatorname{int}_{M^n}(\tau*L)$ such that $h_1(v*x*\partial\tau)=v*w*\partial\tau$. Since $B\cap(\tau*L)\subset\partial(\tau*L)$ and $\tau*L\subset V$, then $h$ fixes $B\cup(M^n-V)$.

Since $v*x*\partial\tau\subset U$ and $v*w*\partial\tau=v*\tau=\sigma$, then $h_1(U)\supset\sigma$. So $h_1(U)\supset A$. ∎

It is convenient for formalize the following fact.

3.2 Suppose $A$, $B$ and $C$ are polyhedra, $B$ is a closed subpolyhedron of $A$, $A \searrow B$, and $g: A \to C$ is a proper PL map. If $S(g) \subset B$, then $g(A) \searrow g(B)$.

<u>Proof:</u> Let $K$ and $L$ be complexes triangulating $A$ and $C$, respectively, so that $g$ maps each simplex of $K$ linearly into a simplex of $L$. Using 2.2, let $K'$ be a subdivision of $K$ $\ni$ a subcomplex of $K'$ triangulates $B$ and $K' \searrow^{s} K'|B$. Then $\exists$ a sequence $K'_0 = K'|B, K'_1, \ldots, K'_k = K'$ of subcomplexes of $K'$ $\ni$ $K'_i \searrow^{es} K'_{i-1}$ for $i = 1, 2, \ldots, k$.

Since $g$ is proper and $S(g) \subset B$, then $g(B)$ is a closed subset of $C$, $g$ embeds $A - B$ and $g(B) \cap g(A-B) = \emptyset$. Thus $g$ embeds each simplex of $K' - K'|B$ and $g(|K'_{i-1}|)$ can't "obstruct" the elementary collapse of $g(|K'_i|)$ to $g(|K'_{i-1}|)$ for $i = 1, 2, \ldots, k$. Therefore $g(|K'_i|) \searrow g(|K'_{i-1}|)$ for $i = 1, 2, \ldots, k$. So $g(|K'|) = g(A) \searrow g(|K'|B|) = g(B)$. ∎

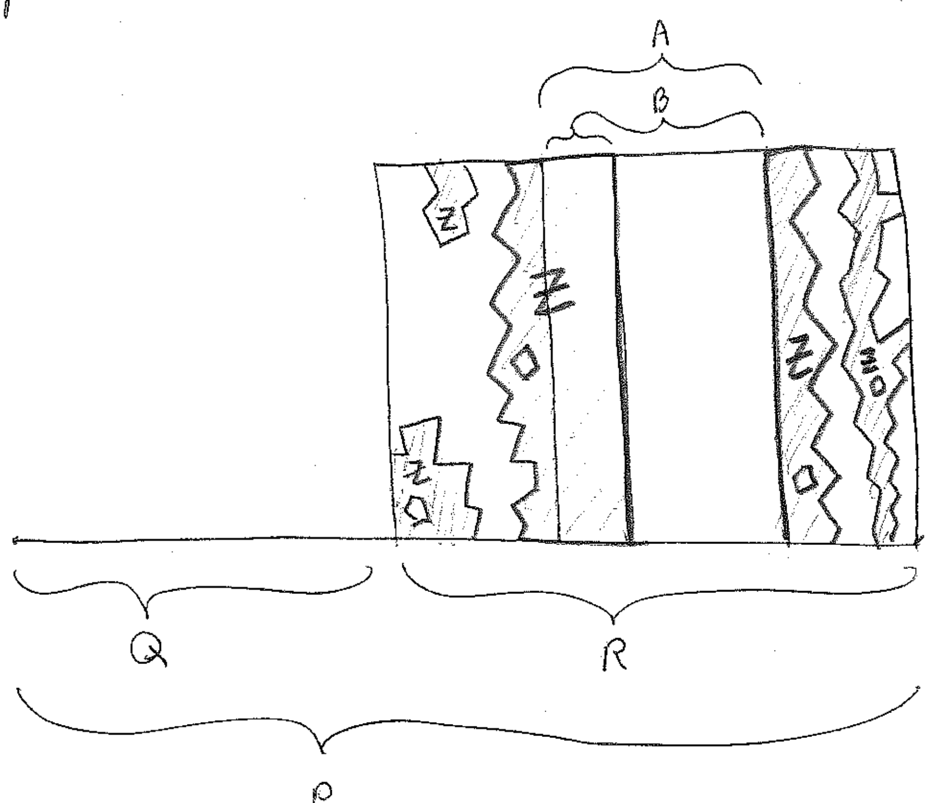
We now come to the principal result of this section.

3.3 The Engulfing Lemma:

Given the scenery, suppose $A$ and $B$ are closed subpolyhedra of $R$, $Z$ is a closed subpolyhedron of $X$, and $f$ is a PL map $\ni$

$$(A\times I)\cap[(Q\cap R)\times I\cup Z]=B\times I \qquad (\text{whence } B=A\cap[Q\cup\pi(Z)])$$

and $S(f|Y\cup Z\cup(A\times I))\subset Z$. Let $U$ be an open neighborhood of $f(Y\cup Z)$ in $M^n$. Then $\forall$ open neighborhood $G$ of $1|M^n$ in $M^n\times M^n$, $\exists$ a PL ambient isotopy $h$ of $M^n$ which fixes $f(Y\cup Z)$ $\ni$ $h_1(U)\supset f(Y\cup Z\cup(A\times I))$ and the $h$-track of each point of $M^n$ is either a singleton, or is contained in $G(f(x\times I))$ for some $x\in A-B$.



<u>Remarks</u>: If the set $Z$ has the additional property that $\pi^{-1}\pi Z = Z$, then following the literature (e.g. [B,1]) we call $Z$ a <u>shadow</u> for $f \mid Y \cup Z \cup (A \times I)$. We find the requirement that $\pi^{-1}\pi Z = Z$ too restrictive for one of the applications we wish to make of the Engulfing Lemma.

The proof of the Engulfing Lemma makes no use of the hypotheses $r \leq n-3$ and $\dim Q \leq n-3$. These hypotheses are necessary only for the results of later sections of this paper.

The proof ~~of~~ the Engulfing Lemma consists of engulfing the track of the "infinite collapse" $Y \cup Z \cup (A \times I) \searrow Y \cup Z$.

<u>Proof:</u> We begin by obtaining a collection $\mathcal{H}$ of open subsets of $M^n$ with the following two properties

(1) $\forall\, x \in A-B,\ \exists\, H \in \mathcal{H} \ni f(x \times I) \subset H$ and

(2) if $H_0, H_1, \ldots, H_r \in \mathcal{H}$ and $H_{i-1} \cap H_i \neq \emptyset$ for $1 \leq i \leq r$, then $\exists\, x \in A-B \ni H_0 \cup H_1 \cup \cdots \cup H_r \subset G\, f(x \times I)$.

To this end let $M' = M^n - f(B \times I)$, let $M''$ denote the quotient space obtained from $M'$ by identifying each $f(x \times I)$ to a point, for $x \in A-B$, and let $q: M' \to M''$

denote the quotient map. $M''$ has the largest topology with respect to which $q$ is continuous. Thus a subset $S$ of $M''$ is open / closed in $M''$ if and only if $q^{-1}(S)$ is open / closed in $M'$. We assert that $q: M' \to M''$ is a closed map. For let $K$ be a closed subset of $M'$ and let $y \in M'' - q(K)$.

If $y \notin q f((A-B) \times I)$, then $U = M^{a} - [K \cup f(A \times I)]$ $= M' - [K \cup f((A-B)\times I)]$ is an open neighborhood of $q^{-1}(y)$ in $M'$ $\ni$ $q(U)$ is an open neighborhood of $y$ in $M''$ (because $q^{-1}q(U) = U$) and $q(U) \cap q(K) = \emptyset$. If $y = q f(x \times I)$ for some $x \in A - B$, then $x$ has an open neighborhood $N$ in $A - B \ni f(N \times I) \cap K = \emptyset$; thus $U = M^{a} - [K \cup f((A-N)\times I)] = M' - [K \cup f(((A-B)-N)\times I)]$ is an open neighborhood of $q^{-1}(y)$ in $M'$ $\ni$ $q(U)$ is an open neighborhood of $y$ in $M''$ (because $q^{-1}q(U) = U$) and $q(U) \cap q(K) = \emptyset$. Thus, in no case is $y$ a limit point of $q(K)$. This proves that $q(K)$ is a closed subset of $M''$. Our assertion follows. Consequently, we see from 1.10 that $M''$ is a separable metric space.

For each $y \in M''$, let

$$G'(y) = \begin{cases} G(q^{-1}(y)) - f(A \times I) & \text{if } y \in M'' - qf((A-B)\times I) \\ M' - q^{-1}q(M' - G(q^{-1}(y))) & \text{if } y \in qf((A-B)\times I) \end{cases}$$

Then for each $y \in M''$: (1) $G'(y)$ is an open neighborhood of $q^{-1}(y)$ in $M'$; (2) $G'(y) \subset G(q^{-1}(y))$; (3) if $q^{-1}(y) \cap f((A-B)\times I) = \emptyset$, then $G'(y) \cap f((A-B)\times I) = \emptyset$; and (4) $q\,G'(y)$ is an open neighborhood of $y$ in $M''$ (because $q^{-1}q\,G'(y) = G'(y)$).
Now 1.4 provides an open cover $\mathcal{H}''$ of $M'' \ni$ if $H_0, H_1, \dots, H_r \in \mathcal{H}''$ and $H_{i-1} \cap H_i \neq \emptyset$ for $1 \leq i \leq r$, then $H_0 \cup H_1 \cup \dots \cup H_r \subset q\,G'(y)$ for some $y \in M''$. Let $\mathcal{H} = \{q^{-1}(H) : H \in \mathcal{H}'' \text{ and } H \cap qf((A-B)\times I) \neq \emptyset\}$.
Now (1) if $x \in A-B$, then $\exists H \in \mathcal{H}'' \ni qf(x \times I) \in H$; whence $f(x \times I) \subset q^{-1}(H) \in \mathcal{H}$. Moreover (2), if $q^{-1}(H_0), q^{-1}(H_1), \dots, q^{-1}(H_r) \in \mathcal{H}$ and $q^{-1}(H_{i-1}) \cap q^{-1}(H_i) \neq \emptyset$ for $1 \leq i \leq r$, then $H_{i-1} \cap H_i \neq \emptyset$ for $1 \leq i \leq r$; So $\exists y \in M'' \ni H_0 \cup H_1 \cup \dots \cup H_r \subset q\,G'(y)$. Hence $q^{-1}(H_0) \cup q^{-1}(H_1) \cup \dots \cup q^{-1}(H_r) \subset q^{-1}q\,G'(y) = G'(y) \subset G(q^{-1}(y))$.
Since $H_i \cap qf((A-B)\times I) \neq \emptyset$ for $0 \leq i \leq r$, then $G'(y) \cap f((A-B)\times I) \neq \emptyset$; whence $\exists x \in A-B \ni q^{-1}(y) = f(x \times I)$. Thus
$q^{-1}(H_0) \cup q^{-1}(H_1) \cup \dots \cup q^{-1}(H_r) \subset G(f(x \times I))$.

Next for each $x \in A-B$, let $N_x$ be an open neighborhood of $x$ in $A-B \ni f(N_x \times I) \subset H$ for some $H \in \mathcal{H}$.

Since $f(B \times I) \subset \mathcal{U}$, then there is a closed subpolyhedron $C$ of $A \ni B \subset \operatorname{int}_A C$ and $f(C \times I) \subset \mathcal{U}$. (To obtain $C$: $\forall x \in B$, let $V_x$ be an open neighborhood of $x$ in $A \ni f(V_x \times I) \subset \mathcal{U}$. Then $\{V_x : x \in B\} \cup \{A-B\}$ is an open cover of $A$. Thus by 2.1, there is a triangulation $K$ of $A$; a subcomplex of which triangulates $B \ni \forall \alpha \in K$, either $\exists x \in B \ni |\operatorname{star}(\alpha, K)| \subset V_x$ or $|\operatorname{star}(\alpha, K)| \subset A-B$. Let $C = \bigcup \{\alpha \in K : \alpha \cap B \neq \emptyset\}$.)

Let $D = \operatorname{cl}_A(A-C)$. By 2.1, there is a triangulation $T$ of $D \ni \forall \alpha \in T$, $\exists x \in A-B \ni |\operatorname{star}(\alpha, T)| \subset N_x$.

Let $T''$ be a second barycentric subdivision of $T$. For $k = 0, 1, \dots, r$, let $\{v_1^k, v_2^k, v_3^k, \dots\}$ be a list of the barycenters of the $k$-simplices of $T$. For $k = 0, 1, \dots, r$ and $i = 1, 2, 3, \dots$, let $S_i^k = |\operatorname{star}(v_i^k, T'')|$. Then for $k = 0, 1, \dots, r$, $\{S_i^k : i = 1, 2, 3, \dots\}$ is a discrete collection of compact subsets of $D$; and $\bigcup_{k=0}^{r} \bigcup_{i \geq 1} S_i^k = D$.

For $k = 0, 1, \dots, r$ and $i = 0, 1, 2, \dots$: $\exists \alpha_i^k \in T \ni v_i^k \in \operatorname{int} \alpha_i^k$; $\exists x_i^k \in A-B \ni |\operatorname{star}(\alpha_i^k, T)| \subset N_{x_i^k}$, so that $S_i^k \subset N_{x_i^k}$; and $\exists H_i^k \in \mathcal{H} \ni f(N_{x_i^k} \times I) \subset H_i^k$, whence $f(S_i^k \times I) \subset H_i^k$.

For each $k=0,1,\dots,r$, since $\{S^k_i : i=1,2,3,\dots\}$ is a discrete collection of compact subsets of $D$, and $f$ is proper, then $\{f(S^k_i \times I) : i=1,2,3,\dots\}$ is a discrete collection of compact subsets of $M^n$; so there is a discrete collection $\{V^k_i : i=1,2,3,\dots\}$ of open subsets of $M^n \ni$ $f(S^k_i \times I) \subset V^k_i \subset H^k_i$ for $i=1,2,3,\dots$.

Let $C^{-1}=C$; and for $k=0,1,\dots,r$, let $C^k = C^{-1} \cup \bigcup_{j=0}^{k} \left(\bigcup_{i\geq 1} S^j_i\right)$. Then $C^r = A$

Now we shall obtain a sequence $h^0, h^1, \dots, h^r$ of PL ambient isotopies of $M$ such that for $k=0,1,\dots,r$, $h^k$ fixes $f(Y \cup Z \cup (C^{k-1}\times I)) \cup (M^n - (\bigcup_{i\geq 1} V^k_i))$, $h^k_1(h^{k-1}_1 \circ \dots \circ h^0_1(U)) \supset f(Y \cup Z \cup (C^k \times I))$, and the $h^k$ track of each point of $M^n$ is either a singleton or lies in some $H^k_i$ for $i \in \{1,2,3,\dots\}$.

Assume $k \in \{0,1,\dots,r\}$ and assume we already have $h^0, h^1, \dots, h^{k-1}$ with the desired properties. We show how to construct $h^k$. Let $i \in \{1,2,3,\dots\}$. Since

$Y \cup Z \cup (C^{k-1}\times I) \cup (S^k_i \times I) \searrow Y \cup Z \cup (C^{k-1}\times I)$, and $S(f|Y\cup Z\cup (A\times I))\subset Z$, then 3.2 implies

$$f(Y\cup Z\cup (C^{k-1}\times I)\cup (S_i^k\times I)) \searrow f(Y\cup Z\cup (C^{k-1}\times I)).$$

By inductive hypothesis, $f(Y\cup Z\cup (C^{k-1}\times I))\subset h_1^{k-1}\circ\cdots\circ h_1^0(U)$. Also $f(S_i^k\times I)\subset V_i^k$. Now 3.1 provides a PL ambient isotopy $h_i^k$ of $M^n$ which fixes $f(Y\cup Z\cup (C^{k-1}\times I))\cup (M^n - V_i^k)$ $\ni$

$$(h_i^k)_1(h_1^{k-1}\circ\cdots\circ h_1^0(U))\supset f(Y\cup Z\cup (C^{k-1}\times I)\cup (S_i^k\times I)).$$

Now the discreteness of the collection $\{V_i^k : i=1,2,3,\ldots\}$ allows us to define the PL ambient isotopy $h^k$ of $M^n$ with the desired properties by letting

$$h^k = \begin{cases} h_i^k & \text{on } V_i^k\times I \quad \text{for } i=1,2,3\ldots \\ 1 & \text{on } (M^n - (\bigcup_{i\geq 1} V_i^k))\times I \end{cases}$$

Finally we define the PL ambient isotopy $h$ of $M^n$ by letting

$$h_t = h^k_{(r+1)t-k}\circ h_1^{k-1}\circ\cdots\circ h_1^0 \quad \text{for } t\in\left[\tfrac{k}{r+1},\tfrac{k+1}{r+1}\right] \text{ and } k=0,1,\ldots,r.$$

It is straightforward to verify that $h$ is a PL ambient isotopy of $M^n$ which fixes $f(Y\cup Z)$ such that $h_1(U)\supset f(Y\cup Z\cup (A\times I))$.

For $x\in M^n$, the $h$-track of $x$ is the union of the $h^0$-track of $x$, the $h^1$-track of $h_1^0(x)$, the $h^2$-track of $h_1^1 h_1^0(x)$, ..., and the $h^r$-track of $h_1^{r-1}\circ\cdots\circ h_1^1\cdot h_1^0(x)$.

Since a non-singleton $h^k$-track is contained in $H_i^k$ for some $i = 1, 2, 3, \ldots$, we deduce that the non-singleton $h$ track of a point $x$ of $M^n$ is contained in the union of $r+1$ or fewer elements $H_0, H_1, \ldots, H_s$ of $\mathcal{H}^*$ (where $s \leq r$) such that $H_{i-1} \cap H_i \neq \emptyset$ for $1 \leq i \leq r$.

Therefore $\exists\, y \in A - B \ni H_0 \cup H_1 \cup \cdots \cup H_s \subset Gf(y \times I)$; so the $h$-track of $x$ lies in $Gf(y \times I)$. ■

# 4. THE APPROXIMATION LEMMAS

## 4.1. The Codimension $\geq 4$ Approximation Lemma:

Given the scenery, suppose $r \leq n-4$. Then $\forall$ open neighborhood $G$ of $1|M^n$ in $M^n \times M^n$, $\exists$ a proper PL map $g: X \to M^n$ and $\exists$ a closed subpolyhedron $Z$ of $X \ni g(x0) = x$ for every $x \in P$, $g \subset G \circ f$, $S(g) \subset Z$, $\pi^{-1}\pi(Z) = Z$, and $\dim Z < r$.

(Here $Z$ is a shadow for $g$.)

Proof: By 1.2 and 2.6, $\exists$ a proper PL map $g: X \to M$ and a triangulation $T$ of $X \ni$ a subcomplex of $T$ triangulates $P$, $g|P\times 0 = f|P \times 0$, $g \subset G \circ f$ and $g$ is in general position with respect to $T$. Thus

$g(x0) = f(x0) = x$ for every $x \in P$, and

$\dim S(g) \leq \max\{2(r+1)-n, (r+1)+(n-3)-n\} \leq (r+1)+(n-3)-n = r-2$

because $r \leq n-4$. Let $Z = \pi^{-1}\pi(S(g))$. Then

$S(g) \subset Z$, $\pi^{-1}\pi(Z) = Z$ and $\dim Z \leq \dim S(g) + 1 \leq r-1$. ∎

4.2 Corollary: Given the scenery, suppose $r \leq n-4$, and suppose $G$, $g$ and $Z$ are as prescribed in 4.1. Let $U$ be an open neighborhood of $g(Y \cup Z)$ in $M^n$. Then $\exists$ a PL ambient isotopy $h$ of $M^n$ which fixes $Q \ni h_1(U) \supset P$ and the $h$-track of each point of $M^n$ is either a singleton or is contained in $G \circ G(f(x \times I))$ for some $x \in R$.

Proof: We apply 3.3,

| | | | | |
|---|---|---|---|---|
| substituting | $g$ | $R$ | $R \cap [Q \cup \pi(Z)]$ | $Z$ |
| for | $f$ | $A$ | $B$ | $Z$ |

We obtain a PL ambient isotopy $h$ of $M^n$ which fixes $g(Y \cup Z) \ni h_1(U) \supset g(Y \cup Z \cup (R \times I))$ and the $h$-track of each point of $M^n$ is either a singleton or is contained in $G\,g(x \times I)$ for some $x \in R - [Q \cup \pi(Z)]$. Since $Q = g(Q \times 0) \subset g(Y)$, then $h$ fixes $Q$. Since $Y \cup Z \cup (R \times I) = X \supset P \times 0$, then $h_1(U) \supset g(P \times 0) = P$.

If $x \in R$, then $g(x \times I) \subset G f(x \times I)$ because $g \subset G \circ f$; whence $G\,g(x \times I) \subset G \circ G\, f(x \times I)$. Hence the $h$-track of each point of $M^n$ is either a singleton or lies in $G \circ G\, f(x \times I)$ for some $x \in R$. ▮

## 4.3 The Codimension = 3 Approximation Theorem

Given the scenery, suppose $r = n-3$. Then $\forall$ open neighborhood $G$ of $1|M^n$ in $M^n \times M^n$, $\exists$ a proper PL map $g : X \to M^n$ and $\exists$ closed subpolyhedra $Z$ of $X$ and $A_1$ and $A_2$ of $R$ $\ni$

$g(x,0) = x$ for every $x \in P$, $g \subset G \circ f$, $\pi^{-1}\pi(Z) = Z$, $\dim Z < r$,

$A_1 \cup A_2 = R$, $\pi(Z) \cap R \subseteq A_1$, $S(g|Y \cup (A_1 \times I)) \subset Z$,

and $S(g|Y \cup (A_1 \times 0) \cup (A_2 \times I)) \subset (Y \cap Z) \cup (A_1 \times 0) \cup [(A_1 \cap A_2) \times I]$.

(Here $Z$ is a shadow for $g|Y \cup (A_1 \times I)$.)

<u>Proof</u>: We shall call a triangulation $K$ of $X$ <u>cylindrical</u> if $\pi : X \to P$ is a simplicial map from $K$ to some triangulation of $P$. Since $\pi : X \to P$ is a proper PL map, then every triangulation of $X$ has a cylindrical subdivision.

The first part of the proof consists of six steps in which we obtain a proper PL map $g : X \to M^n$ approximating $f$ and triangulations $T_0$, $T_1$, $T$ of $X$ $\ni$

(a) $g(x,0) = x$ for every $x \in P$ and $g \subset G \circ f$;

(b) $T_0$ is a cylindrical triangulation of $X$ $\ni$ subcomplexes of $T_0$ triangulate $Q \times 0$, $R \times 0$, $R \times 1$, $R \times I$, $(Q \cap R) \times 0$, $(Q \cap R) \times 1$ and $(Q \cap R) \times I$;

(c) $T_1$ is a subdivision of $T_0$ $\ni$ $g$ is in general position with respect to $T_1$ ;

(d) $T$ is a subdivision of $T_1$ $\ni$ subcomplexes of $T$ triangulate $S(g)$ and $S_3(g)$, $S_3(g) \subset |T^{r-2}| \cap S(g)$, and $g$ is a simplicial map from $T$ to some triangulation of $M^n$.

(e) if $\alpha \in T | S(g)$, then $\pi|\alpha$ is injective ; and

(f) if $\alpha \in T | S(g)$ and $\dim \alpha = r-1$, then $\operatorname{int}\alpha \subset \operatorname{int} R \times \operatorname{int} I$ and $\pi(\operatorname{int}\alpha) \cap \pi(S(g) - \operatorname{int}\alpha) = \phi$.

$T_0$ is obtained in Step 1, $T_1$ is obtained in Step 2, (e) is achieved in Step 3, (f) is achieved in Step 4, $g : X \to M^n$ is defined in Step 5, and $T$ is obtained in Step 6. Steps 3 and 4 are the most involved.

<u>Step 1</u>: Let $H$ be an open neighborhood of $1M^n$ in $M^n \times M^n \ni H \circ H \subset G$. Let $T_0$ be a cylindrical triangulation of $X \ni$

(a) subcomplexes of $T_0$ triangulate $Q \times 0$, $R \times 0$, $R \times 1$, $R \times I$, $(Q \cap R) \times 0$, $(Q \cap R) \times 1$, $(Q \cap R) \times I$ ; and

(b) for $(x,t) \in \alpha \in T_0$, $f(\pi(\alpha) \times t) \subset H(f(x,t))$.

We achieve (b) as follows: Let $J$ be an open neighborhood of $1|M^n$ in $M^n \times M^n \ni J \circ J^{-1} \in H$. For $(x,t) \in X$, let $U(x,t)$ be an open neighborhood of $(x,t)$ in $X \ni f(U(x,t)) \subset J(f(x,t))$. For $x \in P$, let $N(x)$ be an open neighborhood of $x$ in $P \ni$ $\pi^{-1}(N(x)) \subset \bigcup\{U(x,t) : (x,t) \in \pi^{-1}(x)\}$. Finally assume $T_0$ is a cylindrical subdivision of $X$ so fine that $\forall \alpha \in T_0$, $\exists x \in P \ni \pi(\alpha) \subset N(x)$. Now suppose $(x,t) \in \alpha \in T_0$. Then $\exists x_1 \in P \ni$ $\pi(\alpha) \subset N(x_1)$ and $\exists (x_1,t_1) \in \pi^{-1}(x_1) \ni N(x_1) \times t \subset U(x_1,t_1)$. Thus $(x,t) \in \pi(\alpha) \times t \subset U(x_1,t_1)$. So $f(x,t) \in f(\pi(\alpha)\times t) \subset J(f(x_1,t_1))$. Hence $f(x_1,t_1) \in J^{-1}(f(x,t))$. Consequently $f(\pi(\alpha)\times t) \subset J(f(x_1,t_1)) \subset J \circ J^{-1}(f(x,t)) \subset H(f(x,t))$.

<u>Step 2</u>: Let $T_1$ be a subdivision of $T_0$ and let $f_1 : X \to M^n$ be a proper PL map $\ni$

(a) $f_1$ is in general position with respect to $T_1$;

(b) $f_1(x,0) = x$ for every $x \in P$; and

(c) $f_1 \in H \circ f$.

Since $f|P \times 0$ is a PL embedding and $f(x,0) = x$ for every $x \in P$, we need only invoke 2.6 and 1.9.

Next let $T_2$ be a subdivision of $T_1 \ni$

(a) $\forall \alpha \in T_1$, $\partial\alpha$ is triangulated as a full subcomplex of $T_2$ (thus, if $\alpha \in T_1$, $\beta \in T_2$ and $\operatorname{int}\beta \subset \operatorname{int}\alpha$, then $\operatorname{int}\alpha$ contains a vertex of $\beta$);

(b) subcomplexes of $T_2$ triangulate $S(f_1)$ and $S_3(f_1)$; and

(c) $f_1$ is a simplicial map from $T_2$ to some triangulation of $M^n$.

We remark that any subdivision of the first barycentric subdivision of $T_1$ will satisfy (a). We invoke 2-3 to obtain (c).

<u>Step 3</u>: Let $h_1 : X \to X$ be a PL homeomorphism and let $T_3$ be a subdivision of $T_1$ $\ni$

(a) $h_1$ is a simplicial isomorphism from $T_2$ to $T_3$;

(b) $h_1(\alpha) = \alpha$ for every $\alpha \in T_1$;

(c) $h_1$ is "level-preserving": $\tau \circ h_1 = \tau$;

(d) $h_1 | P \times 0 = 1 | P \times 0$; and

(e) $\forall \alpha \in T_2 | S(f_1)$, $\pi | h_1(\alpha)$ is injective.

$h_1$ merely shifts the vertices of $T_2$ slightly "horizontally" to make the simplices of $T_2 | S(f_1)$ map injectively under $\pi$. Here are the details:

For $v \in T_2^0$, let $\alpha(v) \in T_1 \ni v \in \operatorname{int}\alpha(v)$.

By 2.7, $\forall v \in T_2^0$, $\exists$ an open neighborhood $N_v$ of $v$ in $X \ni$

if $h^0 : T_2^0 \to X \ni h^0(v) \in N_v \cap \operatorname{int}\alpha(v)$ for each $v \in T_2^0$, then $\exists$ a subdivision $T_2'$ of $T_1 \ni (T_2')^0 = h^0(T_2^0)$, and $h^0$ extends ~~uniquely~~ to a PL homeomorphism $h$ of $X$ which is a simplicial isomorphism of $T_2$ to $T_2'$ and which maps each simplex of $T_1$ onto itself. If $h^0 : T_2^0 \to X \ni h^0(v) \in N_v \cap \operatorname{int}\alpha(v)$ for each $v \in T_2^0$, $\tau \circ h^0 = \tau$ and $h^0(v) = v$ for every $v \in T_2^0 \cap (P \times 0)$, then we shall call $h^0$ an <u>admissible vertex shift</u>.

We seek a PL homeomorphism $h_1$ of $X$ which is the unique extension to $X$ of a particular admissible vertex shift $h_1^0$. To obtain $h_1^0$, we let $\beta_1, \beta_2, \beta_3, \dots$ be a list of the simplices of $T_2 | S(f_1)$ so that $\beta_i \subset \beta_j \Rightarrow i \le j$. $h_1^0$ is the limit of a sequence of admissible vertex shifts $\sigma_0^0 = 1 | T_2^0$, $\sigma_1^0$, $\sigma_2^0$, $\sigma_3^0$, $\dots \ni$ for $i \ge 1$, $\sigma_i^0$ moves only the vertices of $\beta_1, \beta_2, \dots, \beta_i$; $\sigma_i^0$ agrees with $\sigma_{i-1}^0$ except on a single vertex of $\beta_i$; and if $\sigma_i$ is the unique extension of $\sigma_i^0$ to $X$, then $\pi | \sigma_i(\beta_j)$ is injection for $1 \le j \le i$.

We construct the sequence $\sigma_0^0 = 1 | T_2^0$, $\sigma_1^0$, $\sigma_2^0$, $\sigma_3^0$, $\dots$ inductively. Assume we have $\sigma_{i-1}^0$. Let $\alpha \in T_1 \ni \operatorname{int}\beta_i \subset \operatorname{int}\alpha$.

Then $\sigma_{i-1}(\operatorname{int}\beta_i) \subset \operatorname{int}\alpha$, because $\sigma_{i-1}(\alpha) = \alpha$. Since a full subcomplex of $T_2$ triangulates $\partial\alpha$, then $\operatorname{int}\alpha$ contains a vertex $v$ of $\beta_i$; whence $\alpha(v) = \alpha$.

We will define $\sigma_i^0$ to agree with $\sigma_{i-1}^0$ on every vertex of $T_2$ except $v$. We note that $\sigma_{i-1}^0(v) \in N_v \cap \operatorname{int}\alpha$.

Since $T_0$ is cylindrical, and $\alpha$ is contained in a simplex of $T_0$, then $\pi(\alpha)$ is a convex cell in $P$. We regard $\pi(\alpha)$ as linearly embedded in the Euclidean space $\mathbb{R}^r$; and for any subset $S$ of $\pi(\alpha)$, we let $\Lambda(S)$ denote the intersection of $\pi(\alpha)$ with the plane in $\mathbb{R}^r$ generated by $S$:

$$\Lambda(S) = \pi(\alpha) \cap \left\{ \sum_{j=0}^{\ell} t_j x_j \in \mathbb{R}^r : x_j \in S \text{ and } t_j \in \mathbb{R} \text{ for } 0 \le j \le \ell, \text{ and } \sum_{j=0}^{\ell} t_j = 1 \right\}.$$

Whenever $1 \le j \le i$ and $v \underset{\neq}{\in} \beta_j$, let $1 \le k(j) < j \ni$ $\beta_{k(j)} \underset{\neq}{\subset} \beta_j$ and $\beta_j = v * \beta_{k(j)}$. Then whenever $1 \le j < i$ and $v \underset{\neq}{\in} \beta_j$, we have $\pi\,\sigma_{i-1}^0(v) \notin \Lambda(\pi\,\sigma_{i-1}(\beta_{k(j)}))$ because $\pi|\,\sigma_{i-1}(\beta_j)$ is injective; hence $\sigma_{i-1}^0(v) \notin \pi^{-1}\Lambda(\pi\,\sigma_{i-1}(\beta_{k(j)}))$.

Also $\pi|\,\sigma_{i-1}(\beta_{k(i)})$ is injective. Now if $\pi\,\sigma_{i-1}^0(v) \notin \Lambda(\pi\,\sigma_{i-1}(\beta_{k(i)}))$, then $\pi|\,\sigma_{i-1}(\beta_i)$ is injective and we can let $\sigma_i^0 = \sigma_{i-1}^0$. Assume $\pi\,\sigma_{i-1}^0(v) \in \Lambda(\pi\,\sigma_{i-1}(\beta_{k(i)}))$.

Since $f_i$ is in general position with respect to $T_1$ and $\beta_i \subset S(f_1)$, then $\dim \beta_i \le \dim \alpha + (r+1) - n = \dim \alpha + (n-2) - n = \dim \alpha - 2$.

So $\dim \Lambda(\pi\sigma_{i-1}(\beta_{k(i)})) = \dim \pi\sigma_{i-1}(\beta_{k(i)}) = \dim \beta_{k(i)} \le \dim \alpha - 3$.

Thus $\alpha \cap \pi^{-1}\Lambda(\pi\sigma_{i-1}(\beta_{k(i)}))$ is a "vertical" convex cell in $\alpha$ of dimension $\le \dim \alpha - 2$. On the other hand, $\alpha \cap \tau^{-1}\tau(\sigma^0_{i-1}(v))$ is a "horizontal" convex cell in $\alpha$ of dimension $\ge \dim \alpha - 1$. So we can choose $\sigma^0_i(v)$ in $N_v \cap \operatorname{int}\alpha \cap \tau^{-1}\tau(\sigma^0_{i-1}(v))$ so that

$\sigma^0_{i-1}(v) \notin \bigcup\{\pi^{-1}\Lambda(\pi\sigma_{i-1}(\beta_{k(j)})) : 1 \le j \le i \text{ and } v \underset{\neq}{\in} \beta_j\}$.

Then $\sigma^0_i$ is an admissible vertex shift which agrees with $\sigma^0_{i-1}$ except on the single vertex $v$ of $\beta_i$, and $\pi/\sigma_i(\beta_j)$ is injective for $1 \le j \le i$.

Since the vertices of any given simplex of $T_2$ are moved by only finitely many members of the sequence $\sigma^0_0 = 1|T_2^0, \sigma^0_1, \sigma^0_2, \sigma^0_3, \ldots$, then this sequence converges to an admissible vertex shift $h^0_1 \ni$ if $h_1$ is the unique extension of $h^0_1$ to $X$, then $\pi|h_1(\beta_i)$ is injective for $i = 1, 2, 3, \ldots$. $T_3$ is the subdivision of $T_1 \ni T_3^0 = h^0_1(T^0_2)$ and $h_1$ is a simplicial map from $T_2$ to $T_3$. The admissibility of $h^0_1$ implies $\tau h_1 = \tau$ and $h_1|P\times 0 = 1|P\times 0$.

Define the proper PL map $f_2 : X \to M^n$ by $f_2 = f_1 \circ h_1^{-1}$. Then

(a) $f(x,0) = x$ for every $x \in P$;

(b) $f_2$ is in general position with respect to $T_1$;

(c) $S(f_2) = h_1(S(f_1))$, $S_3(f_2) = h_1(S_3(f_1))$, and subcomplexes of $T_3$ triangulate $S(f_2)$ and $S_3(f_2)$.

(d) $\forall \alpha \in T_3 | S(f_2)$, $\pi|\alpha$ is injective.

<u>Step 4</u>: Let $T_4$ be a cylindrical subdivision of $T_3$. Let $T_5$ be the first barycentric subdivision of $T_4$. (Thus subcomplexes of $T_5$ triangulate $S(f_2)$ and $S_3(f_2)$.) Let $h_2 : X \to X$ be a PL homeomorphism and let $T_6$ be a subdivision of $T_1$ $\ni$

(a) $h_2$ is a simplicial isomorphism from $T_5$ to $T_6$;

(b) $h_2(\alpha) = \alpha$ for every $\alpha \in T_1$;

(c) $h_2$ is "level-preserving": $\tau \circ h_2 = \tau$;

(d) $h_2 | P \times 0 = 1 | P \times 0$;

(e) $\forall \alpha \in T_4 | S(f_2)$, $\pi | h_2(\alpha)$ is injective; and

(f) $\forall \alpha \in T_4 | S(f_2)$, if $\dim \alpha \geq r-1$, then $\pi h_2(\operatorname{int} \alpha) \cap \pi h_2 (S(f_2) - \operatorname{int} \alpha) = \emptyset$.

Since $T_4$ is cylindrical, the projection of the interiors of distinct $r-1$ simplices of $T_4|S(f_2)$ are either identical or disjoint. $h_2$ merely shifts the barycenters of the $r-1$ simplices of $T_4|S(f_2)$ slightly horizontally to make the projections of their interiors disjoint. Here are the details:

Let $\{\gamma_{ij} : 1 \le i < \infty,\ 1 \le j \le k(i)\}$ be a list of the ~~simpl~~ $r-1$ simplices of $T_4|S(f_2) \ni \pi(\gamma_{i_1, j_1}) = \pi(\gamma_{i_2, j_2})$ if and only if $i_1 = i_2$. For $1 \le i < \infty$ and $1 \le j \le k(i)$, let $\alpha_{ij} \in T_1 \ni \operatorname{int} \gamma_{ij} \subset \operatorname{int} \alpha_{ij}$; then $r-1 = \dim \gamma_{ij} \le \dim \alpha_{ij} + (r+1) - n = \dim \alpha_{ij} + (n-2) - n = \dim \alpha_{ij} - 2$, because $f_2$ is in general position with respect to $T_1$; so $\dim \alpha_{ij} = r+1$; hence $\exists\, \beta_{ij} \in T_4 \ni \gamma_{ij} \subset \beta_{ij}$, $\operatorname{int} \beta_{ij} \subset \operatorname{int} \alpha_{ij}$ and $\dim \beta_{ij} = r+1$. For $1 \le i < \infty$ and $1 \le j \le k(i)$, let $v_{ij}$ be the barycenter of $\gamma_{ij}$.

Since $T_4$ is cylindrical, then $\pi(T_4) = \{\pi(\alpha) : \alpha \in T_4\}$ is a triangulation of $P$. For $1 \le i < \infty$, let $\gamma_i = \pi(\gamma_{i1})$ $(= \pi(\gamma_{ij})$ for $1 \le j \le k(i))$ and $\beta_i = \pi(\beta_{i1})$; then $\gamma_i, \beta_i \in \pi(T_4)$, $\dim \gamma_i = r-1$ because $\pi|\gamma_{i1}$ is injective, and $\dim \beta_i = r$ because $\dim \beta_{i1} = r+1$. For $1 \le i < \infty$, let $v_i$ be the barycenter of $\gamma_i$ and let $u_i$ be the barycenter of $\beta_i$; then $v_i = \pi(v_{ij})$ for $1 \le j \le k(i)$.

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By 2.7, for $1 \le i < \infty$ and $1 \le j \le k(i)$, $\exists$ an open neighborhood $N_{ij}$ of $v_{ij}$ in $X \ni$ if $h^0 : T_5{}^0 \to X \ni$ $h^0$ fixes $T_5{}^0 - \{v_{ij} : 1 \le i < \infty,\ 1 \le j \le k(i)\}$ and $h^0(v_{ij}) \in N_{ij} \cap \operatorname{int} \alpha_{ij} \ni$ for $1 \le i < \infty$, $1 \le j \le k(i)$, then $\exists$ a subdivision $T_5'$ of $T_1 \ni$ $(T_5')^0 = h^0(T_5^0)$ and $h^0$ extends uniquely to a PL homeomorphism $h$ of $X$ which is a simplicial isomorphism from $T_5$ to $T_5'$ and which maps each simplex of $T_1$ onto itself.

For $1 \le i < \infty$, let $N_i$ be an open neighborhood of $v_i$ in $P \ni$ $N_i \times \tau(v_{ij}) \subset N_{ij} \cap \operatorname{int} \alpha_{ij}$ for $1 \le j \le k(i)$. For $1 \le i < \infty$, let $w_{i1}, w_{i2}, \ldots, w_{ik(i)}$ be <u>distinct</u> points of $\operatorname{int}(u_i * v_i) \cap N_i$. Define $h_2^0 : T_5{}^0 \to X$ by $h_2^0(v_{ij}) = (w_{ij}, \tau(v_{ij}))$ for $1 \le i < \infty$, $1 \le j \le k(i)$ and $h_2^0 = 1$ on $T_5{}^0 - \{v_{ij} : 1 \le i < \infty,\ 1 \le j \le k(i)\}$. Since $h_2^0(v_{ij}) \in N_i \times \tau(v_{ij}) \subset N_{ij} \cap \operatorname{int} \alpha_{ij}$ for $1 \le i < \infty$, $1 \le j \le k(i)$, then $\exists$ a subdivision $T_6$ of $T_1 \ni$ $T_6{}^0 = h_2^0(T_5^0)$ and $h_2^0$ extends uniquely to a PL homeomorphism $h_2$ of $X$ which is a simplicial isomorphism from $T_5$ to $T_6$, and which maps each simplex of $T_1$ onto itself. Moreover $\tau \circ h_2 = \tau$ because $\tau \circ h_2^0 = \tau$. For $1 \le i < \infty$, $1 \le j \le k(i)$, since $v_{ij} \in \operatorname{int} \alpha_{ij}$ and $\dim \alpha_{ij} = r+1$, then $0 < \tau(v_{ij}) < 1$; hence $h_2^0 = 1$ on $T_2^0 \cap (P \times 0)$;

therefore $h_2 | P \times 0 = 1 | P \times 0$. Since $f_2$ is in general position with respect to $T_1$, then $\dim S(f_2) \le 2(r+1) - n = r + (n-3) + 2 - n = r - 1$. If $\alpha \in T_4 | S(f_2)$ and $\dim \alpha < r-1$, then $h_2(\alpha) = \alpha$, so that $\pi | h_2(\alpha)$ is injective. For $1 \le i < \infty$ and $1 \le j \le k(i)$, $h_2(\gamma_{ij}) = (w_{ij}, \tau(v_{ij})) * \partial \gamma_{ij}$ and $\pi h_2(\gamma_{ij}) = w_{ij} * \partial \gamma_{i}$; it follows from the choice of $w_{ij}$ that $\pi | h_2(\gamma_{ij})$ is injective and $\pi h_2(\operatorname{int} \gamma_{ij}) \cap \pi h_2(S(f_2) - \operatorname{int} \gamma_{ij}) = \emptyset$. This verifies (e) and (f).

<u>Step 5</u>: Define the proper PL map $g : X \to M^n$ by $g = f_2 \circ h_2^{-1}$. Then

(a) $g(x,0) = x$ for every $x \in P$ and $g \in G \circ f$;

(b) $g$ is in general position with respect to $T_1$;

(c) $S(g) = h_2(S(f_2))$, $S_3(g) = h_2(S_3(f_2))$, and subcomplexes of $T_6$ triangulate $S(g)$ and $S_3(g)$;

(d) $\forall \alpha \in T_6 | S(g)$, $\pi | \alpha$ is injective; and

(e) $\forall \alpha \in T_6 | S(g)$, if $\dim \alpha = r-1$, then $\pi(\operatorname{int} \alpha) \cap \pi(S(g) - \operatorname{int} \alpha) = \emptyset$.

We prove $g \in G \circ f$. $g = f_2 \circ h_2^{-1} = f_1 \circ h_1^{-1} \circ h_2^{-1}$.

If $(x,t) \in \alpha \in T_1$, then $h_1^{-1} h_2^{-1}(x,t) \in \alpha$ and $\tau h_1^{-1} h_2^{-1}(x,t) = t$. So $h_1^{-1} h_2^{-1}(x,t) \in \pi(\alpha) \times t$. Consequently

$g(x,t) = f_1 h_2^{-1} h_1^{-1}(x,t) \in f_1(\pi(\alpha) \times t) \subset H(f(\pi(\alpha) \times t)) \subset H \circ H(f(x,t)) \subset G(f(x,t))$.

We also prove (e): Let $\alpha \in T_6 | S(g) \ni \dim \alpha = r-1$. Then $\exists \beta \in T_4 | S(f_2) \ni \dim \beta = r-1$ and $\alpha \subset h_2(\beta)$. Now $S(g) - \operatorname{int} \alpha = h_2(S(f_2) - \operatorname{int} \beta) \cup (h_2(\beta) - \operatorname{int} \alpha)$. Since $\pi h_2(\operatorname{int} \beta) \cap \pi h_2(S(f_2) - \operatorname{int} \beta) = \emptyset$, then $\pi(\operatorname{int} \alpha) \cap \pi h_2(S(f_2) - \operatorname{int} \beta) = \emptyset$. Since $\pi | h_2(\beta)$ is injective, then $\pi(\operatorname{int} \alpha) \cap \pi(h_2(\beta) - \operatorname{int} \alpha) = \emptyset$. It follows that $\pi(\operatorname{int} \alpha) \cap \pi(S(g) - \operatorname{int} \alpha) = \emptyset$.

<u>Step 6</u>: Let $T$ be a subdivision of $T_6 \ni g$ is a simplicial map from $T$ to some triangulation of $M^n$. Then
(a) subcomplexes of $T$ triangulate $S(g)$ and $S_3(g)$ and $S_3(g) \subset |T^{r-2}| \cap S(g)$;
(b) if $\alpha \in T | S(g)$, then $\pi | \alpha$ is injective; and
(c) if $\alpha \in T | S(g)$ and $\dim \alpha = r-1$, then $\operatorname{int} \alpha \subset \operatorname{int} R \times \operatorname{int} I$ and $\pi(\operatorname{int} \alpha) \cap \pi(S(g) - \operatorname{int} \alpha) = \emptyset$.

We prove $S_3(g) \subset |T^{r-2}| \cap S(g)$: Clearly $S_3(g) \subset S(g)$. Since $g$ is in general position with respect to $T_1$, then $\dim S(g) \leq 2(r+1) - n = r + (n-3) + 2 - n = r - 1$. Hence $\dim S_3(g) \leq (r-1) + (r+1) - n = r + (n-3) - n = r - 3$. Since a subcomplex of $T$ triangulates $S_3(g)$, then $S_3(g) \subset |T^{r-3}| \subset |T^{r-2}|$. Thus $S_3(g) \subset |T^{r-2}| \cap S(g)$.

We prove (c): Let $\alpha \in T|S(g) \ni \dim \alpha = r-1$. Let $\beta \in T_1 \ni \operatorname{int}\alpha \subset \operatorname{int}\beta$. Since $g$ is in general position with respect to $T_1$, then $r-1 = \dim\alpha \leq \dim\beta + (r+1) - n = \dim\beta + (n-2) - n = \dim\beta - 2$. So $\dim\beta = r+1$. Hence $\operatorname{int}\beta \subset \operatorname{int}_p R \times \operatorname{int} I$. Thus $\operatorname{int}\alpha \subset \operatorname{int}_p R \times \operatorname{int} I$. Next let $\gamma \in T_6 \ni \operatorname{int}\alpha \subset \operatorname{int}\gamma$. Then $\gamma \in T_6 | S(g)$ and $\dim\gamma \leq r-1$. Therefore $\pi(\operatorname{int}\alpha) \cap \pi(S(g) - \operatorname{int}\gamma) = \phi$ because $\operatorname{int}\alpha \subset \operatorname{int}\gamma$; and $\pi(\operatorname{int}\alpha) \cap \pi(\gamma - \operatorname{int}\alpha) = \phi$ because $\pi|\gamma$ is injective. Since $S(g) - \operatorname{int}\alpha = (S(g) - \operatorname{int}\gamma) \cup (\gamma - \operatorname{int}\alpha)$, it follows that $\pi(\operatorname{int}\alpha) \cap \pi(S(g) - \operatorname{int}\alpha) = \phi$.

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The second part of the proof consists of three more steps in which we obtain the closed subpolyhedra $Z$ of $X$ and $A_1$ and $A_2$ of $R$ with the desired properties. $Z$ is defined in Step 7, $A_1$ and $A_2$ are obtained in Step 8, and their properties are established in Step 9.

<u>Step 7</u>: Define the closed subpolyhedron $Z$ of $X$ by $Z = \pi^{-1}\pi(|T^{r-2}| \cap S(g))$. Then

(a) $|T^{r-2}| \cap S(g) \subset Z$;

(b) $\pi^{-1}\pi(Z) = Z$;

(c) $\dim Z \leq r-1$; and

(d) if $\alpha \in T|S(g)$ and $\dim\alpha = r-1$, then $\pi(\operatorname{int}\alpha) \cap \pi(Z) = \phi$.

(a), (b) and (c) are obvious.

We prove (d): Let $\alpha \in T|S(g) \ni \dim \alpha = r-1$. Then $|T^{r-2}| \cap S(g) \subset S(g) - \operatorname{int}\alpha$ and $\pi(\operatorname{int}\alpha) \cap \pi(S(g) - \operatorname{int}\alpha) = \emptyset$. So $\emptyset = \pi(\operatorname{int}\alpha) \cap \pi(|T^{r-2}| \cap S(g)) = \pi(\operatorname{int}\alpha) \cap \pi(Z)$.

Step 8: Let $\mathcal{E} = \{\alpha \in T|S(g) : \dim \alpha = r-1\}$. It is clear from the properties of $g$ and $T$ that $g$ identifies the elements of $\mathcal{E}$ in pairs. Thus we can partition $\mathcal{E}$ into two subsets: $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2 \ni$ $\mathcal{E}_1 \cap \mathcal{E}_2 = \emptyset$ and for $i = 1$ or $2$, $g / \bigcup\{\operatorname{int}\alpha : \alpha \in \mathcal{E}_i\}$ is injective. For $i = 1, 2$, define $E_i = \bigcup\{\pi(\operatorname{int}\alpha) : \alpha \in \mathcal{E}_i\}$ and $\overline{E}_i = \bigcup\{\pi(\alpha) : \alpha \in \mathcal{E}_i\}$; therefore $\overline{E}_i = \operatorname{cl} E_i$.
We shall choose $A_1$ and $A_2$ to be closed subpolyhedra of $R \ni$

(a) $A_1 \cup A_2 = R$;

(b) $\pi(Z) \cap R \subset A_1$;

(c) $A_1 \cap E_2 = \emptyset$; and

(d) $A_2 \cap E_1 = \emptyset$.

Here is a method of obtaining $A_1$ and $A_2$: Let $L$ be a triangulation of $R \ni$ full subcomplexes of $L$ triangulate $Q \cap R$, $\pi(Z) \cap R$, $\overline{E}_1$ and $\overline{E}_2$.
(For example, let $T'$ be a cylindrical subdivision of $T$.

Then $\pi(T') = \{\pi(\alpha) : \alpha \in T'\}$ is a triangulation of $P \ni$ subcomplexes of $\pi(T')$ triangulate $R$, $Q \cap R$, $\pi(Z) = \pi(|T^{k-2}| \cap S(g))$, $\overline{E}_1$ and $\overline{E}_2$. Let $\pi(T')'$ denote the first barycentric subdivision of $\pi(T')$. Then we can set $L = \pi(T')' | R$.)

Let $L'$ denote the first barycentric subdivision of $L$.

Define $A_1 = \cup\{\beta \in L' : \beta \cap E_2 = \emptyset\}$ and define $A_2 = \cup\{\beta \in L' : \beta \cap E_2 \neq \emptyset\}$.

Then (a) and (c) are obvious.

The proof of (b) is easy: Let $x \in \pi(Z) \cap R$. Then $\exists\, \beta \in L' \ni x \in \beta \subset \pi(Z) \cap R$. We assert that $\beta \cap E_2 = \emptyset$. For let $\alpha \in \mathcal{E}_2$. Then $\pi(\mathrm{int}\,\alpha) \cap \beta \subset \pi(\mathrm{int}\,\alpha) \cap \pi(Z) = \emptyset$. Since $E_2 = \cup\{\pi(\mathrm{int}\,\alpha) : \alpha \in \mathcal{E}_2\}$, our assertion is established. It follows that $\beta \subset A_1$. So $x \in A_1$. This proves (b).

We now prove (d) by contradiction: Assume $A_2 \cap E_1 \neq \emptyset$. Then $\exists\, \beta \in L' \ni \beta \cap E_2 \neq \emptyset$ and $\beta \cap E_1 \neq \emptyset$. Let $\alpha \in L \ni \beta < \alpha$. For $i = 1,2$, $\emptyset \neq \beta \cap E_i \subset \alpha \cap E_i \subset \alpha \cap \overline{E}_i$; so since a full subcomplex of $L$ triangulates $\overline{E}_i$, then $\alpha_i = \alpha \cap \overline{E}_i$ is a non-empty face of $\alpha$.

Since $L'$ is the first barycentric subdivision of $L$, then $\exists\, \gamma_0, \gamma_1, \ldots, \gamma_k \in L \ni$ if $v_i$ is the barycenter of $\gamma_i$ for $0 \le i \le k$, then $\beta = v_0 * v_1 * \cdots * v_k$ and

 

$\gamma_0 \subsetneqq \gamma_1 \subsetneqq \cdots \subsetneqq \gamma_k \subset \alpha$. For $i = 1, 2$: note that

$v_j \in \alpha_i \Rightarrow \gamma_j \subset \alpha_i \Rightarrow \gamma_0, \gamma_1, \ldots, \gamma_j \subset \alpha_i \Rightarrow v_0, v_1, \ldots, v_j \in \alpha_i$!

so $\exists\, j(i) \in \{0, 1, \ldots, k\} \ni \{0, 1, \ldots, j(i)\} = \{j \in \{0, 1, \ldots, k\} : v_j \in \alpha_i\}$;

then $\beta \cap \alpha_i = v_0 * v_1 * \cdots * v_{j(i)}$. First assume $\underline{j(1) \leq j(2)}$.

Then $\beta \cap \alpha_1 = v_0 * v_1 * \cdots * v_{j(1)} \subset v_0 * v_1 * \cdots * v_{j(2)} = \beta \cap \alpha_2$.

Since $\beta = \beta \cap \alpha$, then $\beta \cap E_1 = \beta \cap \alpha \cap E_1 \subset \beta \cap \alpha_1$.

Thus $\beta \cap E_1 \subset \beta \cap \alpha_1 \subset \beta \cap \alpha_2 \subset \overline{E}_2$; so $\beta \cap E_1 \subset \overline{E}_2$.

Since $E_1 \cap \overline{E}_2 = \emptyset$, we see that $\beta \cap E_1 = \emptyset$, a contradiction.

Second assume $\underline{j(2) \leq j(1)}$. Then $\beta \cap \alpha_2 \subset \beta \cap \alpha_1$.

Since $\beta = \beta \cap \alpha$, then $\beta \cap E_2 = \beta \cap \alpha \cap E_2 \subset \beta \cap \alpha_2$.

Thus $\beta \cap E_2 \subset \beta \cap \alpha_2 \subset \beta \cap \alpha_1 \subset \overline{E}_1$; so $\beta \cap E_2 \subset \overline{E}_1$.

Since $\overline{E}_1 \cap E_2 = \emptyset$, we see that $\beta \cap E_2 = \emptyset$, a contradiction.

<u>Step 9</u>: Here we verify that

(a) $S(g \mid Y \cup (A_1 \times I)) \subset Z$; and

(b) $S(g \mid Y \cup (A_1 \times 0) \cup (A_2 \times I)) \subset (Y \cap Z) \cup (A_1 \times 0) \cup [(A_1 \cap A_2) \times I]$

Proof of (a): If $\alpha \in \mathcal{E}_2$, then $\operatorname{int}\alpha \cap [Y \cup (A_1 \times I)] = \emptyset$:
$\operatorname{int}\alpha \cap Y = \emptyset$ because $\operatorname{int}\alpha \subset \operatorname{int}_p R \times \operatorname{int} I$, and
$\operatorname{int}\alpha \cap (A_1 \times I) = \emptyset$ because $\pi(\operatorname{int}\alpha) \cap A_1 \subset E_2 \cap A_1 = \emptyset$.
It follows from the properties of $g$ and $T$ that
$S(g|Y \cup (A_1 \times I)) \subset |T^{r-2}| \cap S(g)$. Thus
$S(g|Y \cup (A_1 \times I)) \subset Z$.

Proof of (b): If $\alpha \in \mathcal{E}_1$, then
$\operatorname{int}\alpha \cap [Y \cup (A_1 \times 0) \cup (A_2 \times I)] = \emptyset$: $\operatorname{int}\alpha \cap [Y \cup (A_1 \times 0)] = \emptyset$
because $\operatorname{int}\alpha \subset \operatorname{int}_p R \times \operatorname{int} I$, and $\operatorname{int}\alpha \cap (A_2 \times I) = \emptyset$
because $\pi(\operatorname{int}\alpha) \cap A_2 \subset E_1 \cap A_2 = \emptyset$. It follows from the
properties of $g$ and $T$ that
$S(g|Y \cup (A_1 \times 0) \cup (A_2 \times I)) \subset [Y \cup (A_1 \times 0) \cup (A_2 \times I)] \cap (|T^{r-2}| \cap S(g))$.
Thus $S(g|Y \cup (A_1 \times 0) \cup (A_2 \times I)) \subset [Y \cup (A_1 \times 0) \cup (A_2 \times I)] \cap Z$.
So $S(g|Y \cup (A_1 \times 0) \cup (A_2 \times I)) \subset (Y \cap Z) \cup (A_1 \times 0) \cup [(A_2 \times I) \cap Z]$
Since $\pi(Z) \cap R \subset A_1$, then $(A_2 \times I) \cap Z \subset (R \times I) \cap Z \subset A_1 \times I$;
so $(A_2 \times I) \cap Z \subset (A_1 \times I) \cap (A_2 \times I) = (A_1 \cap A_2) \times I$. Therefore
$S(g|Y \cup (A_1 \times 0) \cup (A_2 \times I)) \subset (Y \cup Z) \cup (A_1 \times 0) \cup [(A_1 \cap A_2) \times I]$. ▨

4.4 <u>Corollary</u>: Given the scenery, suppose $r = n-3$, and suppose $G$, $g$, $Z$, $A_1$ and $A_2$ are as prescribed in 4.3. Let $U$ be an open neighborhood of $g(Y \cup Z)$ in $M^n$. Then $\exists$ a PL ambient isotopy $h$ of $M^n$ which fixes $Q \ni$ $h_1(U) \supset P$ and the $h$-track of each point of $M^n$ is either a singleton or is contained in $G \circ G(f(\{x,y\} \times I))$ for some $x, y \in R$.

<u>Proof</u>: We apply 3.3 twice. In the first application, we

| | | | | |
|---|---|---|---|---|
| substitute | $g$ | $A_1$ | $A_1 \cap [Q \cup \pi(Z)]$ | $Z$ |
| for | $f$ | $A$ | $B$ | $Z$. |

We obtain a PL ambient isotopy $h^1$ of $M^n$ which fixes $g(Y \cup Z) \ni h^1_1(U) \supset g(Y \cup Z \cup (A_1 \times I))$ and the $h^1$-track of each point of $M^n$ is either a singleton or is contained in $G\, g(x \times I)$ for some $x \in A_1 - [Q \cup \pi(Z)]$.

In the second application of 3.3, we

| | | | | | |
|---|---|---|---|---|---|
| substitute | $g$ | $A_2$ | $A_2 \cap (Q \cup A_1)$ | $(Y \cap Z) \cup (A_1 \times 0) \cup [(A_1 \cap A_2) \times I]$ | $h^1_1(U)$ |
| for | $f$ | $A$ | $B$ | $Z$ | $U$ |

It is easy to verify that

$(A_2 \times I) \cap [(Q \cap R) \times I] = (A_2 \cap Q) \times I$, that $(A_2 \times I) \cap \{(Y \cap Z) \cup (A_1 \times 0) \cup [(A_1 \cap A_2) \times I]\} = (A_1 \cap A_2) \times I$, and that $h_1^1(U) \supset g(Y \cup (A_1 \times 0) \cup [(A_1 \cap A_2) \times I])$. So this application of 3.3 is valid. We obtain a PL ambient isotopy $h^2$ of $M^n$ which fixes $g(Y \cup (A_1 \times 0) \cup [(A_1 \cap A_2) \times I])$ ∋ $h_1^2(h_1^1(U)) \supset g(Y \cup (A_1 \times 0) \cup (A_2 \times I))$ and the $h^2$-track of each point of $M^n$ is either a singleton or is contained in $G\, g(x \times I)$ for some $x \in A_2 - (Q \cup A_1)$. We remark that the set $(Y \cap Z) \cup (A_1 \times 0) \cup [(A_1 \cap A_2) \times I]$ which is substituted for $Z$ in this application of 3.3 is not invariant under $\pi^{-1} \circ \pi$; hence it is not a shadow for $g | Y \cup (A_1 \times 0) \cup (A_2 \times I)$.

Define the PL ambient isotopy $h$ of $M^n$ by

$$h_t = \begin{cases} h^1_{2t} & \text{for } 0 \le t \le \frac{1}{2} \\ h^2_{2t-1} \circ h^1_1 & \text{for } \frac{1}{2} \le t \le 1 \end{cases}$$

Since $Q = g(Q \times 0) \subset g(Y)$, then $h^1$ and $h^2$ both fix $Q$; thus $h$ fixes $Q$. Since $A_1 \cup A_2 = R$, then $Y \cup (A_1 \times 0) \cup (A_2 \times I) \supset (Q \times 0) \cup (R \times 0) = P \times 0$; thus $\overset{h_1(U)=}{h_1^2(h(U))} \supset g(P \times 0) = P$.

If $x, y \in R$, then $g(\{x,y\} \times I) \subset G\, f(\{x,y\} \times I)$ because $g \in G \circ f$; thus $G\, g(\{x,y\} \times I) \subset G \circ G\, f(\{x,y\} \times I)$. The $h$-track of a

point $z \in M^n$ is the union of the $h'$-track of $z$ and the $h_1$-track of $h_1'(z)$. Thus the $h$-track of a point of $M^n$ is either a singleton or is contained in
$Gg(x \times I) \cup Gg(y \times I) = Gg(\{x, y\} \times I) \subset G \circ G f(\{x, y\} \times I)$
for some $x \in A_1 - (Q \cup \pi(Z)) \subset R$ and some $y \in A_2 - (Q \cup A_1) \subset R$. ∎

# 5. THE ENGULFING THEOREMS

Let $M^n$ be a boundaryless PL $n$-manifold. Suppose $r$ is an integer with $0 \le r \le n-3$, $W \subset V \subset U$ are open subsets of $M^n$, and $\mathcal{T}$ is a collection of subsets of $M^n$. We say <u>(finite) $r$ complexes in $U$ can be pulled into $V$ along $\mathcal{T}$ rel $W$</u> if whenever $P \supset Q$ are closed subpolyhedra of $M^n$ $\ni$ $P \subset U$, $Q \subset W$, (cl$(P-Q)$ is compact,) $\dim Q \le n-3$ and $\dim \text{cl}(P-Q) \le r$, then there is a proper homotopy $\varphi: P \times I \to M^n$ $\ni$ $\varphi(x,t) = x$ for every $(x,t) \in (P \times 0) \cup (Q \times I)$, $\varphi(P \times 1) \subset V$, and $\forall x \in P$, $\exists T \in \mathcal{T} \ni \varphi(x \times I) \subset T$.

## 5.1 A Simple Engulfing Theorem:

<u>Hypothesis</u>: Let $M^n$ be a boundaryless PL $n$-manifold. Suppose $r$ is an integer with $0 \le r \le n-3$, $U$ is an open subset of $M^n$, and $\mathcal{T}$ is a collection of subsets of $M^n$ $\ni$ finite $r$-complexes in $M^n$ can be pulled into $U$ along $\mathcal{T}$ rel $U$.

<u>Conclusion</u>: If $P \supset Q$ are closed subpolyhedra of $M^n$ $\ni$ $Q \subset U$, cl$(P-Q)$ is compact, $\dim Q \le n-3$ and $\dim \text{cl}(P-Q) \le r$, then $\forall$ open neighborhood $G$ of $1 \mid M^n$ in $M^n \times M^n$, $\exists$ a compactly supported PL ambient isotopy $h$ of $M^n$ which fixes $Q$ $\ni$ $h_1(U) \supset P$ and the $h$-track of each point of $M^n$ is either a singleton or lies in the $G$-neighborhood of the union of

$$\left\{\begin{array}{lll} r+1 & \text{if} & 0 \le r \le n-4 \\ r+2 & \text{if} & r = n-3 \end{array}\right\} \text{ elements of } \mathcal{T}.$$

Proof: The proof is by induction on $r$ beginning with $r=-1$, in which case there is nothing to prove. Inductively assume that the theorem is valid if $r$ is replaced by $r-1$. Let the hypothesis of the theorem be given and suppose $P \supset Q$ are closed subpolyhedra of $M^n \ni Q \subset \mathcal{U}$, $cl(P-Q)$ is compact, $\dim Q \leq n-3$ and $\dim cl(P-Q) \leq r$. Let $G$ be an open neighborhood of $1|M^n$ in $M^n \times M^n$.

Since finite $r$ complexes in $M^n$ can be pulled into $\mathcal{U}$ along $\mathcal{T}$ rel $\mathcal{U}$, then $\exists$ a proper homotopy $\varphi: P\times I \to M^n \ni$ $\varphi(x,t) = x$ for every $(x,t) \in (P\times 0) \cup (Q\times I)$, $\varphi(P\times 1) \subset \mathcal{U}$ and $\forall x \in P$, $\exists T \in \mathcal{T} \ni \varphi(x \times I) \subset T$. Let $R = cl(P-Q)$, $X = (Q\times 0) \cup (R\times I)$, $Y = (Q\times 0) \cup [(Q\cap R)\times I] \cup (R\times 1)$, and define the proper map $f: X \to M^n$ by $f = \varphi|X$. Then $f(x,0) = x$ for every $x \in P$, $f(Y) \subset \mathcal{U}$ and $\forall x \in R$, $\exists T \in \mathcal{T} \ni f(x\times I) \subset T$. Let $\pi: X \to P$ denote the restriction to $X$ of the projection $P\times I \to P$. Choose $H$ to be an open neighborhood of $1|M^n$ in $M^n\times M^n \ni H\circ H \subset G$, $H(f(Y)) \subset \mathcal{U}$, and if $C$ is a compact subset of $M^n$, then so is $cl\, H(C)$.

First assume $r \leq n-4$. In this case, 4.1 provides a proper PL map $g: X \to M^n$ and a closed subpolyhedron $Z$ of $X \ni$ $g(x,0) = x$ for every $x \in P$, $g \subset H\circ f$, $S(g) \subset Z$, $\pi^{-1}\pi(Z) = Z$ and $\dim Z < r$.

Now $g(Y \cup Z) \supset g(Y)$ are closed subpolyhedra of $M^n \ni g(Y) \subset Hf(Y) \subset U$, $cl\,[g(Y \cup Z) - g(Y)]$ is compact because it is contained in $g\left(cl(P-Q) \times I\right)$, $\dim g(Y) \leq \dim Y \leq n-3$, and $\dim cl\,[g(Y \cup Z) - g(Y)] \leq \dim g(Z) \leq \dim Z \leq r-1$. Hence by inductive hypothesis, $\exists$ a compactly supported PL ambient isotopy $h^1$ of $M^n$ which fixes $g(Y) \ni h^1_1(U) \supset g(Y \cup Z)$ and the $h^1$-track of each point of $M^n$ is either a singleton or lies in the $G$-neighborhood of the union of $r$ elements of $\mathcal{T}$.

Next we invoke 4.2 to obtain a PL ambient isotopy $h^2$ of $M^n$ which fixes $Q \ni h^2_1(h^1_1(U)) \supset P$ and the $h^2$-track of each point of $M^n$ is either a singleton or lies in $H \circ H(f(x \times I))$ for some $x \in R$. Since $h^2$ fixes $M^n - H \circ H(f(R \times I))$, since $H \circ H(f(R \times I)) \subset cl H(cl H(f(R \times I)))$, and since the choice of $H$ guarantees that $cl H(cl H(f(R \times I)))$ is compact, then $h^2$ has compact support. Since $H \circ H \subset G$ and $\forall x \in R, \exists T \in \mathcal{T} \ni f(x \times I) \subset T$, then the $h^2$-track of each point of $M^n$ is either a singleton or lies in the $G$-neighborhood of an element of $\mathcal{T}$.

Finally define the PL ambient isotopy $h$ of $M^n$ by

$$h_t = \begin{cases} h^1_{2t} & \text{for } 0 \leq t \leq \frac{1}{2} \\ h^2_{2t-1} \circ h^1_1 & \text{for } \frac{1}{2} \leq t \leq 1 \end{cases}$$

$h$ has compact support, because both $h'$ and $h^2$ have compact support. Since $Q = g(Q\times 0) \subset g(Y)$, then both $h'$ and $h^2$ fix $Q$; so $h$ fixes $Q$. $h_1(U) = h_1^2 \circ h_1'(U) \supset P$. Since the $h$-track of a point $x \in M^n$ is the union of the $h^1$-track of $x$ and the $h^2$-track of $h_1'(x)$, then the $h$-track of a point of $M^n$ is either a singleton or lies in the $G$-neighborhood of the union of $r+1$ elements of $\mathcal{J}$.

<u>Second</u> <u>assume</u> <u>$r = n-3$</u>. In this case, 4.3 provides a proper PL map $g: X \to M^n$ and closed subpolyhedra $Z$ of $X$ and $A_1$ and $A_2$ of $R \ni g(x,0) = x$ for every $x \in P$, $g \subset G\circ f$, $\pi^{-1}\pi(Z) = Z$, $\dim Z < r$, $A_1 \cup A_2 = R$, $\pi(Z) \cap R \subset A_1$, $S(g|Y \cup (A_1\times I)) \subset Z$, and $S(g|Y\cup (A_1\times 0) \cup (A_2 \times I)) \subset (Y\cap Z) \cup (A_1\times 0) \cup [(A_1\cap A_2)\times I]$.

As in the preceding case, the inductive hypothesis implies $\exists$ a compactly supported PL ambient isotopy $h'$ of $M^n$ which fixes $g(Y) \ni h_1'(U) \supset g(Y\cup Z)$ and the $h'$-track of each point of $M^n$ is either a singleton or lies in the $G$-neighborhood of the union of $r$ elements of $\mathcal{J}$.

Next we invoke 4.4 to obtain a PL ambient isotopy $h^2$ of $M^n$ which fixes $Q \ni h_1^2(h_1'(U)) \supset P$ and the $h^2$-track of each point of $M^n$ is either a singleton or is contained in $H\circ H(f(\{x,y\}\times I))$ for some $x, y \in R$.

It follows, as in the preceding case, that $h^2$ has compact support. Since $H \circ H \subset G$ and $\forall x \in P$, $\exists T \in \mathcal{T} \ni f(x \times I) \subset T$, then the $h^2$-track of each point of $M^n$ is either a singleton or lies in the $G$-neighborhood of the union of <u>two</u> elements of $\mathcal{T}$.

We define the PL ambient isotopy $h$ of $M^n$ as in the preceding case; and as in the preceding case, $h$ has compact support, $h$ fixes $Q$ and $h_1(U) \supset P$. Since the $h$-track of a point $x \in M^n$ is the union of the $h^1$-track of $x$ and the $h^2$-track of $h_1^1(x)$, then the $h$-track of a point of $M^n$ is either a singleton or lies in the $G$-neighborhood of the union of $r+2$ elements of $\mathcal{T}$. ∎

5.2 A Complicated-But-Useful Engulfing Theorem:

Hypothesis: Let $M^n$ be a boundaryless PL $n$-manifold. Suppose $r$ is an integer with $0 \le r \le n-3$,

$$V_{r+1} \subset V_r \subset \cdots \subset V_0 \subset U_r \subset \cdots \subset U_0 \subset U_{-1}$$

are open subsets of $M^n$, and for $i = 0, 1, \ldots, r$ $\mathcal{J}_i$ is a collection of subsets of $U_{i-1}$ $\ni$ $i$-complexes in $U_i$ can be pulled into $V_i$ along $\mathcal{J}_i$ rel $V_{i+1}$.

Conclusion: If $P \supset Q$ are closed subpolyhedra of $M^n \ni$ $P \subset U_r$, $Q \subset V_{r+1}$, $\dim Q \le n-3$ and $\dim cl(P-Q) \le r$, then $\forall$ open neighborhood $G$ of $1|M^n$ in $M^n \times M^n$, $\exists$ a PL ambient isotopy $h$ of $M^n$ which fixes $Q \ni h_1(V_0) \supset P$ and the $h$-track of each point of $M^n$ is either a singleton or lies in the $G$-neighborhood of the union of

$\begin{cases} \text{one element of each of } \mathcal{J}_0, \mathcal{J}_1, \ldots, \mathcal{J}_r \text{ if } 0 \le r \le n-4, \text{ and} \\ \text{one element of each of } \mathcal{J}_0, \mathcal{J}_1, \ldots, \mathcal{J}_{r-1} \text{ and two elements of } \mathcal{J}_r \text{ if } r = n-3. \end{cases}$

Moreover, if $cl(P-Q)$ is compact, then $h$ may be chosen to have compact support.

Proof: The proof is by induction on $r$, beginning with $r=-1$, in which case there is nothing to prove. Inductively assume that the theorem is valid if $r$ is replaced by $r-1$. Let the hypothesis of the theorem be given and suppose

$P \supset Q$ are closed subpolyhedra of $M^n \ni P \subset U_r$, $Q \subset V_{r+1}$, $\dim Q \leq n-3$ and $\dim cl(P-Q) \geq r$. Let $G$ be an open neighborhood of $1|M^n$ in $M^n \times M^n$.

Since $r$-complexes in $U_r$ can be pulled into $V_r$ along $\mathcal{T}_r$ rel $V_{r+1}$, then $\exists$ a proper homotopy $\varphi: P \times I \to M^n \ni \varphi(x,t) = x$ for every $(x,t) \in (P \times 0) \cup (Q \times I)$, $\varphi(P \times 1) \subset V_r$ and $\forall x \in P, \exists T \in \mathcal{T}_r \ni \varphi(x \times I) \subset T$. Let $R = cl(P-Q)$, $X = (Q \times 0) \cup (R \times I)$, $Y = (Q \times 0) \cup [(Q \cap R) \times I] \cup (R \times 1)$, and define the proper PL map $f: X \to M^n$ by $f = \varphi | X$. Then $f(x,0) = x$ for every $x \in P$, $f(Y) \subset V_r$. and $\forall x \in R, \exists T \in \mathcal{T}_r \ni f(x \times I) \subset T$. Since $\bigcup \mathcal{T}_r \subset U_{r-1}$, then $f(X) \subset U_{r-1}$. Let $\pi: X \to P$ denote the restriction to $X$ of the projection $P \times I \to P$. Choose $H$ to be an open neighborhood of $1|M^n$ in $M^n \times M^n \ni H \circ H \subset G$, $Hf(X) \subset U_{r-1}$, $Hf(Y) \subset V_r$ and if $C$ is a compact subset of $M^n$, then so is $cl\, H(C)$.

<u>First assume $r \leq n-4$</u>. In this case 4.1 provides a proper PL map $g: X \to M^n$ and a closed subpolyhedron $Z$ of $X \ni g(x,0) = x$ for every $x \in P$, $g \subset H \circ f$, $S(g) \subset Z$, $\pi^{-1}\pi(Z) = Z$ and $\dim Z < r$.

Now $g(Y \cup Z) \supset g(Y)$ are closed subpolyhedra of $M^n \ni g(Y \cup Z) \subset Hf(Y \cup Z) \subset Hf(X) \subset U_{r-1}$, $g(Y) \subset Hf(Y) \subset V_r$,

$\dim g(Y) \le \dim Y \le n-3$, and $\dim cl[g(Y\cup Z)-g(Y)] \le \dim g(Z) \le \dim Z \le r-1$. Hence by inductive hypothesis, $\exists$ a PL ambient isotopy $h^1$ of $M^n$ which fixes $g(Y) \ni h_1^1(V_0) \supset g(Y\cup Z)$ and the $h^1$-track of each point of $M^n$ is either a singleton or lies in the $G$-neighborhood of the union of one element of each of $\mathcal{T}_0, \mathcal{T}_1, \dots, \mathcal{T}_{r-1}$. Moreover, if $R$ is compact, then $cl[g(Y\cup Z)-g(Y)]$ is compact because $cl[g(Y\cup Z)-g(Y)] \subset g(R\times I)$, whence we can assume that $h^1$ has compact support.

Next we invoke 4.2 to obtain a PL ambient isotopy $h^2$ of $M^n$ which fixes $Q \ni h_1^2(h_1^1(V_0)) \supset P$ and the $h^2$-track of each point of $M^n$ is either a singleton or lies in $H\circ H(f(x\times I))$ for some $x\in R$. Since $H\circ H \subset G$ and $\forall x\in R, \exists T\in \mathcal{T}_r \ni f(x\times I)\subset T$, then the $h^2$-track of each point of $M^n$ is either a singleton or lies in the $G$-neighborhood of an element of $\mathcal{T}_r$. Moreover, $h^2$ fixes $M^n - H\circ H(f(R\times I))$, and $H\circ H(f(R\times I)) \subset clH(clH(f(R\times I)))$; so if $R$ is compact, then the choice of $H$ guarantees that $clH(clH(f(R\times I)))$ is compact, which implies that $h^2$ has compact support.

Finally define the PL ambient isotopy $h$ of $M^n$ by

$$h_t = \begin{cases} h^1_{2t} & \text{for } 0\le t\le \frac{1}{2} \\ h^2_{2t-1}\circ h^1_1 & \text{for } \frac{1}{2}\le t\le 1 \end{cases}$$

Since $Q = g(Q \times 0) \subset g(Y)$, then both $h^1$ and $h^2$ fix $Q$; so $h$ fixes $Q$. $h_1(V_0) = h^2_1 h^1_1(V_0) \supset P$. Since the $h$-track of a point $x \in M^n$ is the union of the $h^1$-track of $x$ and the $h^2$-track of $h^1_1(x)$, then the $h$-track of a point of $M^n$ is either a singleton or lies in the $G$-neighborhood of the union of one element of each of $\mathcal{J}_0, \mathcal{J}_1, \ldots, \mathcal{J}_r$. Moreover, if $R = cl(P-Q)$ is compact, then we may assume that both $h^1$ and $h^2$ have compact support, whence $h$ has compact support.

<u>Second assume $r = n-3$.</u> In this case 4.3 provides a proper PL map $g: X \to M^n$ and closed subpolyhedra $Z$ of $X$ and $A_1$ and $A_2$ of $R \ni g(x0) = x$ for every $x \in P$, $g \subset G$of, $\pi^{-1}\pi(Z) = Z$, $\dim Z < r$, $A_1 \cup A_2 = R$, $\pi(Z) \cap R \subset A_1$, $S(g | Y \cup (A_1 \times I)) \subset Z$, and $S(g | Y \cup (A_1 \times 0) \cup (A_2 \times I)) \subset (Y \cap Z) \cup (A_1 \times 0) \cup [(A_1 \cap A_2) \times I]$.

As in the preceding case, the inductive hypothesis provides a PL ambient isotopy $h^1$ of $M^n$ which fixes $g(Y) \ni h^1_1(V_0) \supset g(Y \cup Z)$ and the $h^1$-track of each point of $M^n$ is either a singleton or lies in the $G$-neighborhood of the union of one element of each of $\mathcal{J}_0, \mathcal{J}_1, \ldots, \mathcal{J}_{r-1}$. Moreover, if $R$ is compact, then as in the preceding case, we may assume that $h^1$ has compact support.

Next we invoke 4.4 to obtain a PL ambient isotopy $h^2$ of $M^n$ which fixes $Q \ni h^2_1(h^1_1(V_0)) \supset P$ and the $h^2$-track of each point of $M^n$ is either a singleton or is contained in $H \circ H(f(\{x,y\} \times I))$ for some $x, y \in R$. Since $H \circ H < G$ and $\forall v \in R, \exists T \in \mathcal{T}_r \ni f(v \times I) \subset T$, then the $h^2$-track of each point of $M^n$ is either a singleton or lies in the $G$-neighborhood of two elements of $\mathcal{T}_r$. Moreover, if $R$ is compact, then it follows as in the preceding case, that $h^2$ has compact support.

We define the PL ambient isotopy $h$ of $M^n$ as in the preceding case; and as in the preceding case, $h$ fixes $Q$ and $h_1(V_0) \supset P$. Since the $h$-track of a point $x \in M^n$ is the union of the $h^1$-track of $x$ and the $h^2$-track of $h^1_1(x)$, then the $h$-track of a point of $M^n$ is either a singleton or lies in the $G$-neighborhood of the union of one element of each of $\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_{r-1}$ and two elements of $\mathcal{T}_r$. If $R = cl(P-Q)$ is compact, then as in the preceding case, $h$ can be chosen to have compact support. ∎